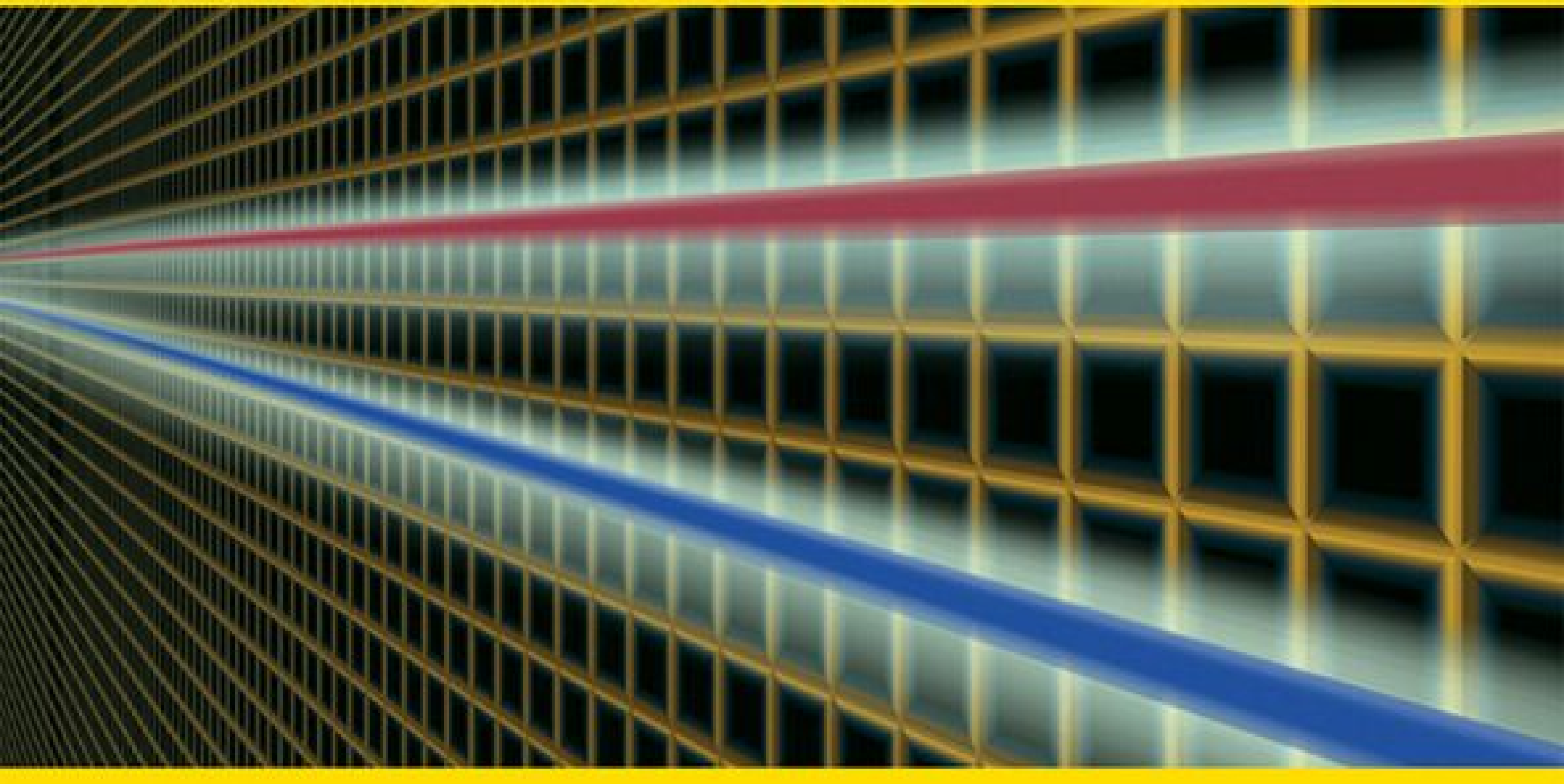
Mathematical Finance

Deterministic and Stochastic Models

Jacques Janssen
Raimondo Manca
Ernesto Volpe



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Preface

This book, written as a treatise on mathematical finance, has two parts: deterministic and stochastic models.

The first part of the book, managed by Ernesto Volpe di Prignano, aims to give a complete presentation of the concepts and models of classical and modern mathematical finance in a mainly deterministic environment. Theoretical aspects and economic, bank and firm applications are developed.

The most important models are presented in detail after the formalization of an axiomatic theory of preferences. This performs the definition of "interest" and the financial regimes, which are the basis of financial evaluation and control the models. They are applied by means of clarifying examples with the solutions often obtained by *Excel* spreadsheet.

Chapter 1 shows how the fundamental definitions of the classical financial theory come from the microeconomic theory of subjective preferences, which afterwards become objective on the basis of the market agreements. In addition, the concepts of interest such as the price of other people's money availability, of financial supply and the indifference curve are introduced.

Chapter 2 develops a strict mathematical formalization on the financial laws of interest and discount, which come from the postulates defined in Chapter 1. The main properties, i.e. decomposability and uniformity in time, are shown.

Chapter 3 shows the most often used financial law in practice. The most important parametric elements, such as interest rates, intensities and their relations, are defined. Particular attention is given to the compound interest and discount laws in different ways. They find wide application in all the pluriennial financial operations.

Chapter 4 gives the concept of discrete time financial operation as a set of financial supplies, of operation value, of fair operation, of retrospective and prospective reserve at a given time, of the usufruct and bare ownership. In addition, a detailed classification of the financial projects based on their features is given. The decision and choice methods among projects are deeply developed. In the appendix to this chapter, a short summary of simple numerical methods, particularly useful to find the project internal rate, is reported.

Chapter 5 discusses all versions of the annuity operations in detail, as a particular case of financial movement with the same sign. The annuity evaluations are given using the compound or linear regime.

Chapter 6 is devoted to management mathematical procedures of financial operations, such as loan amortizations in different usual cases, the funding, the returns and the redemption of the bonds. Many *Excel* examples are developed. the final section is devoted to bond evaluations depending on a given rate or on the other hand to the calculus of return rates on bond investments.

In Chapters 7 and 8, the financial theory is reconsidered assuming variable interest rates following a given term structure. Thus, Chapter 7 defines spot and forward structures and contracts, the implicit relations among the parameters and the transforming formulae as well. Such developments are carried out with parameters referred to real and integer times following the market custom. Chapter 8 discusses the methods developed in Chapters 5 and 6 using term structures.

Chapter 9 is devoted to definition and calculus of the main duration indexes with examples. In particular, the importance of the so-called "duration" is shown for the approximate calculus of the relative variation of the value depending on the rate. However, the most relevant "duration" application is given in the classical immunization theory, which is developed in detail, calculating the optimal time of realization and showing in great detail the Fisher-Weil and Redington theorems.

The second part of the book, managed by Jacques Janssen and Raimondo Manca, aims to give a modern and self-contained presentation of the main models used in so-called *stochastic finance* starting with the seminal development of Black, Scholes and Merton at the beginning of the 1970s. Thus, it provides the necessary follow up of our first part only dedicated to the deterministic financial models.

However, to help in assuring the self-containment of the book, the first four chapters of the second part provide a summary of the basic tools on probability and stochastic processes, semi-Markov theory and Itô's calculus that the reader will need in order to understand our presentation.

Chapter 10 briefly presents the basic tools of probability and stochastic processes useful for finance using the concept of *trajectory* or *sample path* often representing the time evolution of asset values in stock exchanges.

Chapters 11 and 12 summarize the main aspects of Markov and semi-Markov processes useful for the following chapters and Chapter 13 gives a strong introduction to stochastic or Itô's calculus, being fundamental for building stochastic models in finance and their understanding.

With Chapter 14, we really enter into the field of stochastic finance with the full development of classical models for option theory including a presentation of the Black and Scholes results and also more recent models for exotic options.

Chapter 15 extends some of these results in a semi-Markov modeling as developed in Janssen and Manca (2007).

With Chapter 16, we present another type of problem in finance, related to interest rate stochastic models and their application to bond pricing. Classical models such as the Ornstein-Uhlenbeck-Vasicek, Cox-Ingersoll-Ross and Heath-Jarrow-Morton models are fully developed .

Chapter 17 presents a short but complete presentation of Markowitz theory in portfolio management and some other useful models.

Chapter 18 is one of the most important in relation to Basel II and Solvency II rules as it gives a full presentation of the value at risk, called VaR, methodology and its extensions with practical illustrations.

Chapter 19 concerns one of the most critical risks encountered by banks: credit or default risk problems. Classical models by Merton, Longstaff and Schwartz but also more recent ones such as homogenous and non-homogenous semi-Markov models are presented and used for building ratings and following the time evolution.

Finally, Chapter 20 is entirely devoted to the presentation of Markov and semi-Markov reward processes and their application in an important subject in finance, called stochastic annuity.

As this book is written as a treatise in mathematical finance, it is clear that it can be read in sections in a variety of sequences, depending on the main interest of the reader.

This book addresses a very large public as it includes undergraduate and graduate students in mathematical finance, in economics and business studies, actuaries, financial intermediaries, engineers but also researchers in universities and RD departments of banking, insurance and industry.

Readers who have mastered the material in this book will be able to manage the most important stochastic financial tools particularly useful in the application of the rules of governance in the spirit of Basel II for banks and financial intermediaries and Solvency II for insurance companies.

Many parts of this book have been taught by the three authors in several universities: Université Libre de Bruxelles, Vrije Universiteit Brussel, University of West Brittany (EURIA) (Brest), Télécom-Bretagne (Brest), Paris 1 (La Sorbonne) and Paris VI (ISUP) Universities, ENST-Bretagne, University of Strasbourg, Universities of Rome (La Sapienza), Napoli, Florence and Pescara.

Our common experience in the field of solving financial problems has been our main motivation in writing this treatise taking into account the remarks of colleagues, practitioners and students in our various lectures.

We hope that this work will be useful for all our potential readers to improve their method of dealing with financial problems, which always are fascinating.

Part I

Deterministic Models

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Chapter 1

Introductory Elements to Financial Mathematics

1.1. The object of traditional financial mathematics

The object of traditional financial mathematics is the formalization of the exchange between monetary amounts that are payable at different times and of the calculations related to the evaluation of the obligations of financial operations regarding a set of monetary movements.

The reasons for such movements vary and are connected to: personal or corporate reasons, patrimonial reasons (i.e. changes of assets or liabilities) or economic reasons (i.e. costs or revenues). These reasons can be related to initiatives regarding any kind of goods or services, but this branch of applied mathematics considers only the monetary counterpart for cash or assimilated values[1].

The evaluations are founded on equivalences between different amounts, paid at different times in certain or uncertain conditions. In the first part of this book we will cover financial mathematics in a deterministic context, assuming that the monetary income and outcome movements (to which we will refer as "payment" with no distinction) will happen and in the prefixed amount. We will not consider in

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1 The reader familiar with book-keeping concepts and related rules knows that each monetary movement has a real counterpart of opposite movement: a payment at time x (negative financial amount) finds the counterpart in the opening of a credit or in the extinction of a debt. In the same way, a cashing (positive financial amount) corresponds to a negative patrimonial variation or an income for a received service. The position considered here, in financial mathematics, looks to the undertaken relations and the economic reasons for financial payments.

this context *decision theory in uncertain conditions*, which contains *actuarial mathematics* and more generally the *theory of random financial operations*[2].

We suppose that from now, unless otherwise specified, the deterministic hypotheses are valid, assuming then – in harmony with the rules of commonly accepted economic behavior – that:

a) the ownership of a capital (a monetary) amount is advantageous, and everyone will prefer to have it instead of not having it, whatever the amount is;

b) the temporary availability of someone else's capital or of your own capital is a favorable service and has a cost; it is then fair that whoever has this availability (useful for purchase of capital or consumer goods, for reserve funds, etc.) pays a price, proportional to the amount of capital and to the time element (the starting and closing dates of use, or only its time length).

The amount for the aforementioned price is called *interest.* The parameters used for its calculation are calculated using the rules of economic theory.

1.2. Financial supplies. Preference and indifference relations

1.2.1. *The subjective aspect of preferences*

Let us call *financial supply* a *dated amount*, that is, a prefixed amount to place at a given payment deadline. A supply can be formally represented as an ordinate couple *(X,S)* where *S* = *monetary amount* (transferred or accounted from one subject to another) and *X* = *time of payment.*

Referring to one of the contracting parties, *S* has an algebraic sign which refers to the cash flow; it is positive if it is an income and negative if it is an outcome, and the unit measure depends on the chosen currency. Furthermore, the time (or instant) can be represented as abscissas on an oriented temporal axis so as to have chronological order. The time origin is an instant fixed in a completely discretionary

2 In real situations, which are considered as deterministic, the stochastic component is present as a pathologic element. This component can be taken into account throughout the increase of some earning parameter or other artifices rather than introducing probabilistic elements. These elements have to be considered explicitly when uncertainty is a fundamental aspect of the problem (for example, in the theory of stochastic decision making and in actuarial mathematics). We stress that in the recent development of this subject, the aforementioned distinction, as well as the distinction between "actuarial" and "financial" mathematics, is becoming less important, given the increasing consideration of the stochastic aspect of financial problems.

way and the measure unit is usually a year (but another time measure can be used). Therefore, even the times *X*, *Y*, etc., have an algebraic sign, which is negative or positive according to their position with respect to the time origin. It follows that "$X<Y$" means "time *X* before time *Y*".

From a geometric viewpoint, we introduce in the plane $\Sigma^{(2)}$ the Cartesian orthogonal reference system *OXS* (with abscissas *X* and ordinate *S*). $\Sigma^{(2)}$ is then made of the points $P \equiv [X,S]$ that represent the supply (X,S), that is the amount *S* dated in *X*.

As a consequence of the postulates a) and b), the following operative criteria can be derived:

c) given two financial supplies (X,S_1) and (X,S_2) at the same maturity date X, the one with the higher (algebraically speaking) amount is preferred;

d) given two financial supplies (X,S) and (Y,S) with the same amount S and valued at instant Z before both X and Y, if $S>0$ (that is, from the cashing viewpoint) the supply for which the future maturity is closer to Z is preferred; if $S<0$ (that is, from the paying viewpoint) the supply with future maturity farther from Z is preferred. More generally $\forall Z$[3], with two supplies having the same amount, the person who cashes (who pays) prefers the supply with prior (with later) time of payment.

Formulations c) and d) express criteria of absolute preference in the financial choices and clarify the meaning of interest. In fact, referring to a loan, where the *lender* gives to the *borrower* the availability of part of his capital and its possible use for the duration of the loan, the lender would perform a disadvantageous operation (according to postulate a) and b) and criteria c) if, when the borrower gives back the borrowed capital at the fixed maturity date, he would not add a generally positive amount to the lender, which we called *interest,* as a payment for the financial service.

The decision maker's behavior is then based on preference or indifference criteria, which is subjective, in the sense that for them there is *indifference* between two supplies if neither is preferred.

To provide a better understanding of these points, we can observe that:

– the decision maker expresses a judgment of *strong preference*, indicated with $\succ$, of the supply (X_1,S_1) compared to (X_2,S_2) if he considers the first one more advantageous than the second; we then have $(X_1,S_1)\succ(X_2,S_2)$;

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3 It is known that the symbol $\forall$ has the meaning "for all".

– the decision maker expresses a judgment of *weak preference*, indicated with $\succeq$, of the supply (X_1,S_1) compared to (X_2,S_2), if he does not consider the second one more advantageous than the first; we then have $(X_1,S_1) \succeq (X_2,S_2)$[4].

The amplitude of the set of supplies comparable with a given supply for a preference judgment depends on the criteria on which the judgment is based.

Criteria c) and d) make it possible to establish a preference or no preference of (X_0,S_0), but only with respect to a subset of all possible supplies, as we show below.

From a geometric point of view, let us represent the given supply (X_0,S_0) on the plane $\Sigma^{(2)}$, with reference system *OXS*, by the point $P_0 \equiv [X_0,S_0]$. Then, considering the four quadrants adjacent to P_0, based only on criteria c) and d), it turns out that:

1) Comparing $S_0>0$ to supplies with a positive amount, identified by the points P_i (i=1,…,4) (see Figure 1.1), being incomes, it is convenient to anticipate their collection. Therefore, all points $P_2 \equiv [X_2,S_2]$ in the 2nd quadrant (NW) are preferred to P_0 because they have income S_2 greater than S_0 and are available at time X_2 previous to time X_0; whereas P_0 is preferred to all points $P_4 \equiv [X_4,S_4]$ in the 4th quadrant (SE) because they have income S_4 smaller than S_0 and are available at time X_4 later than X_0; it is not possible to conclude anything about the preference between P_0 and points $P_1 \equiv [X_1,S_1]$ in the 1st quadrant (NE) or points $P_3 \equiv [X_3,S_3]$ in the 3rd quadrant (SW).

