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Minghui Zhu
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Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments

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Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments

Springer

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ISSN 2191-8112 ISSN 2191-8120 (electronic)
SpringerBriefs in Electrical and Computer Engineering
ISSN 2192-6786 ISSN 2192-6794 (electronic)
SpringerBriefs in Control, Automation and Robotics
ISBN 978-3-319-19071-6 ISBN 978-3-319-19072-3 (eBook)
DOI 10.1007/978-3-319-19072-3

Library of Congress Control Number: 2015939436

Mathematics Subject Classification: 93-02

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media
(www.springer.com)

To Danying and my parents

Minghui Zhu

To Jorge, Jon Nicolás, and Alexandra

Sonia Martínez

Preface

Smart communication, computing, sensing, and actuation devices are increasingly permeating through our world in an unstoppable manner. These technological advances are fostering the emergence of a variety of large-scale networked systems and applications, including multivehicle networks, the smart grid, smart buildings, medical device networks, intelligent transportation systems, and social networks. It has been an efficient practice to abstract these complex systems as *multi-agent networks*. In particular, each agent in the network represents a strategic entity and is able to communicate, sense, compute, and autonomously react to surrounding changes. The interactions among the agents allow them to solve problems beyond their individual capabilities, resulting in a whole that is certainly more than the sum of its parts.

In order to ensure that the network performs at an optimal level, agents face the problem of choosing the best option among a set of candidates. *Distributed optimization-based control* (DOC, for short) provides a holistic and mathematically rigorous framework to entail network-wide optimal decision making and control. In particular, desired network-wide behavior is encoded as a DOC problem where agents seek for different subobjectives and are required to obey inhomogeneous constraints of physical dynamics and decision choices. This class of problems is characterized by a number of salient features. First, the network consists of a large number of geographically distributed agents. Second, due to information privacy, the agents may not be willing to disclose their own components which define the DOC problem. Third, the agents are expected to self-adapt to internal faults and external changes. Given these features, the top-down frameworks in classic centralized and hierarchical approaches are not well suited for the needs of DOC. It necessitates bottom-up paradigms, i.e, the synthesis of distributed algorithms which allow the agents to coordinate with others via autonomous actions and local interactions resulting into an emerging network-wide behavior that globally optimizes the problem of interest. Bottom-up paradigms are characterized by that the desired global behavior emerges from local actions and interactions.

In a number of engineering applications, agents are required to operate in dynamically changing, uncertain, and hostile environments. Take multivehicle networks as an example. Due to a limited communication bandwidth, underwater vehicles can only exchange information intermittently and thus intervehicle communication topologies frequently change over time. Ground vehicles may be commanded to perform surveillance missions in a region where the environmental information is not provided in advance. In addition, aerial vehicles operate far away from base stations and thus can be compromised by human adversaries who may attack the cyber infrastructures. In order to ensure the high performance and high confidence of multi-agent networks, DOC should explicitly take into account the unforeseeable elements during the algorithm design and performance analysis.

An Outline of the Book

This book aims at a concise and in-depth exposition of specific algorithmic solutions for DOC and their performance analysis. We focus on addressing the particular challenges induced by the environmental complexities: topological dynamics, environmental uncertainties, and cyber adversaries via integrating miscellaneous ideas and tools from Dynamic Systems, Control Theory, Graph Theory, Optimization, Game Theory, and Markov Chains. To achieve this goal, we organize the book in the following way:

Chapter 1 presents a summary of mathematical tools for DOC used in this book. We start with the consensus problem, a canonical problem in multi-agent networks. In particular, we introduce the matrix representation of multi-agent networks as well as the algorithms and convergence results for static and dynamic average consensus. After this, we present a concise introduction to convex optimization and noncooperative game theory. We conclude with a treatment of Markov chains and stochastic stability.

Chapter 2 studies a class of generic distributed convex optimization problems. In particular, each agent is associated with a private objective function and a private convex constraint set. Meanwhile, all the agents are subject to a pair of global inequality and equality constraints. The key feature of the problem is that all the component functions depend upon a global decision variable. The agents aim to agree upon two global quantities: (1) a global minimizer of the sum of all private objective functions, simultaneously enforcing all the given constraints; (2) the induced optimal value.

Chapter 3 investigates a game theoretic solution of an optimal sensor deployment problem. In particular, a set of mobile visual sensors are self-deployed in a geographically extended environment to accomplish a variety of Intelligence, Surveillance and Reconnaissance (ISR) missions, such as environmental monitoring, source seeking, and target assignment. The key feature of the problem is that the environmental distribution function is unknown *a priori* but its values can be measured on site.

Chapter 4 considers attack-resilient distributed formation control of operator-vehicle networks. Through communication infrastructures, human operators remotely control a group of vehicles such that the vehicle team is able to finish the given cooperative mission, e.g., formation achieving. The key feature of the problem is that the communication network is compromised by external cyber attackers who aim to abort the cooperative mission.

The Intended Audience

The intended audience of the book consists of first-year or second-year graduate students in Control, Robotics, Decision Making, Optimization, and Distributed Algorithms from Aerospace Engineering, Computer Science, Electrical Engineering, Mechanical Engineering, and Operations Research. The students are assumed to have a basic background in Mathematical Analysis, Probability Theory, Stochastic Processes, Control Theory, Decision Theory, and Numerical Computation. Yet we hope that the students who do not have a sufficient background can still capture essential ideas. The researchers in Control, Robotics, Decision Making, Optimization, and Distributed Algorithms may also find the book useful as a reference.

Acknowledgments

Over the years, we have been greatly benefiting from the collaboration with a number of persons. We would like to particularly thank Francesco Bullo, Jorge Cortés, Emilio Frazzoli, Jie Huang, Sertac Karaman, Na Li, and fellow professors and students at both UC San Diego and MIT. When he started the academic career at Penn State, the first author got valuable advice and kindly help from many persons, including Kultegin Aydin, Constantino Lagoa, Peng Liu, Ahmad Mirzaei, Shashi Phoha, Padma Raghavan, Asok Ray, David Riley, and Uday Shanbhag. We also would like to thank Miroslav Krstic for encouraging us to write this book. The first author is grateful for support from the Department of Defense. The second author is grateful for support from the National Science Foundation and Department of Defense.

March 2015

Minghui Zhu
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Chapter 1
Preliminaries

In this chapter, we present a set of instrumental results for DOC. In particular, we introduce the so-called consensus problem, which can be considered to be the simplest DOC problem. This will be followed by a concise introduction to convex optimization and noncooperative game theory. At the end of this chapter, we summarize a set of results for Markov chains and their stochastic stability.

1.1 Basic Notations

We introduce a set of basic notations which will be used throughout the book. $\mathbb{R}$ represents the set of real numbers and $\mathbb{Z}_+$ stands for the set of nonnegative integers. $\|\cdot\|$ is the 2-norm in the Euclidean space. We let the function $[\cdot]^+ : \mathbb{R}^s \to \mathbb{R}^s_{\geq 0}$ denote the projection operator onto the nonnegative orthant in $\mathbb{R}^s$. For any vector $c \in \mathbb{R}^r$, we denote $|c| \triangleq (|c_1|, \ldots, |c_r|)^T$. $\mathbf{1}_N$ is the vector in $\mathbb{R}^N$ with all ones.

The affine hull of set S is defined as $\text{aff}(S) \triangleq \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in S,\ \theta_1 + \cdots + \theta_k = 1\}$. The convex hull of set S is defined as $\text{co}(S) \triangleq \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in S,\ \theta_i \geq 0,\ \theta_1 + \cdots + \theta_k = 1\}$. Given a set S, $\text{diag}(S) \triangleq \{(i, j) \in S \times S \mid i \neq j\}$.

1.2 The Consensus Problem

The consensus problem addresses the question of how agents can agree upon a quantity of interest via a distributed algorithm involving local agent computations and interagent communications. One can view the consensus problem as the simplest DOC problem, where the agents aim to minimize their maximum deviation from agreement. The study on consensus is beneficial for understanding how information constraints limit or not the attainment of networkwide objectives. Further, consensus algorithms serve as building blocks of more sophisticated protocols; e.g., distributed

M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*, SpringerBriefs in Control, Automation and Robotics, DOI 10.1007/978-3-319-19072-3_1

estimation, formation control, and task assignment. A special case of particular interest, namely distributed average consensus (or distributed averaging), aims to compute the average of the values generated by different agents.

1.2.1 Algebraic Graph Theory

We review next some basic notions of algebraic graph theory following standard texts as [1, 2]. This will help us formulate the consensus problem and its algorithmic solutions.

A *directed weighted graph* of order N is defined as $\mathscr{G} \triangleq (V, \mathscr{E}, A)$, where V is a finite set of N elements or *nodes*, $\mathscr{E} \subset V \times V \setminus \mathrm{diag}(V)$ is a set of ordered pairs of nodes called the *edge set*, and $A \triangleq [a^i_j] \in \mathbb{R}^{N\times N}$ is the *adjacency matrix* with entries $a^i_j \geq 0$ or weight assigned to the pair $(j, i) \in V \times V \setminus \mathrm{diag}(V)$. Here, $a^i_j > 0$ if and only if $(j, i) \in \mathscr{E}$. The *in-neighbors* of node i are the nodes in the set $\mathscr{N}_i \triangleq \{j \in V \mid (j, i) \in \mathscr{E} \text{ and } j \neq i\}$. Node j is called an *out-neighbor* of node i if $i \in \mathscr{N}_j$. A directed graph $\mathscr{G}$ is said to be *undirected* if $(i, j) \in \mathscr{E}$ implies $(j, i) \in \mathscr{E}$. When clear from the context, we will use the notation $\mathscr{G} = (V, \mathscr{E})$ for a directed weighted graph and refer to it as a *digraph*. A *path* of $\mathscr{G}$ is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is an edge of the graph. A directed weighted graph $\mathscr{G}$ is said to be *strongly connected* if, for any pair of $i, j \in V$, there is a path connecting i and j. For undirected graphs, strong connectivity is simply referred to as *connectivity*. Figure 1.1 is an illustrative example of a directed weighted graph which is also strongly connected. Here, an edge (j, i) is represented by an arrow from j to i. For undirected graphs, a pair of arrows between two nodes can be replaced by a simple link. A directed *tree* is a digraph where any two nodes are connected by exactly one path. A *spanning tree*

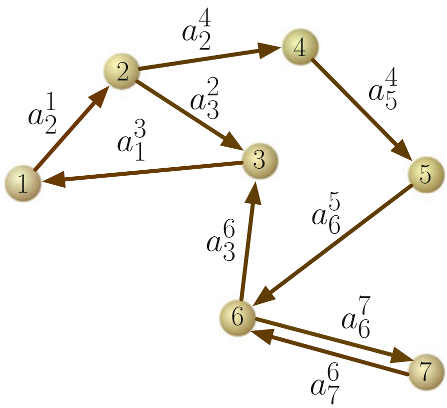

Fig. 1.1 An illustrative example of a directed weighted graph which is strongly connected

of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph.

1.2.2 Network Model

We consider a set of agents labeled by $i \in V \triangleq \{1, \ldots, N\}$. Agent interactions at time $k \geq 0$ are modeled by means of a directed weighted graph $\mathscr{G}(k) \triangleq (V, \mathscr{E}(k), A(k))$. Therefore, these interactions can potentially change over time. In the following, we make the following assumptions on the network graphs or interaction topologies:

Assumption 1.1 (*Nondegeneracy*) There exists a constant $\alpha > 0$ such that $a^i_i(k) \geq \alpha$, and $a^i_j(k)$, for $i \neq j$, satisfies $a^i_j(k) \in \{0\} \cup [\alpha, \ 1]$, for all $k \geq 0$.

Assumption 1.2 (*Double stochasticity*) It holds that $\sum_{j \in V} a^i_j(k) = 1$ for all $i \in V$ and $k \geq 0$, and $\sum_{i \in V} a^i_j(k) = 1$ for all $j \in V$ and $k \geq 0$.

Assumption 1.3 (*Periodic strong connectivity*) There is an integer $B > 0$ such that, for all $k_0 \geq 0$, the directed graph $(V, \bigcup_{k=0}^{B-1} \mathscr{E}(k_0 + k))$ is strongly connected where $\bigcup_{k=0}^{B-1} \mathscr{E}(k_0 + k) \triangleq \mathscr{E}(k_0) \cup \mathscr{E}(k_0 + 1) \cdots \cup \mathscr{E}(k_0 + k)$.

Intuitively speaking, the weight $a^i_j(k)$ is a measure of the influence exerted by agent j onto the computation of agent i at time k. With this, the nondegeneracy Assumption 1.1 indicates that the influence of any neighbor onto agent i is nontrivial. The double stochasticity Assumption 1.2 then means that the total influence of agent i's in-neighbors is identical to that of agent i's out-neighbors. This assumption is necessary for achieving the average consensus; otherwise, the constant sum property (1.2) does not hold. The periodic strong connectivity Assumption 1.3 essentially means that any agent i can influence any other agent $j \neq i$ nontrivially in a finite time.

1.2.3 The Static Average Consensus Problem

We consider here the *static average consensus problem* where agents aim to converge to the average of their initial states $x^i(0) \in \mathbb{R}$. The classic DISTRIBUTED AVERAGING ALGORITHM solves this problem employing a memoryless diffusion process: at each time instant, each agent receives the current estimates from neighbors and updates its own estimate by a value in the convex hull of all of them. More precisely, the update rule for agent i is given by:

$$x^i(k+1) = \sum_{j \in V} a^i_j(k) x^j(k), \qquad k \geq 0, \tag{1.1}$$

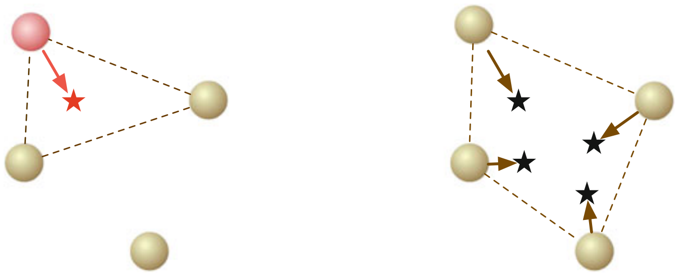

Fig. 1.2 Illustration of one iteration of the DISTRIBUTED AVERAGING ALGORITHM. On the *left*, the agent with the *red* estimate computes a value (marked with a *star*) in the convex hull of neighbors' estimates. On the *right*, the set of updates (marked with *stars*) for all agents are in the convex hull of their initial values

see Fig. 1.2. Here, we would like to provide an intuition of why DISTRIBUTED AVERAGING ALGORITHM (1.1) converges asymptotically. Notice that a property of the algorithm is that it maintains the sum of initial values constant all the time; i.e., the following holds for all $k \geq 0$:

$$\sum_{i \in V} x^i(k) = \sum_{i \in V} x^i(0), \tag{1.2}$$

and the maximum deviation diminishes; i.e.,

$$\lim_{k \to +\infty} D(k) = 0, \tag{1.3}$$

where $D(k) \triangleq M(k) - m(k)$ with $M(k) \triangleq \max_{i \in V} x^i(k)$ and $m(k) \triangleq \min_{i \in V} x^i(k)$. Property (1.2) trivially holds under the double stochasticity Assumption 1.2. The combination of the nondegeneracy Assumption 1.1 and the periodic strong connectivity Assumption 1.3 ensures that at any time instant k, the value of agent i with $x^i(k) = M(k)$ can reach agent j with $x^j(k) = m(k)$ within next $(N-1)B$ iterations. The nondegeneracy Assumption 1.1 further ensures that $D(k + (N-1)B)$ is smaller than $D(k)$ by a constant factor and thus $D(k)$ decreases at an exponential rate. The convergence result is formally stated in the following theorem:

Theorem 1.1 *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2 and the periodic strong connectivity Assumption 1.3 hold. Then, the* DISTRIBUTED AVERAGING ALGORITHM *converges asymptotically as characterized by* $\lim_{k \to +\infty} \|x^i(k) - \frac{1}{N} \sum_{j \in V} x^j(0)\| = 0$, *for all* $i \in V$.

Remark 1.1 In other words, the previous theorem guarantees that the average consensus value is reached. If this requirement is relaxed; i.e., the consensus value does not have to be necessarily the average of the initial states, then property (1.2) is not necessary and the double stochasticity Assumption 1.2 in Theorem 1.1 can be weakened into the following row stochasticity:

Assumption 1.4 (*Row stochasticity*) It holds that $\sum_{j\in V} a^i_j(k) = 1$ for all $i \in V$ and $k \geq 0$.

•

Next, let us examine the convergence rate of the DISTRIBUTED AVERAGING ALGORITHM. Consider the disagreement function $V(x) \triangleq \sum_{i\in V}(x^i - \frac{1}{N}\sum_{j\in V} x^j)^2$. The following theorem establishes that the convergence time of the DISTRIBUTED AVERAGING ALGORITHM is of order $(\frac{N^2}{\alpha})B$.

Theorem 1.2 ([3]) *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. Then, there is a constant $c > 0$ such that for any $\varepsilon > 0$, the following holds for the* DISTRIBUTED AVERAGING ALGORITHM:

$$V(x(k)) \leq \varepsilon V(x(0)), \quad \forall k \geq c\left(\frac{N^2}{\alpha}\right) B \log\left(\frac{1}{\varepsilon}\right).$$

In particular, if one chooses $a^i_j(k) = \frac{1}{|\mathscr{N}_i(k)|}$, then $\alpha \geq \frac{1}{N}$ and the convergence time in Theorem 1.2 is of order N^3B. If the network topology is fixed; i.e., $A(k) = A$ for all $k \geq 0$, the convergence rate can be alternatively characterized in terms of the eigenvalues of A. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the set of eigenvalues of A, sorted by nonincreasing magnitudes. The double stochasticity Assumption 1.2 implies that $\lambda_1 = 1$ with associated eigenvector $\mathbf{1}_N$. As [4, 5], we define a convergence rate for DISTRIBUTED AVERAGING ALGORITHM (1.1) as follows:

$$\rho \triangleq \sup_{x(0)\notin \operatorname{diag}\mathbb{R}^N} \lim_{k\to+\infty} \left(\frac{\|x(k) - \frac{1}{N}\sum_{j\in V} x^j(0)\mathbf{1}_N\|}{\|x(0) - \frac{1}{N}\sum_{j\in V} x^j(0)\mathbf{1}_N\|}\right)^{\frac{1}{k}}.$$

The following theorem identifies the relation between the convergence rate of DISTRIBUTED AVERAGING ALGORITHM and the *essential spectral radius* of A: $\rho_{\text{ess}}(A) \triangleq \max\{|\lambda_2|, |\lambda_N|\}$.

Theorem 1.3 ([4, 5]) *It holds that $\rho_{\text{ess}}(A) < 1$.*

1.2.4 The Dynamic Average Consensus Problem

A variety of missions require autonomous agents to operate in dynamic environments where each agent is influenced by local time-varying signals. A fundamental problem is how the agents distribute their real-time data over the network and further keep track of the average of individually measured time-varying signals. This problem is referred to as the *dynamic average consensus problem* in opposition to the static average consensus problem of the previous section.

To define a possible solution, consider that each agent synchronously measures a local continuous signal $r_i : \mathbb{R} \to \mathbb{R}$ at every integer multiple $k = th$ of a time unit $h > 0$. By induction, we define the nth-order difference of $r_i(k)$, for $n \in \mathbb{N}$, as follows. First, $\Delta^{(1)} r_i(k) \equiv \Delta r_i(k) \triangleq r_i(k) - r_i(k-h)$, for $k \geq 0$. Then,

$$\Delta^{(n)} r_i(k) = \Delta^{(n-1)} r_i(k) - \Delta^{(n-1)} r_i(k-h), \qquad n \geq 2, \quad i \in V.$$

In addition, we will denote $\Delta^{(n)} r_{\max}(k) = \max_{i \in V} \Delta^{(n)} r_i(k)$ and $\Delta^{(n)} r_{\min}(k) = \min_{i \in V} \Delta^{(n)} r_i(k)$ for $n \geq 2$. The following is the nth-order DISTRIBUTED DYNAMIC AVERAGING ALGORITHM:

$$\begin{aligned} x_i^{[\ell]}(k+h) &= \sum_{j \in V} a_j^i(k) x_j^{[\ell]}(k) + x_i^{[\ell-1]}(k+h), \qquad \ell \in \{2, \ldots, n\}, \\ x_i^{[1]}(k+h) &= \sum_{j \in V} a_j^i(k) x_j^{[1]}(k) + \Delta^{(n)} r_i(k). \end{aligned} \tag{1.4}$$

The DISTRIBUTED DYNAMIC AVERAGING ALGORITHM can be viewed as a cascade of n-layer first-order DISTRIBUTED AVERAGING ALGORITHM, where each layer is subject to some external inputs and ℓ is the layer index. In particular, for any $\ell \in \{2, \ldots, n\}$, the update rule $\sum_{j \in V} a_j^i(k) x_j^{[\ell]}(k) + x_i^{[\ell-1]}(k+h)$ at the ℓth layer can be viewed as the DISTRIBUTED AVERAGING ALGORITHM $\sum_{j \in V} a_j^i(k) x_j^{[\ell]}(k)$ subject to external inputs $x_i^{[\ell-1]}(k+h)$ which are the states of the $(\ell-1)$th layer. When $\ell = 1$ and $\Delta r_i(k) = 0$, then the DISTRIBUTED DYNAMIC AVERAGING ALGORITHM reduces to the DISTRIBUTED AVERAGING ALGORITHM. The double stochasticity Assumption 1.2 implies that the sums of time-varying signals are maintained at the highest layer; i.e.,

$$\sum_{i \in V} x_i^{[n]}(k+h) = \sum_{i \in V} r_i(k). \tag{1.5}$$

On the other hand, the disagreement among $x_i^{[\ell]}$ for $i \in V$ can be viewed as a disturbance to the consensus operation on $x_i^{[\ell+1]}$. The deviations among $\Delta^{(n)} r_i(k)$ for $i \in V$ can be viewed as external inputs to the consensus operation on $x_i^{[1]}$. By the arguments of input-to-state stability; e.g., in [6–8], one can show that dynamic average consensus is asymptotically achieved if $\lim_{k \to +\infty} \Delta^{(n)} r_i(k) = 0$.

Theorem 1.4 *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. For all* $i \in V$, *choose initial states* $x_i^{[\ell]}(0) = \Delta^{(n-\ell)} r_i(-h)$ *for* $\ell \in$

$\{1, \ldots, n-1\}$ *and* $x_i^{[n]}(0) = r_i(-h)$. *If* $\lim_{k\to+\infty} \Delta^{(n)} r_i(k) = 0$, *then the nth-order* DISTRIBUTED DYNAMIC AVERAGING ALGORITHM *converges as characterized by* $\lim_{k\to+\infty} \|x_i^{[\ell]}(k) - x_j^{[\ell]}(k)\| = 0$, *for all* $i, j \in V$ *and* $\ell \in \{1, \ldots, n\}$.

Remark 1.2 Theorem 1.4 can be readily extended to the case where $r_i(k)$ is a vector by running the nth-order DISTRIBUTED DYNAMIC AVERAGING ALGORITHM for each dimension. •

1.3 Convex Optimization

This section provides a concise exposition of convex optimization. Our presentation mainly follows the books [9–12].

1.3.1 Convex Analysis

A set $X \in \mathbb{R}^n$ is called *convex* if $\alpha x + (1-\alpha)y \in X$, $\forall x, y \in X$ and $\forall \alpha \in [0, 1]$. That is, a set is convex if the line segment connecting any pair of two points in the set belongs to the set. Examples of convex sets include lines, hyperplanes, cones, and convex hulls.

A function $f : X \to \mathbb{R}$ defined over a convex set X is *convex* if the following holds for all $x, y \in X$ and $\alpha \in [0, 1]$:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{1.6}$$

That is, a function f is convex if the line segment connecting any pair of x and y is above the portion of f between x and y. Any convex function is continuous. A convex function $f : X \to \mathbb{R}$ is called *strictly convex* if inequality (1.6) is strict for $\forall x, y \in X$ with $x \neq y$ and $\forall \alpha \in (0, 1)$. The function f is *concave* if $-f$ is convex. Examples of convex functions include quadratic function and exponential function. An affine function is both convex and concave. Readers are referred to [11] for the ways to check the convexity of sets and functions.

Given nonempty, convex, and closed set $X \in \mathbb{R}^n$, the *projection operator* onto X, $P_X : \mathbb{R}^n \to X$, is defined as $P_X[z] = \mathrm{argmin}_{x\in X} \|x - z\|$. Although P_X is well-defined for any nonempty, convex, and closed set X, the projection $P_X[z]$ is easy to compute only for limited cases; e.g., X is a box or Euclidean ball. The following lemma shows that the projection $P_X[z]$ is closer than z to any point y in X.

Lemma 1.1 (Nonexpansiveness property of projection operators) *Let* X *be a nonempty, closed, and convex set in* $\mathbb{R}^n$. *For any* $z \in \mathbb{R}^n$, *the following holds for any* $y \in Z$: $\|P_X[z] - y\|^2 \leq \|z - y\|^2 - \|P_X[z] - z\|^2$.

1.3.2 Constrained Optimization

Let us consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{s.t.} \quad x \in X,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the *objective function* and $X \subseteq \mathbb{R}^n$ is a *constraint set*. A vector x^* is referred to as a *minimizer* or *global minimum* of f over X if $f(x^*) = \inf_{x \in X} f(x)$. Correspondingly, $f(x^*)$ is referred to as the *optimal value*. The following Weierstrass' Theorem provides a sufficient condition such that the global minimum is achievable.

Theorem 1.5 (Weierstrass' Theorem for continuous functions) *If f is continuous and X is compact, then there is at least one global minimum of f over X.*

A vector x^* is a *local minimum* of f over X if $x^* \in X$ and there is some $\varepsilon > 0$ such that $f(x^*) \leq f(x), \forall x \in X$ with $\|x - x^*\| \leq \varepsilon$. In what follows, we will focus on *convex optimization problems*; i.e., problems for which f and X are convex. The convexity of f and X implies that all local minima are also global.

Proposition 1.1 *If X is a convex set and f is a convex function, then any local minimum of f over X is also a global minimum. If f is strictly convex, there is at most one global minimum of f over X.*

1.3.3 Duality Theory

Consider the following constrained optimization problem:

$$\min_{x \in X} f(x), \quad \text{s.t.} \quad g(x) \leq 0, \tag{1.7}$$

where $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^r$ are all convex. We refer to (1.7) as the *primal problem* and f^* as the *primal optimal value* with $f^* = \inf_{x \in X, g(x) \leq 0} f(x)$.

By duality, the constraint function g can be taken into account by augmenting the objective function f with a weighted combination of f and g. We define the *Lagrangian function* as $\mathscr{L} : X \times \mathbb{R}^r \to \mathbb{R}$ such that $\mathscr{L}(x, \mu) \triangleq f(x) + \mu^T g(x)$, and we refer to μ as the *Lagrange multiplier vector* or *Lagrange multiplier*. The *dual function* $q : \mathbb{R}^r \to \mathbb{R}$ is defined as follows:

$$q(\mu) = \begin{cases} \inf_{x \in X} \mathscr{L}(x, \mu), & \mu \geq 0, \\ -\infty, & \text{otherwise}. \end{cases}$$

The dual function can be used to obtain a lower bound on the primal optimal value, as it holds that $q(\mu) \leq \mathscr{L}(x, \mu)$ for all $x \in X$ and $\mu \in \mathbb{R}^r$ and it can be seen that

$q(\mu) \leq f^*$. The best lower bound is obtained by solving the following *dual problem*:

$$\max_{\mu \geq 0} q(\mu), \tag{1.8}$$

with optimal value q^* referred to as the *dual optimal value*. The dual problem is a convex optimization problem consisting of a concave q and a convex constraint set. In particular, we have $q^* \leq f^*$, which is known as the *weak duality* relation. The following proposition provides necessary and sufficient conditions for *strong duality*; i.e., $q^* = f^*$.

Theorem 1.6 (Optimality conditions) *Consider problem* (1.7). *There holds that* $f^* = q^*$ *and* (x^*, μ^*) *are a pair of primal and dual solutions for* (1.7) *and* (1.8) *if and only if*

- $x^* \in X$;
- $\mu^* \geq 0$;
- $x^* \in \operatorname{argmin}_{x \in X} \mathscr{L}(x, \mu^*)$;
- $\mu^*_\ell g_\ell(x^*) = 0$, *for all* $\ell \in \{1, \ldots, r\}$.

In Theorem 1.6, the first and second conditions ensure the feasibility of the primal and dual optimal solutions, respectively. The third condition indicates that the primal optimal solution x^* is a global minimizer of the Lagrangian function $\mathscr{L}$ given the dual optimal solution μ^*. The fourth condition is referred to as the *complementary slackness*; i.e., for any inactive constraint $g_\ell(x^*) < 0$, the corresponding Lagrange multiplier μ^*_ℓ is equal to zero.

The notion of saddle point plays a key role in Lagrangian duality theory. Consider a function $\phi : X \times M \to \mathbb{R}$ where X and M are nonempty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A pair of vectors $(x^*, \mu^*) \in X \times M$ is called a *saddle point* of ϕ over $X \times M$ if $\phi(x^*, \mu) \leq \phi(x^*, \mu^*) \leq \phi(x, \mu^*)$ holds for all $(x, \mu) \in X \times M$.

Remark 1.3 Equivalently, (x^*, μ^*) is a saddle point of ϕ over $X \times M$ if and only if $(x^*, \mu^*) \in X \times M$, and $\sup_{\mu \in M} \phi(x^*, \mu) \leq \phi(x^*, \mu^*) \leq \inf_{x \in X} \phi(x, \mu^*)$. •

The primal and Lagrangian dual optimal solutions can be characterized as the saddle points of the Lagrangian function.

Theorem 1.7 (Lagrangian Saddle point Theorem) *The pair of* (x^*, μ^*) *is a saddle point of the Lagrangian function* $\mathscr{L}$ *over* $X \times \mathbb{R}^m_{\geq 0}$ *if and only if it is a pair of primal and Lagrangian dual optimal solutions and the following* Lagrangian minimax equality *holds:*

$$\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \inf_{x \in X} \mathscr{L}(x, \mu) = \inf_{x \in X} \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathscr{L}(x, \mu).$$

1.4 Noncooperative Game Theory

In this section, we will provide a brief presentation on game theory. The readers can find a throughout introduction in [13–17].

Game theory can be divided into two main categories: noncooperative game theory [13] and cooperative game theory [18]. Noncooperative game theory analyzes the interactions of a number of strategic decision-makers; i.e., players, where the players may have partially or totally conflicting interests and their decisions jointly affect the outcome of a decision process. Cooperative game theory studies situations where players can cooperate to create value by forming coalitions, but also compete to capture value.

Noncooperative games can be further grouped into static games and dynamic games. In static games, the order of players' decisions does not matter. In contrast, the notion of time has a central role in dynamic games. Noncooperative games can also be categorized into discrete and continuous games. In discrete games, each player has a finite number of actions. Continuous games allow players to choose an action from a continuous set. In the remainder of the book, we restrict our attention to static discrete games.

A *static discrete noncooperative game* $\Gamma \triangleq \langle V, \mathscr{A}, U \rangle$ has three components:

1. A *player set* V enumerating players $i \in V \triangleq \{1, \ldots, N\}$.
2. A finite *action set* $\mathscr{A} \triangleq \prod_{i \in V} \mathscr{A}_i$, which is the space of all action vectors, where $s_i \in \mathscr{A}_i$ is the action of player i and a (multiplayer) action $s \in \mathscr{A}$ has components $s_1, \ldots, s_N$.
3. A collection of *utility functions* $U = \{u_i : \mathscr{A} \to \mathbb{R}\}$, where the utility function u_i models player i's preferences over action profiles.

Denote by s_{-i} the action profile of all players other than i, and by $\mathscr{A}_{-i} \triangleq \prod_{j \neq i} \mathscr{A}_j$ the set of action profiles for all players except i. The concept of (pure) Nash equilibrium is the most important one in noncooperative game theory [15] and is defined as follows.

Consider the static discrete noncooperative game Γ. An action profile $s^* \triangleq (s_i^*, s_{-i}^*)$ is a *(pure) Nash equilibrium* (NE) of the game Γ if $\forall i \in V$ and $\forall s_i \in \mathscr{A}_i$, it holds that $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$.

An action profile corresponding to a Nash equilibrium represents a scenario where no player can benefit from unilateral deviations. It is known that Nash equilibrium exists for potential games [19] and, more generally, weakly acyclic games [20]. In particular, in weakly acyclic games, for every action profile, there exists a better-response improvement path leading from that action profile to a Nash equilibrium. However, the existence of Nash equilibrium may not be true for a generic game. In noncooperative games, players seek to maximize their own interest instead of the social interest. Hence, a Nash equilibrium may not be identical to a social optimum. The price of anarchy and price of stability concepts characterize the efficiency loss of Nash equilibria [16, 21]. In particular, we can define a measure of efficiency of each outcome which we call *welfare function* $W : \mathscr{A} \to \mathbb{R}$. The price of anarchy is then defined as the ratio between the worst Nash equilibrium and the social optimum:

$$\mathrm{PoA} = \frac{\min_{a \in E} W(a)}{\max_{a \in \mathscr{A}} W(a)},$$

where $E \subseteq \mathscr{A}$ is the set of Nash equilibria. The price of stability is defined as the ratio between the best Nash equilibrium and the social optimum:

$$\mathrm{PoS} = \frac{\max_{a \in E} W(a)}{\max_{a \in \mathscr{A}} W(a)}.$$

So it is clear that $\mathrm{PoA} \leq \mathrm{PoS} \leq 1$.

Remark 1.4 In this book, we restrict our attention to pure Nash equilibria. For two-player matrix games, mixed Nash equilibrium always exists where each player can choose probability distributions over its own action space [13]. For continuous games, Nash equilibrium exists for supermodular games [22] and convex games [23]. In particular, supermodular games are those characterized by strategic complementarities where, if one player increases action, the others want to do the same. For convex games, Nash equilibrium exists if the utility function of each player is convex in its own action and the action set of each player is compact. •

1.4.1 Potential Games

Potential games form an important class of strategic games. The static noncooperative game Γ is an *ordinal potential game* if there exists a potential function $\phi : \mathscr{A} \to \mathbb{R}$ such that for every $i \in V$, $s_{-i} \in \mathscr{A}_{-i}$, and $s_i, s_i' \in \mathscr{A}_i$, it holds that

$$\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) > 0 \Leftrightarrow u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0. \tag{1.9}$$

For ordinal potential games, the change in a player's utility caused by a unilateral deviation is aligned with that of a potential function. Exact potential games are a special class of ordinal potential games where the change in a player's utility caused by a unilateral deviation can be exactly captured by a potential function. The static noncooperative game Γ is an *exact potential game* if there exists a potential function $\phi : \mathscr{A} \to \mathbb{R}$ if for every $i \in V$, for every $s_{-i} \in \mathscr{A}_{-i}$, and for every $s_i, s_i' \in \mathscr{A}_i$, it holds that

$$\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}). \tag{1.10}$$

Congestion games are a special class of potential games [19]. In a congestion game, the utility of each player depends on the resources it chooses and the number of players choosing the same resource.

1.4.2 Constrained Games

In conventional noncooperative game theory, all the actions in $\mathscr{A}_i$ can always be selected by player i in response to other players' actions. However, in the context of motion coordination, the actions available to player i can often be constrained to a state-dependent subset of $\mathscr{A}_i$. For example, collision avoidance type of constraints can be encoded this way. In particular, we denote by $F_i(s_i, s_{-i}) \subseteq \mathscr{A}_i$ the set of feasible actions of player i when the action profile is $s \triangleq (s_i, s_{-i})$. We assume that $s_i \in F_i(s_i, s_{-i})$ for any $s \in \mathscr{A}$ throughout this book. Denote $F(s) \triangleq \prod_{i \in V} F_i(s) \subseteq \mathscr{A}$, $\forall s \in \mathscr{A}$ and $F \triangleq \cup\{F(s) \mid s \in \mathscr{A}\}$. The introduction of F leads naturally to the notion of *constrained static noncooperative game*, $\Gamma_{\text{cons}} \triangleq \langle V, \mathscr{A}, U, F \rangle$, and the following associated concepts. Consider the constrained static noncooperative game Γ_{cons}. An action profile s^* is a *constrained (pure) Nash equilibrium* of the game Γ_{cons} if $\forall i \in V$ and $\forall s_i \in F_i(s_i^*, s_{-i}^*)$, it holds that $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$. The game Γ_{cons} is a constrained exact potential game with potential function $\phi(s)$ if for every $i \in V$, every $s_{-i} \in \mathscr{A}_{-i}$, and every $s_i \in \mathscr{A}_i$, the equality (1.10) holds for every $s_i' \in F_i(s_i, s_{-i})$.

With the assumption of $s_i \in F_i(s_i, s_{-i})$ for any $s \in \mathscr{A}$, we observe that if s^* is a Nash equilibrium of the static noncooperative game Γ, then it is also a constrained Nash equilibrium of the constrained static noncooperative game Γ_{cons}. Hence, any constrained exact potential game with the assumption of $s_i \in F_i(s_i, s_{-i})$ for any $s \in \mathscr{A}$ has at least one constrained Nash equilibrium.

1.5 Markov Chains

In this section, we provide a brief summary of results in the literature of Markov chains [24] and their stochastic stability [20]. We refer the readers to [24] for a comprehensive exposition to Markov chains.

A *discrete-time Markov chain* is a discrete-time stochastic process on a finite (or countable) state space which satisfies the *Markov property*; i.e., the future state depends on its present state, but not the past states. A discrete-time Markov chain is said to be *time-homogeneous* if the probability of going from one state to another is independent of the time when the step is taken. Otherwise, the Markov chain is said to be *time-inhomogeneous*. Since time-inhomogeneous Markov chains include time-homogeneous ones as special cases, we restrict our attention to the former in the remainder of this section.

The evolution of a time-inhomogeneous Markov chain $\{\mathscr{P}_k\}$ can be described by a transition matrix $P(k)$ which encodes the probability of traversing from one state to another at each time k. Consider a Markov chain $\{\mathscr{P}_k\}$ with time-dependent transition matrix $P(k)$ on a finite state space X. Denote by $P(m, n) \triangleq \prod_{t=m}^{n-1} P(k)$, for all $0 \leq m < n$.

The Markov chain $\{\mathscr{P}_k\}$ is *strongly ergodic* if there exists a stochastic vector μ^* such that for any distribution μ on X and any $m \in \mathbb{Z}_+$, it holds that $\lim_{k\to+\infty} \mu^T P(m,k) = (\mu^*)^T$. Strong ergodicity of $\{\mathscr{P}_k\}$ is equivalent to $\{\mathscr{P}_k\}$ being convergent in distribution and can be employed to characterize the long-run properties of Markov chains. The investigation of conditions under which strong ergodicity holds is aided by the following concepts.

Given a matrix M, M_{ij} is the element at the ith row and the jth column. For any $n \times n$ stochastic matrix P, its *coefficient of ergodicity* is defined by $\lambda(P) \triangleq 1 - \min_{1\le \ell,\ell' \le n} \sum_{\tau=1}^{n} \min(P_{\ell\tau}, P_{\ell'\tau})$. The Markov chain $\{\mathscr{P}_k\}$ is *weakly ergodic* if $\forall x, y, z \in X$, $\forall m \in \mathbb{Z}_+$, it holds that $\lim_{k\to+\infty} (P_{xz}(m,k) - P_{yz}(m,k)) = 0$. Weak ergodicity merely implies that $\{\mathscr{P}_k\}$ asymptotically forgets its initial state, but it does not guarantee convergence. For a time-homogeneous Markov chain, there is no distinction between weak ergodicity and strong ergodicity. The following theorem provides the sufficient and necessary condition for $\{\mathscr{P}_k\}$ to be weakly ergodic.

Theorem 1.8 *The Markov chain $\{\mathscr{P}_k\}$ is weakly ergodic if and only if there is a strictly increasing sequence of positive numbers k_ℓ, $\ell \in \mathbb{Z}_+$ such that it holds that* $\sum_{i=0}^{+\infty}(1 - \lambda(P(k_\ell, k_{\ell+1}))) = +\infty$.

We are now ready to present the sufficient conditions for strong ergodicity of the Markov chain $\{\mathscr{P}_k\}$.

Theorem 1.9 *A Markov chain $\{\mathscr{P}_k\}$ is strongly ergodic if the following conditions hold:*

(C1) The Markov chain $\{\mathscr{P}_k\}$ is weakly ergodic.

(C2) For each k, there exists a stochastic vector μ^k on X such that μ^k is the left eigenvector of the transition matrix $P(k)$ with eigenvalue 1.

(C3) The eigenvectors μ^k in (C2) satisfy $\sum_{k=0}^{+\infty}\sum_{z\in X} \|\mu_z^k - \mu_z^{k+1}\| < +\infty$.

Moreover, if $\mu^* = \lim_{k\to+\infty} \mu^k$, then μ^* is the vector in the definition of strong ergodicity.

1.5.1 Stochastic Stability

Let P^0 be the transition matrix of the time-homogeneous Markov chain $\{\mathscr{P}_k^0\}$ on a finite state space X. Furthermore, let P^ε be the transition matrix of a *perturbed Markov chain*, say $\{\mathscr{P}_k^\varepsilon\}$. With probability $1-\varepsilon$, the process $\{\mathscr{P}_k^\varepsilon\}$ evolves according to P^0, while with probability ε, the transitions do not follow P^0.

A family of stochastic processes $\{\mathscr{P}_k^\varepsilon\}$ is called a *regular perturbation* of $\{\mathscr{P}_k^0\}$ if the following holds $\forall x, y \in X$:

(A1) For some $\varsigma > 0$, the Markov chain $\{\mathscr{P}_k^\varepsilon\}$ is irreducible and aperiodic for all $\varepsilon \in (0, \varsigma]$.

(A2) $\lim_{\varepsilon \to 0^+} P_{xy}^\varepsilon = P_{xy}^0$.

(A3) If $P_{xy}^\varepsilon > 0$ for some ε, then there exists a real number $\chi(x \to y) \geq 0$ such that $\lim_{\varepsilon \to 0^+} \frac{P_{xy}^\varepsilon}{\varepsilon^{\chi(x \to y)}} \in (0, +\infty)$.

In (A3), $\chi(x \to y)$ is called the *resistance of the transition* from x to y.

(A1) ensures that for $\varepsilon \in (0, \varsigma]$, there is a unique stationary distribution of $\{\mathscr{P}_k^\varepsilon\}$; i.e., $\mu(\varepsilon)^T P^\varepsilon = \mu(\varepsilon)^T$.

Let $H_1, H_2, \ldots, H_J$ be the recurrent communication classes of the Markov chain $\{\mathscr{P}_k^0\}$. Note that within each class H_ℓ, there is a path of zero resistance from every state to every other. Given any two distinct recurrent communication classes H_ℓ and H_s, consider all paths which start from H_ℓ and end at H_s. Denote by $\chi_{\ell s}$ the least resistance among all such paths.

Now define a complete directed graph $\mathscr{G}$ where there is one vertex ℓ for each recurrent class H_ℓ, and the resistance on the edge (ℓ, s) is $\chi_{\ell s}$. An *ℓ-tree* on $\mathscr{G}$ is a spanning tree such that from every vertex $s \neq \ell$, there is a unique path from s to ℓ. Denote by $G(\ell)$ the set of all *ℓ-trees* on $\mathscr{G}$. The *resistance* of an *ℓ-tree* is the sum of the resistances of its edges. The *stochastic potential* of the recurrent class H_ℓ is the least resistance among all *ℓ-trees* in $G(\ell)$.

Theorem 1.10 *Let $\{\mathscr{P}_k^\varepsilon\}$ be a regular perturbation of $\{\mathscr{P}_k^0\}$. Then the limit $\lim_{\varepsilon \to 0^+} \mu(\varepsilon)$ exists and the limit distribution $\mu(0)$ is a stationary distribution of $\{\mathscr{P}_k^0\}$. The stochastically stable states; i.e., the support of $\mu(0)$, are precisely those states contained in the recurrence classes with minimum stochastic potential.*

Markov chain $\{\mathscr{P}_k^\varepsilon\}$ is obtained by perturbing $\{\mathscr{P}_k^0\}$ and the perturbation magnitude is characterized by ε. Informally speaking, Theorem 1.10 says that, as perturbations diminish, the sequence of stationary probability distributions of perturbed Markov chains converge and the support of the limit distribution is contained in the recurrence classes of $\{\mathscr{P}_k^0\}$ with minimum stochastic potential. It means that the states of minimum stochastic potential of $\{\mathscr{P}_k^0\}$ can be used to predict the behavior of limit distributions of perturbed Markov chains.

1.6 Notes

Consensus roots in Computer Science and plays a fundamental role in parallel and distributed computation [25]. The first consensus algorithm was proposed in [26]. Recently, the emergence of multi-agent networks has attracted researchers from various engineering and scientific disciplines, yielding substantial generalizations of the basic consensus algorithm. It is hard to provide a complete literature review given the vast volume of papers devoted to consensus problems. Here we only list a set of representative papers concerned with different aspects of consensus. The readers are referred to the survey papers [27, 28], the monographs [4, 29, 30] and the special issues [31–34] for more comprehensive literature review. In particular, the papers [35, 36] study continuous-time consensus algorithms, and the papers [37–39] instead focus on discrete-time consensus algorithms. In [40], the authors discuss the asynchronous implementation of consensus algorithms. The convergence rates of consensus algorithms are characterized in [3, 5, 41, 42]. Gossip-based consensus algorithms are investigated in [43, 44]. The paper [45] treats the problem of reaching the consensus state when it is constrained in some given convex set and the paper [46] studies manifold consensus. The papers [47, 48] address how to achieve consensus within a finite time, while the algorithm in [49] allows agents to construct a balanced graph out of a non-balanced one in order to implement consensus algorithms. A number of issues on network unreliability have been addressed, including quantization [50–52], transmission delays [53], event-triggered scheduling [54], and limited data rate [55]. As with static average consensus, one can find continuous-time and discrete-time solutions to the dynamic average consensus problem. In continuous time, the early work of [56] provides a first type of algorithm which is analyzed via frequency domain techniques, and which tracks ramp signals. The approach presented in [57] considers a common reference input for all nodes in the network. A PI dynamic consensus algorithm in continuous time is presented in [58], which is extended in [59] to track a wider class of time-varying signals. An interesting property of the algorithms of [58, 59] is their robustness to initialization errors, which allows algorithm adaptation to sporadically departing agents. The paper [60] proposes a discontinuous control algorithm capable of tracking bounded signals with bounded derivatives. Finally, [61] has introduced continuous-time algorithms to solve a type of dynamic consensus algorithms for multiple vehicles under limited control authority. Discrete-time approaches are more appealing as they can usually handle larger discretization steps. These algorithms include the one presented in this chapter, its extension to fast dynamic consensus [62], and their robustification to errors initialization [63]. Finally, an event-triggered dynamic consensus algorithm is introduced in [64].

New algorithms have significantly extended the application scope of consensus. Some interesting examples include, to name a few, attitude alignment [65, 66], clock synchronization [67], coverage control [68], opinion formation [69], oscillator synchronization [70–73], parameter estimation [74], social learning [75], task assignment [76–78], multi-vehicle formation control [30], biochemical networks [79], ocean sampling [80, 81] and microgrid control [82]. The applications of dynamic average consensus include multi-robot coordination [83], sensor fusion [56, 84, 85], distributed spatial estimation [86, 87], feature-based map merging [88], and distributed tracking [89].

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Chapter 2
Distributed Cooperative Optimization

2.1 Introduction

In this chapter, we consider a general multi-agent optimization problem where the goal is to minimize a global objective function, given as a sum of local objective functions, subject to global constraints, which include an inequality constraint, an equality constraint, and a (state) constraint set. Each local objective function is convex and only known to one particular agent. On the other hand, the inequality (resp. equality) constraint is given by a convex (resp. affine) function and known to all agents. Each node has its own convex constraint set, and the global constraint set is defined as their intersection. This problem is motivated by others in distributed estimation [1, 2], distributed source localization [3], network utility maximization [4], optimal flow control in power systems [5, 6], and optimal shape changes of mobile robots [7]. An important feature of the problem is that the objective and (or) constraint functions depend upon a global decision vector. This requires the design of distributed algorithms where, on one hand, agents can align their decisions through a local information exchange and, on the other hand, the common decisions will coincide with an optimal solution and the optimal value.

More precisely, we study two cases: one in which the equality constraint is absent, and the other in which the local constraint sets are identical. For the first case, we adopt a Lagrangian relaxation approach, define a Lagrangian dual problem and devise the DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM based on the characterization of the primal-dual optimal solutions as the saddle points of the Lagrangian function. The DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM involves each agent updating its estimates of the saddle points via a combination of an average consensus step, a subgradient (or supgradient) step, and a primal (or dual) projection step onto its local constraint set (or a compact set containing the dual optimal set). The DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM is shown to asymptotically converge to a pair of primal-dual optimal solutions under Slater's condition and the periodic strong connectivity

M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*, SpringerBriefs in Control, Automation and Robotics, DOI 10.1007/978-3-319-19072-3_2

assumption. Furthermore, each agent asymptotically agrees on the optimal value by implementing a DISTRIBUTED DYNAMIC AVERAGING ALGORITHM (1.4), which allows a multi-agent system to track time-varying average values.

For the second case, to dispense with the additional equality constraint, we adopt a penalty relaxation approach, while defining a penalty dual problem and devising the DISTRIBUTED PENALTY PRIMAL- DUAL SUBGRADIENT ALGORITHM. Unlike the first case, the dual optimal set of the second case may not be bounded, and thus the dual projection steps are not involved in the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM. It renders that dual estimates and thus (primal) subgradients may not be uniformly bounded. This challenge is addressed by a more careful choice of step-sizes. We show that the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM asymptotically converges to a primal optimal solution and the optimal value under Slater's condition and the periodic strong connectivity assumption.

2.2 Problem Formulation

Consider a network of agents labeled by $V \triangleq \{1, \ldots, N\}$ that can only interact with each other through local communication.

[Objective] We aim to synthesize distributed algorithms which allow the multi-agent group to cooperatively solve the following optimization problem (Fig. 2.1):

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \quad \text{s.t. } g(x) \leq 0, \quad h(x) = 0, \quad x \in \cap_{i=1}^{N} X_i, \tag{2.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is the convex objective function of agent i, $X_i \subseteq \mathbb{R}^n$ is the compact and private convex constraint set of agent i, and x is a global decision vector.

Here we assume that the projection onto the set X_i is easy to compute. Assume that f_i and X_i are private information of agent i, and probably different across agents. The function $g : \mathbb{R}^n \to \mathbb{R}^m$ is known to all the agents with each component g_ℓ, for $\ell \in \{1, \ldots, m\}$, being convex. The inequality $g(x) \leq 0$ is understood component wise; i.e., $g_\ell(x) \leq 0$, for all $\ell \in \{1, \ldots, m\}$, and represents a global inequality constraint. The function $h : \mathbb{R}^n \to \mathbb{R}^\nu$, defined as $h(x) \triangleq Ax - b$ with $A \in \mathbb{R}^{\nu \times n}$, represents a global equality constraint, and is known to all the agents. We denote $X \triangleq \cap_{i=1}^{N} X_i$, $f(x) \triangleq \sum_{i=1}^{N} f_i(x)$, and $Y \triangleq \{x \in \mathbb{R}^n \mid g(x) \leq 0, \quad h(x) = 0\}$. We assume that the set of feasible points is nonempty; i.e., $X \cap Y \neq \emptyset$. Since X is compact and Y is closed, then we can deduce that $X \cap Y$ is compact. The convexity of f_i implies that of f and thus f is continuous. In this way, the optimal value p^* of the problem (2.1) is finite and X^*, the set of primal optimal points, is nonempty. Throughout this chapter, we suppose the following Slater's condition holds:

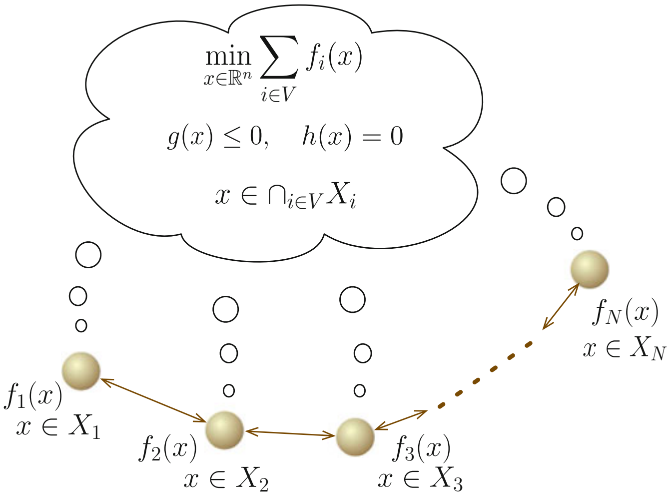

Fig. 2.1 A graphical illustration of problem (2.1)

Assumption 2.1 (*Slater's Condition*) There exists a vector $\bar{x} \in X$ such that $g(\bar{x}) < 0$ and $h(\bar{x}) = 0$. And there exists a relative interior point $\tilde{x}$ of X such that $h(\tilde{x}) = 0$ where $\tilde{x}$ is a relative interior point of X; i.e., $\tilde{x} \in X$ and there exists an open sphere S centered at $\tilde{x}$ such that $S \cap \mathrm{aff}(X) \subset X$ with $\mathrm{aff}(X)$ being the affine hull of X.

In this chapter, we will study two particular cases of Problem (2.1): one in which the global equality constraint $h(x) = 0$ is not included, and the other in which all the local constraint sets are identical. For the case where the constraint $h(x) = 0$ is absent, the Slater's condition 2.1 reduces to the existence of a vector $\bar{x} \in X$ such that $g(\bar{x}) < 0$. Our techniques rely on duality theory in Sect. 1.3.

2.2.1 Subgradient Notions and Notations

In this chapter, we do not assume the differentiability of the problem functions. At the points where functions are not differentiable, the subgradient plays the role of the gradient. For a given convex function $F : \mathbb{R}^r \to \mathbb{R}$ and a point $x \in \mathbb{R}^r$, a *subgradient* of the function F at x is a vector $\mathscr{D}F(x) \in \mathbb{R}^r$ such that the following subgradient inequality holds for any $y \in \mathbb{R}^r$: $\mathscr{D}F(x)^T(y - x) \le F(y) - F(x)$. Similarly, for a given concave function $G : \mathbb{R}^s \to \mathbb{R}$ and a point $\mu \in \mathbb{R}^s$, a *supgradient* of the function G at μ is a vector $\mathscr{D}G(\mu) \in \mathbb{R}^s$ such that the following supgradient inequality holds for any $\lambda \in \mathbb{R}^s$: $\mathscr{D}G(\mu)^T(\lambda - \mu) \ge G(\lambda) - G(\mu)$.

2.3 Case (i): Absence of Equality Constraint

In this section, we study the case of problem (2.1) where the equality constraint $h(x) = 0$ is absent; i.e.,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \quad \text{s.t.} \quad g(x) \leq 0, \quad x \in \cap_{i=1}^{N} X_i. \tag{2.2}$$

In the following, we first provide some preliminary results, including a Lagrangian saddle point characterization of the problem (2.2) and a superset containing the Lagrangian dual optimal set of the problem (2.2). After this, the DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM will be presented along with a summary of its convergence properties.

Overall Strategy and Lagrangian Saddle Point Characterization

First, the problem (2.2) is equivalent to

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{s.t.} \quad N g(x) \leq 0, \quad x \in X,$$

with associated Lagrangian dual problem given by

$$\max_{\mu \in \mathbb{R}^m} q_L(\mu), \quad \text{s.t.} \quad \mu \geq 0.$$

Here, the function $q_L : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$, is defined as $q_L(\mu) \triangleq \inf_{x \in X} \mathscr{L}(x, \mu)$, where $\mathscr{L} : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is the Lagrangian $\mathscr{L}(x, \mu) = f(x) + N\mu^T g(x)$. We denote the Lagrangian dual optimal value of the Lagrangian dual problem by d_L^* and the set of Lagrangian dual optimal points by D_L^*. As is well-known, under the Slater's condition 2.1, the property of strong duality holds; i.e., $p^* = d_L^*$, and $D_L^* \neq \emptyset$.

As explained in Theorem 1.7, saddle points of the Lagrangian correspond to min-max solutions of the primal and dual problems. Assume for simplicity that the Lagrangian is differentiable and there are no other constraints than the ones included already in the Lagrangian. Then, one can define an associated saddle point dynamics (gradient descent in one argument and gradient ascent in the other) as follows. Let $\mathscr{L}_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the function $\mathscr{L}_\mu(x) := \mathscr{L}(x, \mu)$, for μ fixed, and $\mathscr{L}_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ be $\mathscr{L}_x(\mu) := \mathscr{L}(x, \mu)$, for x fixed. Then, the continuous-time saddle point dynamics is given as:

$$\begin{aligned} \dot{x}(t) &= -\nabla \mathscr{L}_{\mu(t)}(x(t), \mu(t)), \\ \dot{\mu}(t) &= \nabla \mathscr{L}_{x(t)}(x(t), \mu(t)). \end{aligned} \tag{2.3}$$

If $\mathscr{L}_{\mu(t)}$ is convex and $\mathscr{L}_{x(t)}$ is concave, these dynamics converge to a saddle point of the Lagrangian from any initial and see [8]. The discrete-time counterpart can be generalized for nondifferentiable Lagrangians and to include additional projections over $x \in X$ [9].

We would like to use a distributed discrete-time version of (2.3) for the multi-agent system by defining related and separated problems for each agent, which then are globally coordinated through a consensus algorithm. To do this, note that, while $\mathcal{L}_x$ is naturally separable as a sum of factors $f_i(x) + g(x)$ for each agent, the dual function q_L is not. Then, our strategy will be to define certain sets M_i for each agent that contain the dual solution set, and, which used with a projection operation on M_i, can converge to a saddle point and a min-max solution.

This following lemma presents some preliminary analysis of saddle points toward this objective.

Lemma 2.1 (Preliminary results on saddle points) *Let M be any superset of D_L^*.*

(a) If (x^*, μ^*) is a saddle point of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$, then (x^*, μ^*) is also a saddle point of $\mathcal{L}$ over $X \times M$.
(b) There is at least one saddle point of $\mathcal{L}$ over $X \times M$.
(c) If $(\check{x}, \check{\mu})$ is a saddle point of $\mathcal{L}$ over $X \times M$, then $\mathcal{L}(\check{x}, \check{\mu}) = p^*$ and $\check{\mu}$ is Lagrangian dual optimal.

Proof (a) It just follows from the definition of saddle point of $\mathcal{L}$ over $X \times M$.
(b) Observe that

$$\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \inf_{x \in X} \mathcal{L}(x, \mu) = \sup_{\mu \in \mathbb{R}^m_{\geq 0}} q_L(\mu) = d_L^*, \quad \inf_{x \in X} \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x, \mu) = \inf_{x \in X \cap Y} f(x) = p^*.$$

Since the Slater's condition 2.1 implies zero duality gap, the Lagrangian minimax equality holds. From Theorem 1.7 it follows that the set of saddle points of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$ is the Cartesian product $X^* \times D_L^*$. Recall that X^* and D_L^* are nonempty, so we can guarantee the existence of the saddle point of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$. Then by (a), we have that (b) holds.
(c) Pick any saddle point (x^*, μ^*) of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$. Since the Slater's condition 2.1 holds, from Theorem 1.7 one can deduce that (x^*, μ^*) is a pair of primal and Lagrangian dual optimal solutions. This implies that

$$d_L^* = \inf_{x \in X} \mathcal{L}(x, \mu^*) \leq \mathcal{L}(x^*, \mu^*) \leq \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*, \mu) = p^*.$$

From Theorem 1.7, we have $d_L^* = p^*$. Hence, $\mathcal{L}(x^*, \mu^*) = p^*$. On the other hand, we pick any saddle point $(\check{x}, \check{\mu})$ of $\mathcal{L}$ over $X \times M$. Then for all $x \in X$ and $\mu \in M$, it holds that $\mathcal{L}(\check{x}, \mu) \leq \mathcal{L}(\check{x}, \check{\mu}) \leq \mathcal{L}(x, \check{\mu})$. By Theorem 1.7, then $\mu^* \in D_L^* \subseteq M$. Recall $x^* \in X$, and thus we have $\mathcal{L}(\check{x}, \mu^*) \leq \mathcal{L}(\check{x}, \check{\mu}) \leq \mathcal{L}(x^*, \check{\mu})$. Since $\check{x} \in X$ and $\check{\mu} \in \mathbb{R}^m_{\geq 0}$, we have $\mathcal{L}(x^*, \check{\mu}) \leq \mathcal{L}(x^*, \mu^*) \leq \mathcal{L}(\check{x}, \mu^*)$. Combining the above two relations gives that $\mathcal{L}(\check{x}, \check{\mu}) = \mathcal{L}(x^*, \mu^*) = p^*$. From Remark 1.3 we see that $\mathcal{L}(\check{x}, \check{\mu}) \leq \inf_{x \in X} \mathcal{L}(x, \check{\mu}) = q_L(\check{\mu})$. Since $\mathcal{L}(\check{x}, \check{\mu}) = p^* = d_L^* \geq q_L(\check{\mu})$, then $q_L(\check{\mu}) = d_L^*$ and thus $\check{\mu}$ is a Lagrangian dual optimal solution. ∎

Remark 2.1 Despite that (c) holds, the reverse of (a) may not be true in general. In particular, x^* may be infeasible; i.e., $g_\ell(x^*) > 0$ for some $\ell \in \{1, \ldots, m\}$. •

An Upper Estimate of the Lagrangian Dual Optimal Set

In what follows, we will find a compact superset of D_L^*. To do that, we define the following primal problem for each agent i:

$$\min_{x \in \mathbb{R}^n} f_i(x), \quad \text{s.t.} \quad g(x) \leq 0, \quad x \in X_i.$$

Due to the fact that X_i is compact and the f_i are continuous, the primal optimal value p_i^* of each agent's primal problem is finite and the set of its primal optimal solutions is nonempty. The associated dual problem is given by

$$\max_{\mu \in \mathbb{R}^m} q_i(\mu), \quad \text{s.t.} \quad \mu \geq 0.$$

Here, the dual function $q_i : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is defined by $q_i(\mu) \triangleq \inf_{x \in X_i} \mathscr{L}_i(x, \mu)$, where $\mathscr{L}_i : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is the Lagrangian function of agent i and given by $\mathscr{L}_i(x, \mu) = f_i(x) + \mu^T g(x)$. The corresponding dual optimal value is denoted by d_i^*. In this way, $\mathscr{L}$ is decomposed into a sum of local Lagrangian functions; i.e., $\mathscr{L}(x, \mu) = \sum_{i=1}^N \mathscr{L}_i(x, \mu)$.

Define now the set-valued map $Q : \mathbb{R}^m_{\geq 0} \to 2^{(\mathbb{R}^m_{\geq 0})}$ by $Q(\tilde{\mu}) = \{\mu \in \mathbb{R}^m_{\geq 0} \mid q_L(\mu) \geq q_L(\tilde{\mu})\}$. Additionally, define a function $\gamma : X \to \mathbb{R}$ by $\gamma(x) = \min_{\ell \in \{1,\ldots,m\}}\{-g_\ell(x)\}$. Observe that if x is a Slater-vector, then $\gamma(x) > 0$. The following lemma is a direct result of Lemma 1 in [10].

Lemma 2.2 (Boundedness of dual solution sets) *The set $Q(\tilde{\mu})$ is bounded for any $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$, and, in particular, for any Slater-vector $\bar{x}$, it holds that* $\max_{\mu \in Q(\tilde{\mu})} \|\mu\| \leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - q_L(\tilde{\mu}))$. □

Notice that $D_L^* = \{\mu \in \mathbb{R}^m_{\geq 0} \mid q_L(\mu) \geq d_L^*\}$. Picking any Slater-vector $\bar{x} \in X$, and letting $\tilde{\mu} = \mu^* \in D_L^*$ in Lemma 2.2 gives that

$$\max_{\mu^* \in D_L^*} \|\mu^*\| \leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - d_L^*). \tag{2.4}$$

Define the function $r : X \times \mathbb{R}^m_{\geq 0} \to \mathbb{R} \cup \{+\infty\}$ by $r(x, \mu) \triangleq \frac{N}{\gamma(x)} \max_{i \in V}\{f_i(x) - q_i(\mu)\}$. By the property of weak duality, it holds that $d_i^* \leq p_i^*$ and thus $f_i(x) \geq q_i(\mu)$ for any $(x, \mu) \in X \times \mathbb{R}^m_{\geq 0}$. Since $\gamma(\bar{x}) > 0$, $r(\bar{x}, \mu) \geq 0$ for any $\mu \in \mathbb{R}^m_{\geq 0}$. With this observation, we pick any $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$ and the following set is well-defined: $\bar{M}_i(\bar{x}, \tilde{\mu}) \triangleq \{\mu \in \mathbb{R}^m_{\geq 0} \mid \|\mu\| \leq r(\bar{x}, \tilde{\mu}) + \theta_i\}$ for some $\theta_i \in \mathbb{R}_{>0}$. Observe that for all $\mu \in \mathbb{R}^m_{\geq 0}$:

$$q_L(\mu) = \inf_{x \in \cap_{i=1}^m X_i} \sum_{i=1}^N (f_i(x) + \mu^T g(x)) \geq \sum_{i=1}^N \inf_{x \in X_i} (f_i(x) + \mu^T g(x)) = \sum_{i=1}^N q_i(\mu). \tag{2.5}$$

Since $d_L^* \geq q_L(\tilde{\mu})$, it follows from (2.4) and (2.5) that

$$\begin{aligned} \max_{\mu^* \in D_L^*} \|\mu^*\| &\leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - q_L(\tilde{\mu})) \leq \frac{1}{\gamma(\bar{x})}\left(f(\bar{x}) - \sum_{i=1}^N q_i(\tilde{\mu})\right) \\ &\leq \frac{N}{\gamma(\bar{x})} \max_{i \in V}\{f_i(\bar{x}) - q_i(\tilde{\mu})\} = r(\bar{x}, \tilde{\mu}). \end{aligned}$$

Hence, we have $D_L^* \subseteq \bar{M}_i(\bar{x}, \tilde{\mu})$ for all $i \in V$.

Note that in order to compute $\bar{M}_i(\bar{x}, \tilde{\mu})$, all the agents have to agree on a common Slater-vector $\bar{x} \in \cap_{i=1}^N X_i$ which should be obtained in a distributed fashion. To handle this difficulty, we now propose a distributed algorithm, namely DISTRIBUTED SLATER- VECTOR COMPUTATION ALGORITHM, which allows each agent i to compute a superset of $\bar{M}_i(\bar{x}, \tilde{\mu})$.

Initially, each agent i chooses a common value $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$; e.g., $\tilde{\mu} = 0$, and computes two positive constants $b_i(0)$ and $c_i(0)$ such that $b_i(0) \geq \sup_{x \in J_i}\{f_i(x) - q_i(\tilde{\mu})\}$ and $c_i(0) \leq \min_{1 \leq \ell \leq m} \inf_{x \in J_i}\{-g_\ell(x)\}$ where $J_i \triangleq \{x \in X_i \mid g(x) < 0\}$.

At every time $k \geq 0$, each agent i updates its estimates by:

$$b_i(k+1) = \max_{j \in \mathcal{N}_i(k) \cup \{i\}} b_j(k), \quad c_i(k+1) = \min_{j \in \mathcal{N}_i(k) \cup \{i\}} c_j(k).$$

We denote $b^* \triangleq \max_{j \in V} b_j(0)$, $c^* \triangleq \min_{j \in V} c_j(0)$ for all $k \geq (N-1)B$, and $M^{[i]}(\tilde{\mu}) \triangleq \{\mu \in \mathbb{R}^m_{\geq 0} \mid \|\mu\| \leq \frac{Nb^*}{c^*} + \theta_i\}$, $J \triangleq \{x \in X \mid g(x) < 0\}$.

Lemma 2.3 (Convergence of the DISTRIBUTED SLATER- VECTOR COMPUTATION ALGORITHM) *Assume that the periodical strong connectivity Assumption 1.3 holds. Consider the sequences of $\{b_i(k)\}$ and $\{c_i(k)\}$ generated by the* DISTRIBUTED SLATER- VECTOR COMPUTATION ALGORITHM. *It holds that after at most $(N-1)B$ steps, all the agents reach the consensus, i.e., $b_i(k) = b^*$ and $c_i(k) = c^*$ for all $k \geq (N-1)B$. Furthermore, we have $M^{[i]}(\tilde{\mu}) \supseteq \bar{M}_i(\bar{x}, \tilde{\mu})$ for $i \in V$.*

Proof It is not difficult to verify achieving max-consensus by the periodical strong connectivity Assumption 1.3. Note that $J \subseteq J_i$, $\forall i \in V$. Hence, we have

$$\max_{i \in V} \sup_{x \in J}\{f_i(x) - q_i(\tilde{\mu})\} \leq \max_{i \in V} \sup_{x \in J_i}\{f_i(x) - q_i(\tilde{\mu})\} \leq b^*,$$

$$\inf_{x \in J} \min_{1 \leq \ell \leq m}\{-g_\ell(x)\} \geq \min_{i \in V} \inf_{x \in J_i} \min_{1 \leq \ell \leq m}\{-g_\ell(x)\} \geq c^*.$$

Since $\bar{x} \in J$, then the following estimate on $r(\bar{x}, \tilde{\mu})$ holds:

$$r(\bar{x}, \tilde{\mu}) \leq \frac{N \sup_{x \in J} \max_{i \in V}\{f_i(x) - q_i(\tilde{\mu})\}}{\inf_{x \in J} \min_{1 \leq \ell \leq m}\{-g_\ell(x)\}} \leq \frac{Nb^*}{c^*}.$$

The desired result immediately follows. ∎

From Lemma 2.3 and the fact that $D_L^* \subseteq \bar{M}_i(\bar{x}, \tilde{\mu})$, we can see that the set of $M(\tilde{\mu}) \triangleq \cap_{i=1}^N M^{[i]}(\tilde{\mu})$ contains D_L^*. In addition, $M^{[i]}(\tilde{\mu})$ and $M(\tilde{\mu})$ are nonempty, compact, and convex. To simplify the notations, we will use the shorthands $M_i \triangleq M^{[i]}(\tilde{\mu})$ and $M \triangleq M(\tilde{\mu})$.

Convexity of the Lagrangian Function

For each $\mu \in \mathbb{R}^m_{\geq 0}$, we define the function $\mathscr{L}_{i\mu} : \mathbb{R}^n \to \mathbb{R}$ as $\mathscr{L}_{i\mu}(x) := \mathscr{L}_i(x, \mu)$. Note that $\mathscr{L}_{i\mu}$ is convex since it is a nonnegative weighted sum of convex functions. For each $x \in \mathbb{R}^n$, we define the function $\mathscr{L}_{ix} : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ as $\mathscr{L}_{ix}(\mu) := \mathscr{L}_i(x, \mu)$. It is easy to check that $\mathscr{L}_{ix}$ is a concave (actually affine) function. Then the Lagrangian function $\mathscr{L}$ is the sum of a collection of convex–concave local functions.

2.3.1 *The* DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM

Here, we introduce the DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM to find a saddle point of the Lagrangian function $\mathscr{L}$ over $X \times M$ and the optimal value. This saddle point will coincide with a pair of primal and Lagrangian dual optimal solutions which is not always the case; see Remark 2.1.

Through the algorithm, at each time k, each agent i maintains the estimate of $(x^i(k), \mu^i(k))$ to the saddle point of the Lagrangian function $\mathscr{L}$ over $X \times M$ and the estimate of $y^i(k)$ to p^*. To produce $x^i(k+1)$ (resp. $\mu^i(k+1)$), agent i takes a convex combination $v_x^i(k)$ (resp. $v_\mu^i(k)$) of its estimate $x^i(k)$ (resp. $\mu^i(k)$) with the estimates sent from its neighboring agents at time k, makes a subgradient (resp. supgradient) step to minimize (resp. maximize) the local Lagrangian function $\mathscr{L}_i$, and takes a primal (resp. dual) projection onto the local constraint X_i (resp. M_i). Furthermore, agent i generates the estimate $y^i(k+1)$ by taking a convex combination $v_y^i(k)$ of its estimate $y^i(k)$ with the estimates of its neighbors at time k and taking one step to track the variation of the local objective function f_i. More precisely, the DISTRIBUTED LAGRANGIAN PRIMAL- DUAL SUBGRADIENT ALGORITHM is described as follows:

Initially, each agent i picks a common $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$ and computes the set M_i with some $\theta_i > 0$ by using the Distributed Slater-vector Computation Algorithm. Agent i then chooses any initial state $x^i(0) \in X_i$, $\mu^i(0) \in \mathbb{R}^m_{\geq 0}$, and $y^i(1) = Nf_i(x^i(0))$.

At every $k \geq 0$, each agent i generates $x^i(k+1)$, $\mu^i(k+1)$ and $y^i(k+1)$ according to the following rules:

$$v_x^i(k) = \sum_{j=1}^{N} a_j^i(k)x^j(k), \quad v_\mu^i(k) = \sum_{j=1}^{N} a_j^i(k)\mu^j(k), \quad v_y^i(k) = \sum_{j=1}^{N} a_j^i(k)y^j(k),$$
$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathscr{D}_x^i(k)], \quad \mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathscr{D}_\mu^i(k)],$$
$$y^i(k+1) = v_y^i(k) + N(f_i(x^i(k)) - f_i(x^i(k-1))),$$

where P_{X_i} (resp. P_{M_i}) is the projection operator onto the set X_i (resp. M_i), the scalars $a_j^i(k)$ are nonnegative weights and the scalars $\alpha(k) > 0$ are step-sizes.[1] We use the shorthands $\mathscr{D}_x^i(k) \equiv \mathscr{D}\mathscr{L}_{iv_\mu^i(k)}(v_x^i(k))$, and $\mathscr{D}_\mu^i(k) \equiv \mathscr{D}\mathscr{L}_{iv_x^i(k)}(v_\mu^i(k))$.

The following theorem summarizes the convergence properties of the DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM where it is guaranteed that agents asymptotically agree upon a pair of primal-dual optimal solutions.

Theorem 2.1 (Convergence of the DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM) *Consider the optimization problem* (2.2). *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. Consider the sequences of* $\{x^i(k)\}$, $\{\mu^i(k)\}$ *and* $\{y^i(k)\}$ *of the* DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM *with the step-sizes* $\{\alpha(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$, $\sum_{k=0}^{+\infty} \alpha(k) = +\infty$, *and* $\sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$. *Then, there is a pair of primal and Lagrangian dual optimal solutions* $(x^*, \mu^*) \in X^* \times D_L^*$ *such that* $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ *and* $\lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0$, *for all* $i \in V$. *Furthermore,* $\lim_{k\to+\infty} \|y^i(k) - p^*\| = 0$, *for all* $i \in V$.

2.3.2 *A Numerical Example for the* DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM

In order to illustrate the performance of the DISTRIBUTED LAGRANGIAN PRIMAL-DUAL SUBGRADIENT ALGORITHM, we here study a numerical example of network utility maximization, e.g., in [4]. Consider five agents and one link where each agent sends data through the link at a rate of z_i, and the link capacity is 5. The global decision vector $x := (z_1 \ z_2 \ z_3 \ z_4 \ z_5)^T \in \mathbb{R}^5$ is the resource allocation vector. Each agent i is associated with a concave utility function $f_i(z_i) := \sqrt{z_i}$, representing the utility

[1] Each agent i executes the update law of $y^i(k)$ for $k \geq 1$.

agent i obtained through sending data at a rate of z_i. Agents aim to maximize the network utility and it can be formulated as follows:

$$\min_{x \in \mathbb{R}^5} \sum_{i \in V} -\sqrt{z_i} \quad \text{s.t.} \quad z_1 + z_2 + z_3 + z_4 + z_5 \leq 5, \quad x \in \cap_{i \in V} X_i, \tag{2.6}$$

where local constraint sets X_i are given by:

$$X_1 := [0.5, \ \ 5.5] \times [0.5, \ \ 5.5] \times [0.5, \ \ 5.5] \times [0.5, \ \ 5.5] \times [0.5, \ \ 5.5],$$
$$X_2 := [0.55, \ \ 5.25] \times [0.55, \ \ 5.25] \times [0.55, \ \ 5.25] \times [0.55, \ \ 5.25] \times [0.55, \ \ 5.25],$$
$$X_3 := [0.5, \ \ 6] \times [0.5, \ \ 6] \times [0.5, \ \ 6] \times [0.5, \ \ 6] \times [0.5, \ \ 6],$$
$$X_4 := [0.5, \ \ 5] \times [0.5, \ \ 5] \times [0.5, \ \ 5] \times [0.5, \ \ 5] \times [0.5, \ \ 5],$$
$$X_5 := [0.525, \ \ 5.75] \times [0.525, \ \ 5.75] \times [0.525, \ \ 5.75] \times [0.525, \ \ 5.75] \times [0.525, \ \ 5.75].$$

One can verify that the optimal solution is $[1 \ \ 1 \ \ 1 \ \ 1 \ \ 1]^T$. We use the Distributed Lagrangian Primal-Dual Subgradient Algorithm to solve problem (2.6) by choosing step-size $\alpha(k) = \frac{1}{k+1}$. Figures 2.2 and 2.3 show the simulation results of the Distributed Lagrangian Primal-Dual Subgradient Algorithm, demonstrating that the agents take 10^4 iterates to agree upon value 1 for z_1 and z_2.

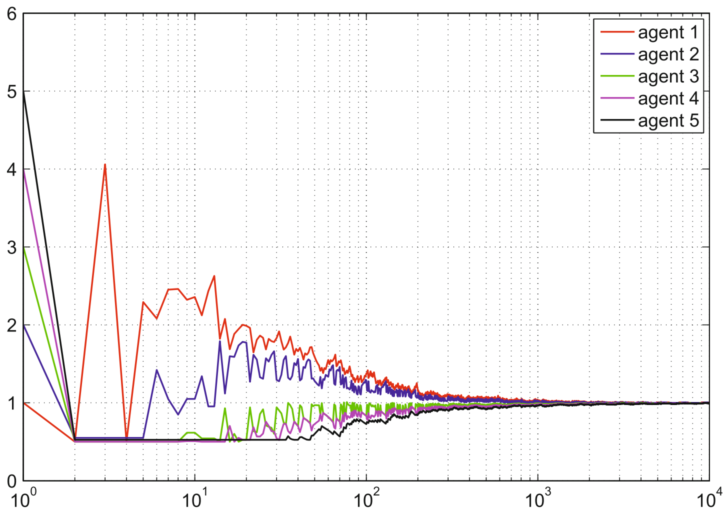

Fig. 2.2 The estimates on z_1 generated by different agents in the Distributed Lagrangian Primal-Dual Subgradient Algorithm (DLPDS)

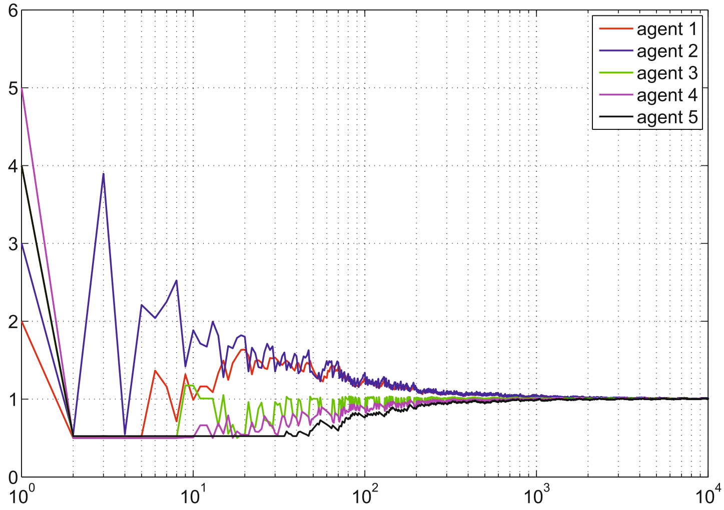

Fig. 2.3 The estimates on z_2 generated by different agents in the Distributed Lagrangian Primal-Dual Subgradient Algorithm (DLPDS)

2.4 Case (ii): Identical Local Constraint Sets

In the previous section, we study the case where the equality constraint is absent in problem (2.1). Here, we turn our attention to another case of problem (2.1), where $h(x) = 0$ is taken into account but we require that local constraint sets are identical; i.e., $X_i = X$ for all $i \in V$. We first adopt a penalty relaxation formulation and provide a penalty saddle point characterization of the primal problem (2.1) with $X_i = X$. We then introduce the Distributed Penalty Primal-Dual Subgradient Algorithm, followed by its convergence properties.

Overall Strategy and a Penalty Saddle Point Characterization

As in the previous section, our strategy will be to define an appropriate dynamics to converge to a saddle point or a min-max solution of the problem. However, to deal with the equality constraint, we will employ a penalty function, which includes positive terms penalizing the violation of the equality and inequality constraints. The identical local constraint sets will also help in guaranteeing the convergence of the method. More precisely, consider the following.

The primal problem (2.1) with $X_i = X$ is trivially equivalent to the following:

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{s.t.} \quad Ng(x) \leq 0, \quad Nh(x) = 0, \quad x \in X, \tag{2.7}$$

with associated penalty dual problem given by

$$\max_{\mu \in \mathbb{R}^m, \lambda \in \mathbb{R}^\nu} q_P(\mu, \lambda), \quad \text{s.t.} \quad \mu \geq 0, \quad \lambda \geq 0. \tag{2.8}$$

Here, the penalty dual function, $q_P : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$, is defined by

$$q_P(\mu, \lambda) \triangleq \inf_{x \in X} \mathscr{H}(x, \mu, \lambda),$$

where $\mathscr{H} : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ is the *penalty function* given by $\mathscr{H}(x, \mu, \lambda) = f(x) + N\mu^T[g(x)]^+ + N\lambda^T|h(x)|$. We denote the penalty dual optimal value by d_P^* and the set of penalty dual optimal solutions by D_P^*. We define the penalty function $\mathscr{H}_i(x, \mu, \lambda) : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ for each agent i as follows: $\mathscr{H}_i(x, \mu, \lambda) = f_i(x) + \mu^T[g(x)]^+ + \lambda^T|h(x)|$. In this way, we have that $\mathscr{H}(x, \mu, \lambda) = \sum_{i=1}^N \mathscr{H}_i(x, \mu, \lambda)$. As proven in the next lemma, the Slater's condition 2.1 ensures zero duality gap and the existence of penalty dual optimal solutions.

Lemma 2.4 (Strong duality and nonemptiness of the penalty dual optimal set) *The values of p^* and d_P^* coincide, and D_P^* is nonempty.*

Proof Consider the auxiliary Lagrangian function $\mathscr{L}_a : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu \to \mathbb{R}$ given by $\mathscr{L}_a(x, \mu, \lambda) = f(x) + N\mu^T g(x) + N\lambda^T h(x)$, with the associated dual problem defined by

$$\max_{\mu \in \mathbb{R}^m, \lambda \in \mathbb{R}^\nu} q_a(\mu, \lambda), \quad \text{s.t.} \quad \mu \geq 0. \tag{2.9}$$

Here, the dual function, $q_a : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu \to \mathbb{R}$, is defined by

$$q_a(\mu, \lambda) \triangleq \inf_{x \in X} \mathscr{L}_a(x, \mu, \lambda).$$

The dual optimal value of problem (2.9) is denoted by d_a^* and the set of dual optimal solutions is denoted D_a^*. Since X is convex, f and g_ℓ, for $\ell \in \{1, \ldots, m\}$, are convex, p^* is finite and the Slater's condition 2.1 holds, it follows from Proposition 5.3.5 in [11] that $p^* = d_a^*$ and $D_a^* \neq \emptyset$. We now proceed to characterize d_P^* and D_P^*. Pick any $(\mu^*, \lambda^*) \in D_a^*$. Since $\mu^* \geq 0$, then

$$\begin{aligned} d_a^* = q_a(\mu^*, \lambda^*) &= \inf_{x \in X}\{f(x) + N(\mu^*)^T g(x) + N(\lambda^*)^T h(x)\} \\ &\leq \inf_{x \in X}\{f(x) + N(\mu^*)^T[g(x)]^+ + N|\lambda^*|^T|h(x)|\} = q_P(\mu^*, |\lambda^*|) \leq d_P^*. \end{aligned} \tag{2.10}$$

On the other hand, pick any $x^* \in X^*$. Then x^* is feasible, i.e., $x^* \in X$, $[g(x^*)]^+ = 0$ and $|h(x^*)| = 0$. It implies that $q_P(\mu, \lambda) \leq \mathscr{H}(x^*, \mu, \lambda) = f(x^*) = p^*$ holds for any $\mu \in \mathbb{R}^m_{\geq 0}$ and $\lambda \in \mathbb{R}^\nu_{\geq 0}$, and thus $d_P^* = \sup_{\mu \in \mathbb{R}^m_{\geq 0}, \lambda \in \mathbb{R}^\nu_{\geq 0}} q_P(\mu, \lambda) \leq p^* = d_a^*$. Therefore, we have $d_P^* = p^*$.

To prove the emptiness of D_P^*, we pick any $(\mu^*, \lambda^*) \in D_a^*$. From (2.10) and $d_a^* = d_P^*$, we can see that $(\mu^*, |\lambda^*|) \in D_P^*$ and thus $D_P^* \neq \emptyset$. ∎

The following is a slight extension of Theorem 1.7 to penalty functions.