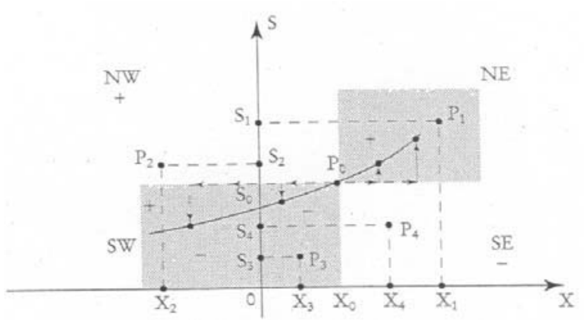

Figure 1.1. *Preferences with positive amounts*

4 The judgment of *weak preference* is equivalent to the merging of *strong preference* of (X_1,S_1) with respect to (X_2,S_2) and of (X_2,S_2) with respect to (X_1,S_1). In other words:

– weak preference = strong preference or indifference;

– indifference = no strong preference of one supply with respect to another.

The economic logic behind the postulates a), b), from which the criteria c), d) follow, implies that the amounts for indifferent supply have the same sign (or are both zero).

2) Comparing $S_0<0$ to supplies with a negative amount, identified by the points P_i (i=1,…,4) (see Figure 1.2), being outcomes, it is convenient to postpone their time of payment. Therefore all points $P_1 \equiv [X_1,S_1]$ in the 1st quadrant (NE) are preferred to P_0 because they have outcome S smaller than S_0 and are payable at time X later than X_0; whereas P_0 is preferred to all points $P_3 \equiv [X_3,S_3]$ in the 3rd quadrant (SW) because they have outcome S_3 greater than S_0 and are payable at time X_3, which is later than X_0. Nothing can be concluded on the preference between P_0 and all points $P_2 \equiv [X_2,S_2]$ of the 2nd quadrant (NW) or all points $P_4 \equiv [X_4,S_4]$ of the 4th quadrant (SE).

Briefly, on the non-shaded area in Figures 1.1 and 1.2 it is possible to establish whether or not there is a strong preference with respect to P_0, while on the shaded area this is not possible.

To summarize, indicating the generic supply *(X,S)* also with point $P \equiv [X,S]$ in the plane *OXS*, we observe that an operator, who follows only criteria c) and d) for his valuation and comparison of financial supplies, can select some supplies *P'* with *dominance* on P_0 (we have dominance of *P'* on P_0 when the operator prefers *P'* to P_0) and other supplies *P''* *dominated* by P_0 (when he prefers P_0 to *P''*), but the comparability with P_0 is *incomplete* because there are infinite supplies *P'''* not comparable with P_0 based on criteria c) and d). To make the comparability of P_0 with the set of all financial supplies *complete*, corresponding to all points in the plane referred to *OXS*, it is necessary to add to criteria c) and d) – which follow from general behavior on the ownership of wealth and the earning of interest – rules which make use of subjective parameters. The search and application of such rules – to fix them external factors must be taken into account, summarized in the "market", making it possible to decide for each supply if it is dominant on P_0, indifferent on P_0 or dominated by P_0 – is the aim of the following discussion.

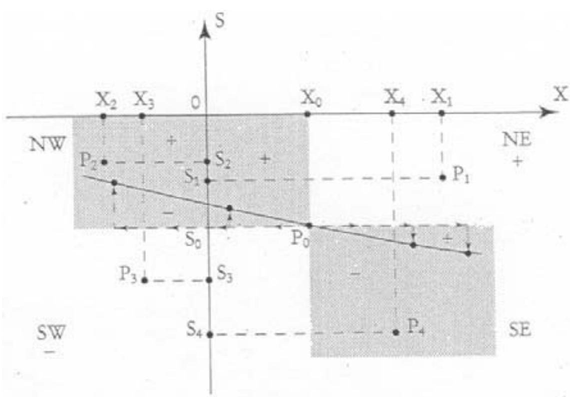

Figure 1.2. *Preferences with negative amounts*

To achieve this aim it is convenient to proceed in two phases:

1) the first phase is to select, in the zone of no dominance (shaded in Figures 1.1 and 1.2), the supplies $P^* \equiv [X^*, S^*]$ with different times of payment from that of P_0 and in indifference relation with P_0;

2) the second phase, according to the transitivity of preferences, is to select the advantageous and disadvantageous preferences with respect to P_0, with any maturity.

In the first phase, we can suppose an opinion poll on the financial operator to estimate the amount B payable in Y that the same operator evaluates in indifference relation, indicated through the symbol $\approx$, with the amount A payable in X. For such an operator we will use:

$$(X,A) \approx (Y,B) \tag{1.1}$$

Given the supply (X,A), on varying Y the curve obtained by the points that indicate the supplies (Y,B) indifferent to (X,A), or satisfying (1.1), is called the *indifference curve characterized by point* $[X,A]$.

From an operative viewpoint, if two points $P' \equiv [X,A]$ and $P'' \equiv [Y,B]$ are located on the same indifference curve, the corresponding supplies (X,A) and (Y,B) are exchangeable without adjustment by the contract parties.

If (1.1) holds, according to criteria c) and d), the amounts A and B have the same sign and $|B|-|A|$ has the same sign of Y-X. The fixation of the indifferent amounts can proceed as follows, as a consequence of the previous geometric results (see Figures 1.1 and 1.2).

Let us denote by $P_0 \equiv [X_0, S_0]$ the point representing the supply for which the indifference is searched. Then:

– if $S_0>0$ (see Figure 1.3), with $X=X_0$, $Y=X_1>X_0$, the rightward movement from P_0 to $A_1 \equiv [X_1, S_0]$ is disadvantageous because of the income delay; to remove such disadvantage the amount of the supply must be increased. The survey, using continuous increasing variations, fixes the amount $S_1>S_0$ which gives the compensation, where P_0 and $P_1 \equiv [X_1, S_1]$, obtained from A_1 moving upwards, and represents indifferent supply (or, in brief, P_1 and P_0 are indifferent points). Instead, if $Y=X_3<X_0$, the leftwards movement from P_0 to $A_3 \equiv [X_3, S_0]$ is advantageous for the income anticipation; therefore, in order to have indifference, there needs to be a decrease in the income from S_0 to S_3, obtained through a survey with downward movement of the indifference point $P_3 \equiv [X_3, S_3]$ with $S_3<S_0$;

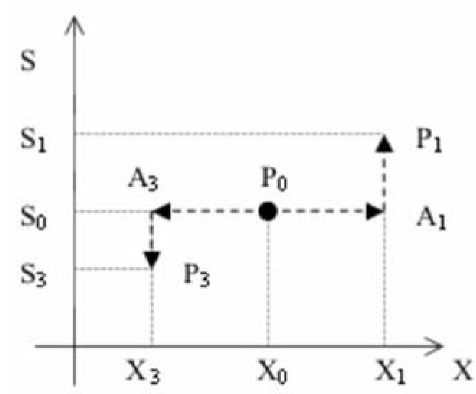

Figure 1.3. *Indifference curve assessment – positive amounts*

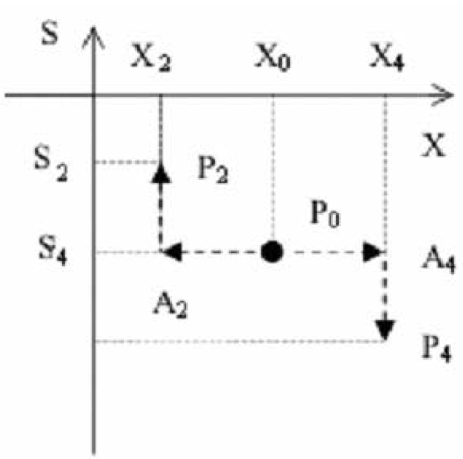

Figure 1.4. *Indifference curve assessment – negative amounts*

– if S_0<0 (see Figure 1.4), since the delay of outcome is advantageous and its anticipation is disadvantageous, proceeding in a similar way starting from $A_2 \equiv [X_2, S_0]$ and $A_4 \equiv [X_4, S_0]$, the points (indifferent to P_0) $P_2 \equiv [X_2, S_2]$, with $X_2 < X_0$, $S_2 > S_0$, are obtained through leftwards and then upwards movement, or $P_4 \equiv [X_4, S_4]$, with $X_4 > X_0$, $S_4 < S_0$, through rightwards and then downwards movement.

Continuously increasing or decreasing the abscissas X_i (i=1,3), we obtain, if S_0>0, a continuous curve with increasing ordinate in the plane *OXS*, resulting from P_0 and the points of type P_1 and P_3, all indifferent to P_0. If S_0<0, the continuous curve resulting by P_0 and the points of type P_4 and P_2 , all indifferent to P_0, obtained by continuously varying X_i (i=2,4), have a decreasing ordinate[5]. However, if P_0 is fixed, these curves of indifference are individualized from P_0 by definition.

We can now define, in general terms, the interest defined in section 1.1, considering only the positive amount. If (1.1) holds with $X<Y$, the exchange between indifferent supplies implies that giving away the availability of amount A from X to Y is fairly compensated by the payment of the amount

$$I = B - A \geq 0. \tag{1.2}$$

We will say that A is the invested *principal*, I is the *interest*, and B is the *accumulated value*, in an operation of *lending* or *investment*.

If (1.1) holds with $X>Y$, the anticipation of the income of A from X to Y is fairly compensated by the payment in Y of the amount

$$D = A - B \geq 0 \tag{1.3}$$

We will say that A is the *capital at maturity*, D is the *discount* and B is the *present value* or *discounted value*, in an operation of *discounting* or *anticipation*.

The second phase is applied in an easy way. It is enough to add, referring to (1.1) in the case A>0, that if a generic $P\equiv[Y,B]$ is indifferent to a fixed $P_0\equiv[X,A]$ then all the points $P'\equiv[Y,B']$ where $B'>B$ are preferred to P_0, while P_0 is preferred to all points $P''\equiv[Y,B'']$ where $B''< B$. This leads to the conclusion that, once the indifference curve through P_0 is built, all the supplies of the type (Y,B') are preferred to the supply (X,A), while the opposite occurs for all supplies of type (Y,B'').

1.2.2. *Objective aspects of financial laws. The equivalence principle*

The previous considerations enable us to give a first empirical formulation of the fundamental "principle of financial equivalence", which is that it is *equivalent*[6] *to a*

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5 If criteria d is removed, supplies with same amount and different time become indifferent and the indifference curves have constant ordinate. All loans without interest made for free are contained in this category.

6 "Equivalent" is often used instead of "indifferent"; if this does not make sense then imagine that *P' equivalent to P"* means that these supplies are in the same equivalence class as in the set theory meaning. For this to be true, other conditions are needed. which we will discuss later.

cash (pay) amount today or to cash (pay) at a later time if there is the cashing (payment) of the interest for such deferment.

In Chapter 2, the indifference curves and the principle of financial indifference will be formalized in objective terms, defining financial factors, rates and intensities for lending and discounting operations, in relation to the possible distribution of interest payments in the deferment period. The equivalence principle will then become objective, assuming the hypothesis that different parties to a financial contract agree in fixing a rule, valid for them, to calculate the equivalent amount B, according to the amount A and the times X, Y.

1.3. The dimensional viewpoint of financial quantities

In financial mathematics, as in physics, it is necessary to introduce, together with numerical measures, a dimensional viewpoint distinguishing between *fundamental quantities* and *derived quantities.*

To describe the laws of mechanics, the oldest of the physical sciences, the following *fundamental quantities* are introduced: length l, time t, mass m, with their units (meter, second, mass-kilogram) and the *derived quantities* are deduced, such as volume l^3, velocity l/t, acceleration l/t^2, force ml/t^2, etc. Their units are derived from those of the fundamental quantity. We then speak about the physical dimension of different quantities, which are completely defined when they are given the dimensions and the numbers which represent the measurement of the given quantity in the unit system.

In financial mathematics we also make a distinction between *fundamental quantities* and *derived quantities.*

The *fundamental quantities* are:

1) *monetary* amount (m), to measure the value of financial transaction in a given unit (i.e., dollar, euro, etc.);

2) *time* (t), to measure the length of the operation and the delay of its maturity in a given unit (i.e. year).

The *derived quantities*, relating to the fundamental quantities based on dimensions, are:

1) *flow*, defined as amount over time (then with dimension m^1t^{-1});

2) *rate*, defined as amount over amount (thus a “pure number”, with dimension m^0t^0);

3) *intensity*, defined as amount over the product of amount multiplied by time (then with dimension m^0t^{-1}).

To clarify:

– *flow* relates the monetary amount to the time interval in which it is produced; a typical flow is the monetary income (i.e.: wages, fees, etc.) expressed as the monetary amount matured in a unit of time as a consequence of the considered operation;

– *rate* relates two amounts which are connected and thus is a "pure number" without dimension; for example, the rate is the ratio between matured interest and invested principal;

– *intensity*, obtained as the ratio between rate and time or flow and amount, takes into account the time needed for the formation of an amount due to another amount; for example, the ratio between interest and invested principal time length of the investment.

This is all summarized in the following *dimensional table* where we go from left to right, dividing by a "time" and from top to bottom, dividing by an "amount".

amount (m^1t^0)	flow (m^1t^{-1})
rate (m^0t^0)	intensity (m^0t^{-1})

Table 1.1. *Financial dimensions*

Chapter 2

Theory of Financial Laws

2.1. Indifference relations and exchange laws for simple financial operations

Let us consider again the indifference relation, indicated by ≈ in (1.1), which depends on the judgment of an economic operator which gives rise to indifferent supplies with the process described in section 1.2.

In a *loan operation* of the amount S at time T the economic operator can calculate the repayment value S' in $T' > T$ such that $(T',S') \approx (T,S)$. Therefore, $S' \geq S$ is calculated according to a function (subjective) of S, T, T' and it is written as

$$S' = f_c\,(S, T; T') \tag{2.1}$$

where f_c is the *accumulation function* (given that in S' the repayment of S and the incorporation of the possible interest is included) that realizes indifference.

In a *discounting operation,* at time $T'' < T'$, of amount S' with maturity T', let $S'' \leq S'$ be the discounted value so that subjectively $(T'',S'') \approx (T',S')$. We then have

$$S'' = f_a\,(S', T'; T'') \tag{2.2}$$

where f_a is the *discounting function* (because S' is discounted at time T'' with a possible reduction due to anticipation of availability) that realizes indifference.

It is obvious that if two operators, one at each side of a loan or discounting contract, want to realize an advantageous contract according to their preference scale, it is not always possible for them to do so.

It can be the case that, in a loan in T' of the principal S', indicating by S''_a the indifferent accumulated amount (= min acceptable) for the *lender* to cash in T'' and by S''_b the indifferent accumulated amount (= max acceptable) for the *borrower* to pay out in T'', if $S''_b < S''_a$ the contract is not stipulated. In the same way, we can prove that, in a discounting operation of the capital S' at maturity T', indicating by S''_a the present indifferent value (= max acceptable) for the *lender* to pay out in $T'' < T'$ and by S''_b the present indifferent value (= min acceptable) for the *borrower* to cash in $T'' < T'$, if $S''_a < S''_b$ the contract is not stipulated.

EXAMPLE 2.1.– Let us suppose that Mr. Robert, who is lending the amount S' at time T' for the period (T',T''), wants to cash in T'' at least $1.09 \cdot S'$. At the same time Mr. George, who is borrowing S' for the same time interval, wants to pay back in T'' no more than $1.07 \cdot S'$. It is obvious that in this way they will not proceed with the loan contract. Indeed:

– with $S'' < 1.07 \cdot S'$, the lender prefers not to lend;

– with $1.07 \cdot S' < S'' < 1.09 \cdot S'$, the lender prefers not to lend and the borrower prefers not to borrow;

– with $S'' > 1.09 \cdot S'$, the borrower prefers not to borrow.

EXAMPLE 2.2.– Let us suppose that Mr. John wants to discount a bill from Mr. Tom, which is amount S' for the time from T' to $T'' < T'$ offering a discounted value not greater than $0.92 \cdot S'$, while Mr. Tom wants to offer this discount for an amount not lower than $0.94 \cdot S'$. It is clear that the contract cannot be reached, because each discounted amount is considered disadvantageous by at least one of the parties.

To further consider the economic theory of market prices, we carry on our analysis using *objective logic* and supposing that the operators, in a specific market, want a fair contract between two supplies *(T,S)* and *(T',S')* in a loan, if their fundamental quantities satisfy equation (2.1); and in the same way, for a discount, which is a type of loan, if equation (2.2) is satisfied. We will now talk about a *fair contract* if equation (2.1) or equation (2.2) is satisfied, but as *favorable* (or *unfavorable*) for one of the parties if the equations are not satisfied. Trade contracts between two supplies *(T',S')* and *(T'',S'')* give rise to *simple financial operations.* As already mentioned in Chapter 1:

– if $T'' > T'$ (= *loan* or *investment*), the parties consider fair the interest $S''-S'$ as the payment for the lending of S' from T' to T'', as delayed payment in T''; then S'' is called *accumulated amount* in T'' of the amount S' lent in T';

– if $T'' < T'$ (= *discount* or *anticipation*), both parties consider fair the interest S'-S'' for the discount of S' from T' to T'', as advance payment in T''; then S'' is called *discounted value* from time T'' of the amount S' to maturity T'[1].

The indifference relation thus assumes a collective value. The function f_c defined in equation (2.1) is an *accumulation law* (or *interest law),* while the function f_a defined in equation (2.2) is a *discount law*. Referring now to the case of positive interest and fixing S and T in equation (2.1), the value S' is an increasing function of T'; fixing S' and T' in equation (2.2), and the value S'' is also an increasing function of T'', because it decreases when T'' decreases.

Applying equation (2.1) and then equation (2.2) with $T'' = T$, we obtain the present value in T of the accumulated amount in T' of S invested in $T \le T'$, given by

$$S^* = f_a\,[\{f_c\,(S,T;T')\},T';T\,] \tag{2.3}$$

If $\forall (S,T,T')$ is $S^* = S$, the f_a neutralizes the effect of f_c, acting as the inverse function, and the following investment or anticipation operation is called the *corresponding operation*; in this case the laws expressed by f_c and f_a are said to be *conjugated.*

Unifying the cases $T \le T'$ and $T>T'$, we can talk of an *exchange law* given by a function f that gives the amount S' payable in T' and exchangeable[2] with S payable in T. It follows that

$$S' = f\,(S,T;T') \tag{2.4}$$

where if $T \le T'$ then $f = f_c$, whereas if $T > T'$ then $f = f_a$.

---

1 Lending and discounting operations are the same thing because in both cases there is an exchange of a lower amount in a previous time for a greater amount in a future time. The only difference is that in the first case the lower and previous amount is fixed, whereas in the second case the greater and future amount is fixed.

2 We will not use “equivalent” – even if it is used in practice – in the cases that we will consider later where ≈ gives rise to an equivalence relation (see footnote 6 of Chapter 1).

Let us consider some properties of the indifference relation ≈:

1) reflexive property

If $\forall(T,S)$ we have $(T,S) \approx (T,S)$, we will say that ≈ satisfies the reflexive property[3];

2) symmetric property

If $\forall(S,T,T')$, from $(T,S) \approx (T',S')$ follows $(T',S') \approx (T,S)$, we will say that ≈ satisfies the symmetric property[4];

3) property of proportional amounts

If $\forall(S,T,T')$, $\forall k>0$, from $(T,S) \approx (T',S')$ follows $(T,kS) \approx (T',kS')$, we will say that ≈ satisfies the property of proportional amounts.

Because of criteria c) and d), if T′-T the amount in T′ exchangeable with S in T is the same as S. Therefore in the set $\mathcal{P}$ of financial supplies the relation ≈ always satisfies the reflexive law. We can then define the exchange law for all three variables as

$$f(S,T;T') = \begin{cases} f_c(S,T;T'), & if\ T < T' \\ S \quad , & if\ T = T' \\ f_a(S,T;T'), & if\ T > T' \end{cases} \tag{2.5}$$

If the symmetric law holds in the considered set P, then

$$S = f_a[\{f_c(S,T,T')\},T',T],\ \forall(S,T,T'),\ T<T'^{\,5} \tag{2.6}$$

In this case, recalling (2.3), the laws f_c and f_a are conjugated, and because of (2.4), (2.5) can be written in the form

$$S = f[\{f(S,T,T')\},T',T],\ \forall(S,T,T') \tag{2.6'}$$

3 Let us recall that a binary relation $\mathcal{R}$ between elements a, b, ... of a set $\mathcal{H}$ satisfies the reflexive law if: $a\mathcal{R}a,\ \forall a \in \mathcal{H}$.

4 Let us recall that a binary relation $\mathcal{R}$ between elements a, b, ... of a set $\mathcal{H}$ satisfies the symmetric law if: $a\mathcal{R}b \Rightarrow b\mathcal{R}a,\ \forall a,b \in \mathcal{H}$.

5 If $T>T'$ is given, f_c and f_a have to be exchanged in (2.6).

which remains valid with the same f if the primed values are changed with the unprimed values and vice versa[6].

If, in the considered set $\mathcal{P}$. the property of proportional amounts holds, f as defined in (2.4) is *linear homogenous* compared to the amount[7].

2.2. Two variable laws and exchange factors

Let us continue the analysis of exchange laws *the reflexive* and *proportional amount properties* assumed to be valid for $\approx$. Due to the second property, it is possible to transform (2.1) in the multiplicative form

$$S' = S \cdot m(T,T'),\ T \leq T' \tag{2.1'}$$

where $m(T,T')$, increasing with respect to T', is called the *accumulation factor* and expresses the *accumulation law* only as a function of the two temporal variables; in the same way it is possible to transform (2.2) in the form

$$S'' = S' \cdot a(T',T''),\ T' \geq T'' \tag{2.2'}$$

where $a(T',T'')$, increasing with respect to T'', is called the *discounted factor* and expresses the *discounting law* only as a function of the two temporal variables. We will now address the *two variables laws.*

The *reflexive law* for $\approx$ is now equivalent to

$$m(T,T) = a(T,T) = 1,\ \forall T \tag{2.7}$$

Furthermore if, using $T'' = T$ in systems (2.1') and (2.2'), we obtain $S'' = S$, i.e. the symmetric property is valid for $\approx$, the laws $m(\cdot)$ and $a(\cdot)$ satisfy

$$m(T,T') \cdot a(T',T) = 1,\ \forall T \leq T' \tag{2.8}$$

---

6 The symmetric case – far from being realistic in the contracts with companies and banks, due to the different conditions and onerousness of the lending market (which leads to costs for the companies) compared to the investment market (which leads to profits for the companies) – can be applied to the contracts between persons or linked companies and, from a theoretical point of view, makes it possible to deal with the two systems in a similar manner.

7 The property of proportional amounts is normally used in theoretical schemes, but should only be used with smaller amounts. The financial profits for the unit of invested capital can change according to the value of the capital and the contractual strength of the investors.

Equation (2.8) shows that *conjugated laws for the same time interval give rise to reciprocal factors.*

When describing (2.4) in detail, we consider *the exchange law of two variables* characterized by *the exchange factor z(X,Y)*, a pure number increasing with respect to *Y,* defined using

$$z(X,Y) \ = \ \begin{cases} m(X,Y)\,, & if \ X < Y \\ \quad 1 \quad , & if \ \ X = Y \\ a(X,Y)\,, & if \ X > Y \end{cases} \tag{2.5'}$$

(2.5′) being a particular case of (2.5).

To summarize, given an indifference relation ≈, the corresponding exchange law expressed by the factor $z(X,Y)$, such that $(X,S_1) \approx (Y,S_2)$, is equivalent to $S_2 = S_1\, z(X,Y)$. The exchange factor $z(X,Y)$ is a function defined for each couple (X,Y) of exchange times, which "brings" the values from X to Y forward (= accumulation) if $X<Y$ and backward (= discounting) if $X>Y$.

We will now assume that

$$z(X,Y) > 0,\ \forall (X,Y) \tag{2.5''}$$

(considering, if needed, only the part of the definition set for the function z where such a condition holds) in order that it cannot be possible that an encashment (payment) can never be indifferent to a payment (encashment) with different time maturity.

In geometric terms, let us consider the Cartesian plane OXY with the points $G \equiv (X,Y)$ with the aforementioned meaning[8]. The exchange factor is then the point function $z(G)$. Because of (2.5'), $z(G)$=1 if G is *on* the bisector of the coordinate axes. Furthermore, if G is *over* the bisector (i.e. if $X<Y$), then $z(G) = m(X,Y) > 1$; otherwise, if G is *under* the bisector (i.e. if $X>Y$), $z(G) = a(X,Y)<1$ and more precisely because of (2.5"): $0<a(X,Y)<1$[9].

---

8 Note that the functions $m(X,Y)$ and $a(X,Y)$ are defined in the disjoint half-planes $X<Y$ and $X>Y$, i.e. over and under the bisector of coordinate axes. It can be useful to extend their definition on the bisector $Y=X$, recalling (2.7) and putting $m(X,X) = a(X,X) = 1$.

9 (2.5') brings to a general formulation of exchange value of two variables, which does not imply the symmetry of financial relations. It follows that the law $z(X,Y)$ can be used to schematize not just the time variability of the cost and profit parameters, but also their difference in investment and discount operations which are of interest to any company. For example, if a company obtains liquid assets through anticipation of future credits and uses

Recalling the considerations of Chapter 1 (especially criteria d) for positive amounts, given that *z(X,Y)* is the exchange value of unitary amount), in the hypothesis of positive returns for the money the contour curves *z(X,Y)* = const. are graphs of strictly increasing functions $Y=\psi(X)$[10].

If relation ≈ expressed by *z(X,Y)* satisfies *the symmetric property*, as a particular case of (2.6') the below condition follows:

$$z(X,Y) \cdot z(Y,X) = 1;\ \forall (X,Y) \tag{2.9}$$

If *z(X,Y)* satisfies (2.9), then it defines a couple of two-variable financial interest and discount laws which are conjugated.

It is obvious that if the indifference relation is symmetric, it is enough to be able to define *z(X,Y)* in one of the two half-planes to obtain the value of *z* in the second half-plane using the following rule: *the values of z for points which are symmetric with respect to the bisector are reciprocal.* In this case $z(X,Y) = 1/z(Y,X)$, $\forall (X,Y)$ then the couples of contour curves of accumulation factor $z = k > 1$ and discount factor $z = 1/k < 1$ are functions which are mutually inverse.

2.3. Derived quantities in the accumulation and discount laws

In light of the laws defined in (2.1') and (2.2'), we can deduce the following derived quantities[11].

2.3.1. *Accumulation*

As a function of the *initial accumulation factor*

$$\text{iaf} := m(X,Y) \tag{2.10}$$

---

them in financial operations, and if the parameters *a* and *m* used in such an operation and summarized in *z* are not reciprocal, a non-zero spread is created.

10 In fact if we assume *z(X,Y)* to be continuous and partially differentiable everywhere, it follows that: $\frac{\partial z}{\partial X} < 0$, $\frac{\partial z}{\partial Y} > 0\ \forall (X,Y)$. Therefore, the contour curves *z(X,Y)* = const. are continuous and strictly increasing; they are graphs of functions $Y = \psi(X)$ invertible. In fact, for a theorem on implicit function, it follows that: $\psi'(X) = -\frac{\partial z}{\partial X} / \frac{\partial z}{\partial Y}$, where in the aforementioned hypothesis $\psi(X)$ is continuous and $\psi'(X) > 0$.

11 In this section we will denote with roman capital letters the temporal variables meaning *time* or *epoch* and with small roman letters, variables meaning *length*.

(:= means "equal by definition") – which measures the multiplicative increment from X to $Y>X$ of the invested capital in X. The factor is "initial" because the date X of investment coincides with the beginning of the time interval (X,Y) on which such an increment is measured. We can also define (see Figure 2.1):

– the *initial interest (per period) rate* (= interest on the unitary invested capital in the time interval from X to $Y>X$) is expressed by

$$\text{iir}:= m(X,Y) - 1 \tag{2.11}$$

– the *initial interest (per period) intensity*, expressed by

$$\text{iii}:= \{m(X,Y) - 1\}/(Y\text{-}X) = \{m(X,X+t) - 1\}/\, t \tag{2.12}$$

where $t = Y\text{-}X > 0$.

Alternatively, still using X as the investment time and imposing $X<Y<Z$, the capital increment is measured on a time interval *(Y,Z) subsequent to X*, then *continuing* with respect to interval (X,Y) without disinvesting in Y, we can then generalize and define continuing factors, rates and intensities in the following way:

– the *continuing accumulation factor* from Y to Z (= accumulated amount in $Z=Y+u$, $u>0$, of the unitary accumulated amount in $Y=X+t$, $t>0$, for the investment started in X) is expressed by

$$\text{caf}:= r(X;Y,Z) = m(X,Z)/m(X,Y) = m(X,Y+u)/m(X,Y) \tag{2.13}$$

– the *continuing interest (per period) rate* from Y to Z (= interest for unitary accumulated amount in Y passing from Y to $Z=Y+u$, $u>0$, for the investment started in X) is expressed by

$$\begin{aligned}\text{cir}:= \text{caf} - 1 &= \{m(X,Z) - m(X,Y)\}/m(X,Y) \\ &= \{m(X,X + u) - m(X,Y)\}/m\,(X,Y)\end{aligned} \tag{2.14}$$

– the *continuing interest (per period) intensity* from Y to $Z = Y+u$, $u>0$ is expressed by

$$\text{cii}:= \frac{r(X;Y,Z)-1}{Z-Y} = \frac{m(X,Z)\text{-}m(X,Y)}{(Z\text{-}Y)\; m(X,Y)} = \frac{m(X,Y+u)\text{-}m(X,Y)}{u\;\; m(X,Y)} \tag{2.15}$$

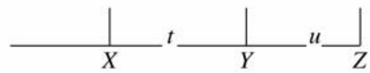

Figure 2.1. *Times in accumulation*

(2.13) is justified if we point out that if the amount K is invested at date X, the accumulated amount in Y has the value $K_Y = K\, m(X,Y)$ while that in Z has the value $K_Z = K\, m(X,Z)$. By definition $r(X;Y,Z)$ satisfies $K_Z = K_Y\, r(X;Y,Z)$. For comparison

$$r(X;Y,Z) = K_Z / K_Y = m(X,Z) / m(X,Y)$$

It is obvious that if X=Y, (2.13), (2.14) and (2.15) become respectively (2.10), (2.11) and (2.12), i.e. the "continuing" quantities become the "initial" quantity. In symbols: $r(Y;Y,Z) = m(Y,Z)$.

Intensity (2.15) is obtained by dividing the partial incremental ratio of function $m(\xi,\eta)$, considered with ξ=X and respect to η from Y to $Y+u$, by $m(X,Y)$. In the hypothesis that $m(\xi,\eta)$ is partially differentiable with respect to η with a continuous derivative in the interesting interval, the right limit of (2.15) then exists when $u \to 0$, which represents *the instantaneous interest intensity*[12] (implying: *continuing*) in Y of an investment started in X, indicated by $\delta(X,Y)$. Using symbols, where "$\log_e$" is indicated with "ln":

$$\delta(X,Y) = \lim_{u\to 0} \frac{m(X,Y+u)-m(X,Y)}{u\;\, m(X,Y)} =$$

$$= \left\{\frac{\partial}{\partial\eta} m(X,\eta)\right\}_{\eta=Y} / \, m(X,Y) = \left\{\frac{\partial}{\partial\eta} \ln m(X,\eta)\right\}_{\eta=Y} \qquad (2.16)$$

Working on the variables ξ, η , with $\xi < \eta$, it can be concluded that $\delta(\xi,\eta)$ is the logarithmic derivative (partial with respect to η) of $m(\xi,\eta)$.

12 It can also be called the *interest force* or (but improperly from a dimensional point of view) *instantaneous interest rate.*

Inverting function δ and the derivative operator in (2.16), the important relationship is obtained for continuing accumulated amount (2.13) as a function of the instantaneous intensity[13]:

$$\frac{m(X,Y+u)}{m(X,Y)} = e^{\int_{Y}^{Y+u} \delta(X,\eta)d\eta} \tag{2.16'}$$

2.3.2. *Discounting*

Let X be the final time of a financial operation (for example, the maturity of a credit). Analogously to accumulation, as a function of the *initial discounting factor*

$$\text{idf:= } a(X,Y) > 0 \tag{2.17}$$

we can also define (see Figure 2.2):

– the *initial per period discounting rate* (= discount for unitary capital at maturity for the anticipation from X to $Y<X$), given by

$$\text{idr:= } 1 - a(X,Y) \tag{2.18}$$

as well as, given $t = X\text{-}Y > 0$:

– the *initial per period discounting intensity*, which can be expressed by:

$$\text{idi:= } \{1 - a(X,Y)\}/(X\text{-}Y) = \{1 - a(X,X - t)\}/\, t \tag{2.19}$$

The dynamic expressions for “continuing discount” for an increment of the length of discount are seldom used, but they have meaning in discounting because of the decrease of the present value in relation to the length of anticipation. Therefore, we also define, in relation to the discount, the *continuing per period intensity* as well as the instantaneous intensity, related to time $X>Y>Z$. Indicating by $u>0$ the length of further discount $Z = Y - u$, we define:

---

13 From (2.16) it follows that, for small u, $m(X,Y)\cdot\delta(X,Y)\Delta u$ linearly approximates $\Delta m = m(X,Y+u) - m(X,Y)$. Furthermore, in the profitable hypothesis of the invested capital, which implies $m(X,X+t)>1$ and increasing with t, the positivity of $\delta(X,\eta)$, $\forall\eta>X$, because of (2.17) and of a well known property of integrals, follows. The opposite is true. A similar conclusion is obtained for the discounting instantaneous intensity, which will be introduced later.