Theorem 2.2 (Penalty Saddle point Theorem) *The pair of* (x^*, μ^*, λ^*) *is a saddle point of the penalty function* $\mathscr{H}$ *over* $X \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$ *if and only if it is a pair of primal and penalty dual optimal solutions and the following* penalty minimax equality *holds:*

$$\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \inf_{x\in X} \mathscr{H}(x, \mu, \lambda) = \inf_{x\in X} \sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathscr{H}(x, \mu, \lambda).$$

Proof The proof is analogous to that of Proposition 6.2.4 in [12], and for the sake of completeness, we provide the details here. It follows from Proposition 2.6.1 in [12] that (x^*, μ^*, λ^*) is a saddle point of $\mathscr{H}$ over $X \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$ if and only if the penalty minimax equality holds and the following conditions are satisfied:

$$\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathscr{H}(x^*, \mu, \lambda) = \min_{x\in X}\{\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathscr{H}(x, \mu, \lambda)\}, \tag{2.11}$$

$$\inf_{x\in X} \mathscr{H}(x, \mu^*, \lambda^*) = \max_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \{\inf_{x\in X} \mathscr{H}(x, \mu, \lambda)\}. \tag{2.12}$$

Notice that $\inf_{x\in X} \mathscr{H}(x, \mu, \lambda) = q_P(\mu, \lambda)$; and if $x \in Y$, then the following holds:

$$\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathscr{H}(x, \mu, \lambda) = f(x),$$

otherwise, $\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathscr{H}(x, \mu, \lambda) = +\infty$. Hence, the penalty minimax equality is equivalent to $d_P^* = p^*$. Condition (2.11) is equivalent to the fact that x^* is primal optimal, and condition (2.12) is equivalent to (μ^*, λ^*) being a penalty dual optimal solution. ■

Convexity of the Penalty Function

Since g_ℓ is convex and $[\cdot]^+$ is convex and nondecreasing, $[g_\ell(x)]^+$ is convex in x for each $\ell \in \{1, \ldots, m\}$. Denote $A \triangleq (a_1^T, \ldots, a_\nu^T)^T$. Since $|\cdot|$ is convex and $a_\ell^T x - b_\ell$ is an affine mapping, then $|a_\ell^T x - b_\ell|$ is convex in x for each $\ell \in \{1, \ldots, \nu\}$.

We denote $w \triangleq (\mu, \lambda)$. For each $w \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$, we define the function $\mathscr{H}_{iw} : \mathbb{R}^n \to \mathbb{R}$ as $\mathscr{H}_{iw}(x) := \mathscr{H}_i(x, w)$. Note that $\mathscr{H}_{iw}(x)$ is convex in x by using the fact that a nonnegative weighted sum of convex functions is convex. For each $x \in \mathbb{R}^n$, we define the function $\mathscr{H}_{ix} : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ as $\mathscr{H}_{ix}(w) := \mathscr{H}_i(x, w)$. It is easy to check that $\mathscr{H}_{ix}(w)$ is concave (actually affine) in w. Then the penalty function $\mathscr{H}(x, w)$ is the sum of convex–concave local functions.

Remark 2.2 The Lagrangian relaxation does not fit into our approach here since the Lagrangian function is not convex in x by allowing λ entries to be negative. •

2.4.1 *The* Distributed Penalty Primal-Dual Subgradient Algorithm

We now proceed to devise the Distributed Penalty Primal- Dual Subgradient Algorithm, which is based on the penalty saddle point Theorem 2.2, to find the optimal value and a primal optimal solution to the primal problem (2.1) with $X_i = X$. The main steps of the Distributed Penalty Primal- Dual Subgradient Algorithm are described as follows.

Initially, agent i chooses any initial state $x^i(0) \in X$, $\mu^i(0) \in \mathbb{R}^m_{\geq 0}$, $\lambda^i(0) \in \mathbb{R}^{\nu}_{\geq 0}$, and $y^i(1) = N f_i(x^i(0))$. After this, at every $k \geq 0$, agent i computes the following convex combinations:

$$v_x^i(k) = \sum_{j=1}^{N} a_j^i(k) x^j(k), \quad v_y^i(k) = \sum_{j=1}^{N} a_j^i(k) y^j(k),$$
$$v_\mu^i(k) = \sum_{j=1}^{N} a_j^i(k) \mu^j(k), \quad v_\lambda^i(k) = \sum_{j=1}^{N} a_j^i(k) \lambda^j(k),$$

and generates $x^i(k+1)$, $y^i(k+1)$, $\mu^i(k+1)$, and $\lambda^i(k+1)$ according to the following:

$$\begin{aligned}
&x^i(k+1) = P_X[v_x^i(k) - \alpha(k) \mathscr{S}_x^i(k)],\\
&y^i(k+1) = v_y^i(k) + N(f_i(x^i(k)) - f_i(x^i(k-1))),\\
&\mu^i(k+1) = v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+, \quad \lambda^i(k+1) = v_\lambda^i(k) + \alpha(k)|h(v_x^i(k))|,
\end{aligned}$$

where P_X is the projection operator onto the set X, the scalars $a_j^i(k)$ are nonnegative weights, and the positive scalars $\{\alpha(k)\}$ are step-sizes.[2] The vector

$$\mathscr{S}_x^i(k) \triangleq \mathscr{D} f_i(v_x^i(k)) + \sum_{\ell=1}^{m} v_\mu^i(k)_\ell \mathscr{D}[g_\ell(v_x^i(k))]^+ + \sum_{\ell=1}^{\nu} v_\lambda^i(k)_\ell \mathscr{D}|h_\ell|(v_x^i(k))$$

is a subgradient of $\mathscr{H}_{iw^i(k)}(x)$ at $x = v_x^i(k)$ where $w^i(k) \triangleq (v_\mu^i(k), v_\lambda^i(k))$.

Given a step-size sequence $\{\alpha(k)\}$, we define the sum over $[0, k]$ by $s(k) \triangleq \sum_{\ell=0}^{k} \alpha(\ell)$, which should satisfy the following; see Remark 2.4 on how to define such a sequence.

Assumption 2.2 (*Step-size assumption*) The step-sizes satisfy

$$\lim_{k \to +\infty} \alpha(k) = 0, \quad \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \quad \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty,$$

[2]Each agent i executes the update law of $y^i(k)$ for $k \geq 1$.

$$\lim_{k\to+\infty}\alpha(k+1)s(k)=0,\quad \sum_{k=0}^{+\infty}\alpha(k+1)^2 s(k)<+\infty,\quad \sum_{k=0}^{+\infty}\alpha(k+1)^2 s(k)^2<+\infty.$$

The following theorem is the main result of this section, characterizing the convergence properties of the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM, where an optimal solution and the optimal value are asymptotically achieved.

Theorem 2.3 (Convergence of the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM) *Consider the problem* (2.1) *with* $X_i = X$*. Let the non-degeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. Consider the sequences of* $\{x^i(k)\}$ *and* $\{y^i(k)\}$ *of the* DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM, *where the step-sizes* $\{\alpha(k)\}$ *satisfy the step-size Assumption 2.2. Then there exists a primal optimal solution* $\tilde{x}\in X^*$ *such that* $\lim_{k\to+\infty}\|x^i(k)-\tilde{x}\|=0$*, for all* $i\in V$*. Furthermore, we have* $\lim_{k\to+\infty}\|y^i(k)-p^*\|=0$*, for all* $i\in V$*.*

Remark 2.3 Observe that $\mu^i(k)\geq 0$, $\lambda^i(k)\geq 0$ and $v_x^i(k)\in X$ (due to the fact that X is convex). Furthermore, $([g(v_x^i(k))]^+, |h(v_x^i(k))|)$ is a supgradient of $\mathcal{H}_{iv_x^i(k)}(w^i(k))$; i.e., the following *penalty supgradient inequality* holds for any $\mu\in\mathbb{R}^m_{\geq 0}$ and $\lambda\in\mathbb{R}^\nu_{\geq 0}$:

$$\begin{aligned}&([g(v_x^i(k))]^+)^T(\mu-v_\mu^i(k))+|h(v_x^i(k))|^T(\lambda-v_\lambda^i(k))\\&\geq \mathcal{H}_i(v_x^i(k),\mu,\lambda)-\mathcal{H}_i(v_x^i(k),v_\mu^i(k),v_\lambda^i(k)).\end{aligned}\tag{2.13}$$

•

Remark 2.4 A step-size sequence that satisfies the step-size Assumption 2.2 is the harmonic series $\{\alpha(k)=\frac{1}{k+1}\}_{k\in\mathbb{Z}_{\geq 0}}$. It is obvious that $\lim_{k\to+\infty}\frac{1}{k+1}=0$, and well-known that $\sum_{k=0}^{+\infty}\frac{1}{k+1}=+\infty$ and $\sum_{k=0}^{+\infty}\frac{1}{(k+1)^2}<+\infty$. We now proceed to check the property of $\lim_{k\to+\infty}\alpha(k+1)s(k)=0$. For any $k\geq 1$, there is an integer $n\geq 1$ such that $2^{n-1}\leq k<2^n$. It holds that

$$\begin{aligned}s(k)&\leq s(2^n)=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right)\\&\leq 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^{n-1}+1}\right)\\&\leq 1+1+1+\cdots+1=n\leq \log_2 k+1.\end{aligned}$$

Then we have $\limsup_{k\to+\infty} \frac{s(k)}{k+2} \le \lim_{k\to+\infty} \frac{\log_2 k+1}{k+2} = 0$. Obviously, $\liminf_{k\to+\infty} \frac{s(k)}{k+2} \ge 0$. Then we have the property of $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$. Since $\log_2 k \le (\log_2 k)^2 < (k+2)^{\frac{1}{2}}$, then

$$\sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k)^2 \le \sum_{k=0}^{+\infty} \frac{(\log_2 k+1)^2}{(k+2)^2} = \sum_{k=0}^{+\infty} \left(\frac{(\log_2 k)^2}{(k+2)^2} + \frac{2\log_2 k}{(k+2)^2} + \frac{1}{(k+2)^2} \right)$$
$$\le \sum_{k=0}^{+\infty} \frac{1}{(k+2)^{\frac{3}{2}}} + \sum_{k=0}^{+\infty} \frac{2}{(k+2)^{\frac{3}{2}}} + \sum_{k=0}^{+\infty} \frac{1}{(k+2)^2} < +\infty.$$

Additionally, we have $\sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k) \le \sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k)^2 < +\infty$. •

2.4.2 *A Numerical Example for the* Distributed Penalty Primal-Dual Subgradient Algorithm

Consider a network with five agents and their objective functions are defined as

$$f_1(x) := \frac{1}{5}\left((a-5)^2 + (b-2.5)^2 + (c-5)^2 + (d+2.5)^2 + (e+5)^2\right),$$
$$f_2(x) := \frac{1}{5}\left((a-2.5)^2 + (b-5)^2 + (c+2.5)^2 + (d+5)^2 + (e-5)^2\right),$$
$$f_3(x) := \frac{1}{5}\left((a-5)^2 + (b+2.5)^2 + (c+5)^2 + (d-5)^2 + (e-2.5)^2\right),$$
$$f_4(x) := \frac{1}{5}\left((a+2.5)^2 + (b+5)^2 + (c-5)^2 + (d-2.5)^2 + (e-5)^2\right),$$
$$f_5(x) := \frac{1}{5}\left((a+5)^2 + (b-5)^2 + (c-2.5)^2 + (d-5)^2 + (e+2.5)^2\right),$$

where the global decision vector $x := [a \quad b \quad c \quad d \quad e]^T \in \mathbb{R}^5$. The global equality constraint function is given by $h(x) := a + b + c + d + e - 5$, and the global constraint set is as follows: $X := [-5, \quad 5] \times [-5, \quad 5] \times [-5, \quad 5] \times [-5, \quad 5] \times [-5, \quad 5]$. Consider the optimization as follows:

$$\min_{x\in\mathbb{R}^5} \sum_{i\in V} f_i(x), \quad \text{s.t.} \quad h(x) = 0, \quad x \in X.$$

One can verify that the optimal solution is $[1 \quad 1 \quad 1 \quad 1 \quad 1]^T$. We employ the Distributed Penalty Primal-Dual Subgradient Algorithm to solve the above optimization problem with the step-size $\alpha(k) = \frac{1}{k+1}$. Its simulation results are included in Figs. 2.4 and 2.5. Observe that the estimates of a and b generated by different agents asymptotically achieve value 1.

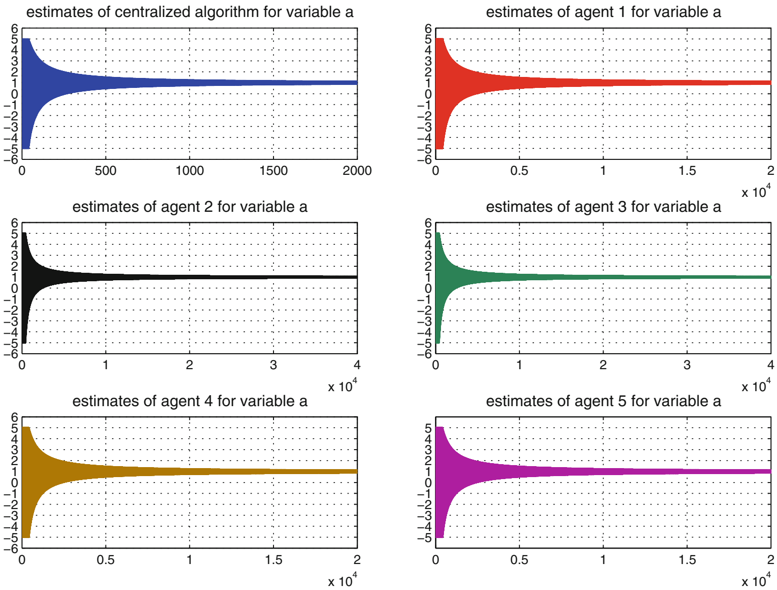

Fig. 2.4 The estimates on a generated by different agents in the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM

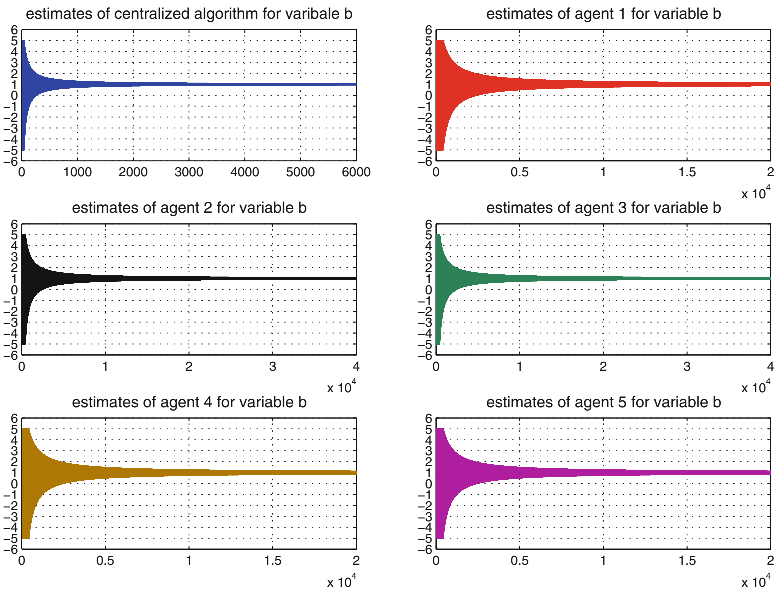

Fig. 2.5 The estimates on b generated by different agents in the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM

2.5 Appendix

We next provide the proofs for the main results, Theorems 2.1 and 2.3, of this chapter. Before doing that, let us state an instrumental result as follows. Consider the following distributed projected subgradient algorithm proposed in [13]: $x^i(k+1) = P_Z[v_x^i(k) - \alpha(k)d_i(k)]$. Denote by $e^i(k) := P_Z[v_x^i(k) - \alpha(k)d_i(k)] - v_x^i(k)$. The following is a slight modification of Lemma 8 and its proof in [13].

Lemma 2.5 *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. Suppose $Z \in \mathbb{R}^n$ is a closed and convex set. Then there exist $\gamma > 0$ and $\beta \in (0, 1)$ such that*

$$\|x^i(k) - \hat{x}(k)\| \leq N\gamma \sum_{\tau=0}^{k-1} \beta^{k-\tau}\{\alpha(\tau)\|d_i(\tau)\| + \|e^i(\tau) + \alpha(\tau)d_i(\tau)\|\} + N\gamma\beta^{k-1}\sum_{i=0}^{N}\|x^i(0)\|.$$

Suppose $\{d_i(k)\}$ is uniformly bounded for each $i \in V$, and $\sum_{k=0}^{+\infty}\alpha(k)^2 < +\infty$, then we have $\sum_{k=0}^{+\infty}\alpha(k)\max_{i\in V}\|x^i(k) - \hat{x}(k)\| < +\infty$.

We start our analysis on Theorems 2.1 and 2.3 by providing the properties of the sequences weighted by $\{\alpha(k)\}$.

Lemma 2.6 (Convergence of weighted sequences) *Let $K \geq 0$. Consider the sequence $\{\delta(k)\}$ defined by $\delta(k) \triangleq \frac{\sum_{\ell=K}^{k-1}\alpha(\ell)\rho(\ell)}{\sum_{\ell=K}^{k-1}\alpha(\ell)}$ where $k \geq K+1$, $\alpha(k) > 0$ and $\sum_{k=K}^{+\infty}\alpha(k) = +\infty$.*

(a) If $\lim_{k\to+\infty}\rho(k) = +\infty$, then $\lim_{k\to+\infty}\delta(k) = +\infty$.
(b) If $\lim_{k\to+\infty}\rho(k) = \rho^*$, then $\lim_{k\to+\infty}\delta(k) = \rho^*$.

Proof (a) For any $\Pi > 0$, there exists $k_1 \geq K$ such that $\rho(k) \geq \Pi$ for all $k \geq k_1$. Then the following holds for all $k \geq k_1 + 1$:

$$\delta(k) \geq \frac{1}{\sum_{\ell=K}^{k-1}\alpha(\ell)}\left(\sum_{\ell=K}^{k_1-1}\alpha(\ell)\rho(\ell) + \sum_{\ell=k_1}^{k-1}\alpha(\ell)\Pi\right)$$
$$= \Pi + \frac{1}{\sum_{\ell=K}^{k-1}\alpha(\ell)}\left(\sum_{\ell=K}^{k_1-1}\alpha(\ell)\rho(\ell) - \sum_{\ell=K}^{k_1-1}\alpha(\ell)\Pi\right).$$

Take the limit on k in the above estimate and we have $\liminf_{k\to+\infty} \delta(k) \geq \Pi$. Since Π is arbitrary, then $\lim_{k\to+\infty} \delta(k) = +\infty$.

(b) For any $\varepsilon > 0$, there exists $k_2 \geq K$ such that $\|\rho(k) - \rho^*\| \leq \varepsilon$ for all $k \geq k_2 + 1$. Then we have

$$\|\delta(k) - \rho^*\| = \|\frac{\sum_{\tau=K}^{k-1} \alpha(\tau)(\rho(\tau) - \rho^*)}{\sum_{\tau=K}^{k-1} \alpha(\tau)}\|$$

$$\leq \frac{1}{\sum_{\tau=K}^{k-1} \alpha(\tau)} \left(\sum_{\tau=K}^{k_2-1} \alpha(\tau)\|\rho(\tau) - \rho^*\| + \sum_{\tau=k_2}^{k-1} \alpha(\tau)\varepsilon \right) \leq \frac{\sum_{\tau=K}^{k_2-1} \alpha(\tau)\|\rho(\tau) - \rho^*\|}{\sum_{\tau=K}^{k-1} \alpha(\tau)} + \varepsilon.$$

Take the limit on k in the above estimate and we have $\limsup_{k\to+\infty} \|\delta(k) - \rho^*\| \leq \varepsilon$. Since ε is arbitrary, then $\lim_{k\to+\infty} \|\delta(k) - \rho^*\| = 0$. ■

2.5.1 *Convergence Analysis of the* Distributed Lagrangian Primal-Dual Subgradient Algorithm

We now proceed to show Theorem 2.1. To do that, we first rewrite the Distributed Lagrangian Primal- Dual Subgradient Algorithm into the following form:

$$x^i(k+1) = v_x^i(k) + e_x^i(k), \quad \mu^i(k+1) = v_\mu^i(k) + e_\mu^i(k), \quad y^i(k+1) = v_y^i(k) + u^i(k),$$

where $e_x^i(k)$ and $e_\mu^i(k)$ are projection errors described by

$$e_x^i(k) \triangleq P_{X_i}[v_x^i(k) - \alpha(k)\mathscr{D}_x^i(k)] - v_x^i(k),$$
$$e_\mu^i(k) \triangleq P_{M_i}[v_\mu^i(k) + \alpha(k)\mathscr{D}_\mu^i(k)] - v_\mu^i(k),$$

and $u^i(k) \triangleq N(f_i(x^i(k)) - f_i(x^i(k-1)))$ is the local input which allows agent i to track the variation of the local objective function f_i. In this manner, the update law of each estimate is decomposed in two parts: a convex sum to fuse the information of each agent with those of its neighbors, plus some local error or input. With this decomposition, all the update laws are put into the same form as the dynamic average consensus algorithm in the Chap. 1. This observation allows us to divide the analysis of the Distributed Lagrangian Primal- Dual Subgradient Algorithm in two steps. First, we show all the estimates asymptotically achieve consensus by utilizing the property that the local errors and inputs are diminishing. Second, we further show that the consensus vectors coincide with a pair of primal and Lagrangian dual optimal solutions and the optimal value.

Lemma 2.7 (Lipschitz continuity of $\mathscr{L}_i$) *Consider $\mathscr{L}_{i\mu}$ and $\mathscr{L}_{ix}$. Then there are $L > 0$ and $R > 0$ such that $\|\mathscr{D}\mathscr{L}_{i\mu}(x)\| \leq L$ and $\|\mathscr{D}\mathscr{L}_{ix}(\mu)\| \leq R$ for each pair of $x \in \text{co}(\cup_{i=1}^N X_i)$ and $\mu \in \text{co}(\cup_{i=1}^N M_i)$. Furthermore, for each $\mu \in \text{co}(\cup_{i=1}^N M_i)$, the function $\mathscr{L}_{i\mu}$ is Lipschitz continuous with Lipschitz constant L over $\text{co}(\cup_{i=1}^N X_i)$, and for each $x \in \text{co}(\cup_{i=1}^N X_i)$, the function $\mathscr{L}_{ix}$ is Lipschitz continuous with Lipschitz constant R over $\text{co}(\cup_{i=1}^N M_i)$.*

Proof Observe that $\mathscr{D}\mathscr{L}_{i\mu} = \mathscr{D} f_i + \mu^T \mathscr{D} g$ and $\mathscr{D}\mathscr{L}_{ix} = g$. Since f_i and g_ℓ are convex, it follows from Proposition 5.4.2 in [11] that ∂f_i and ∂g_ℓ are bounded over the compact $\text{co}(\cup_{i=1}^N X_i)$. Since $\text{co}(\cup_{i=1}^N M_i)$ is bounded, so is $\partial \mathscr{L}_{i\mu}$, i.e., for any $\mu \in \text{co}(\cup_{i=1}^N M_i)$, there exists $L > 0$ such that $\|\partial \mathscr{L}_{i\mu}(x)\| \leq L$ for all $x \in \text{co}(\cup_{i=1}^N X_i)$. Since g_ℓ is continuous (due to its convexity) and $\text{co}(\cup_{i=1}^N X_i)$ is bounded, then g and thus $\partial \mathscr{L}_{ix}$ are bounded, i.e., for any $x \in \text{co}(\cup_{i=1}^N X_i)$, there exists $R > 0$ such that $\|\partial \mathscr{L}_{ix}(\mu)\| \leq R$ for all $\mu \in \text{co}(\cup_{i=1}^N M_i)$.

It follows from the Lagrangian subgradient inequality that

$$\mathscr{D}\mathscr{L}_{i\mu}(x)^T(x' - x) \leq \mathscr{L}_{i\mu}(x') - \mathscr{L}_{i\mu}(x), \quad \mathscr{D}\mathscr{L}_{i\mu}(x')^T(x - x') \leq \mathscr{L}_{i\mu}(x) - \mathscr{L}_{i\mu}(x'),$$

for any $x, x' \in \text{co}(\cup_{i=1}^N X_i)$. By using the boundedness of the subdifferentials, the above two inequalities give that $-L\|x - x'\| \leq \mathscr{L}_{i\mu}(x) - \mathscr{L}_{i\mu}(x') \leq L\|x - x'\|$. This implies that $\|\mathscr{L}_{i\mu}(x) - \mathscr{L}_{i\mu}(x')\| \leq L\|x - x'\|$ for any $x, x' \in \text{co}(\cup_{i=1}^m X_i)$. The proof of the Lipschitz continuity of $\mathscr{L}_{ix}$ is analogous by using the Lagrangian supgradient inequality. ■

The following lemma provides a basic iteration relation used in the convergence proof of the Distributed Lagrangian Primal-Dual Subgradient Algorithm.

Lemma 2.8 (Basic iteration relation) *Let the double stochasticity Assumption 1.2 and the periodic strong connectivity Assumption 1.3 hold. For any $x \in X$, any $\mu \in M$ and all $k \geq 0$, the following estimates hold:*

$$\sum_{i=1}^N \|e_x^i(k) + \alpha(k)\mathscr{D}_x^i(k)\|^2 \leq -\sum_{i=1}^N 2\alpha(k)(\mathscr{L}_i(v_x^i(k), v_\mu^i(k)) - \mathscr{L}_i(x, v_\mu^i(k)))$$
$$+ \sum_{i=1}^N \alpha(k)^2 \|\mathscr{D}_x^i(k)\|^2 + \sum_{i=1}^N \{\|x^i(k) - x\|^2 - \|x^i(k+1) - x\|^2\}, \tag{2.14}$$
$$\sum_{i=1}^N \|e_\mu^i(k) - \alpha(k)\mathscr{D}_\mu^i(k)\|^2 \leq \sum_{i=1}^N 2\alpha(k)(\mathscr{L}_i(v_x^i(k), v_\mu^i(k)) - \mathscr{L}_i(v_x^i(k), \mu))$$
$$+ \sum_{i=1}^N \alpha(k)^2 \|\mathscr{D}_\mu^i(k)\|^2 + \sum_{i=1}^N \{\|\mu^i(k) - \mu\|^2 - \|\mu^i(k+1) - \mu\|^2\}. \tag{2.15}$$

Proof By Lemma 1.1 with $Z = M_i$, $z = v^i_\mu(k) + \alpha(k)\mathscr{D}^i_\mu(k)$ and $y = \mu \in M$, we have that for all $k \geq 0$

$$\begin{aligned}
&\sum_{i=1}^N \|e^i_\mu(k) - \alpha(k)\mathscr{D}^i_\mu(k)\|^2 \leq \sum_{i=1}^N \|v^i_\mu(k) + \alpha(k)\mathscr{D}^i_\mu(k) - \mu\|^2 - \sum_{i=1}^N \|\mu^i(k+1) - \mu\|^2 \\
&= \sum_{i=1}^N \|v^i_\mu(k) - \mu\|^2 + \sum_{i=1}^N \alpha(k)^2 \|\mathscr{D}^i_\mu(k)\|^2 \\
&+ \sum_{i=1}^N 2\alpha(k)\mathscr{D}^i_\mu(k)^T(v^i_\mu(k) - \mu) - \sum_{i=1}^N \|\mu^i(k+1) - \mu\|^2 \\
&\leq \sum_{i=1}^N \alpha(k)^2 \|\mathscr{D}^i_\mu(k)\|^2 + \sum_{i=1}^N 2\alpha(k)\mathscr{D}^i_\mu(k)^T(v^i_\mu(k) - \mu) \\
&+ \sum_{i=1}^N \|\mu^i(k) - \mu\|^2 - \sum_{i=1}^N \|\mu^i(k+1) - \mu\|^2. \qquad (2.16)
\end{aligned}$$

One can show (2.15) by substituting the following Lagrangian supgradient inequality into (2.16):

$$\mathscr{D}^i_\mu(k)^T(\mu - v^i_\mu(k)) \geq \mathscr{L}_i(v^i_x(k), \mu) - \mathscr{L}_i(v^i_x(k), v^i_\mu(k)).$$

Similarly, the equality (2.14) can be shown by using the following Lagrangian subgradient inequality: $\mathscr{D}^i_x(k)^T(x - v^i_x(k)) \leq \mathscr{L}_i(x, v^i_\mu(k)) - \mathscr{L}_i(v^i_x(k), v^i_\mu(k))$. ■

The following lemma shows that the consensus is asymptotically reached.

Lemma 2.9 (Achieving consensus) *Let the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodic strong connectivity Assumption 1.3 hold. Consider the sequences of* $\{x^i(k)\}$, $\{\mu^i(k)\}$, *and* $\{y^i(k)\}$ *of the* Distributed Lagrangian Primal-Dual Subgradient Algorithm *with the step-size sequence* $\{\alpha(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$. *Then there exist* $x^* \in X$ *and* $\mu^* \in M$ *such that*

$$\begin{aligned}
&\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0, \quad \lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0, \quad \forall i \in V, \\
&\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0, \quad \forall i, j \in V.
\end{aligned}$$

Proof Observe that $v^i_x(k) \in \text{co}(\cup_{i=1}^N X_i)$ and $v^i_\mu(k) \in \text{co}(\cup_{i=1}^N M_i)$. Then it follows from Lemma 2.7 that $\|\mathscr{D}^i_x(k)\| \leq L$. From Lemma 2.8 it follows that

$$\sum_{i=1}^{N} \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^{N} \|x^i(k) - x\|^2 + \sum_{i=1}^{N} \alpha(k)^2 L^2$$
$$+ \sum_{i=1}^{N} 2\alpha(k)(\|\mathscr{L}_i(v_x^i(k), v_\mu^i(k))\| + \|\mathscr{L}_i(x, v_\mu^i(k))\|). \tag{2.17}$$

Notice that $v_x^i(k) \in \mathrm{co}(\cup_{i=1}^N X_i)$, $v_\mu^i(k) \in \mathrm{co}(\cup_{i=1}^N M_i)$ and $x \in X$ are bounded. Since $\mathscr{L}_i$ is continuous, then $\mathscr{L}_i(v_x^i(k), v_\mu^i(k))$ and $\mathscr{L}_i(x, v_\mu^i(k))$ are bounded. Since $\{\alpha(k)\}$ diminishes, one can verify that $\lim_{k\to+\infty} \sum_{i=1}^{N} \|x^i(k) - x\|^2$ exists for any $x \in X$. On the other hand, taking limits on both sides of (2.14), we obtain

$$\lim_{k\to+\infty} \sum_{i=1}^{N} \|e_x^i(k) + \alpha(k)\mathscr{D}_x^i(k)\|^2 = 0,$$

and therefore we deduce that $\lim_{k\to+\infty} \|e_x^i(k)\| = 0$ for all $i \in V$. It follows from Theorem 1.4 that $\lim_{k\to+\infty} \|x^i(k) - x^j(k)\| = 0$ for all $i, j \in V$. Combining this with the property that $\lim_{k\to+\infty} \|x^i(k) - x\|$ exists for any $x \in X$, we deduce that there exists $x^* \in \mathbb{R}^n$ such that $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ for all $i \in V$. Since $x^i(k) \in X_i$ and X_i is closed, it implies that $x^* \in X_i$ for all $i \in V$ and thus $x^* \in X$. Similarly, one can show that there is $\mu^* \in M$ such that $\lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0$ for all $i \in V$.

Since $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ and f_i is continuous, then $\lim_{k\to+\infty} \|u^i(k)\| = 0$. It follows from Theorem 1.4 that $\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0$ for all $i, j \in V$. ∎

From Lemma 2.9, we know that the sequences of $\{x^i(k)\}$ and $\{\mu^i(k)\}$ of the Distributed Lagrangian Primal-Dual Subgradient Algorithm asymptotically agree on some point in X and some point in M, respectively. Denote by $\Theta \subseteq X \times M$ the set of such limit points. Denote by the average of primal and dual estimates $\hat{x}(k) \triangleq \frac{1}{N}\sum_{i=1}^N x^i(k)$ and $\hat{\mu}(k) \triangleq \frac{1}{N}\sum_{i=1}^N \mu^i(k)$, respectively. The following lemma further characterizes that the points in Θ are saddle points of the Lagrangian function $\mathscr{L}$ over $X \times M$.

Lemma 2.10 (Saddle point characterization of Θ) *Each point in Θ is a saddle point of the Lagrangian function $\mathscr{L}$ over $X \times M$.*

Proof Denote by $\Delta_x(k) \triangleq \max_{i,j\in V} \|x^j(k) - x^i(k)\|$ the maximum deviation of primal estimates. Notice that

$$\|v_x^i(k) - \hat{x}(k)\| = \|\sum_{j=1}^{N} a_j^i(k)x^j(k) - \sum_{j=1}^{N} \frac{1}{N}x^j(k)\|$$

$$= \| \sum_{j \neq i} a^i_j(k)(x^j(k) - x^i(k)) - \sum_{j \neq i} \frac{1}{N}(x^j(k) - x^i(k)) \|$$
$$\le \sum_{j \neq i} a^i_j(k) \|x^j(k) - x^i(k)\| + \sum_{j \neq i} \frac{1}{N} \|x^j(k) - x^i(k)\| \le 2\Delta_x(k).$$

Denote by the maximum deviation of dual estimates $\Delta_\mu(k) \triangleq \max_{i,j \in V} \|\mu^j(k) - \mu^i(k)\|$. Similarly, we have $\|v^i_\mu(k) - \hat{\mu}(k)\| \le 2\Delta_\mu(k)$.

We will show this lemma by contradiction. Suppose that there is $(x^*, \mu^*) \in \Theta$ which is not a saddle point of $\mathscr{L}$ over $X \times M$. Then at least one of the following equalities holds:

$$\exists x \in X \quad \text{s.t.} \quad \mathscr{L}(x^*, \mu^*) > \mathscr{L}(x, \mu^*), \tag{2.18}$$
$$\exists \mu \in M \quad \text{s.t.} \quad \mathscr{L}(x^*, \mu) > \mathscr{L}(x^*, \mu^*). \tag{2.19}$$

Suppose first that (2.18) holds. Then, there exists $\varsigma > 0$ such that $\mathscr{L}(x^*, \mu^*) = \mathscr{L}(x, \mu^*) + \varsigma$. Consider the sequences of $\{x^i(k)\}$ and $\{\mu^i(k)\}$ which converge respectively to x^* and μ^* defined above. The estimate (2.14) leads to

$$\sum_{i=1}^N \|x^i(k+1) - x\|^2 \le \sum_{i=1}^N \|x^i(k) - x\|^2 + \alpha(k)^2 \sum_{i=1}^N \|\mathscr{D}^i_x(k)\|^2 - 2\alpha(k)$$
$$\times \sum_{i=1}^N (A_i(k) + B_i(k) + C_i(k) + D_i(k) + E_i(k) + F_i(k)),$$

where the notations are given by:

$$A_i(k) = \mathscr{L}_i(v^i_x(k), v^i_\mu(k)) - \mathscr{L}_i(\hat{x}(k), v^i_\mu(k)),$$
$$B_i(k) = \mathscr{L}_i(\hat{x}(k), v^i_\mu(k)) - \mathscr{L}_i(\hat{x}(k), \hat{\mu}(k)),$$
$$C_i(k) = \mathscr{L}_i(\hat{x}(k), \hat{\mu}(k)) - \mathscr{L}_i(x^*, \hat{\mu}(k)), \quad D_i(k) = \mathscr{L}_i(x^*, \hat{\mu}(k)) - \mathscr{L}_i(x^*, \mu^*),$$
$$E_i(k) = \mathscr{L}_i(x^*, \mu^*) - \mathscr{L}_i(x, \mu^*), \quad F_i(k) = \mathscr{L}_i(x, \mu^*) - \mathscr{L}_i(x, v^i_\mu(k)).$$

It follows from the Lipschitz continuity property of $\mathscr{L}_i$; see Lemma 2.7, that

$$\|A_i(k)\| \le L\|v^i_x(k) - \hat{x}(k)\| \le 2L\Delta_x(k), \quad \|B_i(k)\| \le R\|v^i_\mu(k) - \hat{\mu}(k)\| \le 2R\Delta_\mu(k),$$
$$\|C_i(k)\| \le L\|\hat{x}(k) - x^*\| \le \frac{L}{N} \sum_{i=1}^N \|x^i(k) - x^*\|,$$
$$\|D_i(k)\| \le R\|\hat{\mu}(k) - \mu^*\| \le \frac{R}{N} \sum_{i=1}^N \|\mu^i(k) - \mu^*\|,$$
$$\|F_i(k)\| \le R\|\mu^* - v^i_\mu(k)\| \le R\|\mu^* - \hat{\mu}(k)\| + R\|\hat{\mu}(k) - v^i_\mu(k)\|$$

$$\leq \frac{R}{N}\sum_{i=1}^{N}\|\mu^*(k)-\mu^i(k)\| + 2R\Delta_\mu(k).$$

Since $\lim_{k\to+\infty}\|x^i(k)-x^*\|=0$, $\lim_{k\to+\infty}\|\mu^i(k)-\mu^*\|=0$, $\lim_{k\to+\infty}\Delta_x(k)=0$ and $\lim_{k\to+\infty}\Delta_\mu(k)=0$, then all $A_i(k)$, $B_i(k)$, $C_i(k)$, $D_i(k)$, $F_i(k)$ converge to zero as $k\to+\infty$. Then there exists $k_0\geq 0$ such that for all $k\geq k_0$, it holds that

$$\sum_{i=1}^{N}\|x^i(k+1)-x\|^2 \leq \sum_{i=1}^{N}\|x^i(k)-x\|^2 + N\alpha(k)^2L^2 - \varsigma\alpha(k).$$

Following a recursive argument, we have that for all $k\geq k_0$, it holds that

$$\sum_{i=1}^{N}\|x^i(k+1)-x\|^2 \leq \sum_{i=1}^{N}\|x^i(k_0)-x\|^2 + NL^2\sum_{\tau=k_0}^{k}\alpha(\tau)^2 - \varsigma\sum_{\tau=k_0}^{k}\alpha(\tau).$$

Since $\sum_{k=k_0}^{+\infty}\alpha(k)=+\infty$ and $\sum_{k=k_0}^{+\infty}\alpha(k)^2<+\infty$ and $x^i(k_0)\in X_i$, $x\in X$ are bounded, the above estimate yields a contradiction by taking k sufficiently large. In other words, (2.18) cannot hold. Following a parallel argument, one can show that (2.19) cannot hold either. This ensures that each $(x^*,\mu^*)\in\Theta$ is a saddle point of $\mathscr{L}$ over $X\times M$. ■

The combination of (c) in Lemmas 2.1 and 2.10 gives that, for each $(x^*,\mu^*)\in\Theta$, we have that $\mathscr{L}(x^*,\mu^*)=p^*$ and μ^* is Lagrangian dual optimal. We still need to verify that x^* is a primal optimal solution. We are now in the position to show Theorem 2.1 based on two claims.

Proofs of Theorem 2.1:

Claim 2.1 *Each point* $(x^*,\mu^*)\in\Theta$ *is a point in* $X^*\times D_L^*$.

Proof The Lagrangian dual optimality of μ^* follows from (c) in Lemmas 2.1 and 2.10. To characterize the primal optimality of x^*, we define an auxiliary sequence $\{z(k)\}$ by $z(k)\triangleq\frac{\sum_{\tau=0}^{k-1}\alpha(\tau)\hat{x}(\tau)}{\sum_{\tau=0}^{k-1}\alpha(\tau)}$ which is a weighted version of the average of primal estimates. Since $\lim_{k\to+\infty}\hat{x}(k)=x^*$, it follows from Lemma 2.6 (b) that $\lim_{k\to+\infty}z(k)=x^*$.

Since (x^*,μ^*) is a saddle point of $\mathscr{L}$ over $X\times M$, then $\mathscr{L}(x^*,\mu)\leq\mathscr{L}(x^*,\mu^*)$ for any $\mu\in M$; i.e., the following relation holds for any $\mu\in M$:

$$g(x^*)^T(\mu-\mu^*)\leq 0. \tag{2.20}$$

Choose $\mu_a = \mu^* + \min_{i\in V}\theta_i \frac{\mu^*}{\|\mu^*\|}$ where $\theta_i > 0$ is given in the definition of M_i. Then $\mu_a \geq 0$ and $\|\mu_a\| \leq \|\mu^*\| + \min_{i\in V}\theta_i$ implying $\mu_a \in M$. Letting $\mu = \mu_a$ in (2.20) gives that

$$\frac{\min_{i\in V}\theta_i}{\|\mu^*\|} g(x^*)^T \mu^* \leq 0.$$

Since $\theta_i > 0$, we have $g(x^*)^T\mu^* \leq 0$. On the other hand, we choose $\mu_b = \frac{1}{2}\mu^*$ and then $\mu_b \in M$. Letting $\mu = \mu_b$ in (2.20) gives that $-\frac{1}{2}g(x^*)^T\mu^* \leq 0$ and thus $g(x^*)^T\mu^* \geq 0$. The combination of the above two estimates guarantees the property of $g(x^*)^T\mu^* = 0$.

We now proceed to show $g(x^*) \leq 0$ by contradiction. Assume that $g(x^*) \leq 0$ does not hold. Denote $J^+(x^*) \triangleq \{1 \leq \ell \leq m \mid g_\ell(x^*) > 0\} \neq \emptyset$ and $\eta \triangleq \min_{\ell\in J^+(x^*)}\{g_\ell(x^*)\}$. Then $\eta > 0$. Since g is continuous and $v_x^i(k)$ converges to x^*, there exists $K \geq 0$ such that $g_\ell(v_x^i(k)) \geq \frac{\eta}{2}$ for all $k \geq K$ and all $\ell \in J^+(x^*)$. Since $v_\mu^i(k)$ converges to μ^*, without loss of generality, we say that $\|v_\mu^i(k) - \mu^*\| \leq \frac{1}{2}\min_{i\in V}\theta_i$ for all $k \geq K$. Choose $\hat{\mu}$ such that $\hat{\mu}_\ell = \mu_\ell^*$ for $\ell \notin J^+(x^*)$ and $\hat{\mu}_\ell = \mu_\ell^* + \frac{1}{\sqrt{m}}\min_{i\in V}\theta_i$ for $\ell \in J^+(x^*)$. Since $\mu^* \geq 0$ and $\theta_i > 0$, $\hat{\mu} \geq 0$. Furthermore, $\|\hat{\mu}\| \leq \|\mu^*\| + \min_{i\in V}\theta_i$, then $\hat{\mu} \in M$. Equating μ to $\hat{\mu}$ and letting $\mathscr{D}_\mu^i(k) = g(v_x^i(k))$ in the estimate (2.16), the following holds for $k \geq K$:

$$\begin{aligned}
N|J^+(x^*)|\eta \min_{i\in V}\theta_i\alpha(k) &\leq 2\alpha(k)\sum_{i=1}^N \sum_{\ell\in J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu} - v_\mu^i(k))_\ell \\
&\leq \sum_{i=1}^N \|\mu^i(k) - \hat{\mu}\|^2 - \sum_{i=1}^N \|\mu^i(k+1) - \hat{\mu}\|^2 + NR^2\alpha(k)^2 \\
&- 2\alpha(k)\sum_{i=1}^N \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu} - v_\mu^i(k))_\ell. \qquad (2.21)
\end{aligned}$$

Summing (2.21) over $[K, k-1]$ with $k \geq K+1$, dividing by $\sum_{\tau=K}^{k-1}\alpha(\tau)$ on both sides, and using $-\sum_{i=1}^N \|\mu^i(k) - \hat{\mu}\|^2 \leq 0$, we obtain

$$\begin{aligned}
N|J^+(x^*)|\eta \min_{i\in V}\theta_i \leq \frac{1}{\sum_{\tau=K}^{k-1}\alpha(\tau)} \Bigg\{ &\sum_{i=1}^N \|\mu^i(K) - \hat{\mu}\|^2 + NR^2 \sum_{\tau=K}^{k-1}\alpha(\tau)^2 \\
- \sum_{\tau=K}^{k-1} 2\alpha(\tau) &\sum_{i=1}^N \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(\tau))(\hat{\mu} - v_\mu^i(\tau))_\ell \Bigg\}. \qquad (2.22)
\end{aligned}$$

Since $\mu^i(K) \in M_i$, $\hat{\mu} \in M$ are bounded and $\sum_{\tau=K}^{+\infty} \alpha(\tau) = +\infty$, then the limit of the first term on the right-hand side of (2.22) is zero as $k \to +\infty$. Since $\sum_{\tau=K}^{+\infty} \alpha(\tau)^2 < +\infty$, then the limit of the second term is zero as $k \to +\infty$. Since $\lim_{k\to+\infty} v_x^i(k) = x^*$ and $\lim_{k\to+\infty} v_\mu^i(k) = \mu^*$, the following holds:

$$\lim_{k\to+\infty} 2\sum_{i=1}^{N} \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu} - v_\mu^i(k))_\ell = 0.$$

Then it follows from Lemma 2.6 (b) that the limit of the third term is zero as $k \to +\infty$. We have $N|J^+(x^*)|\eta \min_{i\in V} \theta_i \le 0$. Recall that $|J^+(x^*)| > 0$, $\eta > 0$ and $\theta_i > 0$. Then we reach a contradiction, implying that $g(x^*) \le 0$.

Since $x^* \in X$ and $g(x^*) \le 0$, then x^* is a feasible solution and thus $f(x^*) \ge p^*$. On the other hand, since $z(k)$ is a convex combination of $\hat{x}(0), \ldots, \hat{x}(k-1)$ and f is convex, we have the following estimate:

$$\begin{aligned} f(z(k)) &\le \frac{\sum_{\tau=0}^{k-1} \alpha(\tau) f(\hat{x}(\tau))}{\sum_{\tau=0}^{k-1} \alpha(\tau)} \\ &= \frac{1}{\sum_{\tau=0}^{k-1} \alpha(\tau)} \left\{ \sum_{\tau=0}^{k-1} \alpha(\tau)\mathscr{L}(\hat{x}(\tau), \hat{\mu}(\tau)) - \sum_{\tau=0}^{k-1} N\alpha(\tau)\hat{\mu}(\tau)^T g(\hat{x}(\tau)) \right\}. \end{aligned}$$

Recall the following convergence properties:

$$\lim_{k\to+\infty} z(k) = x^*, \quad \lim_{k\to+\infty} \mathscr{L}(\hat{x}(k), \hat{\mu}(k)) = \mathscr{L}(x^*, \mu^*) = p^*,$$
$$\lim_{k\to+\infty} \hat{\mu}(k)^T g(\hat{x}(k)) = g(x^*)^T \mu^* = 0.$$

It follows from Lemma 2.6 (b) that $f(x^*) \le p^*$. Therefore, we have $f(x^*) = p^*$, and thus x^* is a primal optimal point. ■

Claim 2.2 *It holds that* $\lim_{k\to+\infty} \|y^i(k) - p^*\| = 0$.

Proof The following can be proven by induction on k for a fixed $k' \ge 1$:

$$\sum_{i=1}^{N} y^i(k+1) = \sum_{i=1}^{N} y^i(k') + N \sum_{\ell=k'}^{k} \sum_{i=1}^{N} (f_i(x^i(\ell)) - f_i(x^i(\ell-1))). \tag{2.23}$$

Let $k' = 1$ in (2.23) and recall that initial state $y^i(1) = Nf_i(x^i(0))$ for all $i \in V$. Then we have

$$\sum_{i=1}^{N} y^i(k+1) = \sum_{i=1}^{N} y^i(1) + N\sum_{i=1}^{N}(f_i(x^i(k)) - f_i(x^i(0))) = N\sum_{i=1}^{N} f_i(x^i(k)). \tag{2.24}$$

The combination of (2.24) with $\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0$ gives the desired result. ■

2.5.2 *Convergence Analysis of the* Distributed Penalty Primal-Dual Subgradient Algorithm

In order to analyze the Distributed Penalty Primal-Dual Subgradient Algorithm, we first rewrite it into the following form:

$$\mu^i(k+1) = v^i_\mu(k) + u^i_\mu(k), \quad \lambda^i(k+1) = v^i_\lambda(k) + u^i_\lambda(k),$$
$$x^i(k+1) = v^i_x(k) + e^i_x(k), \quad y^i(k+1) = v^i_y(k) + u^i_y(k),$$

where $e^i_x(k)$ is projection error described by

$$e^i_x(k) \triangleq P_X[v^i_x(k) - \alpha(k)\mathscr{S}^i_x(k)] - v^i_x(k),$$

and the quantities $u^i_\mu(k) \triangleq \alpha(k)[g(v^i_x(k))]^+$, $u^i_\lambda(k) \triangleq \alpha(k)|h(v^i_x(k))|$ and $u^i_y(k) = N(f_i(x^i(k)) - f_i(x^i(k-1)))$ represent local inputs. Denote by the maximum deviations of dual estimates $M_\mu(k) \triangleq \max_{i\in V} \|\mu^i(k)\|$ and $M_\lambda(k) \triangleq \max_{i\in V} \|\lambda^i(k)\|$. Before showing Lemma 2.11, we present some useful facts. Since X is compact, and f_i, $[g(\cdot)]^+$ and h are continuous, there exist $F, G^+, H > 0$ such that for all $x \in X$, it holds that $\|f_i(x)\| \le F$ for all $i \in V$, $\|[g(x)]^+\| \le G^+$, and $\|h(x)\| \le H$. Since X is a compact set and f_i, $[g_\ell(\cdot)]^+$, $|h_\ell(\cdot)|$ are convex, then it follows from Proposition 5.4.2 in [11] that there exist $D_F, D_{G^+}, D_H > 0$ such that for all $x \in X$, it holds that $\|\mathscr{D}f_i(x)\| \le D_F$ $(i \in V)$, $m\|\mathscr{D}[g_\ell(x)]^+\| \le D_{G^+}$ $(1 \le \ell \le m)$ and $\nu\|\mathscr{D}|h_\ell|(x)\| \le D_H$ $(1 \le \ell \le \nu)$. Denote by the averages of primal and dual estimates $\hat{x}(k) \triangleq \frac{1}{N}\sum_{i=1}^N x^i(k)$, $\hat{\mu}(k) \triangleq \frac{1}{N}\sum_{i=1}^N \mu^i(k)$ and $\hat{\lambda}(k) \triangleq \frac{1}{N}\sum_{i=1}^N \lambda^i(k)$.

Lemma 2.11 (Diminishing and summable properties) *Suppose the double stochasticity Assumption 1.2 and the step-size Assumption 2.2 hold.*

(a) The following holds:

$$\lim_{k\to+\infty} \alpha(k)M_\mu(k) = 0, \quad \lim_{k\to+\infty} \alpha(k)M_\lambda(k) = 0, \quad \lim_{k\to+\infty} \alpha(k)\|\mathscr{S}^i_x(k)\| = 0.$$

Furthermore, the sequences of $\{\alpha(k)^2 M^2_\mu(k)\}$, $\{\alpha(k)^2 M^2_\lambda(k)\}$, *and* $\{\alpha(k)^2\|\mathscr{S}^i_x(k)\|^2\}$ *are summable.*

(b) The following sequences are summable:

$$\{\alpha(k)\|\hat{\mu}(k) - v^i_\mu(k)\|\}, \{\alpha(k)\|\hat{\lambda}(k) - v^i_\lambda(k)\|\}, \{\alpha(k)M_\mu(k)\|\hat{x}(k) - v^i_x(k)\|\},$$
$$\{\alpha(k)M_\lambda(k)\|\hat{x}(k) - v^i_x(k)\|\}, \{\alpha(k)\|\hat{x}(k) - v^i_x(k)\|\}.$$

Proof (a) Notice that

$$\|v^i_\mu(k)\| = \|\sum_{j=1}^N a^i_j(k)\mu^j(k)\| \leq \sum_{j=1}^N a^i_j(k)\|\mu^j(k)\| \leq \sum_{j=1}^N a^i_j(k)M_\mu(k) = M_\mu(k),$$

where in the last equality we use the double stochasticity Assumption 1.2. Recall that $v^i_x(k) \in X$. This implies that the following holds for all $k \geq 0$:

$$\|\mu^i(k+1)\| \leq \|v^i_\mu(k) + \alpha(k)[g(v^i_x(k))]^+\| \leq \|v^i_\mu(k)\| + G^+\alpha(k) \leq M_\mu(k) + G^+\alpha(k).$$

From here, then we deduce the following recursive estimate on $M_\mu(k+1)$: $M_\mu(k+1) \leq M_\mu(k) + G^+\alpha(k)$. Repeatedly applying the above estimates yields that

$$M_\mu(k+1) \leq M_\mu(0) + G^+s(k). \tag{2.25}$$

Similar arguments can be employed to show that

$$M_\lambda(k+1) \leq M_\lambda(0) + Hs(k). \tag{2.26}$$

Since $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$ and $\lim_{k\to+\infty} \alpha(k) = 0$, we know that

$$\lim_{k\to+\infty} \alpha(k+1)M_\mu(k+1) = 0, \quad \lim_{k\to+\infty} \alpha(k+1)M_\lambda(k+1) = 0.$$

Notice that the following estimate on $\mathscr{S}^i_x(k)$ holds:

$$\|\mathscr{S}^i_x(k)\| \leq D_F + D_{G^+}M_\mu(k) + D_H M_\lambda(k). \tag{2.27}$$

Recall that $\lim_{k\to+\infty} \alpha(k) = 0$, $\lim_{k\to+\infty} \alpha(k)M_\mu(k) = 0$ and $\lim_{k\to+\infty} \alpha(k)M_\lambda(k) = 0$. Then the result of $\lim_{k\to+\infty} \alpha(k)\|\mathscr{S}^i_x(k)\| = 0$ follows. By (2.25), we have

$$\sum_{k=0}^{+\infty} \alpha(k)^2 M^2_\mu(k) \leq \alpha(0)^2 M^2_\mu(0) + \sum_{k=1}^{+\infty} \alpha(k)^2 (M_\mu(0) + G^+s(k-1))^2.$$

It follows from the step-size Assumption 2.2 that $\sum_{k=0}^{+\infty} \alpha(k)^2 M^2_\mu(k) < +\infty$. Similarly, one can show that $\sum_{k=0}^{+\infty} \alpha(k)^2 M^2_\lambda(k) < +\infty$. By using (2.25)–(2.27), we have the following estimate:

$$\sum_{k=0}^{+\infty}\alpha(k)^2\|\mathscr{S}_x^i(k)\|^2 \le \alpha(0)^2(D_F + D_{G^+}M_\mu(0) + D_H M_\lambda(0))^2$$
$$+\sum_{k=1}^{+\infty}\alpha(k)^2(D_F + D_{G^+}(M_\mu(0) + G^+ s(k-1)) + D_H(M_\lambda(0) + Hs(k-1)))^2.$$

Then the summability of $\{\alpha(k)^2\}$, $\{\alpha(k+1)^2 s(k)\}$ and $\{\alpha(k+1)^2 s(k)^2\}$ verifies that of $\{\alpha(k)^2\|\mathscr{S}_x^i(k)\|^2\}$.
(b) Consider the dynamics of $\mu^i(k)$ which is in the same form as the distributed projected subgradient algorithm in [13]. Recall that $\{[g(v_x^i(k))]^+\}$ is uniformly bounded. Then following from Lemma 2.5 in the Appendix 2.5 with $Z = \mathbb{R}^m_{\ge 0}$ and $d_i(k) = -[g(v_x^i(k))]^+$, we have the summability of $\{\alpha(k)\max_{i\in V}\|\hat{\mu}(k) - \mu^i(k)\|\}$. Then $\{\alpha(k)\|\hat{\mu}(k) - v_\mu^i(k)\|\}$ is summable by using the following set of inequalities:

$$\|\hat{\mu}(k) - v_\mu^i(k)\| \le \sum_{j=1}^{N} a_j^i(k)\|\hat{\mu}(k) - \mu^j(k)\| \le \max_{i\in V}\|\hat{\mu}(k) - \mu^i(k)\|, \quad (2.28)$$

where we use $\sum_{j=1}^N a_j^i(k) = 1$. Similarly, it holds that $\sum_{k=0}^{+\infty}\alpha(k)\|\hat{\lambda}(k) - v_\lambda^i(k)\| < +\infty$.

We now consider the evolution of $x^i(k)$. Recall that $v_x^i(k) \in X$. By Lemma 1.1 with $Z = X$, $z = v_x^i(k) - \alpha(k)\mathscr{S}_x^i(k)$ and $y = v_x^i(k)$, we have

$$\|x^i(k+1) - v_x^i(k)\|^2 \le \|v_x^i(k) - \alpha(k)\mathscr{S}_x^i(k) - v_x^i(k)\|^2$$
$$-\|x^i(k+1) - (v_x^i(k) - \alpha(k)\mathscr{S}_x^i(k))\|^2,$$

and thus $\|e_x^i(k) + \alpha(k)\mathscr{S}_x^i(k)\| \le \alpha(k)\|\mathscr{S}_x^i(k)\|$. With this relation, from Lemma 2.5 with $Z = X$ and $d_i(k) = \mathscr{S}_x^i(k)$, the following holds for some $\gamma > 0$ and $0 < \beta < 1$:

$$\|x^i(k) - \hat{x}(k)\| \le N\gamma\beta^{k-1}\sum_{i=0}^{N}\|x^i(0)\| + 2N\gamma\sum_{\tau=0}^{k-1}\beta^{k-\tau}\alpha(\tau)\|\mathscr{S}_x^i(\tau)\|. \quad (2.29)$$

Multiplying both sides of (2.29) by $\alpha(k)M_\mu(k)$ and using (2.27), we obtain

$$\alpha(k)M_\mu(k)\|x^i(k) - \hat{x}(k)\| \le N\gamma\sum_{i=0}^{N}\|x^i(0)\|\alpha(k)M_\mu(k)\beta^{k-1} + 2N\gamma\alpha(k)M_\mu(k)$$
$$\times\sum_{\tau=0}^{k-1}\beta^{k-\tau}\alpha(\tau)(D_F + D_{G^+}M_\mu(\tau) + D_H M_\lambda(\tau)).$$

Notice that the above inequalities hold for all $i \in V$. Then by employing the relation of $ab \le \frac{1}{2}(a^2 + b^2)$ and regrouping similar terms, we obtain

$$\alpha(k)M_\mu(k)\max_{i\in V}\|x^i(k)-\hat{x}(k)\| \le N\gamma\left(\frac{1}{2}\sum_{i=0}^{N}\|x^i(0)\|+(D_F+D_{G^+}+D_H)\sum_{\tau=0}^{k-1}\beta^{k-\tau}\right)$$
$$\times\,\alpha(k)^2M_\mu^2(k)+\frac{1}{2}N\gamma\sum_{i=0}^{N}\|x^i(0)\|\beta^{2(k-1)}$$
$$+\,N\gamma\sum_{\tau=0}^{k-1}\beta^{k-\tau}\alpha(\tau)^2(D_F+D_{G^+}M_\mu^2(\tau)+D_HM_\lambda^2(\tau)).$$

Part (a) gives that $\{\alpha(k)^2M_\mu^2(k)\}$ is summable. Combining this fact with the property of $\sum_{\tau=0}^{k-1}\beta^{k-\tau}\le\sum_{k=0}^{+\infty}\beta^k=\frac{1}{1-\beta}$, then we can say that the first term on the right-hand side in the above estimate is summable. It is easy to check that the second term is also summable. It follows from Part (a) that

$$\lim_{k\to+\infty}\alpha(k)^2(D_F+D_{G^+}M_\mu^2(k)+D_HM_\lambda^2(k))=0,$$

and thus $\{\alpha(k)^2(D_F+D_{G^+}M_\mu^2(k)+D_HM_\lambda^2(k))\}$ is summable. Then Lemma 7 in [13] with $\gamma_\ell=N\gamma\alpha(\ell)^2(D_F+D_{G^+}M_\mu^2(\ell)+D_HM_\lambda^2(\ell))$ ensures that the third term is summable. Therefore, the summability of $\{\alpha(k)M_\mu(k)\max_{i\in V}\|x^i(k)-\hat{x}(k)\|\}$ is guaranteed. Following the same lines in (2.28), one can show the summability of $\{\alpha(k)M_\mu(k)\|v_x^i(k)-\hat{x}(k)\|\}$. Following analogous arguments, we have that $\{\alpha(k)M_\lambda(k)\|v_x^i(k)-\hat{x}(k)\|\}$ and $\{\alpha(k)\|v_x^i(k)-\hat{x}(k)\|\}$ are summable. ■

Remark 2.5 In Lemma 2.11, the assumption of all local constraint sets being identical is utilized to find an upper bound of the convergence rate of $\|\hat{x}(k)-v_x^i(k)\|$ to zero. This property is crucial to establish the summability of expansions pertaining to $\|\hat{x}(k)-v_x^i(k)\|$ in part (b). •

The following is a basic iteration relation of the DISTRIBUTED PENALTY PRIMAL-DUAL SUBGRADIENT ALGORITHM.

Lemma 2.12 (Basic iteration relation) *The following estimates hold for any $x\in X$ and $(\mu,\lambda)\in\mathbb{R}^m_{\ge0}\times\mathbb{R}^\nu_{\ge0}$:*

$$\sum_{i=1}^{N}\|e_x^i(k)+\alpha(k)\mathscr{S}_x^i(k)\|^2\le\sum_{i=1}^{N}\alpha(k)^2\|\mathscr{S}_x^i(k)\|^2$$
$$-\sum_{i=1}^{N}2\alpha(k)(\mathscr{H}_i(v_x^i(k),v_\mu^i(k),v_\lambda^i(k))-\mathscr{H}_i(x,v_\mu^i(k),v_\lambda^i(k)))$$
$$+\sum_{i=1}^{N}(\|x^i(k)-x\|^2-\|x^i(k+1)-x\|^2),\qquad(2.30)$$

$$
\begin{aligned}
0 \leq & \sum_{i=1}^{N}(\|\mu^i(k)-\mu\|^2 - \|\mu^i(k+1)-\mu\|^2) \\
& + \sum_{i=1}^{N}(\|\lambda^i(k)-\lambda\|^2 - \|\lambda^i(k+1)-\lambda\|^2) + \\
& \sum_{i=1}^{N} 2\alpha(k)(\mathscr{H}_i(v_x^i(k), v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}_i(v_x^i(k), \mu, \lambda)) \\
& + \sum_{i=1}^{N} \alpha(k)^2(\|[g(v_x^i(k))]^+\|^2 + \|h(v_x^i(k))\|^2). \qquad (2.31)
\end{aligned}
$$

Proof One can finish the proof by following analogous arguments in Lemma 2.8. ■

Lemma 2.13 (Achieving consensus) *Let us suppose that the nondegeneracy Assumption 1.1, the double stochasticity Assumption 1.2, and the periodical strong connectivity Assumption 1.3 hold. Consider the sequences of* $\{x^i(k)\}$, $\{\mu^i(k)\}$, $\{\lambda^i(k)\}$, *and* $\{y^i(k)\}$ *of the distributed penalty primal-dual subgradient algorithm with the step-size sequence* $\{\alpha(k)\}$ *and the associated* $\{s(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$ *and* $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$. *Then there exists* $\tilde{x} \in X$ *such that* $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ *for all* $i \in V$. *Furthermore,* $\lim_{k\to+\infty} \|\mu^i(k) - \mu^j(k)\| = 0$, $\lim_{k\to+\infty} \|\lambda^i(k) - \lambda^j(k)\| = 0$ *and* $\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0$ *for all* $i, j \in V$.

Proof Similar to (2.16), we have

$$
\begin{aligned}
& \sum_{i=1}^{N} \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^{N} \|x^i(k) - x\|^2 \\
& + \sum_{i=1}^{N} \alpha(k)^2 \|\mathscr{S}_x^i(k)\|^2 + \sum_{i=1}^{N} 2\alpha(k)\|\mathscr{S}_x^i(k)\|\|v_x^i(k) - x\|.
\end{aligned}
$$

Since $\lim_{k\to+\infty} \alpha(k)\|\mathscr{S}_x^i(k)\| = 0$, the proofs of $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ for all $i \in V$ are analogous to those in Lemma 2.9. The remainder of the proofs can be finished by Theorem 1.4 with the properties of $\lim_{k\to+\infty} u_\mu^i(k) = 0$, $\lim_{k\to+\infty} u_\lambda^i(k) = 0$ and $\lim_{k\to+\infty} u_y^i(k) = 0$ (due to $\lim_{k\to+\infty} x^i(k) = \tilde{x}$ and f_i is continuous). ■

We now proceed to show Theorem 2.3 based on five claims.

Proof of Theorem 2.3:

Claim 2.3 *For any* $x^* \in X^*$ *and* $(\mu^*, \lambda^*) \in D_P^*$, *the following sequences are summable:*

$$\left\{\alpha(k)\left[\sum_{i=1}^{N}\mathscr{H}_i(x^*, v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}(x^*, \hat{\mu}(k), \hat{\lambda}(k))\right]\right\},$$

$$\left\{\alpha(k)\left[\sum_{i=1}^{N}\mathscr{H}_i(v_x^i(k), \mu^*, \lambda^*) - \mathscr{H}(\hat{x}(k), \mu^*, \lambda^*)\right]\right\}$$

Proof Observe that

$$\begin{aligned}
&\|\mathscr{H}_i(x^*, v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}_i(x^*, \hat{\mu}(k), \hat{\lambda}(k))\| \\
&\leq \|v_\mu^i(k) - \hat{\mu}(k)\|\|[g(x^*)]^+\| + \|v_\lambda^i(k) - \hat{\lambda}(k)\|\|h(x^*)\| \\
&\leq G^+\|v_\mu^i(k) - \hat{\mu}(k)\| + H\|v_\lambda^i(k) - \hat{\lambda}(k)\|.
\end{aligned} \tag{2.32}$$

By using the summability of $\{\alpha(k)\|\hat{\mu}(k) - v_\mu^i(k)\|\}$ and $\{\alpha(k)\|\hat{\lambda}(k) - v_\lambda^i(k)\|\}$ in Part (b) of Lemma 2.11, we have that the following are summable:

$$\left\{\alpha(k)\sum_{i=1}^{N}\|\mathscr{H}_i(x^*, v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}_i(x^*, \hat{\mu}(k), \hat{\lambda}(k))\|\right\},$$

$$\left\{\alpha(k)\left[\sum_{i=1}^{N}\left(\mathscr{H}_i(x^*, v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}_i(x^*, \hat{\mu}(k), \hat{\lambda}(k))\right)\right]\right\}.$$

Similarly, the following estimates hold:

$$\begin{aligned}
&\|\mathscr{H}_i(v_x^i(k), \mu^*, \lambda^*) - \mathscr{H}_i(\hat{x}(k), \mu^*, \lambda^*)\| \\
&\leq \|f_i(v_x^i(k)) - f_i(\hat{x}(k))\| + \|(\mu^*)^T([g(v_x^i(k))]^+ - [g(\hat{x}(k))]^+)\| \\
&+ \|(\lambda^*)^T(|h(v_x^i(k))| - |h(\hat{x}(k))|)\| \\
&\leq (D_F + D_{G^+}\|\mu^*\| + D_H\|\lambda^*\|)\|v_x^i(k) - \hat{x}(k)\|.
\end{aligned}$$

Then the property of $\sum_{k=0}^{+\infty}\alpha(k)\|\hat{x}(k) - v_x^i(k)\| < +\infty$ in Part (b) of Lemma 2.11 implies the summability of the following sequences:

$$\left\{\alpha(k)\sum_{i=1}^{N}\|\mathscr{H}_i(v_x^i(k), \mu^*, \lambda^*) - \mathscr{H}_i(\hat{x}(k), \mu^*, \lambda^*)\|\right\},$$

$$\left\{\alpha(k)\sum_{i=1}^{N}\left(\mathscr{H}_i(v_x^i(k), \mu^*, \lambda^*) - \mathscr{H}_i(\hat{x}(k), \mu^*, \lambda^*)\right)\right\}.$$

■

Claim 2.4 *Denote the weighted version of $\mathscr{H}_i$ as*

$$\hat{\mathscr{H}_i}(k) \triangleq \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \mathscr{H}_i(v_x^i(\ell), v_\mu^i(\ell), v_\lambda^i(\ell)).$$

The following property holds: $\lim_{k\to+\infty} \sum_{i=1}^{N} \hat{\mathscr{H}_i}(k) = p^*$.

Proof Summing (2.30) over $[0, k-1]$ and replacing x by $x^* \in X^*$ leads to

$$\sum_{\ell=0}^{k-1} \alpha(\ell) \sum_{i=1}^{N} (\mathscr{H}_i(v_x^i(\ell), v_\mu^i(\ell), v_\lambda^i(\ell)) - \mathscr{H}_i(x^*, v_\mu^i(\ell), v_\lambda^i(\ell)))$$
$$\leq \sum_{i=1}^{N} \|x^i(0) - x^*\|^2 + \sum_{\ell=0}^{k-1} \sum_{i=1}^{N} \alpha(\ell)^2 \|\mathscr{S}_x^i(\ell)\|^2. \tag{2.33}$$

The summability of $\{\alpha(k)^2 \|\mathscr{S}_x^i(k)\|^2\}$ in Part (b) of Lemma 2.11 implies that the right-hand side of (2.33) is finite as $k \to +\infty$, and thus

$$\limsup_{k\to\infty} \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \left[\sum_{i=1}^{N} \left(\mathscr{H}_i(v_x^i(\ell), v_\mu^i(\ell), v_\lambda^i(\ell)) - \mathscr{H}_i(x^*, v_\mu^i(\ell), v_\lambda^i(\ell)) \right) \right] \leq 0. \tag{2.34}$$

Pick any $(\mu^*, \lambda^*) \in D_P^*$. It follows from Theorem 2.2 that (x^*, μ^*, λ^*) is a saddle point of $\mathscr{H}$ over $X \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$. Since $(\hat{\mu}(k), \hat{\lambda}(k)) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$, then we have $\mathscr{H}(x^*, \hat{\mu}(k), \hat{\lambda}(k)) \leq \mathscr{H}(x^*, \mu^*, \lambda^*) = p^*$. Combining this relation, Claim 2.3 and (2.34) renders that

$$\limsup_{k\to+\infty} \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \left[\sum_{i=1}^{N} \mathscr{H}_i(v_x^i(\ell), v_\mu^i(\ell), v_\lambda^i(\ell)) - p^* \right]$$
$$\leq \limsup_{k\to+\infty} \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \left[\sum_{i=1}^{N} \left(\mathscr{H}_i(v_x^i(\ell), v_\mu^i(\ell), v_\lambda^i(\ell)) - \mathscr{H}_i(x^*, v_\mu^i(\ell), v_\lambda^i(\ell)) \right) \right]$$
$$+ \limsup_{k\to+\infty} \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \left[\sum_{i=1}^{N} \mathscr{H}_i(x^*, v_\mu^i(\ell), v_\lambda^i(\ell)) - \mathscr{H}(x^*, \hat{\mu}(\ell), \hat{\lambda}(\ell)) \right]$$
$$+ \limsup_{k\to+\infty} \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} (\mathscr{H}(x^*, \hat{\mu}(\ell), \hat{\lambda}(\ell)) - p^*) \leq 0,$$

and thus $\limsup_{k\to+\infty} \sum_{i=1}^{N} \hat{\mathscr{H}_i}(k) \leq p^*$.

On the other hand, $\hat{x}(k) \in X$ (due to the fact that X is convex) implies that $\mathscr{H}(\hat{x}(k), \mu^*, \lambda^*) \geq \mathscr{H}(x^*, \mu^*, \lambda^*) = p^*$. Along similar lines, by using (2.31) with $\mu = \mu^*$, $\lambda = \lambda^*$, and Claim 2.3, we have $\liminf_{k\to+\infty} \sum_{i=1}^{N} \hat{\mathscr{H}_i}(k) \geq p^*$. Then we have the desired relation. ■

Claim 2.5 *Denote by* $\pi(k) \triangleq \sum_{i=1}^{N} \mathscr{H}_i(v_x^i(k), v_\mu^i(k), v_\lambda^i(k)) - \mathscr{H}(\hat{x}(k), \hat{\mu}(k), \hat{\lambda}(k))$. *And we denote the weighted version of* $\mathscr{H}$ *as*

$$\Gamma(k) \triangleq \frac{1}{s(k-1)} \sum_{\ell=0}^{k-1} \alpha(\ell) \mathscr{H}(\hat{x}(\ell), \hat{\mu}(\ell), \hat{\lambda}(\ell)).$$

The following property holds: $\lim_{k \to +\infty} \Gamma(k) = p^*$.

Proof Notice that

$$\begin{aligned}\pi(k) = &\sum_{i=1}^{N} (f_i(v_x^i(k)) - f_i(\hat{x}(k))) + \sum_{i=1}^{N} \left(v_\mu^i(k)^T [g(v_x^i(k))]^+ - v_\mu^i(k)^T [g(\hat{x}(k))]^+ \right) \\ &+ \sum_{i=1}^{N} \left(v_\mu^i(k)^T [g(\hat{x}(k))]^+ - \hat{\mu}(k)^T [g(\hat{x}(k))]^+ \right) \\ &+ \sum_{i=1}^{N} \left(v_\lambda^i(k)^T |h(v_x^i(k))| - v_\lambda^i(k)^T |h(\hat{x}(k))| \right) \\ &+ \sum_{i=1}^{N} \left(v_\lambda^i(k)^T |h(\hat{x}(k))| - \hat{\lambda}(k)^T |h(\hat{x}(k))| \right). \end{aligned} \tag{2.35}$$

By using the boundedness of subdifferentials and the primal estimates, it follows from (2.35) that

$$\begin{aligned}\|\pi(k)\| \le &(D_F + D_{G^+} M_\mu(k) + D_H M_\lambda(k)) \times \sum_{i=1}^{N} \|v_x^i(k) - \hat{x}(k)\| \\ &+ G^+ \sum_{i=1}^{N} \|v_\mu^i(k) - \hat{\mu}(k)\| + H \sum_{i=1}^{N} \|v_\lambda^i(k) - \hat{\lambda}(k)\|. \end{aligned} \tag{2.36}$$

Then it follows from (b) in Lemma 2.11 that $\{\alpha(k)\|\pi(k)\|\}$ is summable. Notice that $\|\Gamma(k) - \sum_{i=1}^{N} \hat{\mathscr{H}}_i(k)\| \le \frac{\sum_{\ell=0}^{k-1} \alpha(\ell)\|\pi(\ell)\|}{s(k-1)}$, and thus $\lim_{k \to +\infty} \|\Gamma(k) - \sum_{i=1}^{N} \hat{\mathscr{H}}_i(k)\| = 0$. The desired result immediately follows from Claim 2.4. ∎

Claim 2.6 *The limit point* $\tilde{x}$ *in Lemma 2.13 is a primal optimal solution.*

Proof Let $\hat{\mu}(k) = (\hat{\mu}_1(k), \ldots, \hat{\mu}_m(k))^T \in \mathbb{R}^m_{\ge 0}$. By the double stochasticity Assumption 1.2, we obtain

$$\sum_{i=1}^{N} \mu^i(k+1) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_j^i(k) \mu^j(k) + \alpha(k) \sum_{i=1}^{N} [g(v_x^i(k))]^+$$

$$= \sum_{j=1}^{N} \mu^j(k) + \alpha(k) \sum_{i=1}^{N} [g(v_x^i(k))]^+.$$

This implies that the sequence $\{\hat{\mu}_\ell(k)\}$ is nondecreasing in $\mathbb{R}_{\geq 0}$. Observe that $\{\hat{\mu}_\ell(k)\}$ is lower bounded by zero. In this way, we distinguish the following two cases:

Case 1: The sequence $\{\hat{\mu}_\ell(k)\}$ is upper bounded. Then $\{\hat{\mu}_\ell(k)\}$ is convergent in $\mathbb{R}_{\geq 0}$. Recall that $\lim_{k\to+\infty} \|\mu^i(k) - \mu^j(k)\| = 0$ for all $i, j \in V$. This implies that there exists $\mu_\ell^* \in \mathbb{R}_{\geq 0}$ such that $\lim_{k\to+\infty} \|\mu_\ell^i(k) - \mu_\ell^*\| = 0$ for all $i \in V$. Observe that $\sum_{i=1}^N \mu^i(k+1) = \sum_{i=1}^N \mu^i(0) + \sum_{\tau=0}^k \alpha(\tau) \sum_{i=1}^N [g(v_x^i(\tau))]^+$. Thus, we have the property of $\sum_{k=0}^{+\infty} \alpha(k) \sum_{i=1}^N [g_\ell(v_x^i(k))]^+ < +\infty$, further implying the property of $\liminf_{k\to+\infty} [g_\ell(v_x^i(k))]^+ = 0$. Since $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ for all $i \in V$, then it holds that $\lim_{k\to+\infty} \|v_x^i(k) - \tilde{x}\| = 0$, implying $[g_\ell(\tilde{x})]^+ = 0$.

Case 2: The sequence $\{\hat{\mu}_\ell(k)\}$ is not upper bounded. Since $\{\hat{\mu}_\ell(k)\}$ is nondecreasing, then $\hat{\mu}_\ell(k) \to +\infty$. It follows from Claim 2.5 and (a) in Lemma 2.6 that it is impossible that $\mathscr{H}(\hat{x}(k), \hat{\mu}(k), \hat{\lambda}(k)) \to +\infty$. Assume that $[g_\ell(\tilde{x})]^+ > 0$. Then we have

$$\begin{aligned}\mathscr{H}(\hat{x}(k), \hat{\mu}(k), \hat{\lambda}(k)) &= f(\hat{x}(k)) + N\hat{\mu}(k)^T [g(\hat{x}(k))]^+ + N\lambda(k)^T |h(\hat{x}(k))| \\ &\geq f(\hat{x}(k)) + \hat{\mu}_\ell(k)[g_\ell(\hat{x}(k))]^+.\end{aligned} \tag{2.37}$$

Taking limits on both sides of (2.37) and we obtain:

$$\liminf_{k\to+\infty} \mathscr{H}(\hat{x}(k), \hat{\mu}(k), \hat{\lambda}(k)) \geq \limsup_{k\to+\infty} (f(\hat{x}(k)) + \hat{\mu}_\ell(k)[g_\ell(\hat{x}(k))]^+) = +\infty.$$

Then we reach a contradiction, implying that $[g_\ell(\tilde{x})]^+ = 0$.

In both cases, we have $[g_\ell(\tilde{x})]^+ = 0$ for any $1 \leq \ell \leq m$. By utilizing similar arguments, we can further prove that $|h(\tilde{x})| = 0$. Since $\tilde{x} \in X$, then $\tilde{x}$ is feasible and thus $f(\tilde{x}) \geq p^*$. On the other hand, since $\frac{\sum_{\ell=0}^{k-1} \alpha(\ell)\hat{x}(\ell)}{\sum_{\ell=0}^{k-1} \alpha(\ell)}$ is a convex combination of $\hat{x}(0), \ldots, \hat{x}(k-1)$ and $\lim_{k\to+\infty} \hat{x}(k) = \tilde{x}$, then Claim 2.5 and (b) in Lemma 2.6 implies that

$$\begin{aligned}p^* = \lim_{k\to+\infty} \Gamma(k) &= \lim_{k\to+\infty} \frac{\sum_{\ell=0}^{k-1} \alpha(\ell)\mathscr{H}(\hat{x}(\ell), \hat{\mu}(\ell), \hat{\lambda}(\ell))}{\sum_{\ell=0}^{k-1} \alpha(\ell)} \\ &\geq \lim_{k\to+\infty} f\left(\frac{\sum_{\ell=0}^{k-1} \alpha(\ell)\hat{x}(\ell)}{\sum_{\ell=0}^{k-1} \alpha(\ell)}\right) = f(\tilde{x}).\end{aligned}$$

Hence, we have $f(\tilde{x}) = p^*$ and thus $\tilde{x} \in X^*$. ■

Claim 2.7 *It holds that* $\lim_{k\to+\infty}\|y^i(k)-p^*\|=0$.

Proof The proof follows the same lines in Claim 2.2 of Theorem 2.1 and thus is omitted here. ■

2.6 Notes

Distributed optimization traces back to 1970s. In [14], the classic dual decomposition approach is proposed to the class of distributed optimization problems characterized by separable component functions. This approach has been successfully applied to handle network utility maximization (NUM) in; e.g., [4, 15, 16]. In [17, 18], the authors develop a general framework for parallel and distributed computation over a set of processors.

Recently, diffusion consensus algorithms have been integrated into distributed optimization to address the nonseparability in component functions and dynamic changes of network topologies. In particular, distributed projected subgradient algorithms are proposed in [13] to address non-smooth multi-agent optimization with constraint sets. The paper [19] comes up with Distributed Lagrangian Primal-Dual Subgradient Algorithm and Distributed Penalty Primal-Dual Subgradient Algorithm to further address inequality and equality constraints. The results developed in [19] are extended to solve a class of distributed nonconvex optimization problems in [20]. All the algorithms aforementioned are discrete-time. The continuous-time counterparts are investigated in [21–23]. In [24], a distributed continuous-time algorithm with discrete-time communication is proposed. Random network and state-dependent topologies are investigated in [25, 26] respectively.

There have been a number of other interesting algorithms for distributed optimization. The authors in [27, 28] apply the second-order Newton method to distributed optimization. The paper [29] studies the dual averaging algorithm and the papers [30, 31] investigate the algorithm of Alternating Direction Method of Multipliers. Distributed Nesterov gradient algorithms are developed in [32] to accelerate the convergence. In [33], the authors aim to minimize a sequence of dynamically changing convex functions. In [34, 35], the authors investigate the robustness of distributed algorithms against external disturbances. In [36], game design is utilized to address distributed optimization. In [37], the authors propose a distributed algorithm to compute Pareto optimal solutions of multiobjective optimization problems.

Distributed Lagrangian Primal-Dual Subgradient Algorithm and Distributed Penalty Primal-Dual Subgradient Algorithm presented in this chapter are built on saddle point dynamics. For a convex–concave function, continuous-time saddle point dynamics is proved in [8] to converge globally towards a saddle point. Recently, [9] presents (discrete-time) primal-dual subgradient methods which relax the differentiability of [8] and further incorporate state constraints. The method in [8] is adopted by [38, 39] to study a distributed optimization problem on fixed graphs where objective functions are separable.

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Chapter 3
Game Theoretic Optimal Sensor Deployment

3.1 Introduction

In noncooperative games, players are self-interested and aim to maximize their own utilities. The selfishness makes the decision-making of players naturally distributed. This attractive feature is evident in a large number of game theoretic learning algorithms, e.g., better and best reply dynamics. As a result, game theoretic learning provides a powerful arsenal for the synthesis of efficient distributed algorithms to multi-agent networks. This chapter will discuss one of its applications to sensor deployment.

There is a widespread belief that continuous and pervasive monitoring will be possible in the near future with large numbers of networked, mobile, and wireless sensors. Thus, we are witnessing an intense research activity that focuses on the design of efficient control mechanisms for these systems. In particular, decentralized algorithms would allow sensor networks to react autonomously to changes in the environment with minimal human supervision.

A substantial body of research on sensor networks has concentrated on simple sensors that can collect scalar data, e.g., temperature, humidity, or pressure data. Here, a main objective is the design of algorithms that can lead to optimal collective sensing through efficient motion control and communication schemes. However, scalar measurements can be insufficient in many situations, e.g., in automated surveillance or traffic monitoring applications. In contrast, data-intensive sensors such as cameras can collect visual data that are rich in information, thus having tremendous potential for monitoring applications, but at the cost of higher processing overhead.

Precisely, this chapter aims to solve a coverage optimization problem taking into account part of the sensing/processing trade-off. Coverage optimization problems have mainly been formulated as cooperative problems where each sensor benefits from sensing the environment as a member of a group. However, sensing may also require expenditure, e.g., the energy consumed or the time spent by image processing algorithms in visual networks. Because of this, we endow each sensor with a utility

M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*, SpringerBriefs in Control, Automation and Robotics, DOI 10.1007/978-3-319-19072-3_3

function that quantifies this trade-off, formulating a coverage problem as a variation of congestion games in [1]. The coverage game we consider here is shown to be a (constrained) exact potential game. A number of learning rules, e.g., better (or best) reply dynamics and adaptive play, have been proposed to reach Nash equilibria in potential games. In these algorithms, each player must have access to the utility values induced by alternative actions. However, *this information is unaccessible* in our problem setup because of the information constraints caused by unknown rewards, motion, and sensing limitations. To tackle this challenge, we develop two distributed payoff-based learning algorithms where each sensor only remembers its own utility values and actions played during the last two plays. Our algorithms extend the use of the payoff-based learning dynamics first novelly proposed in [2, 3].

In the first algorithm, at each time step, each sensor repeatedly updates its action synchronously, either trying some new action in the state-dependent feasible action set or selecting the action which corresponds to a higher utility value in the most recent two time steps. As the algorithm for the special identical interest games in [3], the first algorithm employs a diminishing exploration rate. The dynamically changing exploration rate renders the algorithm a time-inhomogeneous Markov chain, and allows for the convergence in probability to the set of (constrained) Nash equilibria, from which no agent is willing to unilaterally deviate.

The second algorithm is asynchronous. At each time step, only one sensor is active and updates its state by either trying some new action in the state-dependent feasible action set or selecting an action according to a Gibbs-like distribution from those played in the last two time steps when it was active. The algorithm is shown to be convergent in probability to the set of global maxima of a coverage performance metric. Rather than maximizing the associated potential function in [2], the second algorithm optimizes the sum of local utility functions which captures better a global trade-off between the overall network benefit from sensing and the total energy the network consumes. By employing a diminishing exploration rate, our algorithm is guaranteed to have stronger convergence properties than the ones in [2].

3.2 Problem Formulation

We refer the reader to the basic game-theoretic concepts introduced in Chap. 1, Sect. 1.4. These will allow us to formulate subsequently an optimal coverage problem for mobile visual sensor networks as a repeated multi-player game.