– the *continuing discounting factor from Y to Z* (= present value in $Z<Y$ of the present unitary value in $Y<X$ of the capital at maturity in X, then of amount $1/a(X,Y)$), expressed by:

$$\text{cdf:}= a(X,Z)/a(X,Y) = a(X,Y\text{-}u)/a(X,Y) \tag{2.20}$$

– the *continuing discounting rate* from Y to Z (= discount for the anticipation from Y to Z of the present unitary value in Y<X of a capital with maturity in X, then of amount 1/a(X,Y)), expressed by:

$$\text{cdr:}=1-\text{cdf} = \frac{a(X,Y)-a(X,Z)}{a(X,Y)} = \frac{a(X,Y)-a(X,Y-u)}{a(X,Y)} \tag{2.21}$$

– the *continuing discounting intensity from Y to Z*, expressed by:

$$\text{cdi:}=\frac{1-fsp}{Y-Z} = \frac{a(X,Y)-a(X,Z)}{(Y-Z)\;a(X,Y)} = \frac{a(X,Y-u)-a(X,Y)}{-u\;\;a(X,Y)} \tag{2.22}$$

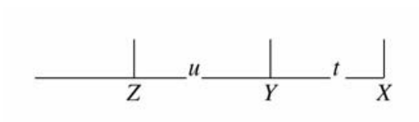

Figure 2.2. *Times in discounting*

Considering the limit as already calculated for the instantaneous interest intensity, it is possible to obtain:

– the *instantaneous discounting intensity* in Y, indicated by $\theta(X,Y)$ and given by:

$$\theta(X,Y)=\left\{\frac{\partial}{\partial\eta}a(X,\eta)\right\}_{\eta=Y} / a(X,Y) = \left\{\frac{\partial}{\partial\eta}\ln a(X,\eta)\right\}_{\eta=Y} \tag{2.23}$$

As $\theta(X,Y)$ is the logarithmic derivative (partial with respect to $Y \leq X$) of $a(X,Y)$, by inverting the process we obtain, $\forall Z < Y$,

$$\frac{a(X,Z)}{a(X,Y)} = e^{\int_Y^Z \theta(X,\eta)d\eta} = e^{-\int_Z^Y \theta(X,\eta)d\eta} \tag{2.24}$$

2.4. Decomposable financial laws

2.4.1. *Weak and strong decomposability properties: equivalence relations*

In the case of the financial law of two variables, we consider the meaning and the consequences of the *decomposability* property, which was introduced by Cantelli.

We have decomposability in an accumulation (or discounting) operation when investing (or discounting) a given capital available at time X, we have the same accumulated amount (or present value) in Z, both if we realize and reinvest immediately the obtained value in a intermediate time Y, or if we continue the financial operation. To summarize, decomposability means *invariance of the result with respect to interruptions of the financial operation.*

With reference to the *interest law* $m(X,Y)$, which follows from relation ≈, and to the three times X, Y, Z, with $X<Y<Z$, let S_2 be the realized accumulated amount in Y of S_1 invested in X; moreover, let S_3 be the accumulated amount in Z of S_2 immediately reinvested in Y. Instead S'_3 is the accumulated amount Z after only one accumulation of S_1 from X to Z. Due to (2.1')

$$S_2 = S_1\, m(X,Y);\; S_3 = S_2\, m(Y,Z);\; S'_3 = S_1\, m(X,Z)\,. \tag{2.25}$$

If

$$S_3 = S'_3,\ \forall (S_1, X<Y<Z) \tag{2.26}$$

the interest law is decomposable. It follows from (2.25) that (2.26) is equivalent to

$$m(X,Y)\, m(Y,Z) = m(X,Z) \tag{2.27}$$

which expresses the decomposability condition for an interest law in terms of accumulation factors.

In the same way, referring to the *discount law* $a(X,Y)$ following ≈ and recalling (2.2'), if $X > Y > Z$ we can define the following discounted values starting from S_1, payable in X:

$$S_2 = S_1\, a(X,Y);\; S_3 = S_2\, a(Y,Z);\; S'_3 = S_1\, a(X,Z) \tag{2.28}$$

If

$$S_3 = S'_3, \forall(S_1, X>Y>Z) \tag{2.29}$$

the discount law is decomposable and because of (2.28) the decomposability condition for this law can be written as

$$a(X,Y)\, a(Y,Z) = a(X,Z) \tag{2.30}$$

Until now, we have defined in *weak* form the decomposability of single laws in accumulation or discounting operations, considering the times X, Y, Z in increasing or decreasing order. This signifies that we require the *prospective transitivity* or respectively the *retrospective transitivity* to the indifference relations, which give rise to the laws.[14] In this case we will talk of *weak decomposability.*

If instead the previous considerations are related to an exchange law following an indifference relation $\approx$ and expressed by the factors $z(X,Y)$ defined in (2.5'), we can think of extending the decomposability relation in (2.25) and (2.26) for any order of payment times. So the relation $\approx$ satisfies the *strong decomposability* property, which bi-implies

$$\{(X,S_1) \approx (Y,S_2)\} \cap \{(Y,S_2) \approx (Z,S_3)\} \Rightarrow (X,S_1) \approx (Z,S_3), \forall(X,Y,Z) \tag{2.31}$$

and then the following condition on the exchange factors:

$$z(X,Y)\, z(Y,Z) = z(X,Z), \;\; \forall(X,Y,Z) \tag{2.32}$$

The following result holds:

THEOREM A.– *If and only if for the exchange law the strong decomposability is valid, the relation $\approx$ is reflexive, symmetric and transitive, then it is an equivalence relation, which we denote by $\mathcal{E}$.*

Proof

Sufficiency: the strong decomposability implies (2.32); putting $Y = Z$ we obtain the reflexivity; putting $Z = X$ we obtain the symmetry; the transitivity is obvious.

Necessity: if $\approx\, = \mathcal{E}$, the unitary amount in X is exchangeable with $z(X,Z)$ in Z and also with $z(X,Y)$ in Y, which is exchangeable with $z(X,Y)\, z(Y,Z)$ in Z (whatever order

14 Let us recall that a binary relation $\mathcal{R}$ on a set $\mathcal{H}$ satisfies the *transitivity property* if $(a\, \mathcal{R}\, b) \cap (b\, \mathcal{R} c) \Rightarrow a\, \mathcal{R}\, c$, $\forall$ $a,b,c \in \mathcal{H}$.

may be among X, Y and Z because of the symmetry property); then (2.32) is also valid if $X = Z$ or if $Y = Z$.

Note: this argument could be developed, in a more formally complicated, but equipollent way, based on relation (2.31).

Considering the relation between weak decomposability (WD) and strong decomposability (SD), it is obvious that the condition of SD implies WD, when X, Y, Z are in increasing or decreasing order from which there are only accumulation or discounting respectively. However, the WD does not imply SD in other cases, when both an accumulation and a discounting occur together. Then, if SD holds, the properties of an equivalence are immediately verified. In fact, considering $X < Z < Y$ (analogously we could consider $X > Z > Y$), the SD expressed by (2.32) gives rise to

$$m(X,Y)\, a(Y,Z) = m(X,Z) \tag{2.33}$$

and the WD following the SD also implies $m(X,Z)\, m(Z,Y) = m(X,Y)$, or, for (2.33), $m(Z,Y) = m(X,Y)/m(X,Z) = 1/a(Y,Z)$, or also that

$$m(Z,Y)\, a(Y,Z) = 1, \ \forall Y<Z \tag{2.34}$$

Then, because of the generic choice of times, m and a are conjugate laws, the financial relation is symmetric, as well as transitive, but also reflexive (it is enough to impose $Y = Z$ in (2.33) obtaining $a(Z,Z) = 1$ and then for (2.34), $m(Z,Z) = 1$). Therefore, the relation is an equivalence; the opposite also holds.

Let us summarize as follows. Given an indifference relation $\approx$ in the hypothesis of proportional amount, the strong decomposability, expressed by (2.32) for the exchange factor $z(X,Y)$, implies that $\approx$ is reflexive, symmetric and transitive, and then it is an equivalence indicated by $\mathcal{E}$. In this case, the derived interest and discount laws are decomposable and conjugated to each other.

EXAMPLE 2.3.– An investor with liquid assets invests the amount S_1 at time X until time Y in a term deposit. A prospectively decomposable accumulation law with accumulation factor $m(X,Y)$ is applied and a refund of $S_2 = S_1\, m(X,Y)$ is expected. At time Z (with $X < Z < Y$) the investor needs liquidity, but he cannot use the capital in the term deposit; therefore, the accumulated amount, given by

$$S'_3 = S_1\, m(X,Z) = S_2 \frac{m(X,Z)}{m(X,Y)} = \frac{S_2}{m(Z,Y)}$$

is not available (as when the capital is invested in a bank account); it is only possible to transfer the credit S_2 with a bank advance, applying a retrospectively decomposable discounting law to *a(Y,Z)*. In practice, in these cases the laws *m(Z,Y)* and *a(Y,Z)* are not conjugated, i.e. (2.34) does not hold. Thus, we do not have strong decomposability of the resulting exchange law, even if the laws *m* and *a* are weakly decomposable. We usually have $a(Y,Z) < 1/m(Z,Y)$, i.e. the cost for discount is greater than that resulting from applying the conjugate law of that regulating the deposit. It follows that $S_3 < S'_3$ and $S'_3 - S_3$ is the cost due to the locking up of capital S_1 until *Y*. The SD would cause $S'_3 = S_3$ and would avoid such cost.

2.4.2. *Equivalence classes: characteristic properties of decomposable laws*

Based on theorem A, if an indifference relation $\approx$ gives rise to a strongly decomposable exchange law, it is an equivalence relation $\mathcal{E}$[15] between all elements *(T,S)* of the set $\mathcal{H}$ of supplies, which makes it possible to separate such supplies into equivalence classes. Each class is made up of financially equivalent supplies, but which are indifferent. However, two supplies in different classes are not equivalent because it is possible to express a judgment of strong preference. Each class is characterized by an *abstract*, made up of the intrinsic financial value of its supplies.

By geometrically representing the supplies *(T,S)* on the plane *OTS*, a class of equivalent supplies is identified by a curve, a locus of points $P \equiv [T,S]$, corresponding to equivalent supplies. The infinite curves do not have common points. In addition:

1) for each point in the plane there is one and only one curve, which is a locus of equivalent points;

2) such curves are the graph of functions $S = \varphi(T)$ (continuous and differentiable, under suitable hypotheses) and, if the postulate on money return holds, increasing where positive, decreasing where negative.

The classes of equivalent supplies on the basis of an SD, i.e. the elements of the quotient set $\mathcal{H}/\mathcal{E}$, form a *totally ordered set*, because the elements of each couple are comparable for a weak preference judgment $\succeq$, using the meaning specified in section 1.2. Moving monotonically towards the classes (= curve in the plane *OTS*), the intrinsic financial value of the supplies improves in one sense (but gets worse in

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15 It is well known that equivalence relations E on the elements of a set H make it possible to stratify these elements in equivalence classes, such that each element is only in one class. Each class is characterized by an *abstract* common to its elements, indicating by *quotient set* H/E the set whose elements are the *abstracts*.

the other sense)[16]. It follows that the SD laws, on the basis of stratification in equivalence classes, allow a global, rather than just local, comparison between

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16 We set out the definition of some properties that are applied in the set of financial supplies. Let $\mathcal{H}$ be a set and $\mathcal{R}$ a binary relation between elements a, b, c, ...$\in \mathcal{H}$. The following properties can hold for $\mathcal{R}$ (where = means coincidence between elements, ~ means negation, $\cup$ means union or logic sum and $\cap$ means intersection or logic product):
1) *reflexive property*: $a\mathcal{R}\,a$, $\forall a \in \mathcal{H}$; 2) *symmetric property*: $a\mathcal{R}\,b \rightarrow b\mathcal{R}\,a$, $\forall$ a,b $\in \mathcal{H}$; 3) *transitive property*: $(a\mathcal{R}\,b)\cap((b\mathcal{R}\,c) \rightarrow a\mathcal{R}\,c$; $\forall$a,b,c $\in \mathcal{H}$; 4) *non-reflexive property*: $\sim(a\mathcal{R}\,a)$, $\forall a \in \mathcal{H}$; 5) *anti-symmetric property*: $(a\mathcal{R}\,b)\cap(b\mathcal{R}a) \rightarrow a=b$; $\forall$a,b $\in\mathcal{H}$; 6) *asymmetric property*: $a\mathcal{R}\,b \Rightarrow \sim(b\mathcal{R}\,a)$, $\forall$a,b $\in \mathcal{H}$; 7) *completeness property*: $(a\mathcal{R}\,b)\cup(b\mathcal{R}\,a)$ is certainly verified, i.e. at least one of $(a\mathcal{R}\,b)$ and $(b\mathcal{R}\,a)$, $\forall a\neq b \in \mathcal{H}$ holds.

We have already talked about the first three properties, pointing out that a binary relation between elements of $\mathcal{H}$ is an equivalence relation $\mathcal{E}$ if for every choice of elements the symmetric, reflexive and transitive properties hold. When $\forall$a,b $\in \mathcal{H}$ it is verified that $(a\mathcal{R}\,b) \cup\sim(a\mathcal{R}\,b)$, and then $(a\mathcal{R}\,b)$ is an event, in the logic meaning, referred to elements of $\mathcal{H}$. We give the following definition regarding ordering. A binary relation $\mathcal{R}$ on the set $\mathcal{H}$ is called a *relation of partial order* if for each element in $\mathcal{H}$ the reflexive, anti-symmetric and transitive properties hold. $\mathcal{H}$ is then said to be *partially ordered.* With this hypothesis, if all elements of $\mathcal{H}$ are comparable two by two (= completeness property), then the relation is called *of total order* and $\mathcal{H}$ is said to be *totally ordered.* A binary relation $\mathcal{R}$ on the set $\mathcal{H}$ is said to be *almost ordered or preordered (total or partial*, if it is comparable or not) if the reflexive or transitive properties hold when it is then called *almost ordered (partially or totally).* Briefly, an *order relation* $\mathcal{O}$ brings to a classification which do not consider "equal elements" while a *almost order relation* $\mathcal{QO}$ allows "equal elements".

Note that if on the set $\mathcal{H}$ a total $\mathcal{QO}$ relation holds, the completeness relation is satisfied, i.e. however chosen b$\in \mathcal{H}$, $\forall a \in \mathcal{H}$, a$\neq$b, it certainly satisfied that $(a\mathcal{QO}b)\cup(b\mathcal{QO}a)$. Given that $(a\mathcal{QO}\,b)\cup(b\mathcal{QO}\,a) = [(a\mathcal{QO}\,b)\cap((b\mathcal{QO}\,a)]\cup[(a\mathcal{QO}\,b)\,(\sim(b\mathcal{QO}\,a)]\cup[\sim(a\mathcal{QO}\,b)\,((b\mathcal{QO}\,a)]$ and that the three possibilities written between square brackets in the second term are incompatible, they make a partition. More briefly, the completeness derived from the totality of $\mathcal{QO}$ is equivalent to the possibility of the realization of $\sim(a\mathcal{QO}\,b)\cap\sim(b\mathcal{QO}\,a)$.

Let us now consider the equivalence relation $\mathcal{E}$, such that $a\mathcal{E}\,b$ if the first possibility is true i.e. $(a\mathcal{QO}\,b)\cap(b\mathcal{QO}\,a)$; in such a case we write a$\approx$b. If the second or third possibility is true, we write respectively $a \prec b$ and $b \prec a$. Relation $\prec$ (or $\succ$) is said to be a *(strong) preference*, characterized by the asymmetric property. Writing $a \succ b$ is equivalent to $b \prec a$. In conclusion, as a consequence of the relation $\mathcal{QO}$ in $\mathcal{H}$, of the three possibilities, a$\approx$b, $a \prec b$, $a \succ b$, one and only one is verified. With a fixed $\mathcal{E}$, the quotient set $\mathcal{H}/\mathcal{E}$, i.e. the set of the supplies, i.e. due to transitivity they make it possible to extend to any number of supplies on the plane *OTS* the preference or indifference relations introduced in Chapter 1 with respect to a given supply.

It is easy to give a method for such a comparison, verifying the existence of total order in $\mathcal{H}$. It is enough to identify the classes using supplies that have the same maturity T_0; then class α identified by (T_0,S'_0) is preferred to class β identified by (T_0,S''_0) if $S'_0 > S''_0$; β is preferred to α if $S'_0 < S''_0$; α and β are equivalent if $S'_0 = S''_0$.

Let us consider some characteristic properties of decomposable laws of two variables, which proceed from the following theorems.