3.2.1 Coverage Game

3.2.1.1 Mission Space

We consider a convex 2-D mission space that is discretized into a (squared) lattice. We assume that each square of the lattice has unit dimensions. Each square will be labeled with the coordinate of its center $q = (q_x, q_y)$, where $q_x \in [q_{x_{\min}}, q_{x_{\max}}]$ and $q_y \in [q_{y_{\min}}, q_{y_{\max}}]$, for some integers $q_{x_{\min}}, q_{y_{\min}}, q_{x_{\max}}, q_{y_{\max}}$. Denote by $\mathcal{Q}$ the collection of all squares of the lattice.

We now define an associated location graph $\mathcal{G}_{\text{loc}} \triangleq (\mathcal{Q}, E_{\text{loc}})$ where $((q_x, q_y), (q_{x'}, q_{y'})) \in E_{\text{loc}}$ if and only if $|q_x - q_{x'}| + |q_y - q_{y'}| = 1$ for $(q_x, q_y), (q_{x'}, q_{y'}) \in \mathcal{Q}$. Note that the graph $\mathcal{G}_{\text{loc}}$ is undirected; i.e., $(q, q') \in E_{\text{loc}}$ if and only if $(q', q) \in E_{\text{loc}}$. The set of neighbors of q in E_{loc} is given by $\mathcal{N}_q^{\text{loc}} \triangleq \{q' \in \mathcal{Q} \setminus \{q\} \mid (q, q') \in E_{\text{loc}}\}$. We assume that the location graph $\mathcal{G}_{\text{loc}}$ is fixed and connected, and denote its diameter by D.

Agents are deployed in $\mathcal{Q}$ to detect certain events of interest. As agents move in $\mathcal{Q}$ and process measurements, they will assign a numerical value $W_q \geq 0$ to the events in each square with center $q \in \mathcal{Q}$. If $W_q = 0$, then there is no significant event at the square with center q. The larger the value of W_q is, the more interest the set of events at the square with center q is of. Later, the amount W_q will be identified with the benefit of observing the point q. In this setup, we assume the values W_q to be constant in time. Furthermore, W_q is not a priori knowledge to the agents, but the agents can measure this value through sensing the point q.

3.2.1.2 Modeling of the Visual Sensor Nodes

Each mobile agent i is modeled as a point mass in $\mathcal{Q}$, with location $a_i \triangleq (x_i, y_i) \in \mathcal{Q}$. Each agent has mounted a pan-tilt-zoom camera, and can adjust its orientation and focal length.

The visual sensing range of a camera is directional, limited-range, and has a finite angle of view. Following a geometric simplification, we model the visual sensing region of agent i as an annulus sector in the 2-D plane; see Figs. 3.1 and 3.2.

The visual sensor footprint is completely characterized by the following parameters: the position of agent i, $a_i \in \mathcal{Q}$, the camera orientation, $\theta_i \in [0, 2\pi)$, the

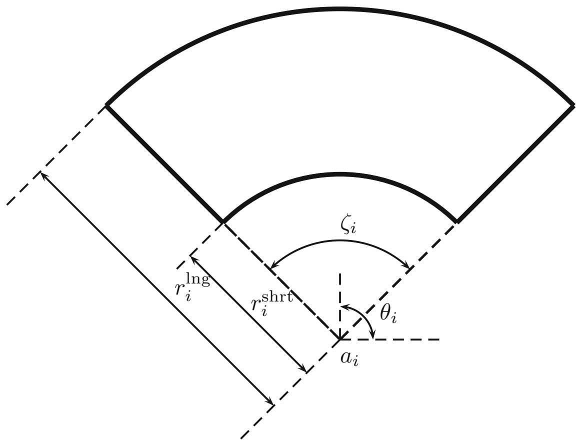

Fig. 3.1 Visual sensor footprint

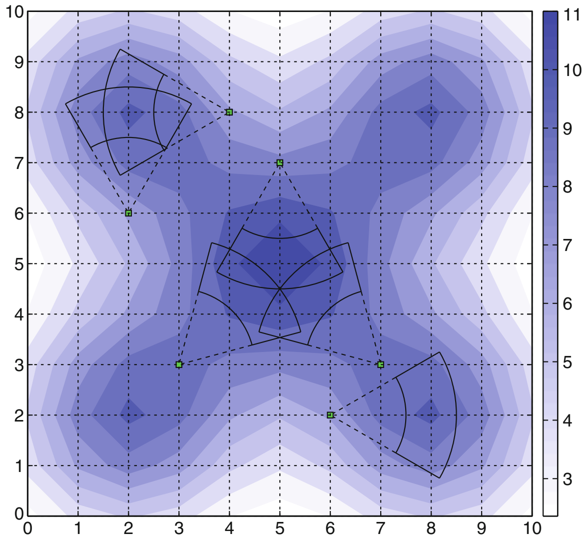

Fig. 3.2 A configuration of the mobile sensor network

camera angle of view, $\zeta_i \in [\alpha_{\min}, \alpha_{\max}]$, and the shortest range (resp. longest range) between agent i and the nearest (resp. farthest) object that can be recognized from the image, $r_i^{\text{shrt}} \in [r_{\min}, r_{\max}]$ (resp. $r_i^{\text{lng}} \in [r_{\min}, r_{\max}]$). The parameters r_i^{shrt}, r_i^{lng}, ζ_i can be tuned by changing the focal length FL_i of agent i's camera. In this way, $c_i := (\text{FL}_i, \theta_i) \in [0, \text{FL}_{\max}] \times [0, 2\pi)$ is the camera control vector of agent i. In what follows, we will assume that c_i takes values in a finite subset $\mathscr{C} \subset [0, \text{FL}_{\max}] \times [0, 2\pi)$. An agent action is thus a vector $s_i \triangleq (a_i, c_i) \in \mathscr{A}_i \triangleq \mathscr{Q} \times \mathscr{C}$, and a multi-agent action is denoted by $s = (s_1, \ldots, s_N) \in \mathscr{A} \triangleq \Pi_{i=1}^N \mathscr{A}_i$.

Let $\mathscr{D}(a_i, c_i)$ be the visual sensor footprint of agent i. Now we can define a proximity sensing graph[1] $\mathscr{G}_{\text{sen}}(s) \triangleq (V, E_{\text{sen}}(s))$ as follows: the set of neighbors of agent i, $\mathscr{N}_i^{\text{sen}}(s)$, is given as $\mathscr{N}_i^{\text{sen}}(s) \triangleq \{j \in V \backslash \{i\} \mid \mathscr{D}(a_i, c_i) \cap \mathscr{D}(a_j, c_j) \cap \mathscr{Q} \neq \emptyset\}$.

Each agent is able to communicate with others to exchange information. We assume that the communication range of agents is $2r_{\max}$. This induces a $2r_{\max}$-disk communication graph $\mathscr{G}_{\text{comm}}(s) \triangleq (V, E_{\text{comm}}(s))$ as follows: the set of neighbors of agent i is given by $\mathscr{N}_i^{\text{comm}}(s) \triangleq \left\{j \in V \backslash \{i\} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \leq (2r_{\max})^2\right\}$. Note that $\mathscr{G}_{\text{comm}}(s)$ is undirected and that $\mathscr{G}_{\text{sen}}(s) \subseteq \mathscr{G}_{\text{comm}}(s)$.

The motion of agents will be limited to a neighboring point in $\mathscr{G}_{\text{loc}}$ at each time step. Thus, an agent feasible action set will be given by $\mathscr{F}_i(a_i) \triangleq (\{a_i\} \cup \mathscr{N}_{a_i}^{\text{loc}}) \times \mathscr{C}$.

3.2.1.3 Coverage Game

We now proceed to formulate a coverage optimization problem as a constrained strategic game. For each $q \in \mathscr{Q}$, we denote $n_q(s)$ as the cardinality of the set

[1] See [4] for a definition of proximity graph.

$\{k \in V \mid q \in \mathscr{D}(a_k, c_k) \cap \mathscr{Q}\}$; i.e., the number of agents which can observe the point q. The "profit" given by W_q will be equally shared by agents that can observe the point q. The benefit that agent i obtains through sensing is thus defined by $\sum_{q \in \mathscr{D}(a_i,c_i)\cap\mathscr{Q}} \frac{W_q}{n_q(s)}$.

On the other hand, and as argued in [5], the processing of visual data can incur a higher cost than that of communication. This is in contrast with scalar sensor networks, where the communication cost dominates. With this observation, we model the energy consumption of agent i by $f_i(c_i) \triangleq \frac{1}{2}\zeta_i((r_i^{\text{lng}})^2 - (r_i^{\text{shrt}})^2)$. A similar energy model is used in [6] and references therein. This measure corresponds to the area of the visual sensor footprint and can serve to approximate the energy consumption or the cost incurred by image processing algorithms.

We will endow each agent with a utility function that aims to capture the above sensing/processing trade-off. In this way, we define a utility function for agent i by

$$u_i(s) = \sum_{q \in \mathscr{D}(a_i,c_i)\cap\mathscr{Q}} \frac{W_q}{n_q(s)} - f_i(c_i).$$

Note that the utility function u_i is local over the visual sensing graph $\mathscr{G}_{\text{sen}}(s)$; i.e., u_i is only dependent on the actions of $\{i\} \cup \mathscr{N}_i^{\text{sen}}(s)$. With the set of utility functions $U_{\text{cov}} = \{u_i\}_{i\in V}$, and feasible action set $\mathscr{F}_{\text{cov}} = \Pi_{i=1}^N \bigcup_{a_i\in\mathscr{A}_i} \mathscr{F}_i(a_i)$, we now have all the ingredients to introduce the coverage game $\Gamma_{\text{cov}} \triangleq \langle V, \mathscr{A}, U_{\text{cov}}, \mathscr{F}_{\text{cov}}\rangle$. This game is a variation of the congestion games introduced in [1].

Lemma 3.1 *The coverage game Γ_{cov} is a constrained exact potential game with potential function*

$$\phi(s) = \sum_{q\in\mathscr{Q}} \sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} - \sum_{i=1}^{N} f_i(c_i).$$

Proof The proof is a slight variation of that in [1]. Consider any $s \triangleq (s_i, s_{-i}) \in \mathscr{A}$ where $s_i \triangleq (a_i, c_i)$. We fix $i \in V$ and pick any $s_i' = (a_i', c_i')$ from $\mathscr{F}_i(a_i)$. Denote $s' \triangleq (s_i', s_{-i})$, $\Omega_1 \triangleq (\mathscr{D}(a_i, c_i)\backslash\mathscr{D}(a_i', c_i')) \cap \mathscr{Q}$ and $\Omega_2 \triangleq (\mathscr{D}(a_i', c_i')\backslash\mathscr{D}(a_i, c_i)) \cap \mathscr{Q}$. Observe that

$$\begin{aligned}
&\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) \\
&= \sum_{q\in\Omega_1} (\sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} - \sum_{\ell=1}^{n_q(s')} \frac{W_q}{\ell}) + \sum_{q\in\Omega_2} (-\sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} + \sum_{\ell=1}^{n_q(s')} \frac{W_q}{\ell}) - f_i(c_i) + f_i(c_i') \\
&= \sum_{q\in\Omega_1} \frac{W_q}{n_q(s)} - \sum_{q\in\Omega_2} \frac{W_q}{n_q(s')} - f_i(c_i) + f_i(c_i') \\
&= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})
\end{aligned}$$

where in the second equality we utilize the fact that for each $q \in \Omega_1$, $n_q(s) = n_q(s') + 1$, and each $q \in \Omega_2$, $n_q(s') = n_q(s) + 1$. ∎

We denote by $\mathscr{E}(\Gamma_{\text{cov}})$ the set of constrained NEs of Γ_{cov}. It is worth mentioning that $\mathscr{E}(\Gamma_{\text{cov}}) \neq \emptyset$ due to the fact that Γ_{cov} is a constrained exact potential game.

Remark 3.1 The assumptions of our problem formulation admit several extensions. For example, it is straightforward to extend our results to non-convex 3-D spaces. This is because the results that follow can also handle other shapes of the sensor footprint; e.g., a complete disk, a subset of the annulus sector. On the other hand, note that the coverage problem can be interpreted as a target assignment problem—here, the value $W_q \geq 0$ would be associated with the value of a target located at the point q. •

3.2.2 Our Objective

In our coverage problem, we assume that W_q is unknown to all the sensors in advance. Furthermore, due to the restrictions of motion and sensing, each agent is unable to obtain the information of W_q if the point q is outside its sensing range. In addition, the utility of each agent depends on the group strategy. These information constraints render that each agent is unable to access the utility values induced by alternative actions. Thus the action-based learning algorithms, e.g., better (or best) reply learning algorithm and adaptive play learning algorithm cannot be employed to solve our coverage games. It motivates us to design distributed learning algorithms which only require the payoff received.

> [Objective] We aim to design and analyze distributed payoff-based algorithms which allow sensors to identify optimal configuration.

3.2.3 Notations

In the following, we will use the Landau symbol, O, as in $O(\varepsilon^{\iota})$, for some $\iota \geq 0$. This implies that $0 < \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^{\iota})}{\varepsilon^{\iota}} < +\infty$. We denote $\text{diag}(\mathscr{A}) \triangleq \{(s, s) \in \mathscr{A}^2 |\ s \in \mathscr{A}\}$ and $\text{diag}(\mathscr{E}(\Gamma_{\text{cov}})) \triangleq \{(s, s) \in \mathscr{A}^2 |\ s \in \mathscr{E}(\Gamma_{\text{cov}})\}$.

Consider $a, a' \in \mathscr{Q}^N$ where $a_i \neq a'_i$ and $a_{-i} = a'_{-i}$ for some $i \in V$. The transition $a \to a'$ is feasible if and only if $(a_i, a'_i) \in E_{\text{loc}}$. A feasible path from a to a' consisting of multiple feasible transitions is denoted by $a \Rightarrow a'$. Let $\diamond a \triangleq \{a' \in \mathscr{Q} |\ a \Rightarrow a'\}$ be the reachable set from a.

Let $s = (a, c), s' = (a', c') \in \mathscr{A}$ where $a_i \neq a'_i$ and $a_{-i} = a'_{-i}$ for some $i \in V$. The transition $s \to s'$ is feasible if and only if $s'_i \in \mathscr{F}_i(a)$. A feasible path from s to s' consisting of multiple feasible transitions is denoted by $s \Rightarrow s'$. Finally, $\diamond s \triangleq \{s' \in \mathscr{A} |\ s \Rightarrow s'\}$ will be the reachable set from s.

3.3 Distributed Learning Algorithms

In this section, we come up with two distributed payoff-based learning algorithms, say the Coverage Learning Algorithm and the Asynchronous Coverage Learning Algorithm. We then present their convergence properties. Relevant and related algorithms include the payoff-based learning algorithms proposed in [2] and [3].

3.3.1 The Coverage Learning Algorithm

For each $k \geq 1$ and $i \in V$, we define $\tau_i(k)$ as follows: $\tau_i(k) = k$ if $u_i(s(k)) \geq u_i(s(k-1))$, otherwise, $\tau_i(k) = k - 1$. Here, $s_i(\tau_i(k))$ is the more successful action of agent i in last two steps. The Coverage Learning Algorithm is formally stated in the following table:

1: [Initialization:] At $k = 0$, all agents are uniformly placed in $\mathscr{Q}$. Each agent i uniformly chooses its camera control vector c_i from the set $\mathscr{C}$, communicates with agents in $\mathscr{N}_i^{\text{sen}}(s(0))$, and computes $u_i(s(0))$. At $k = 1$, all the agents keep their actions.

2: [Update:] At each time $k \geq 2$, each agent i updates its state according to the following rules:

- Agent i chooses the exploration rate $\varepsilon(k) = k^{-\frac{1}{N(D+1)}}$ with D being the diameter of the location graph $\mathscr{G}_{\text{loc}}$, and computes $s_i(\tau_i(k))$.
- With probability $\varepsilon(k)$, agent i experiments, and chooses the temporary action $s_i^{\text{tp}} \triangleq (a_i^{\text{tp}}, c_i^{\text{tp}})$ uniformly from the set $\mathscr{F}_i(a_i(k)) \setminus \{s_i(\tau_i(k))\}$.
- With probability $1 - \varepsilon(k)$, agent i does not experiment, and sets $s_i^{\text{tp}} = s_i(\tau_i(k))$.
- After s_i^{tp} is chosen, agent i moves to the position a_i^{tp} and sets the camera control vector to c_i^{tp}.

3: [Communication and computation:] At position a_i^{tp}, each agent i sends the information $\mathscr{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathscr{Q}$ to agents in $\mathscr{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}^{\text{tp}})$. After that, each agent i identifies the quantity $n_q(s^{\text{tp}})$, for each $q \in \mathscr{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathscr{Q}$, and computes the utility $u_i(s_i^{\text{tp}}, s_{-i}^{\text{tp}})$ and the feasible action set of $\mathscr{F}_i(a_i^{\text{tp}})$.

4: Repeat Steps 2 and 3.

Remark 3.2 A variation of the Coverage Learning Algorithm corresponds to $\varepsilon(k) = \varepsilon \in (0, \frac{1}{2}]$ constant for all $k \geq 2$. If this is the case, we will refer to the algorithm as the Homogeneous Coverage Learning Algorithm. Later, the convergence analysis of the Coverage Learning Algorithm will be based on the analysis of the Homogeneous Coverage Learning Algorithm. •

Denote the space $\mathscr{B} \triangleq \left\{(s, s') \in \mathscr{A} \times \mathscr{A} \mid s_i' \in \mathscr{F}_i(a_i),\ \forall i \in V\right\}$. Observe that $z(k) \triangleq (s(k-1), s(k))$ in the Coverage Learning Algorithm constitutes a time-inhomogeneous Markov chain $\{\mathscr{P}_k\}$ on the space $\mathscr{B}$. The following theorem implies that the COVERAGE LEARNING ALGORITHM asymptotically converges to the set of $\mathscr{E}(\Gamma_{\rm cov})$ in probability.

Theorem 3.1 *Consider the Markov chain* $\{\mathscr{P}_k\}$ *induced by the* COVERAGE LEARNING ALGORITHM. *It holds that* $\lim_{k\to+\infty}\mathbb{P}(z(k)\in \mathrm{diag}(\mathscr{E}(\Gamma_{\rm cov})))=1$.

The proofs of Theorem 3.1 are provided in Sect. 3.4.

Remark 3.3 An algorithm is proposed for the general class of weakly acyclic games (including potential games as special cases) in [3], and is able to find an NE with an arbitrarily high probability by choosing an arbitrarily small and fixed exploration rate ε in advance. However, it is difficult to derive an analytic relation between the convergent probability and the exploration rate. For the special case of identical interest games (all players share an identical utility function), the authors in [3] exploit a diminishing exploration rate and obtain a stronger result of convergence in probability. This motivates us to utilize a diminishing exploration rate in the COVERAGE LEARNING ALGORITHM which allows for convergence to the set of NEs in probability. In the algorithm for weakly acyclic games in [3], each player may execute the baseline action which depends on all the past plays. As a result, the algorithm for weakly acyclic games in [3] cannot be utilized to solve our problem because the baseline action may not be feasible when the state-dependent constraints are present. It is worth mentioning that the paper [3] also investigates a case where the utility values are corrupted by noises. •

3.3.2 *The* ASYNCHRONOUS COVERAGE LEARNING ALGORITHM

Lemma 3.1 shows that the coverage game $\Gamma_{\rm cov}$ is a constrained exact potential game with potential function $\phi(s)$. However, this potential function is not a straightforward measure of the network coverage performance. On the other hand, the objective function $U_g(s)\triangleq\sum_{i\in V}u_i(s)$ captures the trade-off between the overall network benefit from sensing and the total energy the network consumes, and thus can be perceived as a more natural coverage performance metric. Denote by $S^*\triangleq\left\{s\mid \mathrm{argmax}_{s\in\mathscr{A}}U_g(s)\right\}$ the set of global maximizers of $U_g(s)$. In this part, we present the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM which is convergent in probability to the set S^*.

Before that, we first introduce some notations for the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM. Denote by the space $\mathscr{B}'$ as follows:

$$\mathscr{B}'\triangleq\left\{(s,s')\in\mathscr{A}\times\mathscr{A}\mid s_{-i}=s'_{-i},\ s'_i\in\mathscr{F}_i(a_i)\ \text{for some}\ i\in V\right\}.$$

For any $s^0,s^1\in\mathscr{A}$ with $s^0_{-i}=s^1_{-i}$ for some $i\in V$, we denote

$$\Delta_i(s^1,s^0)\triangleq\frac{1}{2}\sum_{q\in\Omega_1}\frac{W_q}{n_q(s^1)}-\frac{1}{2}\sum_{q\in\Omega_2}\frac{W_q}{n_q(s^0)},$$

where $\Omega_1 \triangleq \mathscr{D}(a_i^1, c_i^1)\backslash\mathscr{D}(a_i^0, c_i^0) \cap \mathscr{Q}$ and $\Omega_2 \triangleq \mathscr{D}(a_i^0, c_i^0)\backslash\mathscr{D}(a_i^1, c_i^1) \cap \mathscr{Q}$, and

$$\begin{aligned}
&\rho_i(s^0, s^1) \triangleq u_i(s^1) - \Delta_i(s^1, s^0) - u_i(s^0) + \Delta_i(s^0, s^1),\\
&\Psi_i(s^0, s^1) \triangleq \max\{u_i(s^0) - \Delta_i(s^0, s^1), u_i(s^1) - \Delta_i(s^1, s^0)\},\\
&m^* \triangleq \max_{(s^0,s^1)\in\mathscr{B}, s_i^0\neq s_i^1} \{\Psi_i(s^0, s^1) - (u_i(s^0) - \Delta_i(s^0, s^1)), \frac{1}{2}\}.
\end{aligned}$$

It is easy to check that $\Delta_i(s^1, s^0) = -\Delta_i(s^0, s^1)$ and $\Psi_i(s^0, s^1) = \Psi_i(s^1, s^0)$. Assume that at each time instant, one of the agents becomes active with equal probability. This can be realized by employing the asynchronous time model proposed in [7] where each node has a clock which ticks according to a rate 1 Poisson process. For this reason, we will refer the following algorithm to be asynchronous. Denote by $\gamma_i(k)$ the last time instant before t when agent i was active. We then denote $\gamma_i^{(2)}(k) \triangleq \gamma_i(\gamma_i(k))$. The main steps of the Asynchronous Coverage Learning Algorithm are described in the following.

1: [Initialization:] At $k = 0$, all agents are uniformly placed in $\mathscr{Q}$. Each agent i uniformly chooses the camera control vector c_i from the set $\mathscr{C}$, and then communicates with agents in $\mathscr{N}_i^{\text{sen}}(s(0))$ and computes $u_i(s(0))$. Furthermore, each agent i chooses $m_i \in (2m^*, Km^*]$ for some $K \geq 2$. At $k = 1$, all the sensors keep their actions.

2: [Update:] Assume that agent i is active at time $k \geq 2$. Then agent i updates its state according to the following rules:

- Agent i chooses the exploration rate $\varepsilon(k) = k^{-\frac{1}{(D+1)(K+1)m^*}}$.
- With probability $\varepsilon(k)^{m_i}$, agent i experiments and uniformly chooses $s_i^{\text{tp}} \triangleq (a_i^{\text{tp}}, c_i^{\text{tp}})$ from the action set $\mathscr{F}_i(a_i(k)) \setminus \{s_i(k), s_i(\gamma_i^{(2)}(k) + 1)\}$.
- With probability $1 - \varepsilon(k)^{m_i}$, agent i does not experiment and chooses s_i^{tp} according to the following probability distribution:

$$\mathbb{P}(s_i^{\text{tp}} = s_i(k)) = \frac{1}{1 + \varepsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}},$$

$$\mathbb{P}(s_i^{\text{tp}} = s_i(\gamma_i^{(2)}(k) + 1)) = \frac{\varepsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}}{1 + \varepsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}}.$$

- After s_i^{tp} is chosen, agent i moves to the position a_i^{tp} and sets its camera control vector to be c_i^{tp}.

3: [Communication and computation:] At position a_i^{tp}, the active agent i initiates a message to agents in $\mathscr{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}(k))$. Then each agent $j \in \mathscr{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}(k))$ sends the information of $\mathscr{D}(a_j^{\text{tp}}, c_j^{\text{tp}}) \cap \mathscr{Q}$ to agent i. After receiving such information, agent i identifies the quantity $n_q(s_i^{\text{tp}}, s_{-i}(k))$ for each $q \in \mathscr{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathscr{Q}$, computes the utility $u_i(s_i^{\text{tp}}, s_{-i}(k))$, $\Delta_i((s_i^{\text{tp}}, s_{-i}(k)), s(\gamma_i(k) + 1))$, and the feasible action set of $\mathscr{F}_i(a_i^{\text{tp}})$.

4: Repeat Steps 2 and 3.

Remark 3.4 A variation of the Asynchronous Coverage Learning Algorithm corresponds to $\varepsilon(k) = \varepsilon \in (0, \frac{1}{2}]$ constant for all $k \geq 2$. If this is the case, we will refer to the algorithm as the Homogeneous Asynchronous Coverage Learning Algorithm. Later, we will base the convergence analysis of the Asynchronous Coverage Learning Algorithm on that of the Homogeneous Asynchronous Coverage Learning Algorithm. •

As in the Coverage Learning Algorithm, $z(k) \triangleq (s(t-1), s(k))$ in the Asynchronous Coverage Learning Algorithm constitutes a time-inhomogeneous Markov chain $\{\mathscr{P}_k\}$ on the space $\mathscr{B}'$. The following theorem implies that the Asynchronous Coverage Learning Algorithm asymptotically converges to the set of S^* with probability one.

Theorem 3.2 *Consider the Markov chain* $\{\mathscr{P}_k\}$ *induced by the* Asynchronous Coverage Learning Algorithm *for the game* Γ_{cov}. *Then it holds that* $\lim_{k\to+\infty} \mathbb{P}(z(k) \in \text{diag}(S^*)) = 1$.

The proofs of Theorem 3.2 are provided in Sect. 3.4.

Remark 3.5 A synchronous payoff-based, log-linear learning algorithm is proposed in [2] for potential games in which players aim to maximize the potential function of the game. As mentioned before, the potential function is not suitable to act as a coverage performance metric. As opposed to [2], the Asynchronous Coverage Learning Algorithm instead seeks to optimize a different function $U_g(s)$ perceived as a natural network performance metric. Furthermore, the Asynchronous Coverage Learning Algorithm exploits a diminishing step-size, and this choice allows for convergence to the set of global optima in probability. On the other hand, convergence in [2] is to the set of NE with arbitrarily high probability. Theoretically, our result is stronger than that of [2] by choosing an arbitrarily small and fixed exploration rate in advance. •

3.4 Convergence Analysis

In this section, we prove Theorems 3.1 and 3.2 by appealing to the Theory of Resistance Trees in [8] and the results in strong ergodicity in [9]. Relevant papers include [2, 3] where the Theory of Resistance Trees in [8] are novelly utilized to study the class of payoff-based learning algorithms, and [10–12] where the strong ergodicity theory is employed to characterize the convergence properties of time-inhomogeneous Markov chains.

3.4.1 *Convergence Analysis of the* COVERAGE LEARNING ALGORITHM

We first utilize Theorem 1.10 to characterize the convergence properties of the associated HOMOGENEOUS COVERAGE LEARNING ALGORITHM. This is essential for the analysis of the COVERAGE LEARNING ALGORITHM.

Observe that $z(k) \triangleq (s(k-1), s(k))$ in the HOMOGENEOUS COVERAGE LEARNING ALGORITHM consists of a time-homogeneous Markov chain $\{\mathscr{P}_k^\varepsilon\}$ on the space $\mathscr{B}$. Consider $z, z' \in \mathscr{B}$. A feasible path from z to z' consisting of multiple feasible transitions of $\{\mathscr{P}_t^\varepsilon\}$ is denoted by $z \Rightarrow z'$. The reachable set from z is denoted as $\diamond z \triangleq \{z' \in \mathscr{B} \mid z \Rightarrow z'\}$.

Lemma 3.2 *$\{\mathscr{P}_k^\varepsilon\}$ is a regular perturbation of $\{\mathscr{P}_k^0\}$.*

Proof Consider a feasible transition $z^1 \to z^2$ with $z^1 \triangleq (s^0, s^1)$ and $z^2 \triangleq (s^1, s^2)$. Then we can define a partition of V as $\Lambda_1 \triangleq \left\{i \in V \mid s_i^2 = s_i^{\tau_i(0,1)}\right\}$ and $\Lambda_2 \triangleq \left\{i \in V \mid s_i^2 \in \mathscr{F}_i(a_i^1) \setminus \{s_i^{\tau_i(0,1)}\}\right\}$. The corresponding probability is given by

$$P^\varepsilon_{z^1 z^2} = \prod_{i \in \Lambda_1} (1-\varepsilon) \times \prod_{j \in \Lambda_2} \frac{\varepsilon}{|\mathscr{F}_i(a_i^1)| - 1}. \tag{3.1}$$

Hence, the resistance of the transition $z^1 \to z^2$ is $|\Lambda_2| \in \{0, 1, \ldots, N\}$ since

$$0 < \lim_{\varepsilon \to 0^+} \frac{P^\varepsilon_{z^1 z^2}}{\varepsilon^{|\Lambda_2|}} = \prod_{j \in \Lambda_2} \frac{1}{|\mathscr{F}_i(a_i^1)| - 1} < +\infty.$$

We have that (A3) holds. It is not difficult to see that (A2) holds, and we are now in a position to verify (A1). Since $\mathscr{G}_{\text{loc}}$ is undirected and connected, and multiple sensors can stay in the same position, then $\diamond a^0 = \mathscr{Q}^N$ for any $a^0 \in \mathscr{Q}$. Since sensor i can choose any camera control vector from $\mathscr{C}$ at each time, then $\diamond s^0 = \mathscr{A}$ for any $s^0 \in \mathscr{A}$. It implies that $\diamond z^0 = \mathscr{B}$ for any $z^0 \in \mathscr{B}$, and thus the Markov chain $\{\mathscr{P}_k^\varepsilon\}$ is irreducible on the space $\mathscr{B}$.

It is easy to see that any state in diag($\mathscr{A}$) has period 1. Pick any $(s^0, s^1) \in \mathscr{B} \setminus \text{diag}(\mathscr{A})$. Since $\mathscr{G}_{\text{loc}}$ is undirected, then $s_i^0 \in \mathscr{F}_i(a_i^1)$ if and only if $s_i^1 \in \mathscr{F}_i(a_i^0)$. Hence, the following two paths are both feasible:

$$(s^0, s^1) \to (s^1, s^0) \to (s^0, s^1)$$
$$(s^0, s^1) \to (s^1, s^1) \to (s^1, s^0) \to (s^0, s^1).$$

Hence, the period of the state (s^0, s^1) is 1. This proves aperiodicity of $\{\mathscr{P}_k^\varepsilon\}$. Since $\{\mathscr{P}_k^\varepsilon\}$ is irreducible and aperiodic, then (A1) holds. ■

Lemma 3.3 *For any* $(s^0, s^0) \in \text{diag}(\mathscr{A}) \setminus \text{diag}(\mathscr{E}(\Gamma_{\text{cov}}))$*, there is a finite sequence of transitions from* (s^0, s^0) *to some* $(s^*, s^*) \in \text{diag}(\mathscr{E}(\Gamma_{\text{cov}}))$ *that satisfies*

$$\mathscr{L} \triangleq (s^0, s^0) \overset{O(\varepsilon)}{\rightarrow} (s^0, s^1) \overset{O(1)}{\rightarrow} (s^1, s^1) \overset{O(\varepsilon)}{\rightarrow} (s^1, s^2)$$
$$\overset{O(1)}{\rightarrow} (s^2, s^2) \overset{O(\varepsilon)}{\rightarrow} \cdots \overset{O(\varepsilon)}{\rightarrow} (s^{\tau-1}, s^\tau) \overset{O(1)}{\rightarrow} (s^\tau, s^\tau)$$

where $(s^\tau, s^\tau) = (s^*, s^*)$ *for some* $\tau \geq 1$.

Proof If $s^0 \notin \mathscr{E}(\Gamma_{\text{cov}})$, there exists a sensor i with an action $s_i^1 \in \mathscr{F}_i(a_i^0)$ such that $u_i(s^1) > u_i(s^0)$, where $s_{-i}^0 = s_{-i}^1$. The transition $(s^0, s^0) \rightarrow (s^0, s^1)$ happens when only sensor i experiments, and its corresponding probability is $(1-\varepsilon)^{N-1} \times \frac{\varepsilon}{|\mathscr{F}_i(a_i^0)|-1}$. Since the function ϕ is the potential function of the game Γ_{cov}, we have that $\phi(s^1) - \phi(s^0) = u_i(s^1) - u_i(s^0)$ and thus $\phi(s^1) > \phi(s^0)$.

Since $u_i(s^1) > u_i(s^0)$ and $s_{-i}^0 = s_{-i}^1$, the transition $(s^0, s^1) \rightarrow (s^1, s^1)$ occurs when all sensors do not experiment, and the associated probability is $(1-\varepsilon)^N$.

We repeat the above process and construct the path $\mathscr{L}$ with length $\tau \geq 1$. Since $\phi(s^i) > \phi(s^{i-1})$ for $i = \{1, \ldots, \tau\}$, then $s^i \neq s^j$ for $i \neq j$ and thus the path $\mathscr{L}$ has no loop. Since $\mathscr{A}$ is finite, then τ is finite and thus $s^\tau = s^* \in \mathscr{E}(\Gamma_{\text{cov}})$. ■

A direct result of Lemma 3.2 is that for each ε, there exists a unique stationary distribution of $\{\mathscr{P}_k^\varepsilon\}$, say $\mu(\varepsilon)$. We now proceed to utilize Theorem 1.10 to characterize $\lim_{\varepsilon \to 0^+} \mu(\varepsilon)$.

Proposition 3.1 *Consider the regular perturbation* $\{\mathscr{P}_k^\varepsilon\}$ *of* $\{\mathscr{P}_k^0\}$*. Then the limit of* $\lim_{\varepsilon \to 0^+} \mu(\varepsilon)$ *exists and the limiting distribution* $\mu(0)$ *is a stationary distribution of* $\{\mathscr{P}_k^0\}$*. Furthermore, the stochastically stable states (i.e., the support of* $\mu(0)$*) are contained in the set* $\text{diag}(\mathscr{E}(\Gamma_{\text{cov}}))$.

Proof Notice that the stochastically stable states are contained in the recurrent communication classes of the unperturbed Markov chain that corresponds to the HOMOGENEOUS COVERAGE LEARNING ALGORITHM with $\varepsilon = 0$. Thus the stochastically stable states are included in the set $\text{diag}(\mathscr{A}) \subset \mathscr{B}$. Denote by $T_{\min}$ the minimum resistance tree and by h_v the root of $T_{\min}$. Each edge of $T_{\min}$ has resistance $0, 1, 2, \ldots$ corresponding to the transition probability $O(1), O(\varepsilon), O(\varepsilon^2), \ldots$. The state z' is the *successor* of the state z if and only if $(z, z') \in T_{\min}$. Like Theorem 3.2 in [3], our analysis will be slightly different from the presentation in 1.5. We will construct $T_{\min}$ over states in the set $\mathscr{B}$ (rather than $\text{diag}(\mathscr{A})$) with the restriction that all the edges leaving the states in $\mathscr{B} \setminus \text{diag}(\mathscr{A})$ have resistance 0. The stochastically stable states are not changed under this difference.

Claim 3.1 *For any* $(s^0, s^1) \in \mathscr{B} \setminus \text{diag}(\mathscr{A})$*, there is a finite path*

$$\mathscr{L}' \triangleq (s^0, s^1) \overset{O(1)}{\rightarrow} (s^1, s^2) \overset{O(1)}{\rightarrow} (s^2, s^2)$$

where $s_i^2 = s_i^{\tau_i(0,1)}$ *for all* $i \in V$.

Proof These two transitions occur when all agents do not experiment. The corresponding probability of each transition is $(1-\varepsilon)^N$. ■

Claim 3.2 *The root* h_v *belongs to the set* $\mathrm{diag}(\mathscr{A})$.

Proof Suppose that $h_v = (s^0, s^1) \in \mathscr{B} \setminus \mathrm{diag}(\mathscr{A})$. By Claim 3.1, there is a finite path $\mathscr{L}' \triangleq (s^0, s^1) \overset{O(1)}{\rightarrow} (s^1, s^2) \overset{O(1)}{\rightarrow} (s^2, s^2)$. We now construct a new tree T' by adding the edges of the path $\mathscr{L}'$ into the tree $T_{\min}$ and removing the redundant edges. The total resistance of added edges is 0. Observe that the resistance of the removed edge exiting from (s^2, s^2) in the tree $T_{\min}$ is at least 1. Hence, the resistance of T' is strictly lower than that of $T_{\min}$, and we get to a contradiction. ■

Claim 3.3 *Pick any* $s^* \in \mathscr{E}(\Gamma_{\mathrm{cov}})$ *and consider* $z \triangleq (s^*, s^*)$, $z' \triangleq (s^*, \tilde{s})$ *where* $\tilde{s} \neq s^*$. *If* $(z, z') \in T_{\min}$, *then the resistance of the edge* (z, z') *is some* $\tau \geq 2$.

Proof Suppose the deviator in the transition $z \rightarrow z'$ is unique, say i. Then the corresponding transition probability is $O(\varepsilon)$. Since $s^* \in \mathscr{E}(\Gamma_{\mathrm{cov}})$ and $\tilde{s}_i \in \mathscr{F}_i(a_i^*)$, we have that $u_i(s_i^*, s_{-i}^*) \geq u_i(\tilde{s}_i, \tilde{s}_{-i})$, where $s_{-i}^* = \tilde{s}_{-i}$.

Since $z' \in \mathscr{B} \setminus \mathrm{diag}(\mathscr{A})$, it follows from Claim 3.2 that the state z' cannot be the root of $T_{\min}$ and thus has a successor z''. Note that all the edges leaving the states in $\mathscr{B} \setminus \mathrm{diag}(\mathscr{A})$ have resistance 0. Then none of the experiments are in transition $z' \rightarrow z''$ and $z'' = (\tilde{s}, \hat{s})$ for some $\hat{s}$. Since $u_i(s_i^*, s_{-i}^*) \geq u_i(\tilde{s}_i, \tilde{s}_{-i})$ with $s_{-i}^* = \tilde{s}_{-i}$, we have $\hat{s} = s^*$ and thus $z'' = (\tilde{s}, s^*)$. Similarly, the state z'' must have a successor z''' and $z''' = z$. We then obtain a loop in $T_{\min}$ which contradicts that $T_{\min}$ is a tree.

It implies that at least two sensors experiment in the transition $z \rightarrow z'$. Thus the resistance of the edge (z, z') is at least 2. ■

Claim 3.4 *The root* h_v *belongs to the set* $\mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$.

Proof Suppose that $h_v = (s^0, s^0) \notin \mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$. By Lemma 3.3, there is a finite path $\mathscr{L}$ connecting (s^0, s^0) and some $(s^*, s^*) \in \mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$. We now construct a new tree T' by adding the edges of the path $\mathscr{L}$ into the tree $T_{\min}$ and removing the edges that leave the states in $\mathscr{L}$ in the tree $T_{\min}$. The total resistance of added edges is τ. Observe that the resistance of the removed edge exiting from (s^i, s^i) in the tree $T_{\min}$ is at least 1 for $i \in \{1, \ldots, \tau - 1\}$. By Claim 3.3, the resistance of the removed edge leaving from (s^*, s^*) in the tree $T_{\min}$ is at least 2. The total resistance of removed edges is at least $\tau + 1$. Hence, the resistance of T' is strictly lower than that of $T_{\min}$, and we get to a contradiction. ■

It follows from Claim 3.4 that the states in $\mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$ have minimum stochastic potential. Since Lemma 3.2 shows that Markov chain $\{\mathscr{P}_\tau^\varepsilon\}$ is a regularly perturbed Markov process, Proposition 3.1 is a direct result of Theorem 1.10. ■

We are now ready to show Theorem 3.1.

Proof of Theorem 3.1

Claim 3.5 *Condition (C2) in Theorem 1.9 holds.*

Proof For each $k \geq 0$ and each $z \in X$, we define the numbers

$$\sigma_z(\varepsilon(k)) \triangleq \sum_{T \in G(z)} \prod_{(x,y) \in T} P^{\varepsilon(k)}_{xy}, \quad \sigma^k_z = \sigma_z(\varepsilon(k))$$

$$\mu_z(\varepsilon(k)) \triangleq \frac{\sigma_z(\varepsilon(k))}{\sum_{x \in X} \sigma_x(\varepsilon(k))}, \quad \mu^k_z = \mu_z(\varepsilon(k)).$$

Since $\{\mathscr{P}^\varepsilon_k\}$ is a regular perturbation of $\{\mathscr{P}^0_k\}$, it is irreducible and thus $\sigma^k_z > 0$. As Lemma 3.1 of Chap. 6 in [13], one can show that $(\mu^k)^T P^{\varepsilon(k)} = (\mu^k)^T$. Therefore, condition (C2) in Theorem 1.9 holds. ■

Claim 3.6 *Condition (C3) in Theorem 1.9 holds.*

Proof We now proceed to verify condition (C3) in Theorem 1.9. To do that, let us first fix k, denote $\varepsilon = \varepsilon(k)$, and study the monotonicity of $\mu_z(\varepsilon)$ with respect to ε. We write $\sigma_z(\varepsilon)$ in the following form:

$$\sigma_z(\varepsilon) = \sum_{T \in G(z)} \prod_{(x,y) \in T} P^\varepsilon_{xy} = \sum_{T \in G(z)} \prod_{(x,y) \in T} \frac{\alpha_{xy}(\varepsilon)}{\beta_{xy}(\varepsilon)} = \frac{\alpha_z(\varepsilon)}{\beta_z(\varepsilon)} \tag{3.2}$$

for some polynomials $\alpha_z(\varepsilon)$ and $\beta_z(\varepsilon)$ in ε. With (3.2) in hand, we have that $\sum_{x \in X} \sigma_x(\varepsilon)$ and thus $\mu_z(\varepsilon)$ are ratios of two polynomials in ε; i.e., $\mu_z(\varepsilon) = \frac{\varphi_z(\varepsilon)}{\beta(\varepsilon)}$ where $\varphi_z(\varepsilon)$ and $\beta(\varepsilon)$ are polynomials in ε. The derivative of $\mu_z(\varepsilon)$ is given by

$$\frac{\partial \mu_z(\varepsilon)}{\partial \varepsilon} = \frac{1}{\beta(\varepsilon)^2}(\frac{\partial \varphi_z(\varepsilon)}{\partial \varepsilon}\beta(\varepsilon) - \varphi_z(\varepsilon)\frac{\partial \beta(\varepsilon)}{\partial \varepsilon}).$$

Note that the numerator $\frac{\partial \varphi_z(\varepsilon)}{\partial \varepsilon}\beta(\varepsilon) - \varphi_z(\varepsilon)\frac{\partial \beta(\varepsilon)}{\partial \varepsilon}$ is a polynomial in ε. Denote by $\iota_z \neq 0$ the coefficient of the leading term of $\frac{\partial \varphi_z(\varepsilon)}{\partial \varepsilon} - \varphi_z(\varepsilon)\frac{\partial \beta(\varepsilon)}{\varepsilon}$. The leading term dominates $\frac{\partial \varphi_z(\varepsilon)}{\partial \varepsilon} - \varphi_z(\varepsilon)\frac{\partial \beta(\varepsilon)}{\varepsilon}$ when ε is sufficiently small. Thus there exists $\varepsilon_z > 0$ such that the sign of $\frac{\partial \mu_z(\varepsilon)}{\partial \varepsilon}$ is the sign of ι_z for all $0 < \varepsilon \leq \varepsilon_z$. Let $\varepsilon^* = \max_{z \in X} \varepsilon_z$.