THEOREM B.– *Referring to definitions* (2.10) *and* (2.13)*, an interest law is weakly decomposable if and only if, for each choice of subsequent times X<Y<Z, the initial accumulation factor from Y to Z is equal to the continuing accumulation factor from Y to Z of an accumulation started in X. In symbols: r(X;Y,Z)=r(Y;Y,Z)=m(Y,Z). Therefore, the decomposability implies independence of r(X;Y,Z) from the time of investment, and vice versa. There is an analogous condition in relation to the discount factors* (2.17) *and* (2.20)*, for each choice of time X>Y>Z holds for a weakly decomposable discount law.*

THEOREM C.– *An interest law is weakly decomposable if and only if the instantaneous intensity* δ(*X,T*)*, continuous by hypothesis, does not depend on the initial time X but only on the current time T. The analogous condition on the intensity* θ(*X,T*) *holds for a weakly decomposable discount law. Under the same condition necessary and sufficient on the instantaneous intensity of interest and*

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equivalence classes with respect to $\mathcal{E}$ of the elements in $\mathcal{H}$, results totally ordered because between the classes {a}, {b} identified by a, b only one relation holds: {a}={b}, $\{a\} \prec \{b\}$, $\{a\} \succ \{b\}$. To summarize, an almost order relation (total) on $\mathcal{H}$ induces an equivalence relation $\mathcal{E}$ and then an order (total) relation on $\mathcal{H}/\mathcal{E}$.

In financial applications it follows that if the exchange law applicable to the supplies $(T,S) \in \mathcal{H}$ is strongly separable and then follows from an equivalence relation $\mathcal{E}$, then:

1) There is an *almost order* between each supply $\in \mathcal{H}$ (total if the law is applicable to all supplies) where between two supplies or there is indifference or one is preferred (strongly). There is then the possibility of "equals" or indifference.
2) There is *order* (total in the same hypothesis) between supply equivalence classes, elements of $\mathcal{H}/\mathcal{E}$, where between two different classes there is always a strong preference relation, regarding each pair of supplies each taken in a class. In formula, $\{a\} \prec \{b\} \Rightarrow a \prec b$, $\forall$ ($a \in \{a\}$, $b \in \{b\}$).

discount, a strong decomposability of an exchange law specified by the factor identified by (2.5') *which satisfies* (2.9) *can be verified.*

THEOREM D.– *An exchange law specified by the factor identified by* (2.5') *which satisfies* (2.9) *is strongly decomposable if and only if there exists an increasing function h(T) such that*

$$z(X,Y) = \frac{h(Y)}{h(X)}, \quad \forall(X,Y) \tag{2.35}$$

Given z(·) = m(·), (2.35) $\forall(X \le Y)$ *gives a WD condition for an interest law (= of prospective transitivity for ≈); furthermore,* (2.35), $\forall(X \ge Y)$, *and given z(·) = a(·), is WD condition for a discount law (= of retrospective transitivity for ≈). If ≈ is not symmetric, i.e.* (2.9) *is not valid, we have weak decomposability of interest and the discount law is not conjugated following ≈ if and only if there exist two different functions* $h_1(T)$ *and* $h_2(T)$ *such that* (2.35) *holds where:* $h(T) = h_1(T)$ *if* $X \le Y$; $h(T) = h_2(T)$ *if* $X > Y$ [17].

Briefly, theorems C and D show that: 1) a characteristic property of strongly decomposable exchange laws is the coincidence of interest and discount intensity in

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17 The proofs of theorems B, C and D are as follows:

– theorem B is proved by noticing that, with respect to the interest (or discount) laws, the equality between *m(Y,Z)* and *m(X,Z)/m(X,Y)* (or between *a(Y,Z)* and *a(X,Z)/a(X,Y)*) bi-implies (2.28) or (2.31);

– theorem C is proved, with respect to interest laws, by noticing that because of (2.17) and of theorem B the decomposability of law *m* is equivalent to the identity chain:

$$e^{\int_Y^Z \delta(X,\eta)d\eta} = \frac{m(X,Z)}{m(X,Y)} = m(Y,Z) = e^{\int_Y^Z \delta(Y,\eta)d\eta}, \forall(X<Y<Z)$$

which, because of the arbitrariness of time, bi-implies $\delta(X,\eta) = \delta(Y,\eta)$, i.e. because of the same arbitrariness, an intensity depends only on current time. An analogous proof holds in regard to the condition on the intensity $\theta(X,T)$ to have decomposability of the discount law, $\forall(X>Y>Z)$, and on the intensity condition $\delta(X,T) = \theta(X,T) = \delta(T)$ to have strong decomposability of the exchange law $z(X,T)$, $\forall(X,Y,Z)$;

– theorem D for exchange law is proved by noticing that:

sufficient condition: if there is *h(T)* verifying (2.30), clearly *z(X,X)* = 1, ∀*X*, and then (2.9) and (2.33) hold so that $\approx = \mathcal{E}$ and the exchange law is strongly decomposable,

necessary condition: if *z (X,Y)* identifies a strongly decomposable law, because of theorem C the interest and discount intensity are expressed by the same function $\delta(T)$ and the requested function $h(T)$, which is clearly defined regardless of a multiplicative constant, has the dimension and meaning of an amount valued in *T* and must satisfy the differential equation: $h'(T) = \delta(T)h(T)$, where the general expression $h(T) = k\, e^{\int_{T_0}^T \delta(\eta)d\eta}$ is assumed as having the meaning of valuation in *T*, based on the exchange law *z* of the amount *k* dated at time T_0. Theorem D regarding conditions of weak decomposability is an immediate corollary.

a function $\delta(T)$ which depends only on current time; 2) the exchange factor of a strongly decomposable law assumes the characteristic form

$$z(X,Y)=e^{\int_X^Y \delta(X,\eta)d\eta} \tag{2.36}$$

EXAMPLE 2.4.– Give the following accumulation law

$$m(X,Y)=e^{0.05(Y-X)+0.002(Y+X)(Y-X)}$$

using an instantaneous intensity $\delta(t) = 0.05 + 0.004\,t$, a function only of the current time t, where $m(X,Y)$ is a decomposable law.

Let us verify the decomposability using (2.27). We obtain

$$m(X,Z)=e^{0.05(Z-X)+0.002\left(Z^2-X^2\right)}\ ;\ m(X,Y)=e^{0.05(Y-X)+0.002\left(Y^2-X^2\right)}$$

$$m(Y,Z)=e^{0.05(Z-Y)+0.002\left(Z^2-Y^2\right)}$$

then (2.6) $\forall(X<Y<Z)$ holds.

If we put: $X = 1;\ Y = 5+\dfrac{5}{12} = 5.417;\ Z = 6+\dfrac{1}{12} = 6.083$, it results in

$$\begin{aligned}
m(X,Y) &= e^{0.05\cdot 4.417+0.002\cdot 28.344} = e^{0.277538} = 1.319876\\
m(Y,Z) &= e^{0.05\cdot 0.666+0.002\cdot 7.659} = e^{0.048618} = 1.049819\\
m(X,Z) &= e^{0.05\cdot 5.083+0.002\cdot 36.003} = e^{0.326156} = 1.385632
\end{aligned}$$

and then (summing the exponents of e) (2.6) is verified. Even the alternative expression following theorem B is verified as

$$r(X;Y,Z)=\frac{m(X,Z)}{m(X,Y)}=\frac{1.385632}{1.319876}=1.049819\ =m(Y,Z)=r(Y;Y,Z)$$

EXAMPLE 2.5.– Given, with $Y<Z$,

$$m(Y,Z) = 1+1.06^Z - 1.06^Y$$

satisfying $m(Y,Y)$=1, increasing with Z, decreasing with Y, resulting in: $m(0,Z) = 1.06^Z$. Put S_1=1,450, $Y = 5+\frac{5}{12} = 5.417$, $Z = 6+\frac{1}{12} = 6.083$, it follows that

$$m(Y,Z) = 1 + 1.425396 - 1.371140 = 1.054256$$

and then: S_2 = 1,528.67; initial per period rate = 0.054256; initial per period intensity = 0.081465 years^{-1}.

Given X = 1 it follows, in continuing terms, that:

$$r(X;Y,Z) = \frac{1+1.425396 - 1.06}{1+1.371140 - 1.06} = \frac{1.365396}{1.311140} = 1.04138 \neq m(Y,Z).$$

This financial law is not decomposable. In addition:

– the continuing per period rate is 0.041381;

– the continuing per period intensity is 0.062078 years^{-1}.

2.5. Uniform financial laws: mean evaluations

2.5.1. *Theory of uniform exchange laws*

The hypothesis of *uniformity* (or *homogenity*) *in time* is common in financial practice. In formal terms, an indifference financial relation ≈ is *uniform in time* if:

$$(X,S_1) \approx (Y,S_2) \Rightarrow (X+h,S_1) \approx (Y+h,S_2),\ \forall h \tag{2.37}$$

that is, an indifference relation is not changed by a time translation (i.e. moving X and Y of the same time interval forwards or backwards), as long as the payment times remain in the applicability interval of the financial law.

Assuming the proportionality of amounts, because of (2.37) for the exchange factor $z(X,Y) = S_2 / S_1$ the following property is worth:

$$z(X,Y) = z(X+h,Y+h),\ \forall h \tag{2.38}$$

To summarize: *a uniform relation is characterized by the property that the exchange factor does not change with a rigid time translation such that the time difference $Y - X = (Y+h) - (X+h)$ does not change.*

It follows for the corresponding financial law (which we will call *uniform*) that

$$z(X,Y) \equiv g\,(Y\text{-}X) \equiv g\,(\tau) > 0,\ \forall\tau \tag{2.39}$$

that is, *in a uniform law the exchange factor depends only on the duration (with sign)* $\tau=Y\text{-}X$ *of the financial operation and not just on the times X,Y of the beginning and the end of the operation, considered separately.*

If the relation ≈ is *uniform* and also *symmetric,* the couples of conjugated interest and discount laws are expressed by the factors $g(\tau)$ and $g(-\tau)$[18] satisfying

$$g(\tau)\ g(-\tau) = 1,\ \forall\tau \tag{2.40}$$

If the exchange law $z(X,Y)$ is uniform on time, the contour curves $z(X,Y) = const.$ are lines parallel to the bisector $Y=X$. Furthermore, if ≈ is also symmetric, considering geometrically (2.40), the increasing graph of $g(\tau)$ is such that the opposite values of τ correspond with the reciprocal values of $g(\tau)$. Such factors remain constant respectively on parallel lines equidistant of the bisector $\tau = 0$, from which $z(X,X) = g(0) = 1$ follows.

Often the accumulation and discount factor, instead of being considered unified through $g(\tau)$, are considered separately and expressed as a function of the (absolute) duration $t = |Y–X| = |\tau|$.

Obviously we have:

$$t = \tau\ ,\ \text{if}\ \tau > 0;\ t = -\tau\ ,\ \text{if}\ \tau < 0.$$

We can then put a correspondence between a *uniform* relation ≈, which is characterized by a exchange factor $g(\tau)$, defined $\forall\tau$, and two laws, the former of interest, expressed by an *accumulation factor u(t)*, the latter of discount, expressed by a *discount factor v(t)*, both defined $\forall t \geq 0$ in the following way:

$$\begin{cases} u(t) = g(t) = g(\tau), & if\ \tau = t > 0 \\ u(0) = v(0) = g(0) = 1 & \\ v(t) = g(\tau) = g(-t), & if\ \tau = -t < 0 \end{cases} \tag{2.41}$$

In (2.41), the second equation express the reflexive property of ≈; the first and the third equation express respectively the exchange factor in accumulation and discount. By assuming the usual hypothesis of onerous nature of a loan, $u(t) \geq 1$ is a

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18 More precisely, in an accumulation law, the result is $\tau = Y\text{-}X > 0$ and $g(\tau)$ is *the accumulation factor*, whereas $g(-\tau) = 1/g(\tau)$ is the conjugate *discount factor* from *Y* to *X*. However, in a discount operation, the result is $\tau = Y\text{-}X < 0$ and $g(\tau)$ is *the discount factor* while $g(-\tau) = 1/g(\tau)$ is the conjugated *accumulation factor* from *Y* to *X*.

strictly increasing function of the duration t, and $v(t)$, subject to $0 < v(t) \leq 1$, is a strictly decreasing function of t.

If $\approx$ is also *symmetric,* from (2.40) and (2.41) it follows that:

$$u(t)\,v(t) = 1,\ \forall t > 0 \tag{2.42}$$

that is, the accumulation and the discount factors for a fixed duration t are reciprocal.

It is useful at this point to adopt for the uniform laws and for the exchange factors $u(t)$ and $v(t)$ the definitions and positions introduced for the factors $m(X,Y)$ and $a(X,Y)$. The following table is then obtained[19].

FACTORS, RATES AND INTENSITIES FOR UNIFORM LAWS			
Financial quantity		*Interest laws*	*Discount laws*
I)	*initial accumulation factor* for duration t	$u(t)$	$v(t)$
II)	*initial rate* for duration t	$u(t) - 1$	$1 - v(t)$
III)	*initial intensity* for duration t	$\frac{u(t) - 1}{t}$	$\frac{1 - v(t)}{t}$
IV)	*continuing accumulation factor* for the subsequent duration h after t	$\frac{u(t+h)}{u(t)}$	$\frac{v(t+h)}{v(t)}$
V)	*continuing rate* for the subsequent duration h after t	$\frac{u(t+h)}{u(t)} - 1$	$1 - \frac{v(t+h)}{v(t)}$
VI)	*continuing intensity* for the subsequent duration h after t	$\frac{u(t+h) - u(t)}{h\ u(t)}$	$\frac{v(t) - v(t+h)}{h\ v(t)}$
VII)	*instantaneous intensity* in t (*)	$\delta(t) = \frac{u'(t)}{u(t)}$	$\theta(t) = -\frac{v'(t)}{v(t)}$
(*) (VII) is the limit case of (VI) when $h \to 0$ and assumes the derivability of exchange factors $u(t)$ and $v(t)$. Prime means differentiation. For simplicity, intensities are indicated with the same symbols δ and θ used for those connected with law of two variables.			

Table 2.1. *Factors, rates and intensities for uniform laws*

19 We notice that because of the invariance with translation following (2.39), it is possible and convenient to choose the time origin as X, the "beginning" time of the operation, and to measure time forwards (in interest laws) or backwards (in discount laws) for a time interval of length t.

From definition VII in Table 2.1, which expresses $\delta(t)$ and $-\theta(t)$ as logarithmic derivatives of $u(t)$ and $v(t)$, by inversion it follows that:

$$u(t) = e^{\int_0^t \delta(z)\,dz}\ ;\ v(t) = e^{-\int_0^t \theta(z)\,dz} \tag{2.43}$$

If the uniform interest and discount laws are conjugated (i.e. in the symmetry hypothesis), it results in $\delta(t) = \theta(t)$. In fact, it validates the theorem.

THEOREM.– *The necessary and sufficient condition in order for* (2.42) *to hold is the equality* $\delta(t) = \theta(t)$, $\forall t \geq 0$.

Proof:

Necessity: if (2.42) holds, it follows that $\forall t \geq 0$: $\ln u(t) = -\ln v(t)$ and, differentiating, $\delta(t) = \theta(t)$.

Sufficiency: if $\delta(z) = \theta(z)$, $\forall z \geq 0$, for (2.43) it follows, $\forall t \geq 0$, that

$$u(t)\,v(t) = e^{\int_0^t [(\delta(z)-\theta(z)]\,dz} = 1$$

because the integrand function is identically zero in the interval $(0,t)$.

Examples and applications of laws uniform in time will be shown in Chapter 3.

2.5.2. *An outline of associative averages*

Let us recall the concept of mean, as introduced by Chisini and developed by de Finetti[20], from which the mean of quantities $x_1, x_2, ..., x_n$ with respect to a quantity $y = f(x_1, x_2, ..., x_n)$, which depends univocally on $x_1, x_2,...,x_n$ by the function f, is a value $\hat{x}$ such that:

$$f(\hat{x},\hat{x},...,\hat{x}) = f(x_1, x_2, ...,x_n) \tag{2.44}$$

where if $x_1, x_2,...,x_n$ are replaced by $\hat{x}$, f remains unchanged. In such a way the individuation of mean, which has a summarizing meaning, depends on the considered problem which constitutes a choice criterion.

20 See, for example, de Finetti (1931); Volpe di Prignano (1985).

A mean is said to be *associative* when the same result is obtained, averaging out the given quantities (each with its *weight*) or averaging out the partial averages of their subgroup (each with the total weight of the subgroup). The consequent "associative property" is verified by the center of mass of a distribution of masses concentrated on the point of a line, a center whose abscissa $\bar{x} = \sum_h p_h x_h / \sum_h p_h$ is the weighted arithmetic mean[21] of the abscissas x_h where the masses are put, with weights p_h corresponding to the masses. It can be proved (see the *Nagumohy Kolmogoroff-de Finetti theorem*) that, given the distribution (x_h, p_h), $(h = 1, ..., n)$, the set of its associative averages coincides with the set of transformations of the arithmetic mean through a function $q(x)$ chosen in the class of continuous and strictly monotonic functions. In other words, with $q(x)$ continuous and strictly decreasing or increasing, the number $\hat{x}_q$, solution of the following equation in x

$$q(x) = \sum_h p_h \, q(x_h) / \sum_h p_h \tag{2.45}$$

is an associative average of the values x_h with weights p_h and all the others can be obtained by varying $q(x)$ in the class specified above. Since $q(x)$ has an inverse function $q^{-1}(x)$, we univocally obtain

$$\hat{x}_q = q^{-1}\left(\sum_h p_h \, q(x_h) / \sum_h p_h\right) \tag{2.46}$$

$\hat{x}_q$, called q-average, is invariant for linear transformation on $q(x)$, because it follows from (2.45) that

$$a\,q(x)+b = \sum_h p_h [a\,q(x_h)+b] / \sum_h p_h$$

The more important averages used in applications are associative.[22]

The following properties hold:

1) the geometric mean can be obtained as the limit of the power mean when $k \rightarrow 0$;

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21 If the weights are all equal, the mean is called "simple".