Since $\varepsilon(k)$ strictly decreases to zero, there is a unique finite time instant k^* such that $\varepsilon(k^*) = \varepsilon^*$ (if $\varepsilon(0) < \varepsilon^*$, then $k^* = 0$). Since $\varepsilon(k)$ is strictly decreasing, we can define a partition of X as follows:

$$\Xi_1 \triangleq \{z \in X \mid \mu_z(\varepsilon(k)) > \mu_z(\varepsilon(t+1)), \quad \forall t \in [k^*, +\infty)\},$$

$$\Xi_2 \triangleq \{z \in X \mid \mu_z(\varepsilon(k)) < \mu_z(\varepsilon(t+1)), \quad \forall t \in [k^*, +\infty)\}.$$

We are now ready to verify (C3) of Theorem 1.9. Since $\{\mathscr{P}^\varepsilon_k\}$ is a regular perturbed Markov chain of $\{\mathscr{P}^0_t\}$, it follows from Theorem 1.10 that $\lim_{t \to +\infty} \mu_z(\varepsilon(k)) = \mu_z(0)$, and thus it holds that

$$\sum_{k=0}^{+\infty}\sum_{z\in X}\|\mu_z^k-\mu_z^{k+1}\| = \sum_{k=0}^{+\infty}\sum_{z\in X}|\mu_z(\varepsilon(k))-\mu_z(\varepsilon(k+1))|$$

$$= \sum_{k=0}^{k^*}\sum_{z\in X}|\mu_z(\varepsilon(k))-\mu_z(\varepsilon(k+1))| + \sum_{k=k^*+1}^{+\infty}(\sum_{z\in\Xi_1}\mu_z(\varepsilon(k))-\sum_{z\in\Xi_1}\mu_z(\varepsilon(k+1)))$$

$$+\sum_{k=k^*+1}^{+\infty}(1-\sum_{z\in\Xi_1}\mu_z(\varepsilon(k+1))-(1-\sum_{z\in\Xi_1}\mu_z(\varepsilon(k))))$$

$$= \sum_{k=0}^{k^*}\sum_{z\in X}|\mu_z(\varepsilon(k))-\mu_z(\varepsilon(k+1))| + 2\sum_{z\in\Xi_1}\mu_z(\varepsilon(k^*+1)) - 2\sum_{z\in\Xi_1}\mu_z(0) < +\infty.$$

■

Claim 3.7 *Condition (C1) in Theorem 1.9 holds.*

Proof Denote by $P^{\varepsilon(k)}$ the transition matrix of $\{\mathscr{P}_k\}$. As in (3.1), the probability of the feasible transition $z^1 \to z^2$ is given by

$$P_{z^1z^2}^{\varepsilon(k)} = \prod_{i\in\Lambda_1}(1-\varepsilon(k)) \times \prod_{j\in\Lambda_2}\frac{\varepsilon(k)}{|\mathscr{F}_i(a_i^1)|-1}.$$

Observe that $|\mathscr{F}_i(a_i^1)| \le 5|\mathscr{C}|$. Since $\varepsilon(k)$ is strictly decreasing, there is $t_0 \ge 1$ such that t_0 is the first time when $1-\varepsilon(k) \ge \frac{\varepsilon(k)}{5|\mathscr{C}|-1}$. Then for all $k \ge t_0$, it holds that

$$P_{z^1z^2}^{\varepsilon(k)} \ge (\frac{\varepsilon(k)}{5|\mathscr{C}|-1})^N.$$

Denote $P(m,n) \triangleq \prod_{k=m}^{n-1} P^{\varepsilon(k)}$, $0 \le m < n$. Pick any $z \in \mathscr{B}$ and let $u_z \in \mathscr{B}$ be such that $P_{u_zz}(k,k+D+1) = \min_{x\in\mathscr{B}} P_{xz}(k,k+D+1)$. Consequently, it follows that for all $k \ge t_0$,

$$\min_{x\in\mathscr{B}} P_{xz}(k,k+D+1) = \sum_{i_1\in\mathscr{B}}\cdots\sum_{i_D\in\in\mathscr{B}} P_{u_zi_1}^{\varepsilon(k)}\cdots P_{i_{D-1}i_D}^{\varepsilon(k+D-1)}P_{i_Dz}^{\varepsilon(k+D)}$$

$$\ge P_{u_zi_1}^{\varepsilon(k)}\cdots P_{i_{D-1}i_D}^{\varepsilon(k+D-1)}P_{i_Dz}^{\varepsilon(k+D)} \ge \prod_{i=0}^{D}(\frac{\varepsilon(k+i)}{5|\mathscr{C}|-1})^N \ge (\frac{\varepsilon(k)}{5|\mathscr{C}|-1})^{(D+1)N}$$

where in the last inequality we use that $\varepsilon(k)$ is strictly decreasing. Then we have

$$1-\lambda(P(k,k+D+1)) = \min_{x,y\in\mathscr{B}}\sum_{z\in\mathscr{B}}\min\{P_{xz}(k,k+D+1), P_{yz}(k,k+D+1)\}$$

$$\ge \sum_{z\in\mathscr{B}} P_{u_zz}(k,k+D+1) \ge |\mathscr{B}|(\frac{\varepsilon(k)}{5|\mathscr{C}|-1})^{(D+1)N}.$$

Choose $k_i \triangleq (D+1)i$ and let i_0 be the smallest integer such that $(D+1)i_0 \geq t_0$. Then, we have that:

$$\sum_{i=0}^{+\infty}(1-\lambda(P(k_i,k_{i+1}))) \geq |\mathscr{B}| \sum_{i=i_0}^{+\infty}(\frac{\varepsilon((D+1)i)}{5|\mathscr{C}|-1})^{(D+1)N}$$
$$= \frac{|\mathscr{B}|}{(5|\mathscr{C}|-1)^{(D+1)N}} \sum_{i=i_0}^{+\infty} \frac{1}{(D+1)i} = +\infty. \quad (3.3)$$

Hence, the weak ergodicity property follows from Theorem 1.8. ■

All the conditions in Theorem 1.9 hold. Thus it follows from Theorem 1.9 that the limiting distribution is $\mu^* = \lim_{k\to+\infty}\mu^k$. Note that $\lim_{k\to+\infty}\mu^k = \lim_{k\to+\infty}\mu(\varepsilon(k)) = \mu(0)$ and Proposition 3.1 shows that the support of $\mu(0)$ is contained in the set $\mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$. Hence, the support of μ^* is contained in the set $\mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))$, implying that $\lim_{t\to+\infty}\mathbb{P}(z(k) \in \mathrm{diag}(\mathscr{E}(\Gamma_{\mathrm{cov}}))) = 1$. It completes the proof.

3.4.2 *Convergence Analysis of the* ASYNCHRONOUS COVERAGE LEARNING ALGORITHM

First of all, we employ Theorem 1.10 to study the convergence properties of the associated HOMOGENEOUS ASYNCHRONOUS COVERAGE LEARNING ALGORITHM. This is essential to analyze the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM.

To simplify notations, we will use $s_i(k-1) \triangleq s_i(\gamma_i^{(2)}(k)+1)$ in the remainder of this section. Observe that $z(k) \triangleq (s(k-1), s(k))$ in the HOMOGENEOUS ASYNCHRONOUS COVERAGE LEARNING ALGORITHM constitutes a Markov chain $\{\mathscr{P}_k^{\varepsilon}\}$ on the space $\mathscr{B}'$.

Lemma 3.4 *The Markov chain $\{\mathscr{P}_k^{\varepsilon}\}$ is a regular perturbation of $\{\mathscr{P}_k^{0}\}$.*

Proof Pick any two states $z^1 \triangleq (s^0, s^1)$ and $z^2 \triangleq (s^1, s^2)$ with $z^1 \neq z^2$. We have that $P^{\varepsilon}_{z^1 z^2} > 0$ if and only if there is some $i \in V$ such that $s^1_{-i} = s^2_{-i}$ and one of the following occurs: $s_i^2 \in \mathscr{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}$, $s_i^2 = s_i^1$ or $s_i^2 = s_i^0$. In particular, the following holds:

$$P^{\varepsilon}_{z^1 z^2} = \begin{cases} \eta_1, & s_i^2 \in \mathscr{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}, \\ \eta_2, & s_i^2 = s_i^1, \\ \eta_3, & s_i^2 = s_i^0, \end{cases}$$

where

$$\eta_1 \triangleq \frac{\varepsilon^{m_i}}{N|\mathscr{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}|}, \quad \eta_2 \triangleq \frac{1-\varepsilon^{m_i}}{N(1+\varepsilon^{\rho_i(s^0,s^1)})}, \quad \eta_3 \triangleq \frac{(1-\varepsilon^{m_i}) \times \varepsilon^{\rho_i(s^0,s^1)}}{N(1+\varepsilon^{\rho_i(s^0,s^1)})}.$$

Observe that $0 < \lim_{\varepsilon\to 0^+} \frac{\eta_1}{\varepsilon^{m_i}} < +\infty$. Multiplying the numerator and denominator of η_2 by $\varepsilon^{\Psi_i(s^1,s^0)-(u_i(s^1)-\Delta_i(s^1,s^0))}$, we obtain

$$\eta_2 = \frac{1-\varepsilon^{m_i}}{N} \times \frac{\varepsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))}}{\eta_2'},$$

where $\eta_2' \triangleq \varepsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))} + \varepsilon^{\Psi_i(s^0,s^1)-(u_i(s^0)-\Delta_i(s^0,s^1))}$. Use

$$\lim_{\varepsilon\to 0^+} \varepsilon^x = \begin{cases} 1, & x = 0, \\ 0, & x > 0, \end{cases}$$

and we have

$$\lim_{\varepsilon\to 0^+} \frac{\eta_2}{\varepsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))}} = \begin{cases} \frac{1}{N}, & u_i(s^0)-\Delta_i(s^0,s^1) \neq u_i(s^1)-\Delta_i(s^1,s^0), \\ \frac{1}{2N}, & \text{otherwise.} \end{cases}$$

Similarly, it holds that

$$\lim_{\varepsilon\to 0^+} \frac{\eta_3}{\varepsilon^{\Psi_i(s^0,s^1)-(u_i(s^0)-\Delta_i(s^0,s^1))}} \in \{\frac{1}{2N}, \frac{1}{N}\}.$$

Hence, the resistance of the feasible transition $z^1 \to z^2$, with $z^1 \neq z^2$ and sensor i as the unilateral deviator, can be described as follows:

$$\chi(z^1 \to z^2) = \begin{cases} m_i, & s_i^2 \in \mathscr{F}_i(a^1) \setminus \{s_i^0, s_i^1\}, \\ \Psi_i(s^0,s^1) - (u_i(s^1) - \Delta_i(s^1,s^0)), & s_i^2 = s_i^1, \\ \Psi_i(s^0,s^1) - (u_i(s^0) - \Delta_i(s^0,s^1)), & s_i^2 = s_i^0. \end{cases}$$

Then (A3) in Sect. 1.5 holds. It is straightforward to verify that (A2) in Sect. 1.5 holds. We are now in a position to verify (A1). Since $\mathscr{G}_{\rm loc}$ is undirected and connected, and multiple sensors can stay in the same position, then $\diamond a^0 = \mathscr{Q}^N$ for any $a^0 \in \mathscr{Q}$. Since sensor i can choose any camera control vector from $\mathscr{C}$ at each time, $\diamond s^0 = \mathscr{A}$ for any $s^0 \in \mathscr{A}$. This implies that $\diamond z^0 = \mathscr{B}'$ for any $z^0 \in \mathscr{B}'$, and thus the Markov chain $\{\mathscr{P}_t^\varepsilon\}$ is irreducible on the space $\mathscr{B}'$.

It is easy to see that any state in diag($\mathscr{A}$) has period 1. Pick any $(s^0, s^1) \in \mathscr{B}' \setminus \text{diag}(\mathscr{A})$. Since $\mathscr{G}_{\rm loc}$ is undirected, then $s_i^0 \in \mathscr{F}_i(a_i^1)$ if and only if $s_i^1 \in \mathscr{F}_i(a_i^0)$. Hence, the following two paths are both feasible:

$$(s^0, s^1) \to (s^1, s^0) \to (s^0, s^1)$$
$$(s^0, s^1) \to (s^1, s^1) \to (s^1, s^0) \to (s^0, s^1).$$

Hence, the period of the state (s^0, s^1) is 1. This proves aperiodicity of $\{\mathscr{P}_t^\varepsilon\}$. Since $\{\mathscr{P}_t^\varepsilon\}$ is irreducible and aperiodic, (A1) holds. ■

A direct result of Lemma 3.4 is that for each $\varepsilon > 0$, there exists a unique stationary distribution of $\{\mathscr{P}_k^\varepsilon\}$, say $\mu(\varepsilon)$. From the proof of Lemma 3.4, we can see that the resistance of an experiment is m_i if sensor i is the unilateral deviator. We now proceed to utilize Theorem 1.10 to characterize $\lim_{\varepsilon\to 0^+} \mu(\varepsilon)$.

Proposition 3.2 *Consider the regular perturbed Markov process $\{\mathscr{P}_k^\varepsilon\}$. Then the limit of* $\lim_{\varepsilon\to 0^+} \mu(\varepsilon)$ *exists and the limiting distribution $\mu(0)$ is a stationary distribution of $\{\mathscr{P}_t^0\}$. Furthermore, the stochastically stable states (i.e., the support of $\mu(0)$) are contained in the set* $\mathrm{diag}(S^*)$.

Proof The unperturbed Markov chain corresponds to the HOMOGENEOUS ASYNCHRONOUS COVERAGE LEARNING ALGORITHM with $\varepsilon = 0$. Hence, the recurrent communication classes of the unperturbed Markov chain are contained in the set $\mathrm{diag}(\mathscr{A})$. We will construct resistance trees over vertices in the set $\mathrm{diag}(\mathscr{A})$. Denote by $T_{\min}$ the minimum resistance tree. The remainder of the proof is divided into the following four claims.

Claim 3.8 $\chi((s^0, s^0) \Rightarrow (s^1, s^1)) = m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$ *where $s^0 \neq s^1$ and the transition $s^0 \to s^1$ is feasible with sensor i as the unilateral deviator.*

Proof One feasible path for $(s^0, s^0) \Rightarrow (s^1, s^1)$ is $\mathscr{L} \triangleq (s^0, s^0) \to (s^0, s^1) \to (s^1, s^1)$ where sensor i experiments in the first transition and does not experiment in the second one. The total resistance of the path $\mathscr{L}$ is $m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$ which is at most $m_i + m^*$.

Denote by $\mathscr{L}'$ the path with minimum resistance among all the feasible paths for $(s^0, s^0) \Rightarrow (s^1, s^1)$. Assume that the first transition in $\mathscr{L}'$ is $(s^0, s^0) \to (s^0, s^2)$ where node j experiments and $s^2 \neq s^1$. Observe that the resistance of $(s^0, s^0) \to (s^0, s^2)$ is m_j. Regardless of whether j is equal to i or not, the path $\mathscr{L}'$ must include at least one more experiment to introduce s_i^1. Hence the total resistance of the path $\mathscr{L}'$ is at least $m_i + m_j$. Since $m_i + m_j > m_i + 2m^*$, the path $\mathscr{L}'$ has a strictly larger resistance than the path $\mathscr{L}$. To avoid contradiction, the path $\mathscr{L}'$ must start from the transition $(s^0, s^0) \to (s^0, s^1)$. Similarly, the sequent transition (which is also the last one) in the path $\mathscr{L}'$ must be $(s^0, s^1) \to (s^1, s^1)$ and thus $\mathscr{L}' = \mathscr{L}$. Hence, the resistance of the transition $(s^0, s^0) \Rightarrow (s^1, s^1)$ is the total resistance of the path $\mathscr{L}$; i.e., $m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$. ■

Claim 3.9 *All the edges $((s, s), (s', s'))$ in $T_{\min}$ must consist of only one deviator; i.e., $s_i \neq s_i'$ and $s_{-i} = s_{-i}'$ for some $i \in V$.*

Proof Assume that $(s, s) \Rightarrow (s', s')$ has at least two deviators. Suppose the path $\hat{\mathscr{L}}$ has the minimum resistance among all the paths from (s, s) to (s', s'). Then, $\ell \geq 2$ experiments are carried out along $\hat{\mathscr{L}}$. Denote by i_τ the unilateral deviator in the τ-th experiment $s^{\tau-1} \to s^\tau$ where $1 \leq \tau \leq \ell$, $s^0 = s$ and $s^\ell = s'$. Then the resistance of $\hat{\mathscr{L}}$ is at least $\sum_{\tau=1}^{\ell} m_{i_\tau}$; i.e., $\chi((s^0, s^0) \Rightarrow (s', s')) \geq \sum_{\tau=1}^{\ell} m_{i_\tau}$.

Let us consider the following path on $T_{\min}$:

$$\bar{\mathscr{L}} \triangleq (s^0, s^0) \Rightarrow (s^1, s^1) \Rightarrow \cdots \Rightarrow (s^\ell, s^\ell).$$

From Claim 3.1, we know that the total resistance of the path $\bar{\mathscr{L}}$ is at most $\sum_{\tau=1}^{\ell} m_{i_\tau} + \ell m^*$.

A new tree T' can be obtained by adding the edges of $\bar{\mathscr{L}}$ into $T_{\min}$ and removing the redundant edges. The removed resistance is *strictly* greater than $\sum_{\tau=1}^{\ell} m_{i_\tau} + 2(\ell-1)m^*$ where $\sum_{\tau=1}^{\ell} m_{i_\tau}$ is the lower bound on the resistance on the edge from (s^0, s^0) to (s^ℓ, s^ℓ), and $2(\ell-1)m^*$ is the strictly lower bound on the total resistances of leaving (s^τ, s^τ) for $\tau = 1, \ldots, \ell-1$. The added resistance is the total resistance of $\bar{\mathscr{L}}$ which is at most $\sum_{\tau=1}^{\ell} m_{i_\tau} + \ell m^*$. Since $\ell \geq 2$, we have that $2(\ell-1)m^* \geq \ell m^*$ and thus T' has a strictly lower resistance than $T_{\min}$. This contradicts the fact that $T_{\min}$ is a minimum resistance tree. ■

Claim 3.10 *Given any edge $((s, s), (s', s'))$ in $T_{\min}$, denote by i the unilateral deviator between s and s'. Then the transition $s_i \to s'_i$ is feasible.*

Proof Assume that the transition $s_i \to s'_i$ is infeasible. Suppose the path $\check{\mathscr{L}}$ has the minimum resistance among all the paths from (s, s) to (s', s'). Then, there are $\ell \geq 2$ experiments in $\check{\mathscr{L}}$. The remainder of the proof is similar to that of Claim 3.9. ■

Claim 3.11 *Let h_v be the root of $T_{\min}$. Then, $h_v \in \mathrm{diag}(S^*)$.*

Proof Assume that $h_v = (s^0, s^0) \notin \mathrm{diag}(S^*)$. Pick any $(s^*, s^*) \in \mathrm{diag}(S^*)$. By Claims 3.9 and 3.10, we have that there is a path from (s^*, s^*) to (s^0, s^0) in the tree $T_{\min}$ as follows:

$$\tilde{\mathscr{L}} \triangleq (s^\ell, s^\ell) \Rightarrow (s^{\ell-1}, s^{\ell-1}) \Rightarrow \cdots \Rightarrow (s^1, s^1) \Rightarrow (s^0, s^0)$$

for some $\ell \geq 1$. Here, $s^* = s^\ell$, there is only one deviator, say i_τ, from s^τ to $s^{\tau-1}$, and the transition $s^\tau \to s^{\tau-1}$ is feasible for $\tau = \ell, \ldots, 1$.

Since the transition $s^\tau \to s^{\tau+1}$ is also feasible for $\tau = 0, \ldots, \ell-1$, we obtain the reverse path $\tilde{\mathscr{L}}'$ of $\tilde{\mathscr{L}}$ as follows:

$$\tilde{\mathscr{L}}' \triangleq (s^0, s^0) \Rightarrow (s^1, s^1) \Rightarrow \cdots \Rightarrow (s^{\ell-1}, s^{\ell-1}) \Rightarrow (s^\ell, s^\ell).$$

By Claim 3.8, the total resistance of the path $\tilde{\mathscr{L}}$ is

$$\chi(\tilde{\mathscr{L}}) = \sum_{\tau=1}^{\ell} m_{i_\tau} + \sum_{\tau=1}^{\ell} \{\Psi_{i_\tau}(s^\tau, s^{\tau-1}) - (u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau))\},$$

and the total resistance of the path $\tilde{\mathscr{L}}'$ is

$$\chi(\tilde{\mathscr{L}}') = \sum_{k=1}^{\ell} m_{i_\tau} + \sum_{\tau=1}^{\ell} \Psi_{i_\tau}(s^{\tau-1}, s^\tau) - (u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})).$$

We make the following notations:

$$\Lambda_1' \triangleq (\mathscr{D}(a_{i_\tau}^\tau, r_{i_\tau}^\tau) \backslash \mathscr{D}(a_{i_{\tau-1}}^{\tau-1}, r_{i_{\tau-1}}^{\tau-1})) \cap \mathscr{Q}, \quad \Lambda_2' \triangleq (\mathscr{D}(a_{i_{\tau-1}}^{\tau-1}, r_{i_{\tau-1}}^{\tau-1}) \backslash \mathscr{D}(a_{i_\tau}^\tau, r_{i_\tau}^\tau)) \cap \mathscr{Q}.$$

Observe that

$$\begin{aligned} U_g(s^\tau) - U_g(s^{\tau-1}) &= u_{i_\tau}(s^\tau) - u_{i_\tau}(s^{\tau-1}) - \sum_{q \in \Lambda_1'} W_q(\frac{n_q(s^{\tau-1})}{n_q(s^{\tau-1})} - \frac{n_q(s^{\tau-1})}{n_q(s^\tau)}) \\ &+ \sum_{q \in \Lambda_2'} W_q(\frac{n_q(s^\tau)}{n_q(s^\tau)} - \frac{n_q(s^\tau)}{n_q(s^{\tau-1})}) \\ &= (u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})) - (u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau)). \end{aligned}$$

We now construct a new tree T' with the root (s^*, s^*) by adding the edges of $\tilde{\mathscr{L}}'$ to the tree $T_{\min}$ and removing the redundant edges $\tilde{\mathscr{L}}$. Since $\Psi_{i_\tau}(s^{\tau-1}, s^\tau) = \Psi_{i_\tau}(s^\tau, s^{\tau-1})$, the difference in the total resistances across the trees $\chi(T')$ and $\chi(T_{\min})$ is given by

$$\begin{aligned} &\chi(T') - \chi(T_{\min}) = \chi(\tilde{\mathscr{L}}') - \chi(\tilde{\mathscr{L}}) \\ &= \sum_{\tau=1}^{\ell} -(u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau)) - \sum_{\tau=1}^{\ell} -(u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})) \\ &= \sum_{\tau=1}^{\ell} (U_g(s^\tau) - U_g(s^{\tau-1})) = U_g(s^0) - U_g(s^*) < 0. \end{aligned}$$

This contradicts that $T_{\min}$ is a minimum resistance tree. ■

It follows from Claim 3.4 that the state $h_v \in \text{diag}(S^*)$ has minimum stochastic potential. Then Proposition 3.2 is a direct result of Theorem 1.10. ■

We are now ready to show Theorem 3.2.

Proof of Theorem 3.1

Claim 3.12 *Condition (C2) in Theorem 1.9 holds.*

Proof The proof is analogous to Claim 3.5. ■

Claim 3.13 *Condition (C3) in Theorem 1.9 holds.*

Proof Denote by $P^{\varepsilon(k)}$ the transition matrix of $\{\mathscr{P}_k\}$. Consider the feasible transition $z^1 \to z^2$ with unilateral deviator i. The corresponding probability is given by

$$P^{\varepsilon(k)}_{z^1z^2} = \begin{cases} \eta_1, & s_i^2 \in \mathscr{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}, \\ \eta_2, & s_i^2 = s_i^1, \\ \eta_3, & s_i^2 = s_i^0, \end{cases}$$

where

$$\eta_1 \triangleq \frac{\varepsilon(k)^{m_i}}{N|\mathscr{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}|}, \quad \eta_2 \triangleq \frac{1-\varepsilon(k)^{m_i}}{N(1+\varepsilon(k)^{\rho_i(s^0,s^1)})},$$
$$\eta_3 \triangleq \frac{(1-\varepsilon(k)^{m_i}) \times \varepsilon(k)^{\rho_i(s^0,s^1)}}{N(1+\varepsilon(k)^{\rho_i(s^0,s^1)})}.$$

The remainder is analogous to Claim 3.6. ∎

Claim 3.14 *Condition (C1) in Theorem 1.9 holds.*

Proof Observe that $|\mathscr{F}_i(a_i^1)| \le 5|\mathscr{C}|$. Since $\varepsilon(k)$ is strictly decreasing, there is $t_0 \ge 1$ such that t_0 is the first time when $1-\varepsilon(k)^{m_i} \ge \varepsilon(k)^{m_i}$.

Observe that for all $t \ge 1$, it holds that

$$\eta_1 \ge \frac{\varepsilon(k)^{m_i}}{N(5|\mathscr{C}|-1)} \ge \frac{\varepsilon(k)^{m_i+m^*}}{N(5|\mathscr{C}|-1)}.$$

Denote $b \triangleq u_i(s^1) - \Delta_i(s^1, s^0)$ and $a \triangleq u_i(s^0) - \Delta_i(s^0, s^1)$. Then $\rho_i(s^0, s^1) = b - a$. Since $b - a \le m^*$, then for $k \ge t_0$ it holds that

$$\eta_2 = \frac{1-\varepsilon(k)^{m_i}}{N(1+\varepsilon(k)^{b-a})} = \frac{(1-\varepsilon(k)^{m_i})\varepsilon(k)^{\max\{a,b\}-b}}{N(\varepsilon(k)^{\max\{a,b\}-b}+\varepsilon(k)^{\max\{a,b\}-a})}$$
$$\ge \frac{\varepsilon(k)^{m_i}\varepsilon(k)^{\max\{a,b\}-b}}{2N} \ge \frac{\varepsilon(k)^{m_i+m^*}}{N(5|\mathscr{C}|-1)}.$$

Similarly, for $k \ge t_0$, it holds that

$$\eta_3 = \frac{(1-\varepsilon(k)^{m_i})\varepsilon(k)^{\max\{a,b\}-a}}{N(\varepsilon(k)^{\max\{a,b\}-b}+\varepsilon(k)^{\max\{a,b\}-a})} \ge \frac{\varepsilon(k)^{m_i+m^*}}{N(5|\mathscr{C}|-1)}.$$

Since $m_i \in (2m^*, Km^*]$ for all $i \in V$ and $Km^* > 1$, for any feasible transition $z^1 \to z^2$ with $z^1 \ne z^2$, it holds that

$$P^{\varepsilon(k)}_{z^1z^2} \ge \frac{\varepsilon(k)^{(K+1)m^*}}{N(5|\mathscr{C}|-1)}$$

for all $k \geq t_0$. Furthermore, for all $k \geq t_0$ and all $z^1 \in \text{diag}(\mathscr{A})$, we have that

$$P^{\varepsilon(k)}_{z^1 z^1} = 1 - \frac{1}{N}\sum_{i=1}^{N} \varepsilon(k)^{m_i} = \frac{1}{N}\sum_{i=1}^{N}(1-\varepsilon(k)^{m_i}) \geq \frac{1}{N}\sum_{i=1}^{N} \varepsilon(k)^{m_i} \geq \frac{\varepsilon(k)^{(K+1)m^*}}{N(5|\mathscr{C}|-1)}.$$

Choose $k_i \triangleq (D+1)i$ and let i_0 be the smallest integer such that $(D+1)i_0 \geq t_0$. Similar to (3.3), we can derive the following property:

$$\sum_{\ell=0}^{+\infty}(1-\lambda(P(k_\ell, k_{\ell+1}))) \geq \frac{|\mathscr{B}|}{(N(5|\mathscr{C}|-1))^{(D+1)(K+1)m^*}} \sum_{i=i_0}^{+\infty} \frac{1}{(D+1)i} = +\infty.$$

Hence, the weak ergodicity of $\{\mathscr{P}_k\}$ follows from Theorem 1.8. ■

All the conditions in Theorem 1.9 hold. Thus it follows from Theorem 1.9 that the limiting distribution is $\mu^* = \lim_{k\to+\infty} \mu^k$. Notice the following relation:

$$\lim_{k\to+\infty} \mu^k = \lim_{k\to+\infty} \mu(\varepsilon(k)) = \mu(0),$$

and Proposition 3.2 shows that the support of $\mu(0)$ is contained in the set $\text{diag}(S^*)$. Hence, the support of μ^* is contained in the set $\text{diag}(S^*)$, implying that $\lim_{k\to+\infty} \mathbb{P}(z(k) \in \text{diag}(S^*)) = 1$. It completes the proof.

3.5 Numerical Examples

In this section, we present some remarks along with two numerical examples to illustrate the performance of our algorithms.

Theorems 3.1 and 3.2 guarantees the asymptotic convergence in probability of the proposed algorithms. However, our theoretic results do not provide any estimate of the convergence rates, which could be very slow in practice. This is a consequence of the well-known exploration-exploitation trade-off termed in reinforcement learning; e.g., in [14]. Intuitively, each algorithm starts from a relatively large exploration rate and this allows the algorithm to explore the unknown environment quickly. As time processes, the exploration rate is decreased, allowing each algorithm to exploit the information collected and converge to some desired configuration. In order to avoid being locked-in some undesired configuration, each algorithm requires a very slow exploration decreasing rate. In the numerical examples below, we have chosen suitable exploration rates empirically.

3.5.1 *A Numerical Example of the* Coverage Learning Algorithm

Consider a 10×10 square and each grid is 1×1 and a group of 9 mobile visual sensors are deployed in this area. Note that, given arbitrary sensing range and distribution, it would be difficult to compute an NE. In order to avoid this computational challenge and make our simulation results evident, we make the following assumptions:

1. All the sensors are identical, and each has a fixed sensing range which is a circle of radius 1.5.
2. Each point in this region is associated with a uniform value of 1.

With these two assumptions, it is not difficult to see that any configuration where sensing ranges of sensors do not overlap is an NE at which the global potential function is equal to 81.

In this example, the diameter of the location graph is 20 and $N = 9$. According to our theoretic result, we should choose an exploration rate of $\varepsilon(k) = (\frac{1}{k})^{\frac{1}{189}}$. The exploration rate decreases extremely slowly and the algorithm requires an extremely long time to converge. Instead, we choose $\varepsilon(k) = (\frac{1}{k+2^{10}})^{\frac{1}{2}}$ in our simulation. Figure 3.3 shows the initial configuration of the group where all of the sensors start at the same position. Figure 3.5 presents the configuration at iteration 5000 and it is evident that this configuration is an NE. Figure 3.4 is the evolution of the global potential function which eventually oscillates between 78 and the maximal value of 81. This verifies that the sensors approach the set of NEs.

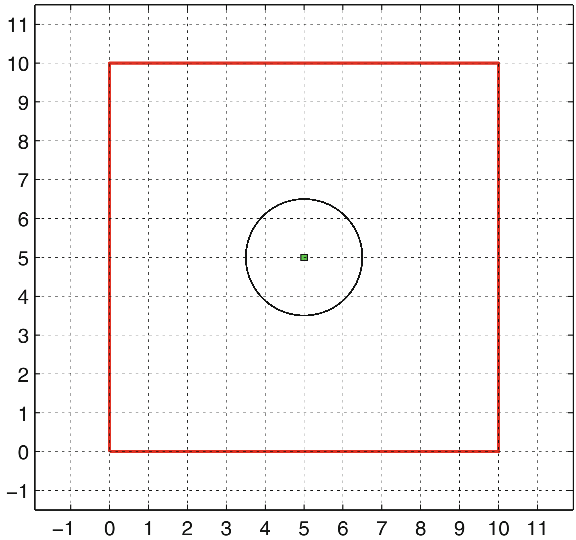

Fig. 3.3 Initial configuration of the network

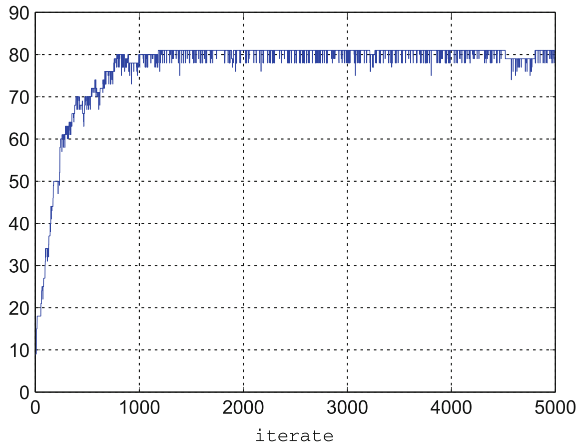

Fig. 3.4 The evolution of the global potential function with a diminishing exploration rate for the COVERAGE LEARNING ALGORITHM

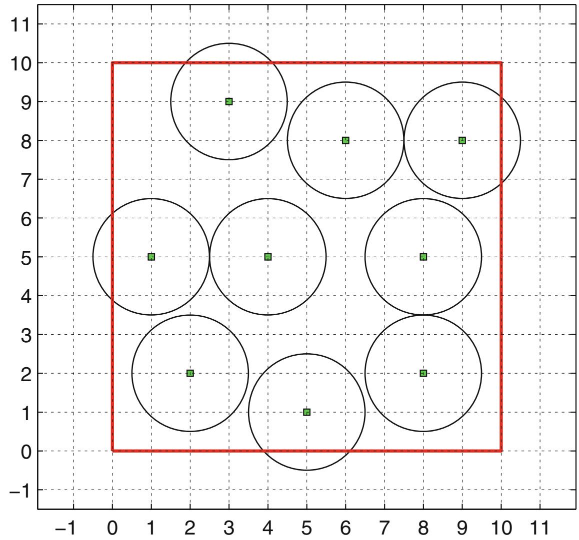

Fig. 3.5 Final network configuration at iteration 5000 of the COVERAGE LEARNING ALGORITHM

As in [2, 3], we will use fixed exploration rates in the COVERAGE LEARNING ALGORITHM which then reduces to the HOMOGENEOUS COVERAGE LEARNING ALGORITHM. Figures 3.6, 3.7 and 3.8 present the evolution of the global potential functions for $\varepsilon = 0.1, 0.01, 0.001$, respectively. When $\varepsilon = 0.1$, the convergence to the neighborhood of the value 81 is the fastest, but its variation is the largest. When $\varepsilon = 0.001$, the convergence rate is slowest. The performance of $\varepsilon = 0.01$ is similar to the diminishing step-size $\varepsilon(k) = (\frac{1}{k+2^{10}})^{\frac{1}{2}}$. This comparison shows that, for both diminishing and fixed exploration rates, we have to empirically choose the exploration rate to obtain a good performance.

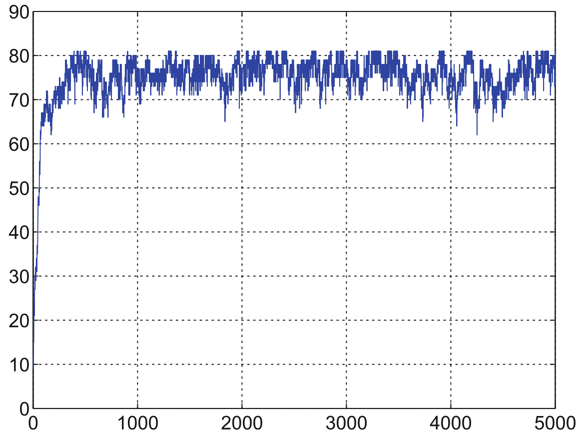

Fig. 3.6 The evolution of the global potential function under COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.1$

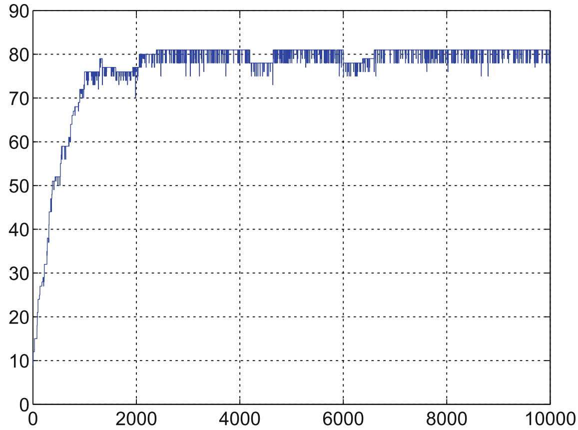

Fig. 3.7 The evolution of the global potential function under COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.01$

3.5.2 A Numerical Example of the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM

We consider a lattice of unit grids and each point is associated with a uniform weight 0.1. There are four identical sensors, and each of them has a fixed sensing range which is a circle of radius 1.5. The global optimal value of U_g is 36. All the sensors start from the center of the region. We run the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM for 50,000 iterations and sample the data every 5 iterations (Fig. 3.9). Figures 3.10, 3.11, 3.12 and 3.13 show the evolution of the global function U_g for the following four cases, respectively: $\varepsilon(k) = \frac{1}{4}(\frac{1}{k+1})^{\frac{1}{4}}$, $\varepsilon = 0.1$, $\varepsilon = 0.01$ and $\varepsilon = 0.001$.

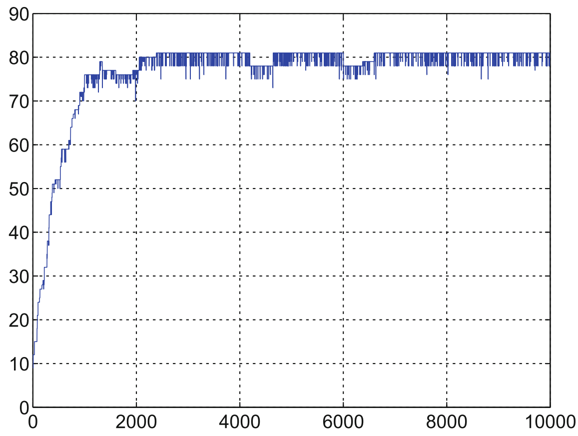

Fig. 3.8 The evolution of the global potential function under COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.001$

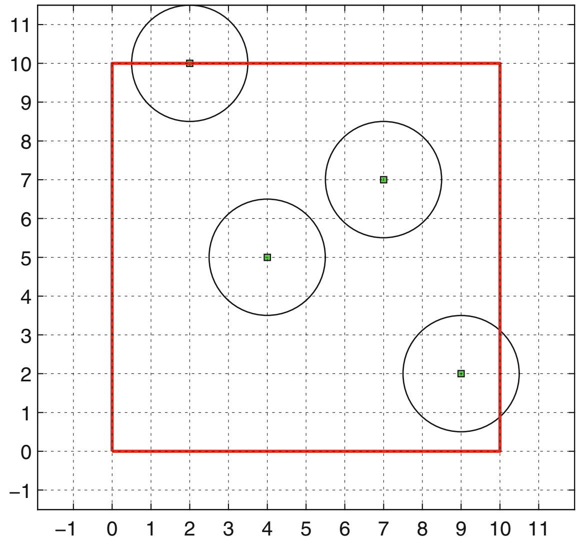

Fig. 3.9 Final configuration of the network at iteration 50,000 of the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM

3.6 Notes

In broad terms, the problem studied in this chapter is related to a bevy of sensor location and planning problems in the Computational Geometry, Geometric Optimization, and Robotics literature. For example, different variations on the (combinatorial) Art Gallery problem include [15–17]. The objective here is to find the optimum number of guards in a non-convex environment so that each point is visible from at least one guard. A related set of references for the deployment of mobile robots with omnidirectional cameras includes [18, 19]. Unlike the Art Gallery classic algorithms, the latter papers assume that robots have local knowledge of the environment and no recollection of the past. Other related references on robot deployment in convex

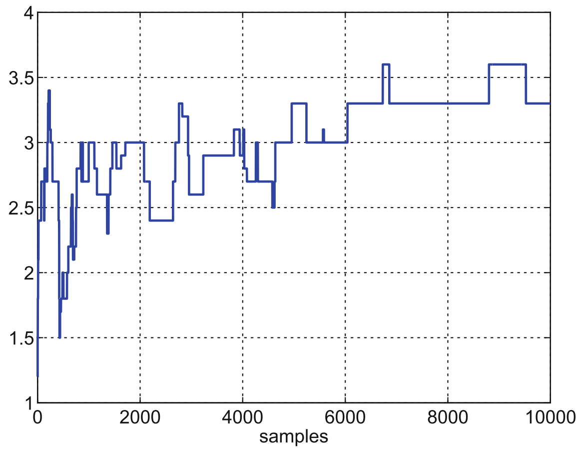

Fig. 3.10 The evolution of the global potential function under the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM with a diminishing exploration rate

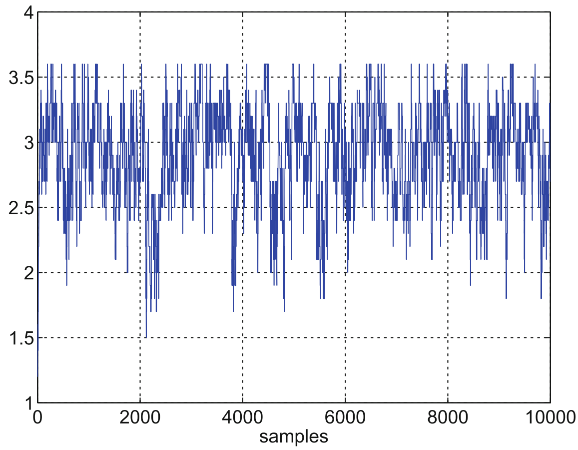

Fig. 3.11 The evolution of the global potential function under the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.1$ is kept fixed

environments include [20] for anisotropic and circular footprints. The paper [21] is an excellent survey on multimedia sensor networks where the state of the art in algorithms, protocols, and hardware is surveyed, and open research issues are discussed in detail. As observed in [22], multimedia sensor networks enhance traditional surveillance systems by enlarging, enhancing, and enabling multi-resolution views. The investigation of coverage problems for static visual sensor networks is conducted in [6, 23, 24].

From the technical point of view, this chapter falls into the framework of game theoretic learning or learning for games. As for discrete games, the classic methods to compute Nash equilibrium include best-response dynamics, better-response dynamics, factitious play, regret matching, logit-based dynamics, replicator dynamics, and see [25–28]. As for continuous games, generalized Nash games consist of an important class and are first formulated in [29]. Since then, great efforts have been

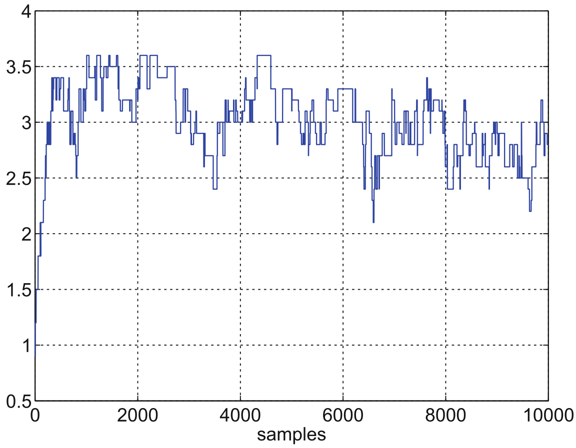

Fig. 3.12 The evolution of the global potential function under the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.01$ is kept fixed

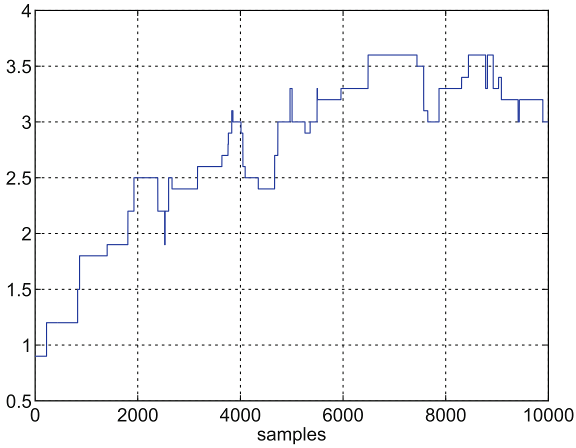

Fig. 3.13 The evolution of the global potential function under the ASYNCHRONOUS COVERAGE LEARNING ALGORITHM when $\varepsilon = 0.001$ is kept fixed

dedicated to studying the existence and structural properties of generalized Nash equilibria in; e.g., [30] and the recent survey paper [31]. A number of algorithms have been proposed to compute generalized Nash equilibria, including ODE-based methods [30], nonlinear Gauss-Seidel-type approaches [32], iterative primal-dual Tikhonov schemes [33], and best-response dynamics [34]. Recently, a self-triggering algorithm is considered in [35]. When the game model is not available in advance, a number of algorithms are proposed to compute Nash equilibrium; e.g., [2, 3, 36, 37] for discrete games and [38–41] for continuous games. The results presented in this chapter are based on our paper [37].

This chapter is restricted to static games. As for dynamic games, differential games consist of an important class where the decisions of each player are restricted by a differential equation. Among the limited number of differential games for which closed-form solutions have been derived are the homicidal-chauffeur and the ladyin-the-lake games [42, 43], which are played in unobstructed environments. For more complicated games, numerical methods must be used to determine solutions, including PDE-based methods in [42, 44, 45], viability-based approaches in [46–48] and level-set methods in [49, 50]. The papers [51–54] study mean-field games where the player number is very large.

As discussed at the beginning of this chapter, game theory recently emerges as a new tool to synthesize efficient distributed coordination algorithms. The connections between distributed control and potential games are elaborated in [55]. The paper [56] discusses utility design for multi-agent networks. In [57], game theory is used in algorithm design of distributed optimization. For multi-vehicle networks, game theory is used in [37, 40] for sensor deployment, [36] for vehicle routing and [58] for robotic motion planning.

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Chapter 4
Distributed Resilient Formation Control

4.1 Introduction

In recent years, unmanned vehicles have been substantially developed and so their markets have been undergoing a dramatic expansion [1]. Without crew on board, unmanned vehicles offer competitive advantages over their manned counterparts such as lower deployment costs and longer lifetime. Thus, unmanned vehicles have been widely deployed in civilian and military settings, including examples of border and road patrol, search and rescue, scientific monitoring in severe climates, firefighting, agriculture, and transportation. In this way, the Federal Aviation Administration is developing the NextGen air transportation system so that unmanned vehicles can be included into the national airspace system.

In particular, the use of unmanned vehicles by (human) operators has been proposed to enhance information sharing and maintain situational awareness. However, this capability comes at the expense of the inherent vulnerability of information technology systems to cyber attacks. The communication between operators and vehicles can be intentionally compromised by (human) adversaries, disrupting the network-wise objective. Since we cannot rule out that adversaries are able to successfully mount attacks, it is of prominent importance to provide resilient solutions that assure mission completion despite the presence of security threats.

The current chapter formulates the problem of distributed constrained formation control against replay attacks in an operator-vehicle adversarial network. In particular, each vehicle is remotely controlled by an operator and its actuation is limited. Vehicles aim to reach the desired formation within a given constraint set through real-time coordination with operators. Each operator-vehicle pair is attacked by an adversary, who is able to produce replay attacks by maliciously and consecutively repeating the control commands for a period of time. The information that operators know about their opponents is limited and restricts to the maximum number, say $\tau_{\max}$, of consecutive attacks each adversary is able to launch. We focus on cyber resilience; that is, we are interested in devising distributed algorithms which ensure the mission

M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*, SpringerBriefs in Control, Automation and Robotics, DOI 10.1007/978-3-319-19072-3_4

completion in the presence of replay attacks. To achieve this goal, we come up with a distributed formation control algorithm which is based on Receding Horizon Control (RHC) and leverages the idea of moving toward target points of [2]. We show that the input and state constraints are always enforced, and the desired formation can be asymptotically achieved provided that the union of communication graphs between operators satisfies certain connectivity assumption. Under the same set of conditions, our proposed algorithm shows an analogous resilience to denial-of-service attacks.

4.2 Problem Formulation

In this section, we first present the architecture of the operator-vehicle network, and the distributed constrained formation control problem of interest. After that, we introduce the model of replay attackers considered in this chapter. This is followed by a description of the prior knowledge operators possess about their rivals and the objective of this chapter.

4.2.1 The Operator-Vehicle Network

Consider a group of vehicles in $\mathbb{R}^d$, for some $d \in \mathbb{Z}_{>0}$, labeled by $i \in V \triangleq \{1, \ldots, N\}$. The dynamics of each vehicle is governed by the following second-order, discrete-time dynamic system:

$$\begin{aligned} p_i(k+1) &= p_i(k) + v_i(k), \\ v_i(k+1) &= v_i(k) + u_i(k), \end{aligned} \tag{4.1}$$

where $p_i(k) \in X \subseteq \mathbb{R}^d$ (resp. $v_i(k) \in \mathbb{R}^d$) is the position (resp. the velocity) of vehicle i, and $u_i(k) \in U \subseteq \mathbb{R}^d$ then stands for its input. Throughout this chapter, we suppose the following on the constraint sets:

Assumption 4.1 (*Constraint sets*) The state constraint set X is convex and compact. The input constraint set U is a box; i.e., $U \triangleq \{u \in \mathbb{R}^d \mid \|u\|_\infty \leq u_{\max}\}$[1] for some $u_{\max} > 0$. •

Each vehicle i is remotely maneuvered by an operator i, and this assignment is one-to-one and fixed. Each vehicle is able to identify its location and velocity, and send this information to its operator through a communication network. Within the vehicle team, vehicles cannot communicate with each other. Each operator, on the one hand, can exchange information with neighboring operators, and on the other hand, deliver control commands to her associated vehicle via the communication network. See Fig. 4.1 for the sketch of the operator-vehicle network.

[1] In this chapter, the notation of $\|\cdot\|$ (resp. $\|\cdot\|_\infty$) stands for the 2-norm (resp. ∞-norm) of vectors.

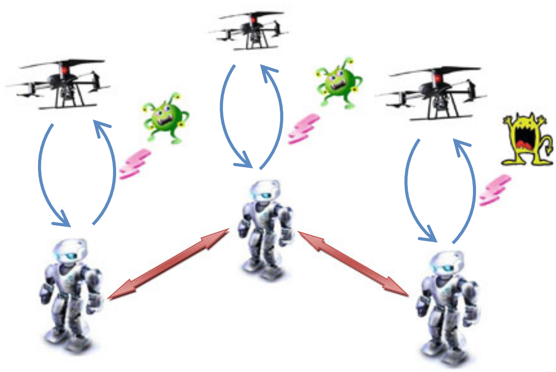


Fig. 4.1 The architecture of the operator-vehicle adversarial network where the operator is represented by the humanoid robot

The mission of the operator-vehicle network is to achieve a desired formation which is characterized by the formation digraph $\mathcal{G}^F \triangleq (V, \mathcal{E}^F)$. Each edge $(i, j) \in \mathcal{E}^F \subseteq V \times V \setminus \text{diag}(V)$, starting from vehicle j and pointing to vehicle i, is associated with a formation vector $\nu_{ij} \in \mathbb{R}^d$. Throughout this chapter, we impose the following on $\mathcal{G}^F$:

Assumption 4.2 (*Formation digraph*) The formation digraph $\mathcal{G}^F$ is strongly connected; i.e., for any pair of $(j, i) \in \mathcal{E}^F$, there is a directed path starting from i and ending up with j. •

Being a member of the team, each operator i is only aware of local formation vectors; i.e., ν_{ij} for $j \in \mathcal{N}_i$ where $\mathcal{N}_i \triangleq \{j \in V \setminus \{i\} \mid (j, i) \in \mathcal{E}^F\}$. The multi-vehicle constrained formation control mission can be formulated as a team optimization problem where each global optimum corresponds to the formation of interest. In particular, the constrained formation control problem is encoded as the following quadratic program:

$$\min_{p \in X^N} \sum_{(i,j) \in \mathcal{E}^F} \|p_i - p_j - \nu_{ij}\|^2,$$

whose solution set, denoted as $X^* \subseteq X^N$, satisfies the following:

Assumption 4.3 (*Feasibility*) The optimal solution set X^* is nonempty. •

We assume that operators and vehicles are synchronized by using a single clock. The interconnection between operators at time $k \geq 0$ will be represented by a directed graph $\mathcal{G}(k) = (V, \mathcal{E}(k))$ where $\mathcal{E}(k) \subseteq \mathcal{E}^F$ is the set of edges. Here $(i, j) \in \mathcal{E}(k)$ if and only if operator i is able to receive the message from operator j at time k. Denote by $\mathcal{N}_i(k) \triangleq \{j \in V \mid (i, j) \in \mathcal{E}(k)\}$ the set of (in-)neighboring operators of operator i at time k. In order to achieve network-wise objectives, interoperator topologies should be sufficiently connected such that decisions of any operator can eventually affect any other one. This is formally stated in the following assumption:

Assumption 4.4 (*Periodic Communication*) There is a positive integer B such that, for any $k \geq 0$, $(i, j) \in \bigcup_{s=0}^{B-1} \mathscr{E}(k+s)$ for any $(i, j) \in \mathscr{E}^F$. •

A direct result of Assumptions 4.2 and 4.4 is that $\bigcup_{s=0}^{B-1} \mathscr{E}(k+s)$ is a superset of $\mathscr{E}^F$, and thus $(V, \bigcup_{s=0}^{B-1} \mathscr{E}(k+s))$ is strongly connected.

4.2.2 Model of Adversaries

We now set out to describe the attacker model we consider in this chapter. A group of N adversaries try to abort the mission of achieving formation in X. An adversary is allocated to attack a specific operator-vehicle pair and this assignment is fixed over time. Thus, we identify adversary i with the operator-vehicle pair i. In this chapter, we consider the class of *replay attacks* where the packages transmitted from operators to vehicles are maliciously repeated by adversaries. In particular, each adversary i is associated with a memory storing past information and its state is denoted by $M_i^a(k)$. If she launches a replay attack at time k, adversary i executes the following: (1) erases the data sent from operator i; (2) delivers the past control command stored in her memory, $M_i^a(k)$, to vehicle i; (3) keeps the state of the memory; i.e., $M_i^a(k+1) = M_i^a(k)$. In this case, $s_i^a(k) = 1$, indicates the occurrence of a replay attack, where the auxiliary variable $s_i^a(k) \in \{0, 1\}$. If she does not produce any replay attack at time k, adversary i intercepts the data, say u_i, sent from operator i and stores it in her memory; $M_i^a(k+1) = u_i$. In this case, $s_i^a(k) = 0$ and u_i is successfully received by vehicle i. Without loss of any generality, we assume that $s_i^a(0) = 0$.

We define the variable $\tau_i^a(k)$ with initial state $\tau_i^a(0) = 0$ to indicate the consecutive number of attacks. The evolution of $\tau_i^a(k)$ is determined in the following way: if $s_i^a(k) = 1$, then $\tau_i^a(k) = \tau_i^a(k-1) + 1$; otherwise, $\tau_i^a(k) = 0$. It is noted that $\tau_i^a(k)$ is reset to zero when adversary i does not replay the data at time k. Hence, for any k with $s_i^a(k) = 0$, $\tau_i^a(k)$ represents the number of consecutive attacks produced by adversary i up to time k since the largest $0 \leq k' < k$ with $s_i^a(k') = 0$.

Each adversary needs to spend certain amount of energy to launch a replay attack. We assume that the energy of adversary i is limited, and adversary i is only able to launch at most $\tau_{\max} \geq 1$ consecutive attacks, i.e.,

Assumption 4.5 (*Maximum number of consecutive attacks*) There is $\tau_{\max} \geq 1$ such that $\max_{i \in V} \sup_{k \geq 0} \tau_i^a(k) \leq \tau_{\max}$. •

Remark 4.1 Replay attacks have been successfully used in the past, and show a number of advantages to an adversary. Stuxnet was the latest cyber attack to control systems. In this accident, Stuxnet exploited replay attacks to compromise a nuclear facility, see [3, 4].

Replay attacks (and denial-of-service attacks in Sect. 4.6) do not require any information of the operator-vehicle network and the algorithm exploited. This is in contrast to false data injection in [5–7] and deception attacks in [8–10]. From the point of view of adversaries, replay attacks (and denial-of-service attacks) are easier to launch, and thus more preferable when they lack the information of the target control systems. Note that replay attacks are less sophisticated than deception attacks and false data injection. However, the discussion in next section demonstrates that replay attacks are still capable of making a mission fail if they are not explicitly taken into account in the algorithm design.

Finally, in comparison with denial-of-service attacks, deception attacks and false data injection, replay attacks demand more memory to store intercepted information. •

Remark 4.2 For the ease of presentation, we assume that only the links from operators to vehicles are compromised. Our proposed algorithm can be readily applied to the scenario where the links from vehicles to operators are attacked. •

4.2.3 A Motivating Scenario

In this section, we use a simple scenario to illustrate the failure of the classic formation control algorithm under replay attacks. For the ease of presentation, we consider the special case: (1) $\nu_{ij} = 0$; (2) the vehicle dynamics is first order; (3) the input and state constraints are absent; i.e., $X = U = \mathbb{R}$. The special case is the consensus or rendezvous problem which has been extensively studied.

The classic consensus algorithm; e.g., in [11], is rephrased to fit in our setup as follows: at each time instant k, operator i receives $p_j(k)$ from neighboring operator $j \in \mathcal{N}_i(k)$, and sends the control command $u_i(k) = \sum_{j\in V} a_{ij}(k)p_j(k) - p_i(k)$ to vehicle i. If $s_i^a(k) = 1$, adversary i sends $M_i^a(k)$ to vehicle i and lets $M_i^a(k+1) = M_i^a(k)$. If $s_i^a(k) = 0$, adversary i then lets $M_i^a(k+1) = u_i(k)$. After receiving the data $u_i(k)$ (if $s_i^a(k) = 0$) or $M_i^a(k)$ (if $s_i^a(k) = 1$), vehicle i implements it and then sends the new location $p_i(k+1) = p_i(k)+u_i(k)$ (if $s_i^a(k) = 0$) or $p_i(k+1) = p_i(k)+M_i^a(k)$ (if $s_i^a(k) = 1$) to operator i.

In the above classic consensus algorithm, it is not difficult to verify that if the event of $s_i^a(k) = 1$ occurs infinitely often for any $i \in V$, then vehicles fail to reach any consensus. Even worse, the maximum deviation of $D(k) \triangleq \max_{i\in V} p_i(k) - \min_{i\in V} p_i(k)$ can be intentionally driven to infinity despite the limitation of $\tau_{\max}$. We further look into a simpler case to illustrate this point.

Consider two operator-vehicle pairs with $p_1(0) \neq p_2(0)$. Assume that the two operators communicate with each other all the time, and the update rule is $\frac{1}{2}(p_i(k) + p_j(k))$. Suppose $\tau_{\max} \geq 2$, and that adversaries adopt a periodic strategy: $s_1(k) = s_2(k) = 0$ if k is a multiple of $\tau_{\max} + 1$; otherwise, $s_1(k) = s_2(k) = 1$. It is not difficult to verify that $D(\kappa(\tau_{\max} + 1)) = \tau_{\max}^{\kappa} D(0)$ for integer $\kappa \geq 1$. Hence $D(k)$ diverges to infinity at a geometric rate of $\tau_{\max}$.

The above discussion yields the following insights: first, the classic consensus algorithm can be easily prevented from reaching consensus by persistently launching replay attacks; second, in the worst case adversaries may be able to drive $D(k)$ to infinity if adversaries know the algorithm and are able to intelligently take advantage of this information; further, if their energy restriction is smaller; i.e., $\tau_{\max}$ is larger, adversaries can speed up the divergence of $D(k)$. These facts evidently motivate the design of new distributed resilient algorithms which explicitly take into account replay attacks.

The detection of replay attacks is not difficult when operators and vehicles are synchronized. A detection scheme consists of attaching a time index to each control command from the operator, and then the vehicle can detect replay attacks by simply comparing the current time instant and the time index of the received command. This simple detection scheme will be employed in our subsequent algorithm design.

4.2.4 Prior Information About Adversaries and Objective

In hostile environments, it would be reasonable to expect that operators have limited information about adversaries. In this chapter, we assume that the only information operator i possesses is the quantity $\tau_{\max}$ or any of its upper bounds. At each time, each operator i makes a decision before her opponent, adversary i. Hence, operator i cannot predict whether adversary i would produce an attack at this time. Our objective is to design a distributed algorithm which ensures formation control under the above informational restriction.

[Objective] Given the only information of $\tau_{\max}$, we aim to devise a distributed algorithm, including the distributed control law $u_i(k)$ for vehicle i, such that $p_i(k) \in X$ and $u_i(k) \in U$ for all $k \geq 0$ and $i \in V$, and it holds that $\lim_{k\to+\infty} \text{dist}(p(k), X^*) = 0$, $\lim_{k\to+\infty} \|v_i(k)\| = 0$.

To conclude this section, we summarize the main notations in Table 4.1 that will be used in Sects. 4.3 and 4.4. In particular, $u_i(k+s|k)$ means the control command of time instant $k+s$ ($s \geq 0$) and this control command is generated at time instant k.

Table 4.1 Summary of common notation used in the sequel

n	The computing horizon
$\mathbf{u}_i(k \to k+n-1\|k)$	The collection of $\{u_i(k+s\|k)\}_{0\leq s\leq n-1}$
$\bar{\mathbf{u}}_i(k \to k+n-1\|k)$	The collection of $\triangleq \{\bar{u}_i(k+s\|k)\}_{0\leq s\leq n-1}$
$\mathbf{K}_i(k \to k+n-1\|k)$	The collection of $\triangleq \{K_i(k+s\|k)\}_{0\leq s\leq n-1}$
$\|\mathscr{N}_i(k)\|$	The cardinality of $\mathscr{N}_i(k)$
$p_{\max}$ (resp. $v_{\max}$)	The upper bound on the position (resp. the velocity)
$\mathbb{P}_X$	The projection operator onto X

4.3 Preliminaries

In this section, we provide both notations and a set of preliminary results that will be used to state our algorithm and analyze its convergence properties in the sequel.

4.3.1 A Coordinate Transformation

We pick any scalar $\beta > 1$, and define the change of coordinates $T : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ such that $T(p_i, v_i, u_i) = (p_i, q_i, \bar{u}_i)$ where $q_i = p_i + \beta v_i$ and $\bar{u}_i = v_i + \beta u_i = \frac{1}{\beta}(q_i - p_i) + \beta u_i$. Applying this coordinate transformation on dynamics (4.1), we obtain:

$$p_i(k+1) = \left(1 - \frac{1}{\beta}\right) p_i(k) + \frac{1}{\beta} q_i(k),$$
$$q_i(k+1) = \left(1 + \frac{1}{\beta}\right) q_i(k) - \frac{1}{\beta} p_i(k) + \beta u_i(k) = q_i(k) + \bar{u}_i(k). \qquad (4.2)$$

We refer to $\bar{u}_i(k)$ as the auxiliary control of vehicle i.

Remark 4.3 Since β is nonzero, then the formation property of $\lim_{k\to+\infty} \text{dist}(p(k), X^*) = 0$ and $\lim_{k\to+\infty} \|v_i(k)\| = 0$ is equivalent to

$$\lim_{k\to+\infty} \text{dist}(q(k), X^*) = 0, \quad \lim_{k\to+\infty} \|p_i(k) - q_i(k)\| = 0, \quad \forall i \in V.$$

This equivalence will be used for the algorithm design and analysis. •

4.3.2 A Constrained Multiparametric Program

In this part, we introduce a constrained multiparametric program which will be used in our distributed resilient formation control algorithms. Given any pair of $u_{\max} > 0$ and $\beta > 1$, we choose a pair of positive constants $v_{\max}$ and $\bar{u}_{\max}$ such that the following holds:

$$v_{\max} + \bar{u}_{\max} \le \beta u_{\max}, \quad \bar{u}_{\max} \le v_{\max}. \qquad (4.3)$$

We then introduce the following notations:

$$\rho \triangleq \min\left\{\frac{1}{2}, \frac{\bar{u}_{\max}}{2p_{\max} + \beta v_{\max}}\right\}, \qquad (4.4)$$
$$W \triangleq \{v_i \in \mathbb{R}^d \mid \|v_i\|_\infty \le v_{\max}\}, \quad \bar{U} \triangleq \{\bar{u}_i \in \mathbb{R}^d \mid \|\bar{u}_i\|_\infty \le \bar{u}_{\max}\},$$

where W (resp. $\bar{U}$) is the constraint set imposed on the velocity v_i (resp. the auxiliary input $\bar{u}_i$) of vehicle i.

Choose $\hat{\rho} \in (0, \rho]$. One can see that a set of positive constants δ, α, and γ can be chosen so that:

$$(1 + (1 - \hat{\rho})^2)\alpha + \hat{\rho}^2\gamma < \min\{2\alpha, \alpha + \gamma\} - \delta. \tag{4.5}$$

The relation (4.5) will be used in the proof of Claim 4.3 of Proposition 4.1.

Remark 4.4 We now proceed to choose *one* set of parameters to satisfy (4.5). Choose $\gamma < \alpha$, then (4.5) is equivalent to $(1 + (1 - \hat{\rho})^2)\alpha + \hat{\rho}^2\gamma < \alpha + \gamma - \delta$ and then

$$\alpha < \frac{1 - \hat{\rho}^2}{(1 - \hat{\rho})^2}\gamma - \frac{1}{(1 - \hat{\rho})^2}\delta.$$

Notice that $\frac{1-\hat{\rho}^2}{(1-\hat{\rho})^2} > 1$. Then one can always choose a set of δ, α, and γ with $\gamma < \alpha$ and δ being sufficiently small to satisfy the above relation. •

With the above notations in place, we then define the following n-horizon optimal control with the state and input constraints X and U (n-OC, for short) parameterized by the vector $(p_i, q_i, z_i, v_i) \in X^3 \times W$:

$$\begin{aligned}
&\min_{\bar{\mathbf{u}}_i \in \mathbb{R}^{d\times n}} \sum_{s=0}^{n-1} \left(\alpha\|z_i - q_i(s)\|^2 + \gamma\|\bar{u}_i(s)\|^2\right) + \alpha\|z_i - q_i(n)\|^2,\\
&\text{such that} \quad q_i(s+1) = q_i(s) + \bar{u}_i(s),\\
&\qquad p_i(s+1) = \left(1 - \frac{1}{\beta}\right) p_i(s) + \frac{1}{\beta} q_i(s),\\
&\qquad \bar{u}_i(s) = K_i(s)(z_i - q_i(s)),\\
&\qquad v_i(s) = \frac{1}{\beta}(q_i(s) - p_i(s)),\\
&\qquad q_i(s+1) \in X, \quad v_i(s) \in W,\\
&\qquad \bar{u}_i(s) \in \bar{U}, \quad 0 \le s \le n-1.
\end{aligned} \tag{4.6}$$

The initial states are given by $p_i(0) = p_i$, $q_i(0) = q_i$, $v_i(0) = v_i$. This problem is defined for sets satisfying Assumption 4.1 and constants α, γ satisfying the condition (4.5). In the n-OC problem (4.6), the state $z_i \in X$ will be some target point defined later. A detailed discussion on problem (4.6) will be given in Remark 4.5.

The following proposition characterizes the solutions to the n-OC and its proof will be given in Sect. 4.5.

Proposition 4.1 (Characterization of the optimal solutions to the n-OC). *There is at least one solution to the n-OC parameterized by the vector $(p_i, q_i, z_i, v_i) \in X^3 \times W$. Consider any of its optimal solution $\bar{\mathbf{u}}_i = (\bar{u}_i(0), \ldots, \bar{u}_i(n-1))^T \in \mathbb{R}^{d\times n}$ with*

$\bar{u}_i(s) = K_i(s)(z_i - q_i(s))$, *for all* $0 \leq s \leq n-1$. *There is a pair of* $\vartheta_{\min}, \vartheta_{\max} \in (0, 1)$ *independent of* $(p_i, q_i, z_i, v_i) \in X^3 \times W$ *such that* $K_i(s) \in [\vartheta_{\min}, \vartheta_{\max}]$.

We will use the n-OC problem in our algorithms for some $(p_i(k), q_i(k), z_i(k), v_i(k)) \in X^3 \times W$, which change for $k \geq 0$. When necessary, we will use the notation $\mathbf{u}_i(k \to k+n-1|k)$, and $\mathbf{K}_i(k \to k+n-1|k)$ to refer to the resulting control sequences.

4.4 Distributed Attack-Resilient Algorithm

In this section, we propose a distributed constrained formation control algorithm. After this, we summarize the algorithm resilience to replay attacks.

4.4.1 Algorithm Statement

In order to play against replay attackers, we exploit the Receding Horizon Control (RHC) methodology to synthesize a distributed algorithm. The usage of RHC in the proposed algorithm is motivated by two salient features of RHC: first, it can *explicitly* handle state and input constraints, which is a unique advantage of RHC; second, it is able to generate suboptimal control laws approximating an associated infinite-horizon optimal control problem. More importantly, RHC is able to produce a sequence of feasible control commands for the next few steps. These commands serve as backup and are used by vehicles in response to replay attacks. As mentioned before, operators cannot predict the occurrence of replay attacks and have to account for the worst case. That is, each operator assumes that her opponent would launch attacks at every time instant, and chooses $n \geq \tau_{\max} + 1$. The distributed algorithm is described as follows.

[*Algorithm Description*] Each vehicle has a memory storing the backup control commands in response to replay attacks. The state of vehicle i's memory is denoted by $M_i^v(k) \in \mathbb{R}^{d \times n}$.

At each time k, operator i receives $p_j(k)$ from operator $j \in \mathcal{N}_i(k)$. Operator i assumes that the vehicles of his/her current neighbors do not move over a finite time horizon of length n, and then identifies $\phi_i(k)$, the target point which minimizes the local formation error of $\sum_{j \in \mathcal{N}_i(k)} \|q_i - p_j(k) - \nu_{ij}\|^2 + \|q_i - p_i(k)\|^2 + \|q_i - q_i(k)\|^2$:

$$\phi_i(k) \triangleq \frac{1}{2 + |\mathcal{N}_i(k)|} \left(q_i(k) + p_i(k) + \sum_{j \in \mathcal{N}_i(k)} (p_j(k) + \nu_{ij}) \right).$$

According to Remark 4.3, the local formation error of $\sum_{j\in\mathcal{N}_i(k)} \|q_i - p_j(k) - \nu_{ij}\|^2 + \|q_i - p_i(k)\|^2 + \|q_i - q_i(k)\|^2$ captures the sum of the distance of $q(k)$ to X^* and the disagreement between p_i and q_i.

If $\nu_{ij} = 0$, then $\phi_i(k)$ is a convex combination of the time-dependent states. If these time-dependent states are in X, so is $\phi_i(k)$. However, the formation vectors ν_{ij} are nonzero, then $\phi_i(k)$ is potentially outside X. In order to enforce the state constraint X, operator i computes the target point $z_i(k)$ via projecting $\phi_i(k)$ onto X, that is, $z_i(k) \triangleq \mathbb{P}_X[\phi_i(k)]$ where $\mathbb{P}_X$ is the projection operator onto the set of X.

After obtaining the target point $z_i(k)$, operator i solves the n-OC parameterized by the vector of $(p_i(k), q_i(k), z_i(k), v_i(k))$, and obtains the auxiliary control sequence $\bar{\mathbf{u}}_i(k \to k+n-1|k)$.[2] Operator i then generates the real control sequence of $\mathbf{u}_i(k \to k+n-1|k)$ by simulating the dynamics of vehicle i over the time frame $[k, k+n]$ as follows:

$$\begin{aligned}
p_i(k+s+1|k) &= \left(1 - \frac{1}{\beta}\right) p_i(k+s|k) + \frac{1}{\beta} q_i(k+s|k),\\
q_i(k+s+1|k) &= q_i(s+k|k) + \bar{u}_i(k+s|k),\\
u_i(k+s|k) &= \frac{1}{\beta}\bar{u}_i(k+s|k) - \frac{1}{\beta^2}(q_i(k+s|k)\\
&\quad - p_i(k+s|k)), \quad 0 \le s \le n-1, \qquad (4.7)
\end{aligned}$$

where $q_i(k|k) = q_i(k)$ and $p_i(k|k) = p_i(k)$. After that, operator i sends the package including $\mathbf{u}_i(k \to k+n-1|k)$ to vehicle i where each element $u_i(k+s|k)$ in the package is labeled by the time index $k+s$ for $0 \le s \le n-1$.

If $s_i^a(k) = 1$, adversary i launches a replay attack, sending the stored command $M_i^a(k)$ to vehicle i, and letting $M_i^a(k+1) = M_i^a(k)$. If $s_i^a(k) = 0$, adversary i then does not produce any attack, but instead intercepts the package containing $\mathbf{u}_i(k \to k+n-1|k)$, and updates her memory as $M_i^a(k+1) = \mathbf{u}_i(k \to k+n-1|k)$.

After receiving the package, vehicle i checks the time index which is $k - \tau_i^a(k)$. If the package is new (i.e., $\tau_i^a(k) = 0$), then vehicle i replaces it in her memory by the new arrival (i.e., $M_i^v(k+1) = \mathbf{u}_i(k \to k+n-1|k)$), implements $u_i(k) = u_i(k|k)$, and sends $p_i(k+1)$ and $v_i(k+1)$ to operator i. If the package is repeated (i.e., $\tau_i^a(k) \ge 1$), then vehicle i implements $u_i(k|k - \tau_i^a(k))$ in its memory, sets $M_i^v(k+1) = M_i^v(k)$, and sends $p_i(k+1)$ and $v_i(k+1)$ to operator i. At the next time $k+1$, every decision maker will repeat the above process.

Remark 4.5 In the n-OC parameterized by the vector of $(p_i(k), q_i(k), z_i(k), v_i(k))$, the solution $\bar{\mathbf{u}}_i(k \to k+n-1|k)$ is a suboptimal controller on steering the state $q_i(k)$ toward the target point $z_i(k)$ on saving the control effort $\bar{u}_i(k)$ in (4.2). The idea of moving toward target points for distributed RHC was first proposed and analyzed in [2]. Like any other RHC law, e.g., in [12, 13], our proposed algorithm requires that

[2]Here we assume the feasibility of the n-OC parameterized by the vector of $(p_i(k), q_i(k), z_i(k), v_i(k))$. Later we will verify this point in Lemma 4.1.

each operator online solves an optimization problem, the n-OC, at each time instant. We will discuss the issue of solving these optimization problems in Sect. 4.6. •

We summarize the distributed REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM in Algorithm 1.

Algorithm 1 Replay-attack resilient formation control

• Initially, operators agree on $\beta > 1$ and a pair of positive constants $v_{\max}$ and $\bar{u}_{\max}$ such that (4.3) holds. In addition, operators agree on a set of positive constants δ, α, and γ such that (4.5) holds.

• At each $k \geq 0$, adversary, operator, and vehicle i execute the following steps:

1. Operator i receives the location $p_j(k)$ from her neighboring operator $j \in \mathcal{N}_i(k)$, and computes the target point $z_i(k)$. Operator i solves the n-OC parameterized by the vector of $(p_i(k), q_i(k), z_i(k), v_i(k))$, and obtains the solution of $\bar{\mathbf{u}}_i(k \to k+n-1|k)$. After that, operator i computes $\mathbf{u}_i(k \to k+n-1|k)$ via (4.7) and sends it to vehicle i.

2. If $s_i^a(k) = 1$, adversary i sends $M_i^a(k)$ to vehicle i, and let $M_i^a(k+1) = M_i^a(k)$. If $s_i^a(k) = 0$, adversary i sets $M_i^a(k+1) = \mathbf{u}_i(k \to k+n-1|k)$.

3. If $\tau_i^a(k) = 0$, then vehicle i sets $M_i^v(k+1) = \mathbf{u}_i(k \to k+n-1|k)$, implements $u_i(k|k)$, and sends $p_i(k+1)$ and $v_i(k+1)$ to operator i. If $\tau_i^a(k) \geq 1$, then vehicle i implements $u_i(k|k-\tau_i^a(k))$ in $M_i^v(k)$, sets $M_i^v(k+1) = M_i^v(k)$, and sends $p_i(k+1)$ and $v_i(k+1)$ to operator i.

4. Repeat for $k = k+1$.

4.4.2 The Resilience Properties

The theorem to follow summarizes the convergence properties of the distributed REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM, whose proof is included in Sect. 4.5.

Theorem 4.1 (Convergence properties of the distributed replay-attack resilient formation control algorithm) *Suppose that Assumptions 4.1, on the constraint sets, 4.2, on the connected formation digraph, 4.3, on problem feasibility, 4.4, on periodic communication, and 4.5, on the maximum number of attacks hold. Let vehicle i start from $(p_i(0), v_i(0))$ with $(p_i(0), p_i(0) + \beta v_i(0)) \in X^2$ and $v_i(0) \in W$ for $i \in V$. Then, the distributed replay-attack resilient formation control algorithm with $n \geq \tau_{\max} + 1$ ensures the following properties:*

[Constraint Satisfaction] $p_i(k) \in X$ and $u_i(k) \in U$ for all $i \in V$ and $k \geq 0$.

[Achieving Formation] It holds that

$$\lim_{k \to +\infty} \operatorname{dist}(p(k), X^*) = 0, \quad \textit{and} \quad \lim_{k \to +\infty} \|v_i(k)\| = 0, \quad i \in V.$$

4.5 Convergence Analysis

In this section, we provide complete analysis of Proposition 4.1 and Theorem 4.1. We start with the proof of Proposition 4.1.

Proof of Proposition 4.1 To simplify notations in the proof, we assume that $k = 0$ and drop the conditional independency on the starting time k; e.g., $q_i(s) = q_i(k+s|k)$.

By (4.2), one can verify that $v_i(s+1)$ is a convex combination of $v_i(s)$ and $\bar{u}_i(s)$ through the following relations:

$$\begin{aligned} v_i(s+1) &= \frac{1}{\beta}(q_i(s+1) - p_i(s+1)) \\ &= \frac{1}{\beta}(q_i(s) + \bar{u}_i(s) - \left(1 - \frac{1}{\beta}\right) p_i(s) - \frac{1}{\beta} q_i(s)) \\ &= \left(1 - \frac{1}{\beta}\right) v_i(s) + \frac{1}{\beta}\bar{u}_i(s). \end{aligned} \tag{4.8}$$

In order to simplify the notations of the n-OC, we define the coordinate transformation $y_i(s) = q_i(s) - z_i(0)$. Then the n-OC parameterized by $(p_i(0), q_i(0), z_i(0), v_i(0)) \in X^3 \times W$ becomes the following one:

$$\begin{aligned} &\min_{\mathbf{K}_i(0\to n-1)\in\mathbb{R}^n} \sum_{s=0}^{n-1} \left(\alpha\|y_i(s)\|^2 + \gamma\|\bar{u}_i(s)\|^2\right) + \alpha\|y_i(n)\|^2, \\ &\text{s.t.}\ \ p_i(s+1) = \left(1 - \frac{1}{\beta}\right) p_i(s) + \frac{1}{\beta}(y_i(s) + z_i(0)), \\ &\qquad y_i(s+1) = y_i(s) + \bar{u}_i(s), \\ &\qquad \bar{u}_i(s) = -K_i(s) y_i(s), \quad 0 \le s \le n-1, \\ &\qquad v_i(s+1) = \left(1 - \frac{1}{\beta}\right) v_i(s) + \frac{1}{\beta}\bar{u}_i(s), \\ &\qquad y_i(s+1) + z_i(0) \in X, \quad v_i(s) \in W, \\ &\qquad \bar{u}_i(s) \in \bar{U}, \quad 0 \le s \le n-1, \end{aligned} \tag{4.9}$$

where $y_i(0) \neq 0$ and we change the decision variable $\bar{\mathbf{u}}_i(0 \to n-1)$ to $\mathbf{K}_i(0 \to n-1)$ for the ease of presentation. The remainder of the proof is divided into the following three claims to characterize the solutions to (4.9).

Claim 4.1 Given $z_i(0) \in \mathbb{R}^d$, the set of $Y \triangleq \{y_i \in \mathbb{R}^d \mid y_i + z_i(0) \in X\}$ is convex.

Proof Pick any $\bar{y}_i$ and $\tilde{y}_i$ from Y, and any $\mu \in [0, 1]$. Since $\bar{y}_i$ and $\tilde{y}_i$ are in Y, and thus $\bar{y}_i + z_i(0)$ and $\tilde{y}_i + z_i(0)$ are in X. Since X is convex, then $\mu(\bar{y}_i + z_i(0)) + (1 - \mu)(\tilde{y}_i + z_i(0)) = \mu\bar{y}_i + (1-\mu)\tilde{y}_i + z_i(0) \in X$. This implies that $\mu\bar{y}_i + (1-\mu)\tilde{y}_i \in Y$ and the convexity of Y follows. •

With Claim 4.1, we are now ready to find a feasible solution to (4.9) which will produce an upper bound of the optimal value of (4.9).

Claim 4.2 Consider the scalar sequence of $\tilde{\mathbf{K}}_i(0 \to n-1) \triangleq \{\tilde{K}_i(s)\}_{0 \le s \le n-1}$. If $\tilde{K}_i(s) \in [0, \rho]$ for $0 \le s \le n-1$ and ρ satisfies (4.4), then $\tilde{\mathbf{K}}_i(0 \to n-1)$ is a feasible solution candidate to (4.9).

Proof Consider (4.9) where $v_i(0) = \tilde{v}_i(0)$, $y_i(0) = \tilde{y}_i(0)$ and $\mathbf{K}_i(0 \to n-1) = \tilde{\mathbf{K}}_i(0 \to n-1)$. Let $\{\tilde{v}_i(s)\}_{0 \le s \le n}$ and $\{\tilde{y}_i(s)\}_{0 \le s \le n}$ be the generated states and $\{\bar{u}_i(s)\}_{0 \le s \le n-1}$ be the produced auxiliary inputs in (4.9).

In order to verify the feasibility of $\tilde{\mathbf{K}}_i(0 \to n-1)$ to (4.9), we will check by induction that the following property, say Constraint Verification (CV, for short), holds for all $0 \le \tau \le n-1$: for $0 \le s \le \tau$, we have that $\tilde{y}_i(s+1) \in Y$, $\tilde{v}_i(s+1) \in W$, and $\bar{u}_i(s) \in \bar{U}$.

Let us start from the case $\tau = 0$. Recall that $z_i(0) \in X$ and $\tilde{y}_i(0) + z_i(0) = q_i(0) \in X$. This implies that 0 and $\tilde{y}_i(0)$ are both in Y. Since Y is convex shown in Claim 4.1 and $\tilde{K}_i(0) \in [0, 1]$, then $\tilde{K}_i(0) \times 0 + (1 - \tilde{K}_i(0)) \times \tilde{y}_i(0) = \tilde{y}_i(1) \in Y$. In addition, we notice the following estimates on $\|\tilde{y}_i(0)\|_\infty$:

$$\|\tilde{y}_i(0)\|_\infty \le \|p_i(0) - z_i(0)\|_\infty + \beta \|\tilde{v}_i(0)\|_\infty \le 2p_{\max} + \beta v_{\max},$$

where $p_{\max}$ (resp. $v_{\max}$) is the uniform bound on X (resp. W). Since $\tilde{K}_i(0) \in [0, \rho]$ and (4.4), then we have

$$\|\bar{u}_i(0)\|_\infty \le \tilde{K}_i(0) \|\tilde{y}_i(0)\|_\infty \le \rho(\beta v_{\max} + 2p_{\max}) \le \bar{u}_{\max},$$

i.e., $\bar{u}_i(0) \in \bar{U}$. Note that $\tilde{v}_i(1)$ is a convex combination of $\tilde{v}_i(0) \in W$ and $\bar{u}_i(0) \in \bar{U} \subseteq W$. Hence, we have $\tilde{v}_i(1) \in W$ and CV holds for $\tau = 0$.

Assume that CV holds for some $0 \le \tau \le n-2$. One can follow the same arguments above by replacing the time instants 0 and 1 with τ and $\tau + 1$, respectively, to show that CV holds for $\tau + 1$. By induction, we conclude that $\tilde{\mathbf{K}}_i(0 \to n-1)$ consists of a feasible solution candidate to (4.9). This completes the proof of Claim 4.2. •

It follows from Claim 4.2 that the n-OC is feasible; that is, one can build a candidate solution by taking $0 \le \tilde{K}_i(s) \le \rho$, with $0 \le s \le n-1$. We now set out to further characterize its optimal solutions.

Claim 4.3 There is a pair of $\vartheta_{\min}$ and $\vartheta_{\max}$ in $(0, 1)$ such that $K_i(s) \in [\vartheta_{\min}, \vartheta_{\max}]$ for any optimal solution $\mathbf{K}_i(0 \to n-1)$ to (4.9).