22 Let us recall the mean of powers of order k, with transformation function $q(x) = x^k$ (the arithmetic mean for $k = 1$, the quadratic mean for $k = 2$, and the harmonic mean for $k = -1$), the geometric mean for $q(x) = \log x$, and the exponential mean for $q(x) = e^{cx}$.

2) with the same data, power means with exponent k give values increasing with k;

3) the inequality between $\hat{x}_q$ and $\bar{x}$ depends on the feature of $q(x)$, resulting:

– $\hat{x}_q > \bar{x}$, if $q(x)$ is increasing convex or decreasing concave,

– $\hat{x}_q < \bar{x}$, if $q(x)$ is increasing concave or decreasing convex.

The concavity and convexity are, as usual, downwards.

The aforementioned properties are shown in Figure 2.3, which explains the calculation of a simple associative average of two elements.

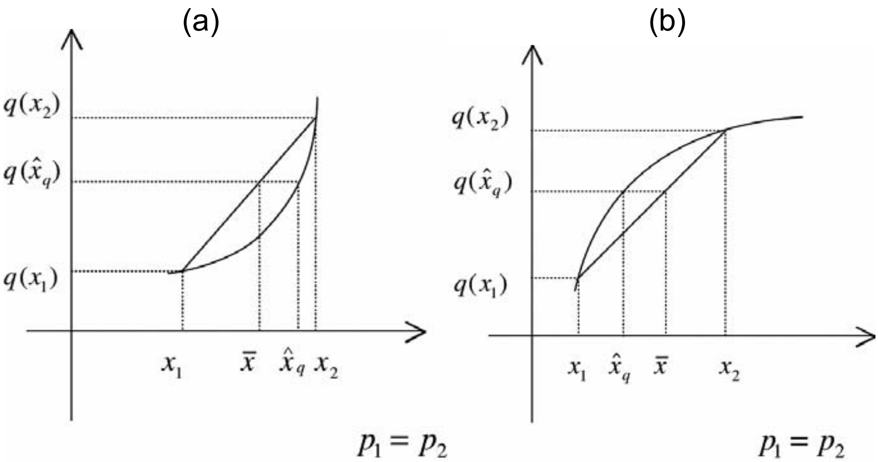

Figure 2.3.a *Associative average with convex q(x)*

Figure 2.3.b *Associative average with concave q(x)*

2.5.3. *Average duration and average maturity*

Let us consider a financial relation ≈ expressed by a law with an always positive intensity, and suppose that the exchange factors $q(t)$ consequent to ≈ are continuous and strictly monotonic of the duration.

Let us also consider the following problem: given the amounts K_1, K_2, ..., K_n accumulated or discounted according to the same exchange law $q(t)$ and the respective durations t_1, t_2, ..., t_n, we want to find the duration $\hat{t}_q$ of investment (or discount) according to $q(t)$ of the amount $K = \sum_h K_h$ so as to have the same interest (or discount) obtainable as with the original operation on the n amounts K_1, $K_2,..., K_n$[23].

Under these assumptions the value $\hat{t}_q$ is univocally determined and is called the *average length* (or *average maturity,* choosing 0 as starting point) of the operation. This makes it possible, having fixed the starting time T_0 (i.e. the beginning of the investment or maturity of the amount to be discounted), to find the *average maturity* $T_1 = T_0 + \hat{t}_q$ (in accumulation) or $T_1 = T_0 - \hat{t}_q$ (in discounting).

The average length $\hat{t}_q$ depends on the choice of $q(t)$ and can be found by using (2.46) with $\hat{x}_q = \hat{t}_q$, $p_h = K_h$, $x_h = t_h$. Based on the financial meaning, in accumulation the interest obtained with the n investments based on the factor $q(t) = u(t)$ for the given times t_h is $\sum_{h=1}^{n} K_h \left[u(t_h) - 1\right]$, while that obtained with only one investment of $\sum_{h=1}^{n} K_h$ for the time t is $(\sum_{h=1}^{n} K_h)\left[u(t) - 1\right]$; these values are the same if $t = \hat{t}_q$. The position is analogous in discounting with $q(t) = v(t)$. This proves the following theorem.

THEOREM.– *In a financial operation of investment or discount of more than one amount with different durations* t_h, *the average length* $\hat{t}_q$ *is associative and coincides with the q-average of the lengths, weighted with the amounts* K_h, *where the transformation function q(t) coincides with the factor u(t) or v(t), respectively in an accumulation or discount operation.*

2.5.4. *Average index of return: average rate*

Let us consider the following problem of averaging. Let us invest for the same duration t the amounts C_1, C_2,...,C_n by using accumulation laws (for simplicity following the same regime) with different returns, based on the factors $u_1(t)$,..., $u_n(t)$. We want to find the accumulation factor that leaves the total interest unchanged for the same duration t. The solution is:

$$\hat{u}(t) = \sum_{h=1}^{n} C_h u_h(t) / \sum_{h=1}^{n} C_h \qquad (2.47)$$

23 The financial operations with n>2 amounts, which are called *complex*, will be discussed in Chapter 4.

The same result is obtained for a discount of length t, with factors $v_h(t)$ applied at maturity to the amounts M_h.

The following theorem is then proved.

THEOREM.– *Applying different exchange factors to different amounts for the same duration t, in accumulation or discount operations, the factor which does not change the returns is the arithmetic average of factors weighted with the amounts.*

If the accumulation factors $u_h(t)$ can be expressed $\forall t$ with the same invertible function $q(i_h;t)$ of the interest rates i_h $(h = 1,...,n)$, the *mean rate* $\hat{i}_q(t)$ is defined by

$$\hat{i}_q(t) = q^{-1}\left\{\sum_{h=1}^{n} C_h \; q(i_h;t) / \sum_{h=1}^{n} C_h\right\} \tag{2.48}$$

In the same way, the mean rate $\hat{d}_q(t)$ of the discount rates d_h $(h = 1,...,n)$ is defined for discounting, using in (2.48) the discount factors $q(d_h;t)$ instead of the accumulation factors $q(i_h;t)$ and the capitals at maturity M_h instead of the invested capitals C_h.

2.6. Uniform decomposable financial laws: exponential regime

We have already shown the practical importance of uniform financial laws. In relation to a *financial regime* – defined as a set of financial laws, based on a common feature and identified in the set by a parameter – it is important to investigate the existence and the properties of regimes which are decomposable and, at the same time, uniform. Hence, given that the financial laws

$$u(t) = e^{\delta t} \quad ; \quad v(t) = e^{-\theta t} \tag{2.49}$$

are called *exponential laws* and, by varying the parameters δ and θ, they constitute the *exponential regime* (often considered in symmetric hypothesis i.e. $\delta = \theta$), the following theorem holds.

THEOREM.– *The exponential regime, characterized by intensities constant in time, is the only one to be decomposable and uniform.*

Proof:

1) If $X \leq Y \leq Z$, given $Y - X = t_1$, $Z - Y = t_2$ and then $Z - X = t_1+t_2$, if $\approx$ is uniform, it follows that: $m(X,Y) = u(t_1)$; $m(Y,Z) = u(t_2)$; $m(X,Z) = u(t_1+t_2)$. From (2.27), because of decomposability, we obtain the following *characterization of decomposable and uniform interest laws:*

$$u(t_1)\,u(t_2) = u(t_1+t_2);\ \forall(t_1 \geq 0,\ t_2 \geq 0) \tag{2.50}$$

2) If $X \geq Y \geq Z$, given $X - Y = t_1$, $Y - Z = t_2$ and then $X - Z = t_1+t_2$, if $\approx$ is uniform, it follows that $a(X,Y) = v(t_1)$; $a(Y,Z) = v(t_2)$; $a(X,Z) = v(t_1+t_2)$. From (2.30), because of decomposability, we obtain the following *characterization of decomposable and uniform discount laws:*

$$v(t_1)\,v(t_2) = v(t_1+t_2);\ \forall(t_1 \geq 0,\ t_2 \geq 0) \tag{2.50'}$$

It is known that in the hypothesis that is valid for $u(t)$ and $v(t)$, the functional equations (2.50) and (2.50') are satisfied only by exponential functions; this proves the theorem[24].

If $\approx$ is uniform and strongly decomposable, and then symmetric, in (2.49) this results in $\delta = \theta$. The exchange factors then assume the form

$$g(t) = e^{\delta t},\ \forall t \in \Re \tag{2.51}$$

which satisfies (2.40). Equation (2.51), which is a particular example of (2.36), summarizes the exponential regime in the symmetric hypothesis and for all choices of δ identifies an exchange law that is strongly decomposable and uniform. Briefly, *the exchange exponential laws, and only those laws, correspond to indifferent relations that are equivalences that are uniform in time*[25].

24 The previous result can be deduced directly by observing that intensity depends on the initial time X and on the current time T, but if the law is decomposable the intensity must depend at the most on T, $\forall X$, while if the law is uniform the intensity must depend at the most on T-X. Then if the law is decomposable and uniform, both principles $\forall X$ being valid, it is necessary and sufficient that the intensity does not depend on any time variables, and it is constant; for compound accumulation laws, which in the continuous case lead to the exponential laws, see Chapter 3.

25 In the strongly decomposable and uniform law, which follows from a relation of uniform equivalence, the curves $S = \varphi(T)$, which correspond to the equivalence classes that are characterized, because of uniformity, by the further property of *invariance by translation*. Therefore, they follow by only one curve, which is translated continuously with a movement rigid and parallel to the time axes. The exponential curves $S = k\,e^{\delta T} = e^{\delta(T-T')}$ (where $T' = ln\ k/\delta$) are obtained, such that all the supplies equivalent to (T_0,S_0), so that all the supplies and only them, are represented by a point on the curve obtained putting $k = S_0\,e^{\delta T_0}$.

Chapter 3

Uniform Regimes in Financial Practice

3.1. Preliminary comments

In this chapter we will consider financial laws widely applied in the practice of investment and discount. One of their common features is the *uniformity in time*, so that the calculation of accumulated and discounted values depends only on the duration of the operation.

It is clear that the return of an operation is measured by a per period rate[1]. In a uniform law, if the rate remains constant for all given periods (we then talk about *flat structure*, in the field of all possible term structures of interest rate, concepts that we will consider later), it is clear that percentage returns remain unchanged wherever the operation is located in the time axis. This does not happen in financial markets, where to be at least approximately realistic, it would be necessary, in order to keep the simplicity of uniform law, to use the flat structure for a relatively short period. If this cannot be done because of the variability of returns with time, it is necessary to use the laws of two variables, characterized by per period rates changing with current time.

We will consider three couples of *uniform financial regimes*[2] that give rise to many infinite families of uniform laws of interest and discount identified by the return parameters.

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1 Let us recall that the per period rate measures the price for the availability of money in the given period.

2 When we distinguish between accumulation and discount, instead of “regime” we can talk about “couple of regimes” of interest and discount.

Because of uniformity we can use 0 as the initial time of the operation of investment (or the maturity of discounts), using small letters for duration (see section 2.5).

3.1.1. *Equivalent rates and intensities*

In any given financial regime of interest or discount, the problem of comparing rates or interest relative to different duration often arises. The following definitions hold.

Two per period interest (or discount) rates for different durations are said to be *equivalent* if they give rise to the same percentage of annual return and then, according to previous definitions, if they follow from the same financial law of interest (or discount).

Two intensities of interest (or discount) for different durations are said to be *equivalent* if they correspond to equivalent rates, and then if they follow from the same financial law of interest (or discount).

Two per period rates, one of interest for the length t' and the other of discount for the length t'', are said to be *equivalent* if they give rise to returns expressed by the annual interest and discount corresponding to conjugate laws. The equivalence for intensities follows from the equivalence for per period rates.

Rates and intensities for the regimes, discussed in the following text, are to be considered “initial”.

3.2. The regime of simple delayed interest (SDI)

Continuing the considerations in section 1.1, we observe that the simplest way to calculate interest on a loan amount C is to consider the interest I proportional both to the principal C and the duration $t = y\text{-}x$ (with no dependence on the initial time x) obtained as:

$$I = C\ i\ t \tag{3.1}$$

Parameter i, which is usually given in percentage form $r\% = r/100$, where $r = 100\ i$, measures the interest for a unitary capital and a unitary time interval. Assuming from now on (unless otherwise stated) that the year is the unit measure for time, i is called the *annual interest rate (delayed)*. The accumulated amount $M = C+I$ after time t is then given by

$$M = C\,(1 + i\,t) \tag{3.2}$$

Relations (3.1) and (3.2) for each choice of C, i, t, are characteristic of the *regime of simple delayed interest (SDI)*, in which interests are paid, or booked, only at the end of the loan of length t.

It follows from (3.1) and (3.2) that for accumulation laws in the SDI regime

– the *accumulation factor* for the length $t = y - x > 0$ is

$$u_t = 1 + i\,t \tag{3.3}$$

– the *per period interest rate* (1.3) for the length t is

$$i_t = i\,t \tag{3.4}$$

– the *per period interest intensity* for the length t is

$$j_t = i_t\,/t = i \tag{3.5}$$

independent of the duration and equal to the annual interest rate[3].

Relation (3.4) gives the equivalent per period rates to a given annual rate i. More generally, for durations that are not alike and different from a year, there exists proportionality between equivalent per period rates and lengths. In symbols, if $i_{t'}$ and $i_{t''}$ are the rates for the length t' and t'', they are equivalent if

$$i_{t'}\,/t' = i_{t''}\,/t'' = I \;\;^{4} \tag{3.6}$$

EXAMPLE 3.1.– If in the SDI regime the quarterly interest ($t' = 1/3$) is 5.25%, the equivalent semi-annual rate ($t''= 1/2$) is: 0.05253/2 = 0.07875, or 7.875%. They both give rise to the annual return $i = 0.1575$ which also measures the intensity.

---

3 Such equality, which is only numeric and not dimensional, is due to the fact that the interest rate is measured annually.

4 $\forall (t',t'')$ members in (3.6) are equal to the per period intensity and then the equivalence of rates gives the equivalence of the per period intensities between them and to i. “per period” refers to the period in which the return matures.

If the intensity changes during the lifetime of the loan, assuming the values $i^{(1)}, \ldots, i^{(n)}$ for the length $t_1,...,t_n$, (where $\sum_{s=1}^{n} t_s = t$), (3.2) can be generalized as:

$$M = C\,(1 + \sum_{s=1}^{n} i^{(s)} t_s) = C\,(1 + \bar{i}\, t) \tag{3.7}$$

where $\bar{i}$ is the arithmetic mean of intensities $i^{(s)}$ weighted with the length t_s.

EXAMPLE 3.2.– We invest €150,000 in the SDI regime obtaining for the first 3 months the annual interest (= intensity) of 5%, for the next 4 months interest of 5.5%, and for the next 6 months interest of 5.2%. The accumulated amount at the end is:

$$M = 150{,}000 \cdot [1 + (0.05 \cdot 3 + 0.055 \cdot 4 + 0.052 \cdot 6)/12] = €158{,}525$$

Exercises on the SDI regime

3.1

Calculate in the SDI regime the interest earned for 6 months on a principal of €1,500,000 at the annual rate of 8.25%.

A. Applying (3.1): I = €61,875

3.2

Calculate in the SDI regime (adopting bank year, with 360 days and each month having 30 days) the accumulated amount of a loan of €2,500,000 and of length 2 years, 6 months and 25 days at the annual rate of 9.5%.

A. Applying (3.2): M = €3,110,243

3.3

Calculate the accumulated amount as in Exercise 3.2, applying the varying interests: 9.5% in the 1st year, 10.5% in the 2nd year, 9% in the 3rd year.

A. Applying (3.3):

$$M = 2{,}500{,}000\ (1 + 0.095 + 0.105 + 0.09\ 205/360) = €3{,}128{,}125$$

The average annual interest for the operation is 0.25125·360/925 = 9.778%.