Proof Let $\{y_i(s)\}_{0 \le s \le n}$ be the states generated by the optimal solution $\mathbf{K}_i(0 \to n-1)$ in (4.9). Pick any $1 \le \tau \le n$ and assume that $y_i(n-\tau) \neq 0$. From Bellman's principle of optimality, we know that the last τ components, $\{K_i(s)\}_{n-\tau \le s \le n-1}$, of $\mathbf{K}_i(0 \to n-1)$ define an optimal solution to the truncated version of n-OC. More precisely, $\{K_i(s)\}_{n-\tau\leq s\leq n-1}$ is an optimal solution to the $(n-\tau)$-OC parameterized by $(p_i(n-\tau), v_i(n-\tau), q_i(n-\tau), z_i(0))$ which is given by:

$$\begin{aligned}
&\min_{\mathbf{K}_i(n-\tau\to n-1)\in\mathbb{R}^\tau} \sum_{s=n-\tau}^{n-1} \left(\alpha\|y_i(s)\|^2 + \gamma\|\bar{u}_i(s)\|^2\right) + \alpha\|y_i(n)\|^2,\\
&\text{s.t. } \; p_i(s+1) = \left(1-\frac{1}{\beta}\right)p_i(s) + \frac{1}{\beta}(y_i(s)+z_i(0)),\\
&\quad y_i(s+1) = y_i(s) + \bar{u}_i(s),\\
&\quad \bar{u}_i(s) = -K_i(s)y_i(s), \quad n-\tau\leq s\leq n-1,\\
&\quad v_i(s+1) = \left(1-\frac{1}{\beta}\right)v_i(s) + \frac{1}{\beta}\bar{u}_i(s),\\
&\quad y_i(s+1)+z_i(0)\in X, \quad v_i(s)\in W,\\
&\quad \bar{u}_i(s)\in\bar{U}, \quad n-\tau\leq s\leq n-1.
\end{aligned} \tag{4.10}$$

Denote by r_τ^* the optimal value of (4.10). It is easy to see that r_τ^* is lower bounded by the sum of the first two running states and the first input; that is:

$$\begin{aligned}
r_\tau^* &\geq \alpha\|y_i(n-\tau)\|^2 + \gamma\|K_i(n-\tau)y_i(n-\tau)\|^2 + \alpha\|(1-K_i(n-\tau))y_i(n-\tau)\|^2\\
&= h(K_i(n-\tau))\|y_i(n-\tau)\|^2,
\end{aligned} \tag{4.11}$$

where $(1-K_i(n-\tau))y_i(n-\tau)$ is the state by applying the auxiliary input $-K_i(n-\tau)y_i(n-\tau)$ to $y_i(n-\tau)$, and $h(\nu)\triangleq\alpha+\gamma\nu^2+\alpha(1-\nu)^2$.

Regarding the function $h(\nu)$, we notice that $h(\nu)$ is quadratic in ν and reaches the minimum at $\frac{\alpha}{\alpha+\gamma}$. Then there is a pair of $\vartheta_{\min}$ and $\vartheta_{\max}$ in $(0,1)$ such that

$$h(\nu)\geq\min\{\alpha+\gamma, 2\alpha\}-\delta, \quad \nu\notin[\vartheta_{\min},\vartheta_{\max}], \tag{4.12}$$

where $\alpha+\gamma=h(1)$, $2\alpha=h(0)$, and δ is given in (4.5).

We now set out to show that $K_i(n-\tau)\in[\vartheta_{\min},\vartheta_{\max}]$. To achieve this, we now construct a solution candidate $\{\tilde{K}_i(s)\}_{n-\tau\leq s\leq n-1}$ to (4.10) where $\tilde{K}_i(s)=\hat{\rho}\in(0,\rho]$ for $n-\tau\leq s\leq n-1$. It follows from Claim 4.2 that $\{\tilde{K}_i(s)\}_{n-\tau\leq s\leq n-1}$ is a feasible solution candidate to (4.9). Let $\tilde{r}_\tau$ be the value of (4.10) generated by $\{\tilde{K}_i(s)\}_{n-\tau\leq s\leq n-1}$. We then have the following relations on $\tilde{r}_\tau$:

$$\begin{aligned}
\tilde{r}_\tau &= \alpha\sum_{\kappa=0}^{\tau}(1-\hat{\rho})^{2\kappa}\|y_i(n-\tau)\|^2 + \gamma\sum_{\kappa=0}^{\tau-1}\hat{\rho}^2(1-\hat{\rho})^{2\kappa}\|y_i(n-\tau)\|^2\\
&\leq \frac{1}{1-(1-\hat{\rho})^2}\left(\alpha(1-(1-\hat{\rho})^4)+\gamma\hat{\rho}^2(1-(1-\hat{\rho})^2)\right)\times\|y_i(n-\tau)\|^2\\
&= \left((1+(1-\hat{\rho})^2)\alpha+\hat{\rho}^2\gamma\right)\|y_i(n-\tau)\|^2.
\end{aligned} \tag{4.13}$$

On the right-hand side of the first equality of (4.13), the first summation is the aggregation sum of running states and the second one is the accumulated control cost.

By (4.11–4.13) and (4.5), one can verify that if $K_i(n-\tau) \notin [\vartheta_{\min}, \vartheta_{\max}]$, then

$$\begin{aligned}\tilde{r}_\tau &= \big((1+(1-\hat{\rho})^2)\alpha + \hat{\rho}^2\gamma\big)\|y_i(n-\tau)\|^2 \\ &< (\min\{\alpha+\gamma, 2\alpha\} - \delta)\|y_i(n-\tau)\|^2 \le r_\tau^*.\end{aligned}$$

That is, $\tilde{r}_\tau < r_\tau^*$, contradicting the optimality of $\{K_i(s)\}_{n-\tau\le s\le n-1}$ for (4.10). Hence, it must be the case that $K_i(n-\tau) \in [\vartheta_{\min}, \vartheta_{\max}] \subset (0,1)$. This holds for any $1 \le \tau \le n$, and thus this completes the proof of Claim 4.3. •

The last claim establishes the result of Proposition 4.1. •

The following lemma shows the property of constraint enforcement in Theorem 4.1.

Lemma 4.1 (Constraint satisfaction and feasibility of the n-OC) *The n-OC parameterized by $(p_i(k), q_i(k), z_i(k), v_i(k)) \in X^3 \times W$ is feasible for all $i \in V$ and $k \ge 1$. In addition, it holds that $p_i(k) \in X$, $v_i(k) \in W$, and $u_i(k) \in U$ for all $i \in V$ and $k \ge 0$.*

Proof It is trivial that $z_i(0) \in X$ is due to the projection operator. Recall that X is convex, and $(p_i(0), q_i(0)) \in X^2$. Since $p_i(1)$ is a convex combination of $p_i(0)$ and $q_i(0)$ by (4.2), thus $p_i(1) \in X$. As a consequence of Claim 4.2 in the proof of Proposition 4.1, the n-OC parameterized by $(p_i(0), q_i(0), z_i(0), v_i(0)) \in X^3 \times W$ is feasible. This ensures that $q_i(1) \in X$, $v_i(1) \in W$, and $\bar{u}_i(0) \in W$. Note that $\|u_i(0)\|_\infty \le \frac{1}{\beta}(\|v_i(0)\|_\infty + \|\bar{u}_i(0)\|_\infty) \le \frac{1}{\beta}(\bar{u}_{\max} + v_{\max}) \le u_{\max}$ by (4.3). Hence, it yields that $u_i(0) \in U$.

The remainder of the proof can be derived by means of induction, and is omitted here. •

With the above instrumental results, we are now ready to characterize the convergence properties of the Replay-Attack Resilient Formation Control Algorithm and complete the proof of Theorem 4.1.

Proof of Theorem 4.1 Consider $i \in V$ and time $k \ge 0$. Note that the control command $u_i(k) = u_i(k|k-\tau_i^a(k))$ is applied to (4.1), or, equivalently, $\bar{u}_i(k)$ is applied to (4.2). Thus, the closed-loop dynamics of (4.2) is given by:

$$\begin{aligned}p_i(k+1) &= \left(1-\frac{1}{\beta}\right)p_i(k) + \frac{1}{\beta}q_i(k),\\ q_i(k+1) &= q_i(k) + \bar{u}_i(k|k-\tau_i^a(k))\\ &= q_i(k) + K_i(k|k-\tau_i^a(k))(z_i(k-\tau_i^a(k)) - q_i(k))\\ &= q_i(k) + K_i(k|k-\tau_i^a(k))(\mathbb{P}_X[\phi_i(k-\tau_i^a(k))] - q_i(k))\\ &= q_i(k) + K_i(k|k-\tau_i^a(k))(\phi_i(k-\tau_i^a(k)) - q_i(k)) + w_i(k). \qquad (4.14)\end{aligned}$$

The term $w_i(k)$ in (4.14) is the error induced by the projection operator $\mathbb{P}_X$ given by:

$$w_i(k) \triangleq K_i(k|k-\tau_i^a(k))(\mathbb{P}_X[\phi_i(k-\tau_i^a(k))] - \phi_i(k-\tau_i^a(k))).$$

Substituting directly the definition of $\phi_i(k-\tau_i^a(k))$ as follows:

$$\phi_i(k-\tau_i^a(k)) = \frac{q_i(k-\tau_i^a(k)) + p_i(k-\tau_i^a(k))}{2+|\mathcal{N}_i(k-\tau_i^a(k))|} + \frac{\sum_{j\in\mathcal{N}_i(k-\tau_i^a(k))}(p_j(k-\tau_i^a(k))+\nu_{ij})}{2+|\mathcal{N}_i(k-\tau_i^a(k))|},$$

into (4.14) leads to the following:

$$\begin{aligned} p_i(k+1) &= \left(1-\frac{1}{\beta}\right)p_i(k) + \frac{1}{\beta}q_i(k), \\ q_i(k+1) &= q_i(k) + K_i(k|k-\tau_i^a(k)) \\ &\quad \times \left\{\frac{1}{2+|\mathcal{N}_i(k-\tau_i^a(k))|}\Big(q_i(k-\tau_i^a(k)) + p_i(k-\tau_i^a(k))\right. \\ &\quad \left. + \sum_{j\in\mathcal{N}_i(k-\tau_i^a(k))}(p_j(k-\tau_i^a(k))+\nu_{ij})\Big) - q_i(k)\right\} + w_i(k). \end{aligned} \tag{4.15}$$

Pick any $p^* \in X^*$, and we define the errors $q_i^e(k) \triangleq q_i(k) - p_i^*$ and $p_i^e(k) \triangleq p_i(k) - p_i^*$. Subtract p_i^* on both sides of (4.15), and we rewrite (4.15) in terms of $p_i^e(k)$ and $q_i^e(k)$ as follows:

$$\begin{aligned} p_i^e(k+1) &= \left(1-\frac{1}{\beta}\right)p_i^e(k) + \frac{1}{\beta}q_i^e(k), \\ q_i^e(k+1) &= (1-K_i(k|k-\tau_i^a(k)))q_i^e(k) \\ &\quad + \frac{K_i(k|k-\tau_i^a(k))}{2+|\mathcal{N}_i(k-\tau_i^a(k))|}q_i^e(k-\tau_i^a(k)) \\ &\quad + \sum_{j\in\mathcal{N}_i(k-\tau_i^a(k))\cup\{i\}}\frac{K_i(k|k-\tau_i^a(k))}{2+|\mathcal{N}_i(k-\tau_i^a(k))|}p_j^e(k-\tau_i^a(k)) + w_i(k), \end{aligned} \tag{4.16}$$

where we use $p_i^* - p_j^* = \nu_{ij}$ for $p^* \in X^*$ in (4.16). By Remark 4.3, we notice that the following consensus property for algorithm (4.16)

$$\lim_{k\to+\infty}\|p_i^e(k) - q_j^e(k)\| = 0, \quad i, j \in V, \tag{4.17}$$

is equivalent to achieving the formation control mission in (4.1). In particular, $\lim_{k\to+\infty}\|p_i^e(k)-q_i^e(k)\|=0$ implies that $\lim_{k\to+\infty}\|p_i(k)-q_i(k)\|=0$. And the property $\lim_{k\to+\infty}\|q_i^e(k)-q_j^e(k)\|=0$ implies the following:

$$\lim_{k\to+\infty}\|(q_i(k)-p_i^*)-(q_j(k)-p_j^*)\| = \lim_{k\to+\infty}\|q_i(k)-q_j(k)-d_{ij}\|=0.$$

We will show that $w_i(k)$ is diminishing in Claim 4.4. In this way, the dynamics of (4.16) are decoupled along different dimensions, and thus we will only consider the scalar case; i.e., $d=1$, for the ease of presentation in the remainder of the proof.

In order to show the consensus property (4.17), we transform the second-order algorithm (4.16) into an equivalent first-order one. To achieve this, we introduce a transformed system with two classes of agents: location agents labeled by $\{1,\ldots,N\}$ and velocity agents labeled by $\{N+1,\ldots,2N\}$. With these, we define the state $x(k)\in\mathbb{R}^{2N}$ in such a way that $x_i(k)=p_i^e(k)$ for location agent $i\in\{1,\ldots,N\}$, $x_i(k)=q_{i-N}^e(k)$ for velocity agent $i\in\{N+1,\ldots,2N\}$. Let $V_T\triangleq\{1,\ldots,2N\}$. Consequently, algorithm (4.16) can be transformed into the following first-order consensus algorithm subject to delays and errors $e_\ell(k)$:

$$\begin{aligned} x_\ell(k+1) &= a_{\ell\ell}(k)x_\ell(k)+\bar{a}_{\ell\ell}(k)x_\ell(k-\tau_\ell^a(k))\\ &\quad+\sum_{\ell'\in V_T\setminus\{\ell\}} a_{\ell\ell'}(k)x_{\ell'}(k-\tau_\ell^a(k))+e_\ell(k), \end{aligned} \tag{4.18}$$

where $e_\ell(k)=0$, $\bar{a}_{\ell\ell}(k)=0$ for $\ell\in\{1,\ldots,N\}$, and $e_\ell(k)=w_{\ell-N}(k)$ for $\ell\in\{N+1,\ldots,2N\}$. Without loss of any generality, we assume that $x_\ell(k)=x_\ell(0)$ for $k=-1,\ldots,-\tau_{\max}$.

The weights in (4.18) induce the communication graph $\mathcal{G}_T(k)\triangleq\{V_T,\mathcal{E}_T(k)\}$ defined as $(\ell,\ell')\in\mathcal{E}_T(k)$ if and only if $a_{\ell\ell'}(k)\neq0$ for $\ell\neq\ell'$. From (4.15), we can see that location agent i and velocity agent i can communicate to each other all the time. This observation in conjunction with Assumptions 4.2, on the connectedness of the formation digraph, and 4.4, on periodic communication, yields that the directed graph $(V_T,\bigcup_{k=0}^{B-1}\mathcal{E}_T(k_0+k))$ is strongly connected for any $k_0\geq0$. Figure 4.2 shows an illustrative example with three operators where operators 1 and 2 communicate when k is odd, and operators 2 and 3 communicate when k is even.

Recall that $K_i(k|k-\tau_i^a(k))\in[\vartheta_{\min},\vartheta_{\max}]\subset(0,1)$ by Proposition 4.1. For (4.18), one can verify that there is $\eta_{\min}\in(0,1)$ such that:

$$a_{\ell\ell}(k)\geq\eta_{\min},\quad a_{\ell\ell'}(k)\neq0\Rightarrow a_{\ell\ell'}(k)\geq\eta_{\min}, \tag{4.19}$$

$$\bar{a}_{\ell\ell}(k)+\sum_{\ell'\in V_T}a_{\ell\ell'}(k)=1, \tag{4.20}$$

where (4.19) is referred to as the nondegeneracy property and (4.20) is referred to as the stochasticity property.

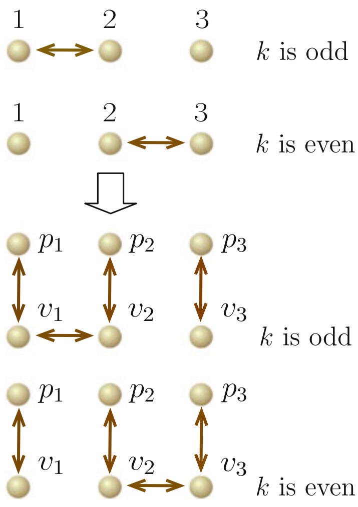


Fig. 4.2 Three agents communicate as indicated by the first two graphs over even and odd times. This translates into the communication graphs below for the transformed system

In order to show the consensus property (4.17), we first show that the error term $e_\ell(k)$ is diminishing.

Claim 4.4 For any $\ell \in \{N+1, \ldots, 2N\}$, it holds that

$$\lim_{k\to+\infty} \|e_\ell(k)\| = 0.$$

Proof Since X is convex and $p_i^* \in X$, then it follows from the projection theorem (e.g., on p. 17 in [14]) that the following holds for $i \in V$:

$$\begin{aligned}\|z_i(k-\tau_i^a(k)) - p_i^*\|^2 &= \|\mathbb{P}_X[\phi_i(k-\tau_i^a(k))] - p_i^*\|^2 \\ &\leq \|\phi_i(k-\tau_i^a(k)) - p_i^*\|^2 - \|w_i(k)\|^2. \end{aligned} \tag{4.21}$$

For the term of $\|\phi_i(k-\tau_i^a(k)) - p_i^*\|^2$, the following relations hold:

$$\begin{aligned}&\|\phi_i(k-\tau_i^a(k)) - p_i^*\|^2 \\ &= \|\frac{q_i^e(k-\tau_i^a(k)) + p_i^e(k-\tau_i^a(k))}{2+|\mathcal{N}_i(k-\tau_i^a(k))|} + \frac{\sum_{j\in\mathcal{N}_i(k-\tau_i^a(k))} p_j^e(k-\tau_i^a(k))}{2+|\mathcal{N}_i(k-\tau_i^a(k))|}\|^2 \\ &\leq \frac{\|q_i^e(k-\tau_i^a(k))\|^2 + \|p_i^e(k-\tau_i^a(k))\|^2}{2+|\mathcal{N}_i(k-\tau_i^a(k))|} + \frac{\sum_{j\in\mathcal{N}_i(k-\tau_i^a(k))} \|p_j^e(k-\tau_i^a(k))\|^2}{2+|\mathcal{N}_i(k-\tau_i^a(k))|}, \end{aligned} \tag{4.22}$$

where in the equality we use $p_i^* - p_j^* = \nu_{ij}$, and in the inequality we use the fact that the function $\|\cdot\|^2$ is a convex function and the Jensen's inequality (e.g., inequality (1.7) on p. 19 in [14]).

Subtract p_i^* on both sides of the update rule for $q_i(k)$ in (4.14), and it renders the following:

$$q_i^e(k+1) = (1 - K_i(k|k - \tau_i^a(k)))q_i^e(k) + K_i(k|k - \tau_i^a(k))(z_i(k - \tau_i^a(k)) - p_i^*).$$

Since $\|\cdot\|^2$ is a convex function, then the following holds:

$$\begin{aligned}\|q_i^e(k+1)\|^2 &\le (1 - K_i(k|k - \tau_i^a(k)))\|q_i^e(k)\|^2 \\ &+ K_i(k|k - \tau_i^a(k))\|z_i(k - \tau_i^a(k)) - p_i^*\|^2. \end{aligned} \tag{4.23}$$

Analogously, one can verify the following relation via the update rule for $p_i(k)$ in (4.15):

$$\|p_i^e(k+1)\|^2 \le (1 - \frac{1}{\beta})\|p_i^e(k)\|^2 + \frac{1}{\beta}\|q_i^e(k)\|^2. \tag{4.24}$$

Recall that $K_i(k|k - \tau_i^a(k)) \in [\vartheta_{\min}, \vartheta_{\max}]$ by Proposition 4.1. Then the combination of (4.21–4.24) establishes that the following holds for all $\ell \in V_T$:

$$\begin{aligned}\|x_\ell(k+1)\|^2 &\le b_{\ell\ell}(k)\|x_\ell(k)\|^2 + \bar{b}_{\ell\ell}(k)\|x_\ell(k - \tau_\ell^a(k))\|^2 \\ &+ \sum_{\ell' \in V_T \setminus \{\ell\}} b_{\ell\ell'}(k)\|x_{\ell'}(k - \tau_\ell^a(k))\|^2 - \vartheta_{\min}\|e_\ell(k)\|^2, \end{aligned} \tag{4.25}$$

where the following properties hold for the weights:

$$b_{\ell\ell} \ge \bar{\eta}_{\min}, \quad b_{\ell\ell'}(k) \ne 0 \Rightarrow b_{\ell\ell'}(k) \ge \bar{\eta}_{\min}, \tag{4.26}$$

$$\bar{b}_{\ell\ell}(k) + \sum_{\ell' \in V_T} b_{\ell\ell'}(k) = 1, \tag{4.27}$$

for some $\bar{\eta}_{\min} \in (0, 1)$.

The iterative relation (4.25) induces the communication graph $\bar{\mathcal{G}}_T(k) \triangleq \{V_T, \bar{\mathcal{E}}_T(k)\}$ where $(\ell, \ell') \in \bar{\mathcal{E}}_T(k)$ if and only if $b_{\ell\ell'}(k) \ne 0$ with $\ell \ne \ell'$. Recall that $K_i(k|k - \tau_i^a(k)) \in [\vartheta_{\min}, \vartheta_{\max}]$ by Proposition 4.1. Then $\mathcal{E}_T(k) = \bar{\mathcal{E}}_T(k)$ and thus the directed graph $(V_T, \bigcup_{k=0}^{B-1} \bar{\mathcal{E}}_T(k_0 + k))$ is strongly connected for any $k_0 \ge 0$.

We denote the maximum value of $\|x_\ell\|^2$ over the interval $[k - \tau_{\max}, k]$ as follows:

$$\Pi(k) \triangleq \max_{0 \le s \le \tau_{\max}} \max_{\ell \in V_T} \|x_\ell(k - s)\|^2.$$

By (4.20) and $\tau_i^a(k) \le \tau_{\max}$, it then follows from (4.25) and (4.27) that the following holds for any $\ell \in V_T$:

$$\|x_\ell(k+1)\|^2 \le \Pi(k) - \vartheta_{\min}\|e_\ell(k)\|^2,$$

and thus, $\Pi(k+1) \le \Pi(k)$; i.e., the sequence of $\{\Pi(k)\}$ is nonincreasing.

We now move to show by contradiction that $\|e_\ell(k)\|^2$ decreases to zero for all $\ell \in \{N+1, \ldots, 2N\}$ via studying the iterative relation (4.25). In particular, we assume that $\|e_\ell(k)\|^2$ is strictly away from zero infinitely often and derive that $\Pi(k)$ could be arbitrarily negative, contradicting $\Pi(k) \geq 0$.

Assume that there is some $\bar{\ell} \in \{N+1, \ldots, 2N\}$ and $\varepsilon > 0$ such that the event $\vartheta_{\min}\|e_{\bar{\ell}}(k)\|^2 \geq \varepsilon$ occurs infinitely often. Denote by the set $\{s_1, s_2, \ldots\}$ the collection of time instants when $\vartheta_{\min}\|e_{\bar{\ell}}(k)\|^2 \geq \varepsilon$ occurs. Without loss of any generality, we assume that $s_1 \geq 2NB+1$, and $s_{\kappa+1} \geq s_\kappa + 2NB + 1$ for $\kappa \geq 1$.

We now consider the time instant s_1. Define the set $\mathscr{D}_0 = \{\bar{\ell}\}$. Since the graph $(V_T, \bigcup_{k=0}^{B-1} \bar{\mathscr{E}}_T(s_1+k))$ is strongly connected, there is a nonempty set $\mathscr{D}_1 \subset V_T \setminus \{\bar{\ell}\}$ of agents such that for all $\ell \in \mathscr{D}_1$, $b_{\ell\bar{\ell}}(k) \neq 0$ occurs at least once during the time frame $[s_1, s_1 + B - 1]$. By induction, a set $\mathcal{D}_{\kappa+1} \subset V_T \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\kappa)$ can be defined by considering those agents $\ell \notin \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\kappa$ where there is some $\ell' \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\kappa$ such that $b_{\ell\ell'}(k) \neq 0$ occurs at least once during the time frame $[s_1 + \kappa B, s_1 + (\kappa+1)B - 1]$. The graph $(V_T, \bigcup_{k=0}^{B-1} \bar{\mathscr{E}}_T(s_1 + \kappa B + k))$ is strongly connected, $\mathcal{D}_{\kappa+1} \neq \emptyset$ as long as $V_T \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\kappa) \neq \emptyset$. Thus, there exists $\mathcal{L} \leq 2N-1$ such that the collection of $\mathscr{D}_0, \ldots, \mathscr{D}_{\mathcal{L}}$ is a partition of V_T.

For each time instant $k \geq s_1 + 1$, we define the set $\Omega(k) \subseteq V_T$ such that $\ell \in \Omega(k)$ if and only if $\|x_\ell(k+1)\|^2 \leq \Pi(s_1) - \bar{\eta}_{\min}^{k-s_1-1}\varepsilon$. One then can verify that the following properties hold for the set $\Omega(k)$:

(P1) If $\ell \in \Omega(K)$, then $\ell \in \Omega(k)$ for all $k \geq K+1$.

(P2) If $\ell' \in \Omega(k)$ and $b_{\ell\ell'}(k) \neq 0$, then $\ell \in \Omega(k+1)$.

In particular, (P1) is a result of the following:

$$\begin{aligned}\|x_\ell(k+1)\|^2 &\leq b_{\ell\ell}(k)\|x_\ell(k)\|^2 - (1 - b_{\ell\ell}(k))\Pi(k) \\ &\leq b_{\ell\ell}(k)\|x_\ell(k)\|^2 - (1 - b_{\ell\ell}(k))\Pi(s_1),\end{aligned}$$

where we use the monotonicity property of $\{\Pi(k)\}$ and the stochasticity property (4.27). Analogously, (P2) is a result of the following:

$$\|x_\ell(k+1)\|^2 \leq b_{\ell\ell'}(k)\|x_{\ell'}(k)\|^2 - (1 - b_{\ell\ell'}(k))\Pi(s_1).$$

One can see that $\mathcal{D}_0 = \{\bar{\ell}\} \subseteq \Omega(s_1+1)$. By (P1), $\mathcal{D}_0 = \{\bar{\ell}\} \subseteq \Omega(k)$ for all $k \geq s_1+1$. Assume that $\mathscr{D}_0 \cup \cdots \cup \mathscr{D}_\kappa \subseteq \Omega(s_1 + 1 + \kappa B)$ for some $0 \leq \kappa \leq \mathcal{L} - 1$. Pick any $\ell \in \mathscr{D}_{\kappa+1}$ for $\kappa \geq 0$. By construction of the set of $\{\mathscr{D}_0, \ldots, \mathscr{D}_{\mathcal{L}}\}$, there is some $\ell' \in \mathscr{D}_0 \cup \cdots \cup \mathscr{D}_\kappa$ such that $b_{\ell\ell'}(k') \neq 0$ at some time $k' \in [s_1 + \kappa B, s_1 + (\kappa+1)B - 1]$. Hence, $\{\ell\} \cup \mathscr{D}_0 \cup \cdots \cup \mathscr{D}_\kappa \subseteq \Omega(k'+1)$, and thus $\mathscr{D}_0 \cup \cdots \cup \mathscr{D}_{\kappa+1} \subseteq \Omega(s_1 + 1 + (\kappa+1)B)$ by (P1) and (P2). By induction, we have $V_T = \Omega(s_1 + 1 + (\mathcal{L}+1)B)$. By (P1), we further have $V_T = \Omega(s_1 + 1 + 2NB)$ and thus:

$$\Pi(s_1 + 1 + 2NB) \leq \Pi(s_1) - \bar{\eta}_{\min}^{2NB}\varepsilon. \tag{4.28}$$

Recall that $s_2 \geq s_1 + 2NB + 1$. By the monotonicity of $\{\Pi(k)\}$, we have

$$\Pi(s_2) \leq \Pi(s_1 + 1 + 2NB) \leq \Pi(s_1) - \bar{\eta}_{\min}^{2NB}\varepsilon. \tag{4.29}$$

Following analogous lines toward (4.29), one can verify the following by induction:

$$\Pi(s_{\kappa+1}) \leq \Pi(s_\kappa) - \bar{\eta}_{\min}^{2NB}\varepsilon, \quad \forall \kappa \geq 1.$$

This further gives that

$$\Pi(s_{\kappa+1}) \leq \Pi(s_1) - \kappa\bar{\eta}_{\min}^{2NB}\varepsilon.$$

Since $\inf_{k\geq 0}\Pi(k) \geq 0$, we reach a contradiction by letting $\kappa \to +\infty$ in the above relation. Consequently, it establishes that $\{e_\ell(k)\}$ diminishes. •

With Claim 4.4 at hand, we are now ready to show the consensus property (4.17).

Claim 4.5 The consensus property (4.17) holds.

Proof We denote

$$M(k) \triangleq \max_{0\leq s\leq\tau_{\max}} \max_{\ell\in V_T} x_\ell(k-s), \tag{4.30}$$

$$m(k) \triangleq \min_{0\leq s\leq\tau_{\max}} \min_{\ell\in V_T} x_\ell(k-s), \tag{4.31}$$

$$D(k) \triangleq M(k) - m(k).$$

To summarize, algorithm (4.18) enjoys the nondegeneracy property (4.19), the stochasticity property (4.20) and the property that the graph $(V_T, \bigcup_{k=0}^{B-1}\mathscr{E}_T(k_0+k))$ is strongly connected for any $k_0 \geq 0$. By using Claim 4.4 and following similar lines toward Corollary 3.1 in [15], we show that the maximum deviation of $D(k)$ is diminishing, and the desired result is established.

Here we provide a sketch of the proof on $D(k)$ being diminishing. Let us fix $\ell \in V_T$ for every time instant k and define $\mathcal{D}_0 = \{\ell\}$. Recall that $(V_T, \bigcup_{k=0}^{B-1}\mathscr{E}_T(k_0+k))$ is strongly connected for any $k_0 \geq 0$. We replace B by $\max\{B, \tau_{\max}\}$ in the paragraph right before Lemma 3.1 in [15] and construct the collection of $\mathscr{D}_0, \ldots, \mathscr{D}_{\mathcal{L}}$ consisting of a partition of V_T with some $\mathcal{L} \leq 2N-1$. Following the same lines in Lemma 3.1 and using the new definitions of $m(k)$ and $M(k)$ in (4.30) and (4.31), one can show that for every $\kappa \in \{1, \ldots, \mathcal{L}\}$, there exists a real number $\eta_\kappa > 0$ such that for every integer $s \in [\kappa\max\{B, \tau_{\max}\}, \ (\mathcal{L}\max\{B, \tau_{\max}\} + \max\{B, \tau_{\max}\} - 1)]$, and $\kappa' \in \mathcal{D}_\kappa$, it holds that for $t = k + s$

$$x_{\kappa'}(t) \geq m(k) + \sum_{q=0}^{s-1} \min_{\ell' \in V_T} e_{\ell'}(k+q) + \eta_\kappa (x_\ell(s) - m(s)),$$

$$x_{\kappa'}(t) \leq M(k) + \sum_{q=0}^{s-1} \max_{\ell' \in V_T} e_{\ell'}(k+q) - \eta_\kappa (M(s) - x_\ell(s)).$$

The remaining of the proofs can be finished by following the same lines in [15] and replacing B by $\max\{B, \tau_{\max}\}$. •

By Remark 4.3, the consensus property (4.17) establishes the desired result. This completes the proof. •

4.6 Discussion

In this section, we discuss several aspects of the distributed replay-attack resilient formation control algorithm and its possible variations.

4.6.1 The Special Case of Consensus

In the constrained formation control problem, we cannot characterize the diminishing rate of projection errors, and this prevents us from finding an estimate of the convergence rate of the REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM. When $\nu_{ij} = 0$, the formation control problem reduces to the consensus (or rendezvous) problem. Since X is convex and $\phi_i(k)$ is a convex combination of states in X, $z_i(k) = \phi_i(k)$ and the projection errors are absent. For this special case, we can guarantee that the algorithm converges at a geometric rate.

Corollary 4.1 *Suppose that $\nu_{ij} = 0$ and Assumptions 4.1, 4.4, and 4.5 hold. Let vehicle i start from $(p_i(0), v_i(0))$ with $(p_i(0), p_i(0) + \beta v_i(0)) \in X^2$ and $v_i(0) \in W$ for $i \in V$. The* REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM *with $n \geq \tau_{\max}$ ensures that the vehicles converge to the consensus at a geometric rate of $(1 - \eta)^{\frac{1}{2NB-1}}$ for some $\eta \in (0, 1)$.*

4.6.2 Resilience to Denial-of-Service Attacks

Consider the class of denial-of-service (DoS) attacks; e.g., in [16–18]. In particular, adversary i produces a DoS attack by erasing the control commands sent from operator i, and vehicle i receives nothing at this time. It is easy for vehicles to detect the occurrence of DoS attacks via verifying the receipt of control commands at each time instant. The REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM can be slightly modified to address the scenario where adversaries launch replay or denial-of-service attacks on the data sent from vehicles to operators. If adversary i produces an attack at time k, then operator i does nothing at this time. In this way, the results of Theorem 4.1 apply as well provided that the computing horizon is larger than the maximum number of consecutive DoS attacks; i.e., $n \geq \tau_{\max} + 1$.

*4.6.3 The Issue of Solving the n-*OC

As in Proposition 4.1, we will focus on the program (4.9) in order to simplify the notations. Now we convert the program (4.9) into a quadratic program through the following steps. By using the relation of $y_i(s) = y_i(0) + \sum_{\tau=0}^{s-1} \bar{u}_i(\tau)$, one can see that $\bar{u}_i(s) = -K_i(s) \prod_{\tau=0}^{s-1}(1 - K_i(\tau)) y_i(0)$ and $y_i(s) = \prod_{\tau=0}^{s-1}(1 - K_i(\tau)) y_i(0)$. We denote $J_i(s) \triangleq K_i(s) \prod_{\tau=0}^{s-1}(1 - K_i(\tau))$. By using $y_i(s) = y_i(0) + \sum_{\tau=0}^{s-1} \bar{u}_i(\tau)$ and $\bar{u}_i(s) = -J_i(s) y_i(0)$, one can simplify (4.9) to the following compact form after some algebraic manipulation:

$$\begin{aligned} &\min_{\mathbf{J}_i(0 \to n-1) \in \mathbb{R}^n} \mathbf{J}_i(0 \to n-1)^T P_i \mathbf{J}_i(0 \to n-1) + y_i(0)^T Q_i \mathbf{J}_i(0 \to n-1) \\ &\text{s.t.} \quad E_i \mathbf{J}_i(0 \to n-1) \leq F_i y_i(0) + G_i z_i(0) + H_i, \end{aligned} \tag{4.32}$$

where a term independent of $\mathbf{J}_i(0 \to n-1)$ has been removed from the original objective function. In (4.32), the matrix P_i is symmetric and positive definite, and the matrices Q_i, E_i, F_i, G_i, and H_i have proper dimensions. One can see that the program (4.32) is a multiparametric quadratic program and a number of existing efficient algorithms can be used to solve it. Given the solution $\mathbf{J}_i(0 \to n-1)$ to (4.32), operator i then computes $\{\bar{u}_i(s)\}_{0 \leq s \leq n-1}$ by using $\bar{u}_i(s) = -J_i(s) y_i(0)$.

4.6.4 Pros and Cons

By exploiting the RHC methodology, the REPLAY-ATTACK RESILIENT FORMATION CONTROL ALGORITHM demonstrates the resilience to replay attacks and denial-of-service attacks. The resilience is achieved under limited information about adversaries; that is, operators are only aware of $\tau_{\max}$, but do not need to know the attacking policy. In addition, the usage of the RHC methodology explicitly guarantees that the state and input constraints are enforced all the time. These attractive advantages stimulate the interesting future direction of extending the usage of the RHC methodology to other cooperative control tasks in the presence of replay and DoS attacks.

On the other hand, we also notice that the resilience of our algorithms comes at the expense of higher computation, communication, and memory costs in comparison with the classic consensus algorithm. In particular, each operator needs to solve a multiparametric program at each time; a sequence of control commands have to be sent to each vehicle; and each vehicle is required to store a sequence of control commands as backup. If X is a polyhedron, the computational burden of solving the multiparametric quadratic program (4.32) can be traded with memory costs by means of explicit model predictive control initiated in [19].

4.6.5 Tradeoff Between Computation, Memory, and Communication Costs

One can trade communication costs with computation costs by exploiting the idea of event/self-triggered control; e.g., in [20]. In particular, each operator increases the computing horizon $n \geq \tau_{\max} + 1$ and, aperiodically computes and sends the control commands. Consider the time instant $k_{i,0} \geq 0$, and assume $s_i(k_{i,0}) = 0$. Then $\bar{\mathbf{u}}_i(k_{i,0} \to k_{i,0} + n - 1)$ is successfully delivered. After that, operator i does not compute and send any control command to vehicle i until the time instant $k_{i,0} + (n - \tau_{\max} - 2)$. Since $k_{i,0} + (n - \tau_{\max} - 2)$, operator i keeps executing Step 1 in the Replay-Attack Resilient Formation Control Algorithm at each time instant until $s_i(k_{i,1}) = 0$ for some $k_{i,1} \in [k_{i,0} + (n - \tau_{\max} - 2), k_{i,0} + n]$. Operator i then repeats the above process after $k_{i,1}$.

Event/self-triggered control only requires operator i to perform local computation and communication to vehicle i at $\{k_{i,\ell}\}_{\ell \geq 0}$. However, on the other hand, self-triggered control increases the size of the n-OC and introduces larger delays into the system, potentially slowing down the convergence rate.

4.7 Numerical Examples

Consider a group of ten vehicles restricted in the area $X \triangleq [-10, 10] \times [-10, 10]$. The input and velocity limits of each vehicle are $u_{\max} = 5$ and $v_{\max} = 2.5$, respectively. We study the following three cases via numerical simulations:

(1) $n = 10$ and $\tau_{\max} = 0$ (no attacks occur);
(2) $n = 10$ and $\tau_{\max} = 10$ (each adversary launches the attacks all the time except the time instants which are the multiples of 10);
(3) $n = 30$ and $\tau_{\max} = 30$ (each adversary launches the attacks all the time except the time instants which are the multiples of 30).

We now proceed to discuss the simulation results. Figure 4.3 is concerned with Case (3), and demonstrates that the vehicles start from four corners of the square X and eventually form the desired configuration at the center of X. Figure 4.4 compares

Fig. 4.3 The vehicle trajectories for Case (3)

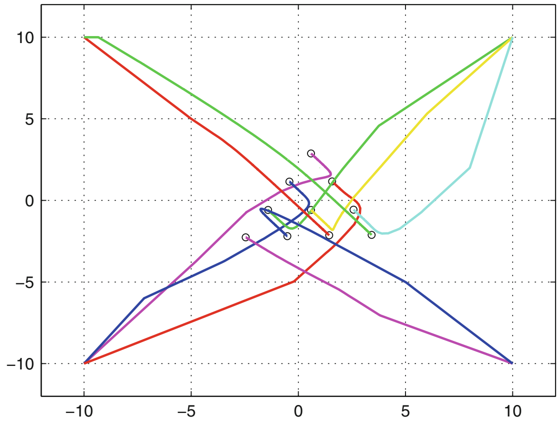
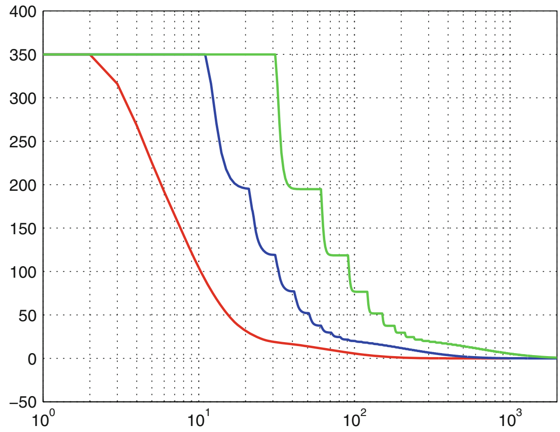


Fig. 4.4 The comparison of error evolution where the *red* (resp. *blue* and *green*) line is for Case (1) (resp. (2) and (3))



the error evolution of three cases. It is evident that a larger $\tau_{\max}$ slows down the convergence rate, and makes the error evolution less smooth. This coincides with the theoretic results in Theorem 4.1 and the intuition that a larger $\tau_{\max}$ would produce a greater damage to the operator-vehicle network.

4.8 Notes

Security is a critical issue for information technology networks. In practice, either reactive or protective mechanisms have been exploited to prevent cyber attacks. Non-cooperative game theory is used as a rigorous mathematical framework that models the interdependency between attackers and administrators, see the (incomplete) list of references [21, 22].

Recently, the topic of security of a class of control systems, namely *cyber-physical systems*, is drawing mounting attention. Three classes of attacks have been investigated: denial-of-service (DoS) attacks, replay attacks, and deception (or false data injection) attacks. Denial-of-service attacks destroy data availability in control systems, and see [16–18]. Replay attacks maliciously repeat transmitted data, and their impact to control systems was first studied in [23]. See Remark 4.1 for a more detailed discussion on the above two classes of attacks. Deception attacks compromise the data integrity of state estimation and control, and see for example [5–10]. In [24], an attack space defined by the adversary's system knowledge, disclosure, and disruption resources is introduced.

Attack detection, attack resilient control, and security economics are three important aspects of the security of CPS. More specifically, attack detection aims to detect the existence of malicious attacks and further identify their actions. This is achieved by designing input–state estimators and see [25–27]. Attack resilient control is to design control laws which can ensure control system performance despite malicious attacks. In [28, 29], the authors exploit pursuit-evasion games to compute optimal evasion strategies for mobile agents in the face of jamming attacks. Event trigger control under denial-of-service attacks is studied in [30]. Our papers [31, 32] discuss distributed attack-resilient formation control of multiple vehicles against DoS, replay, and deception attacks. The paper [33] considers the problem of computing arbitrary functions of initial states in the presence of faulty or malicious agents, whereas [34] focuses on consensus problems. Security economics aims to incentivize heterogeneous stock holders to contribute to the security of CPS. Some papers along this line include [16] and [35]

This chapter employs a distributed receding horizon control methodology for the resilient control of a class of multi-agent systems. Previously, distributed receding horizon control has been applied to solve the problem of stabilizing an *a priori* known common set point for decoupled subsystems [36, 37], coupled subsystems [38], and the problem of reaching consensus in [2, 39]. Our chapter is also relevant to the set of papers concerned with formation control of multiple vehicles; e.g., in [36]. Centralized receding horizon control over unreliable communication networks is studied in [40–42] for package dropouts and in [43, 44] for transmission delays. Recently, a recursive and centralized networked predictive control method based on round-trip time delay is proposed in [45] to compensate for denial-of-service attacks.

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Series Editor's Biographies

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M. Zhu and S. Martínez, *Distributed Optimization-Based Control of Multi-Agent Networks in Complex Environments*, SpringerBriefs in Control, Automation and Robotics, DOI 10.1007/978-3-319-19072-3

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