3.3. The regime of rational discount (RD)

From the SDI laws we can deduce the conjugated discount laws that give rise to reciprocal factors. They fall within the *rational discount (RD)* regime. The discounted value C, payable in x instead of the amount M at maturity $y>x$, is obtained from (3.2), resulting in

$$C = \frac{M}{1 + i\,t} \tag{3.8}$$

Giving the annual interest rate i of the conjugate SDI law, we obtain the RD law for which:

– the *discount factor* for the length t is

$$v_t = 1/(1 + i\,t) \tag{3.9}$$

– the *per period discount rate* for the length t is

$$d_t = \frac{i\,t}{1+i\,t} \tag{3.10}$$

– the *per period discount intensity* for the length t is

$$\rho_t = d_t\,/t = \frac{i}{1+i\,t} \tag{3.11}$$

If the annual discount rate $d = i/(1+i)$ is given[5], from which $i = d/(1-d)$, the previous quantities (3.9), (3.10) and (3.11) are obtained as a function of d:

$$v_t = \frac{1}{1-\frac{d\,t}{1-d}} = \frac{1-d}{1-d(1-t)} \tag{3.9'}$$

$$d_t = 1 - v_t = \frac{d\,t}{1-d(1-t)} \tag{3.10'}$$

5 This law has a trivial interpretation: i is the interest paid at the end of the year on the unitary capital, while d is the discount or the interest paid at the beginning of the year. Then d is the discounted value of i, the relation is obtained from (3.10) posing $t = 1$. It is useful to make use of such arguments based on the financial equivalence's principle.

$$\rho_t = d_t / t = \frac{d}{1 - d(1 - t)} \tag{3.11'}$$

Equation (3.10') gives the per period discount rate for the length t equivalent to the annual discount rate d.

EXAMPLE 3.3

1) If in the RD regime the delayed interest $i = 7.40\%$, using (3.10) we obtain the semi-annual, four-monthly, quarterly and monthly discount rates: 3.5680%, 2.4073%, 1.8164% and 0.6129%.

2) If in the RD regime the advance rate is $d = 6.80\%$, using (3.10') we obtain the semi-annual, four-monthly, quarterly and monthly discount rates: 3.5197%, 2.3743%, 1.7914% and 0.6043%.

3) If in the RD regime the four-monthly discount rate is $d_{1/3} = 2.15\%$, inverting (3.10) with $t = 1/3$ we obtain the equivalent annual rate $i = 3 \cdot 0.0215/0.9785 = 0.065917$. Then the equivalent semi-annual rate $i_{1/2}$ is obtained through (3.10) and it is 3.1907%.

The amount D of the discount on M and the discounted amount C as a function of d are given respectively by

$$D = M\, d_t = \frac{M\, d\, t}{1 - d(1 - t)} \; ; C = M\, v_t = \frac{M\,(1 - d)}{1 - d(1 - t)} \tag{3.12}$$

In Figure 3.1, for an SDI law, the graph of $I = I(t)$ and $M = M(t)$ are shown (see (3.1) and (3.2)) as a function of t. Figure 3.2 shows, for an RD law, the graph $C = C(t)$.

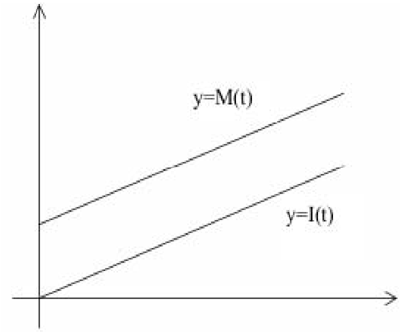

Figure 3.1. *Simple delayed interest*

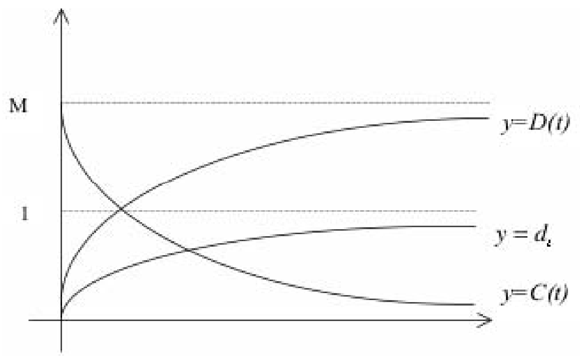

Figure 3.2. *Rational discount*

Comments

All linear laws, including conjugated laws, are used in general for short time periods. For the SDI laws, by indicating as g the number of days in the financial operation, the interest can be written as

$$I = \frac{C\,g}{T/i} \tag{3.13}$$

where T=360 if the "bank year" is used and T=365 if the "calendar year" is used. The numerator in (3.13) takes the name of "number" and the denominator that of "fixed dividend" because it depends only on the rate.

(3.13) is useful for finding the interest on a current account ruled by the SDI law in a given period (bank accounts are typical), because in order to calculate the interest in the considered period it is enough to sum the numbers relative to the days between two changes and divide by the fixed dividend.

Exercises on the RD regime

3.4

Calculate in the RD regime the discount to cash with a 3 month advance a credit of €30,000 at an annual interest rate of 6%.

A. By applying (3.10) and (3.12) with i = 0.06, t = 0.25, the following is obtained

$$d_t = 0.015/1.015 = 0.014778 = 1.4778\%$$

$$D = M\,d_t = €443.35$$

3.5

Calculate in the SD regime the discounted amount at 31 March of an amount of €160,000 payable on 31 August, following the calendar year and applying an annual discount rate of 6%.

A. By applying (3.12) with $d = 0.06$, $t = 153/365 = 0.419178$, $M = 160{,}000$, the following is obtained:

$$C = (160{,}000 \cdot 0.94)/[1-0.06(1-0.419178)] = €155{,}830.6.$$

3.4. The regime of simple discount (SD)

If in the choice of financial regime we consider the problem of discount and – with a symmetric argument that gave rise to the SDI laws – we want to find a regime that gives rise to proportionality between payment and terminal value and anticipation time, we obtain the *simple discount (SD)* regime. In the SD regime, the amount D of discount on a terminal value M for a length t is given by

$$D = M\, d\, t \tag{3.14}$$

Parameter d, which is usually given in percentage r%, where $r = 100\, d$, has the meaning of discount for unitary capital and for a unitary time interval and is called the *annual rate of discount*. The discounted amount $C = M\text{-}D$ at time x, corresponding to the amount M payable at maturity $y = x+t > x$, is given by

$$C = M\,(1 - d\, t) \tag{3.15}$$

From (3.14) and (3.15) it follows that, for a law in the SD regime,

– the *discount factor* for length t of advance is

$$v_t = 1 - d\, t \tag{3.16}$$

– the *per period discount rate* (1.4) for length t is

$$d_t = d \cdot t \tag{3.17}$$

– the *per period discount intensity* for length t is

$$\rho_t = d_t / t = d \tag{3.18}$$

independent of the length, and numerically equal to the given annual discount rate *d*.

As in the SDI regime with (3.4) and (3.5), (3.17) gives the per period discount rate equivalent to the annual rate *d*. More generally, there exists proportionality between equivalent per period rate and length, and

$$d_{t'} / t' = d_{t''} / t'' = d \tag{3.19}$$

results, so we have the independence of the discount intensity from length.

EXAMPLE 3.4.– If in the SD regime the bimonthly rate (t' = 1/6) is 1.25%, the equivalent semi-annual rate (t''= 1/2) is: 0.0125·6/2 = 0.0375 = 3.75%. Both give the percentage of advance annual return d = 7.50%.

Exercises on the SD regime

3.6

Let us assume that a bill of €3,500 has a deadline on 30 September of the year *T*. We ask for the discount at bank Z, in the SD regime at the annual rate of 7% with payment on 25 June of the same year. Not considering transaction costs, calculate the return.

A. Because of (3.15) it is given by

$$C = 3{,}500\ (1 - 0.07\ \frac{97}{360}) = €3{,}433.99.$$

3.7

It has been agreed on the anticipation at 20 May of the amount of €68,000 with maturity at 30 September of the same year, in the SD regime (using the calendar year) and fixing the four-monthly equivalent rate of 2.65%. Calculate the amount of discount.

A. The annual equivalent rate *d* is 0.0265·3 = 0.0795. Using (3.14):

$$D = 68{,}000 \cdot 0.0795\ \frac{133}{365} = €1{,}969.86$$

3.5. The regime of simple advance interest (SAI)

The interest law conjugated to the simple discount gives rise to the regime of *simple advance interest (SAI)*, which is also called the regime of *commercial interest.*

Using the annual advance interest rate d in an SAI law:

– the *accumulation factor* for length t is

$$u_t = 1 / (1 - d\,t) \tag{3.20}$$

i.e. inverse of the factor v_t defined in (3.16);

– the *per period interest rate (delayed)* for length t is[6]

$$i_t = \frac{d\,t}{1 - d\,t} \tag{3.21}$$

– the *interest intensity* for length t is

$$j_t = i_t / t = \frac{d}{1 - d\,t} \tag{3.22}$$

Multiplying (3.20) for a capital C invested in x the accumulated amount is obtained

$$M = C\,u_t \tag{3.20'}$$

at time $y = x+t > x$.

6 Equation (3.21) gives the per period interest rate equivalent to the advance annual rate d or to the delayed annual rate $i = d/(1-d)$ of the conjugate law.

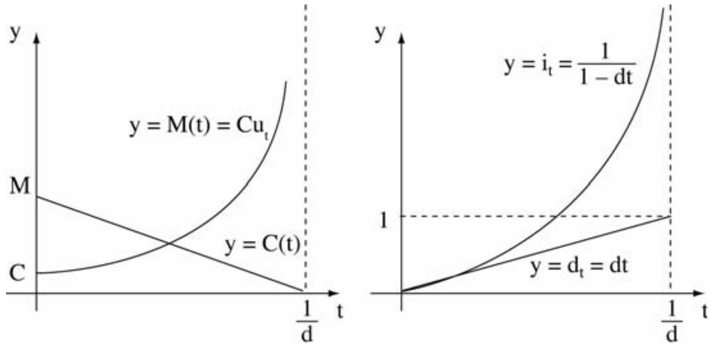

Figure 3.3. *a) Simple discount; b) simple advance interest*

EXAMPLE 3.5

1) In the SAI regime, given the advance rate d = 8.20%, the semi-annual, four-monthly, quarterly, monthly, etc., interest rate can be found using (3.21); 4.2753%, 2.8101%, 2.0929% and 0.6880% respectively are obtained.

2) In the SAI regime, given the delayed rate i = 9.50%, the corresponding rate d is 0.095/1.095 = 0.086758 = 8.6758%, and applying (3.21) the semi-annual, four-monthly, quarterly, monthly, etc., interest rate can be found; 4.5346%, 2.9781%, 2.2170% and 0.7282% respectively are obtained.

3) In the SAI regime, given the four-monthly interest rate $i_{1/3}$ = 2.35%, inverting (3.21) with t = 1/3 the equivalent annual rate d = 3·0.0235/1.0235 = 0.068881 can be found. Then the equivalent semi-annual rate $i_{1/2}$ can be found, using (3.21) to be 3.5669%.

Exercises on the SAI regime

3.8

Calculate the accumulated amount after 20 months of the investment of €120,000 in the SAI regime at the advance annual interest rate of 4.50%, and also the per period equivalent interest rate.

A. By applying (3.20) and (3.20') the following is obtained

$$M = 120{,}000/(1 - 0.045 \cdot 20/12) = €129{,}730.$$

The per period equivalent interest rate is calculated by (3.21) and the following is obtained:

$$0.075/(1 - 0.075) = 8.1081\%.$$

3.9

It is known that an 8 month discount operation in the SD regime at the annual rate $d = 6\%$ gives a discounted amount $C = €155{,}000$. Calculate:

– the capital at maturity;

– the per period discount rate;

– the per period interest rate in the conjugate law.

A. Given that the conjugate law to the applied SD law is an SAI:

– the capital at maturity is calculating using (3.20'):

$$M = 155{,}000/(1 - 0.06\cdot 8/12) = €161{,}458;$$

– the per period discount rate is $0.06\cdot 8/12 = 4\%$;

– the per period interest rate in the conjugate law is $0.04/1.04 = 4.1667\%$.

3.10

Consider the same problem as in Exercise 3.9 but with: $C = €155{,}000$, $d = 6\%$ and $t = 10.75$ (= 10y+9m).

A. The capital at maturity is $M = €436{,}620$, the per period discount rate is 64.50% and the interest rate of the SAI law is 181.69%: note that the spread between the two rates increases. Note that the critical length threshold $t = 1/d$, such that the delayed interest and the accumulated amount diverge, is in this case 1/0.06 years=16 years and 8 months.

3.6. Comments on the SDI, RD, SD and SAI uniform regimes

Each of the two couples of uniform financial regimes considered in sections 3.2 and 3.3 and in sections 3.4 and 3.5 is made of a regime with factors which are linear functions of the length and another regime, which includes the conjugate laws, with factors which are a rational function of the length (their graph is an equilateral hyperbola). We can summarize this by saying that such regimes are made of uniform linear laws and their conjugate.

Let us summarize further properties of and observations about such couples.

3.6.1. *Exchange factors (EF)*

Using the symbols in section 2.4 we indicate by $g(\tau)$ the exchange factor (EF) for the length with sign τ (accumulation if $\tau>0$, discount if $\tau<0$) and we put $t=|\tau|$. If the corresponding laws are conjugate, (2.41) holds; then, $\forall\tau$, $g(\tau)$ and $g(-\tau)$ are reciprocal.

If we consider a couple of SDI and RD conjugate laws, we have, with $\tau = t > 0$:

$$g(\tau) = 1+i\,\tau = 1+i\,t \text{ (SDI) } ; g(-\tau) = 1/g(\tau) = 1/(1+i\,t) \text{ (RD)}$$

If we consider a couple of SD and SAI conjugate laws, we have, with $\tau = -t < 0$:

$$g(\tau) = 1+d\,\tau = 1-d\,t \text{ (SD) } ; g(-\tau) = 1/g(\tau) = 1/(1-d\,t) \text{ (SAI)}$$

3.6.2. *Corrective operations*

We notice, in the example of uniform financial laws considered here, that the operative role is similar to an “offsetting entry” that the conjugate laws have. Indeed, if an investment of C has been agreed with an SDI (or SAI) law for the length t, which gives rise to M, and to cancel such an investment, instead of an offsetting entry, we can restore the previous situation by applying to M the corresponding RD (or SD) factor.

3.6.3. *Initial averaged intensities and instantaneous intensity*

As already mentioned in footnote 3, values (3.5), (3.11), (3.18) and (3.22) are initial averaged intensities in the interval $(0,t)$ for investment or anticipation. The instantaneous intensity[7] in t (time from investment or time to maturity) has another meaning: it is obtained as a limit case of the continuing intensity defined in section 2.3.

Recalling that in the interest laws the instantaneous intensity are obtained from the logarithmic derivatives with respect to t of the exchange factors, the following expression for the instantaneous intensity in t can be easily deduced:

a) SDI (rate i):	$\delta_t = i/(1 + it)$	decreasing with t
RD (rate i'):	$\theta_t = i'/(1 + i't)$	decreasing with t
b) SAI (rate d):	$\delta_t = d/(1 - dt)$	increasing with t

7 Summarizing the definition in Chapter 2, in the accumulation of an investment made in x=0 the instantaneous intensity in y=t>0 is the limit of the per period intensity between y and $y+\Delta y$, with Δy>0, while in the discount of capital with maturity in x=0 the instantaneous intensity at time y=-t<0 is the limit of the per period intensity between y and $y+\Delta y$, with Δy<0.

SD (rate d') $\theta_t = d'/(1 - d't)$ increasing with t[8]

If in a) $i = i'$ or in b) $d = d'$, the corresponding laws are conjugate to each other.

3.6.4. *Average length in the linear law and their conjugates*

By applying the considerations in sections 2.5.2 and 2.5.3, it is easily verified that:

– in the *SDI regime:* the factor $u_t = 1+it$ is linear and then the average length $\hat{t}_q$ is the arithmetic mean of the investment length t_h, weighted with the amounts C_h. It can be verified that the equality between the interests $\sum_{h=1}^{n} C_h i t_h$, obtained with investments on times t_h, (h=1,…,n), and the interests $it\sum_{h=1}^{n} C_h$ obtained with only one investment for time t, can be obtained if and only if $t = \sum_{h=1}^{n} C_h t_h / \sum_{h=1}^{n} C_h$;

– in the *SD regime:* the factor $v_t = 1-dt$ is linear and then the average length $\hat{t}_q$ is the arithmetic mean of the discount length t_h, weighted with the amounts M_h;

– in the *SAI regime:* $\hat{t}_q$ is an associative mean of the length t_h, such that $1-d\hat{t}_q$ is the harmonic mean of the factors $1-dt_h$, weighted with the amounts C_h;

– in the *RD regime:* $\hat{t}_q$ is an associative mean of the length t_h, such that $1+i\hat{t}_q$ is the harmonic mean of the factors $1+it_h$, weighted with the amounts M_h.

3.6.5. *Average rates in linear law and their conjugated laws*

Referring to the symbols introduced in section 2.5.4 and using the same arguments used for the average length, we can deduce that:

– in the *regime SDI:* the average rate is the arithmetic mean of the rates i_h with weights given by the used amounts C_h;

– in the *regime SD:* the average rate is the arithmetic mean of the rates d_h with weights given by the capital at maturity M_h;

---

8 The formal coincidence, due to the analytic properties of the exchange factors, of formulae (3.11) and (3.22) of initial intensity in the RD and SAI regimes with the respective instantaneous intensity does not change the difference between initial intensity, which is a domain function, and instantaneous intensity, which is a point function.

– in their conjugate regime, the average rates are obtained as associative means given by harmonic mean of the exchange factors, i.e. $1+i_h t$ with weights C_h in the RD regime and $1-d_h t$ with weights M_h in the SAI regime.

3.7. The compound interest regime

3.7.1. *Conversion of interests*

Let us reconsider the interest formation with an SDI law, which reflects a spontaneous propensity of the market due to the double proportionality of the interest with respect to the amount of the invested capital and also the length of investment, as (3.1) shows. However, we observe that if the interest is added to the principal at the end of the operation, then there is an asynchrony between the position of the *lender*, which gives his supply continuously (making it possible that other persons use his capital, depriving himself of its profitable use), and the *borrower*, who delays his payment until maturity. Such asynchrony, prejudicial for the lender, is greater the longer the time of investment. Thus, an investor can accept this regime, with equal return rates, only in the short-term (usually not longer than one year)[9].

Briefly, with the SDI regime the earned interest remains unprofitable until the end of the operation. Concerning SDI we can imagine the presence of two accounts: on the first account we book the principal C, giving interest with flow $C\,i$ and then with amount $C\,i\,\Delta t$ for every time of length Δt. However, such interest is booked on the second unprofitable account. At the end of the operation of length t the sums on both accounts, given by C and $I = Cit$, are withdrawn and transferred to the creditor[10]. It is then preferable to consider financial regimes that realize synchrony between the parties making the earned interest profitable. The transferring of earned interest between the unprofitable interest/account and the profitable principal/account, without having to wait until the capital is no longer being used, is called *interest conversion*. When these amounts are available for the creditor, he will be able to cash and use them elsewhere (and then the profitable capital in the original operation will remain unchanged) or he can add them to the capital (giving

9 A rough solution to the damage connected with the asynchrony can be obtained easily with an increment of the interest rate. Furthermore, the fair increment would increase with the length.

10 The SDI process is analogous to those of the following hydraulic scheme. A first tank holds a constant volume C of water; since time 0, by means of an open input tap some water flows into a second tank with a closed output tap; a gear is applied so that the inflow is proportional to C on the basis of the factor i, so we obtain a flow Ci. At time t the output tap is opened and the contents Cit of the second tank are poured into the first tank. All the water $C(1+it)$ is soon withdrawn.

more interest) in the same operation[11]. In this second case, a movement of money is not needed and it is enough to credit the interest in the same profitable principal/account.

It is obvious that intermediate conversions increase the amount, i.e. the lender credit, in t, as is shown below (considering, for simplicity, only one conversion). Let a principal C be invested at time 0 at annual rate i for the length t, with the assumption that the interest is formed using an SDI law but let the interest be converted at time $t_1 = t - t_2 < t$ and keep it invested at the same rate until t. Adding to C the interest Cit_1 earned at time t_1, the amount with the added interest becomes $M(t_1) = C(1+it_1)$ and the amount at term time t reaches the level: $M(t) = C(1+it_1)(1+it_2) = C[(1+it) + i^2 t_1 t_2]$. It is thus proved that an intermediate conversion increases, at the same interest, the final amount: the simple interest for time t_2 is added to the interest Cit_1 earned in the time t_1.

The compound interest regime is characterized by the conversion of simple interests to profitable capital during the operation.

Such a regime can be applied in two ways:

1) *the conversion is made with per period terms*, or more generally *in the discrete scheme*; this is the method used in bank and commercial practice, with conversion at the end of the calendar year, calendar quarter, etc. We will then talk about *discretely compound interest (DCI)*;

2) *the conversion is made continuously* over time, only in this case there is a perfect synchrony between the parties in the contract. We will then talk about *continuously compound interest (CCI).*

3.7.2. *The regime of discretely compound interest (DCI)*

A general approach to the DCI laws leads to the following scheme: the use of principal C for the length $t = t_1+t_2+ \ldots +t_n$ (using a year as measure of time, unless otherwise stated), such that in the sub-period of length t_s the intensities $i^{(s)}$, ($s = 1, \ldots, n$) are used, and at the term of each sub-period the conversion is made. We then obtain the amount in t, given by

$$M(t) = C \prod_{s=1}^{n} (1+i^{(s)} t_s) \tag{3.23}$$

11 The decision will depend on convenience and alternative uses and we will talk about this when discussing investment choices (see Chapter 4).

The product gives the accumulation factor from 0 to t in the DCI law[12].

Let us now consider some particular cases of discrete conversion that are relevant for banking and business application.

Accumulation with annual conversion

Assume in (3.23): $t_s = 1$, $\forall s$, then $t=n \in \mathcal{N}$ ($\mathcal{N}$ = set of natural numbers); $i^{(s)}$ = constant = i. A particular case of this model is for the conversion of interests at the end of the solar year. When we have only one payment C, made at the beginning of first year, the amount at the end of n^{th} year is given by

$$M(n) = C\,(1 + i)^n \tag{3.24}$$

EXAMPLE 3.6.– If €1,263,500 is banked at the beginning of 1998 in a bank account ruled by compound interests, annual conversion, at the annual rate of 4.35%, the terminal value at the 6th year (soon after the 6th conversion) is €1,631,285.

Mixed accumulation with annual conversion

With the hypothesis that the conversion is done on 31 December of each year, the amount $M(t)$ for the use of a principal C for a length t, in between n+2 years (i.e. the final part f_1 of the first year, other n years and the initial part f_2 of the $(n + 2)^{\text{th}}$ year, then $t = f_1 + n + f_2 < n+2$) is given by

$$M(t) = C\,(1 + i\,f_1)\,(1 + i)^n\,(1 + i\,f_2) \tag{3.25}$$

where the simple interest law is applied for a fraction of a year.

To maintain a bank account in which banking and withdrawal are made, we can apply the *direct method* making the algebraic sum of the relative amounts calculated using (3.25) from the time of movement until the common last time t. However, the *scalar method* is more often used, in which the "numbers" are found between subsequent balances in each calendar year and the conversion of interest is made at the end of the year or when the bank account is closed.

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12 Recalling that, due to the conversions, parameters $i^{(s)}$ are intensities and not also annual return rates, investing in 0 the principal C, the amount obtained after the 1st conversion is: $M(t_1) = C(1+i^{(1)}t_1)$ and becomes profitable with intensity $i^{(2)}$; then at the 2nd conversion we obtain: $M(t_2) = C(1+i^{(1)}t_1)(1+i^{(2)}t_2)$. And then, at the n^{th} conversion i.e. at time t, we obtain the result specified in (3.23).

EXAMPLE 3.7.– On 4 September 1996, Mr. John banks €23,500 on a bank account ruled by 4.65% per year, with mixed accumulation and annual conversion. The amount on 20 October 1999 is

$$M = 23500\ (1{+}0.0465{\cdot}118/360)\ (1.0465)^2\ (1{+}0.0465{\cdot}292/360) = €27{,}114.$$

Accumulation with fractional conversion

Let $\forall s$: $t_s = 1/m$ in (3.23), where m-1 ∈ $\mathcal{N}$; $i^{(s)}$ = constant = $j(m)$. We then have the conversion m times per year, where m is called *frequency* of the conversion of interest in profitable capital, indicating j as a function of the conversion frequency. This is the *intensity* parameter, where $K\ j(m)\ \Delta t$ is the interest for the profitable capital K for the length $\Delta t<1/m$. Parameter $j(m)$ is sometimes called the *nominal annual rate, convertible m* times a year or, more briefly, the *annual m/convertible rate.* The fractional conversion is usually used with the frequencies m = 2, 3, 4, 6, 12.

EXAMPLE 3.8.– If m = 4 (= quarterly conversion) and $j(4)$ = 8%/year, the interest on the capital K is $0.08{\cdot}K\Delta t$ for a period $\Delta t \leq 1/4$ and for a quarter the interest is $0.08{\cdot}0.25\ K = 0.02K$. Using C for the capital at the beginning of the year, the amount at the end of the year (after 4 conversions) is

$$C\ (1 + 0.02)^4 = C{\cdot}1.08243216$$

where the effective annual return is measured by i = 8.243216% > 8%.

In the fractional accumulation with frequency m (o *m-fractionated*), if i is the effective annual rate, then

– the *accumulation factor* for the length $1/m$ is: $u_{1/m} = 1+i_{1/m}$;

– the *per period interest rate* for the length $1/m$ is:

$$i_{1/m} = (1 + i)^{1/m} - 1 \tag{3.26}$$

which is found from the equivalence relation between rates: $(1+i_{1/m})^m = 1+i$;

– the *per period interest intensity* for the duration $1/m$ is:

$$j(m) = m\ i_{1/m} \tag{3.26'}$$

and relation $i > j(m)$ can be deduced, if m-1 ∈ $\mathcal{N}$.

EXAMPLE 3.9.– We want to receive a return measure by the annual rate $i = 6.45\%$ with a prefixed use with monthly conversion. Then the monthly rate is $i_{1/12} = 0.522\%$, the corresponding intensity is $j(12) = 6.266802\%$/year and the monthly accumulation factor is $u_{1/12} = 1.00522$.

Mixed accumulation with conversion m times per year

Using the assumption that the conversion is made at the end of each m^{th} of the solar year, if $f_1 < 1/m$ measures the interval between the investment and the first conversion and $f_2 < 1/m$ the interval between the last conversion and the end of the operation, by a generalization of (3.25) and using $t = f_1 + k/m + f_2$, we obtain:

$$M(t) = C(1 + j(m) f_1)\,(1 + j(m)/m)^k\,(1 + j(m) f_2) \tag{3.27}$$

EXAMPLE 3.10.– On 4 September 1996, Mr. Tizio withdraws €23,500 from a bank account ruled by a nominal 4-convertible rate = 4.65%/year, with mixed accumulation quarterly converted. The debt on 20 October 1999 is

$$M = 23500\cdot(1{+}0.0465\cdot 26/360)\cdot(1{+}0.0465/4)^{12}\cdot(1{+}0.0465\cdot 20/360) = €27{,}157.$$

Note: comparing this with the results in Example 3.7 using equal time and rate, the increase of the amount, which goes from €27,114 to €27,157 due to the more frequently interest conversion, will be noticed.

Equivalent rate and intensity in the fractional conversion

Two compound accumulation laws, the first with annual conversion at rate i and the second with m-fractional conversion at per period rate $i_{1/m}$, are called *equivalent* if they give the same annual return. This happens if i and $i_{1/m}$ satisfy (3.26); in this case such rates are said to be *equivalent.*

More generally, two compound accumulation laws, the first with m'-fractional conversion at rate $i_{1/m'}$ and the second with m''-fractional conversion at rate $i_{1/m''}$, are called, for the same reason, *equivalent* if

$$(1+i_{1/m'})^{m'} = (1+i_{1/m''})^{m''} \tag{3.28}$$

and then $i_{1/m'}$ and $i_{1/m''}$ are called *per period equivalent rates*.

An analogous definition for the intensities can be given. Due to (3.26') and (3.28), if

$$(1+\frac{j(m')}{m'})^{m'} = (1+\frac{j(m")}{m"})^{m"} \tag{3.29}$$

is true, then $j(m')$ and $j(m")$ are *equivalent intensities.*

Exercise 3.11

Calculate the per period rates and intensities for the annual, semi-annual, four-monthly, quarterly, bimonthly, monthly, weekly, daily conversion frequencies in the compound regime at the annual rate of 5.27% and the quarterly rate of 1.36%, using Excel.

A. The given frequencies are: m = 1, 2, 3, 4, 6, 12, 52, 360. To obtain the solution we will use an Excel spreadsheet, which is particularly useful for calculating formulae with repeated structures (here varying m), using the “copy and paste” function. This is because in Excel the “copy” operation does not refer to the number in the cell but to the formula written in this cell, which works on the values written in other cells; besides, by “pasting” into another cell the formula is “translated”, i.e. it works on the cells corresponding by translation (unless the command $ is used). For example, if C6 includes a formula depending on the contents of the cells A9 and B10, by copying C6 and pasting in C9, the result is the value of the same formula applied on the contents of the cells A12 and B13: indeed, there is a three cell translation downwards. Consequently, changing data on the cells, all the results are instantaneously changed, which is very advantageous. This should be remembered for all exercises in this book that use Excel.

Using an Excel spreadsheet, using such techniques we will find the solutions based on (3.26), (3.26') and (3.28), (3.29) starting from the given rates 5.27% (annual) and 1.36% (quarterly). The following table is obtained.

CALCULATION OF EQUIVALENT RATES AND INTENSITIES

	Equivalent to rate i = 5.27%		Equivalent to rate $i_{1/4}$ = 1.36%	
m	$i_{1/m}$	*j(m)*	$i_{1/m}$	*j(m)*
1	5.270%	0.05270	5.552%	0.05552
2	2.601%	0.05202	2.738%	0.05477
3	1.727%	0.05180	1.817%	0.05452
4	1.292%	0.05169	1.360%	0.05440
6	0.860%	0.05158	0.905%	0.05428
12	0.429%	0.05147	0.451%	0.05416
52	0.099%	0.05138	0.104%	0.05406
360	0.014%	0.05136	0.015%	0.05404

Table 3.1. *Equivalent rates and intensities*

The Excel instructions are as follows. The first three rows are used for data and titles; D3: 0.0527; G3: 0.0136. The 4th row is empty. The 5th row has the column titles; from the 6th to 13th rows:

– column A (frequency): given frequency;

– column B: empty;

– column C (equivalent rates): C6:= (1+$D$3)^(1/A6)-1; copy C6, then paste on C7 to C13;

– column D (equivalent intensity) D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (equivalent rate): F6:= (1+$G$3)^(4/A6)-1; copy F6, then paste on F7 to F13;

– column G (equivalent intensity): G6:= A6*F6; copy D6, then paste on G7 to G13.

Note: rates are expressed in %; intensities are expressed in unitary form.

Effects of frequency variations

It is instructive to assess the effects on returns connected with a change of the conversion frequency, observing that:

a) if the intensity j, i.e. the flow of interest accruing divided by the updated principal, is fixed (constant in the time), the annual rate i that measures the return of the unitary principal after one year of investment with m equally spaced conversions is given by

$$i = f(j,m) = (1+\frac{j}{m})^m - 1 \tag{3.30}$$

which is a sequence increasing with m;

b) if the annual rate i, i.e. the return of a unitary principal after one year of investment with m equally spaced conversions, is fixed, the intensity j (constant in the time) is given by

$$j = g(i,m) = m[(1+i)^{1/m} - 1] \tag{3.31}$$

which is a sequence decreasing with m.

EXAMPLE 3.11

a) Let the intensity be j = 12% per year, i.e. it is established that within each interval between two subsequent conversions, the interest, which is still unprofitable, on the profitable sum $S(t)$ is earned according to the flow $0.12 \cdot S(t)$; it is then the product of such flow and the considered length Δt. The interest earned after one year is an increasing function of the number m of conversions, each done after $1/m$ of a year, and is given by the product Si, where $i = f(j,m)$ is given, for the usual choices of m, by the values in the 3rd column of Table 3.2 below, obtained using (3.30).

b) Let the delayed annual interest be i = 12%; it is then established that, whatever the number m of conversions in one year, the intensity j (constant in the time) is such to assure at the end of the year of investment and interest return equal to $0.12 \cdot C$, where C is the principal. With the increase of m the intensity $j = g(i,m)$ decreases and assumes, for the usual choices of m, the values in the 4th column in Table 3.2, obtained using (3.31).

The calculations are made on an Excel spreadsheet.

		Problem a) j = 0.12	*Problem b) i = 0.12*
Conversion frequency	*M*	*i, given j*	*j, given i*
Annual	1	0.120000	0.120000
Semi-annual	2	0.123600	0.116601
Four-monthly	3	0.124864	0.115496
Quarterly	4	0.125509	0.114949
Bimonthly	6	0.126162	0.114406
Monthly	12	0.126825	0.113866
Weekly	52	0.127341	0.113452
Daily	360	0.127474	0.113347

Table 3.2. *Correspondence between i and j*

The Excel instructions are as follows. The first three rows are used for data and titles; C3: 0.12; D3: 0.12. The 4th row is empty. The 5th row has column titles. From the 6th to 13th rows:

– column A: conversion frequency;

– column B (frequency): given frequency;

– column C (equiv. annual rat.) C6:= (1+C$3/B6)^B6-1; copy C6, then paste on C7 to C13;

– column D (equiv. intensity) D6:= B6*((1+D$3)^(1/B6)-1); copy D6, then paste on D7 to D13.

3.7.3. *The regime of continuously compound interest (CCI)*

We showed in section 3.7.1 that perfect synchrony of the supplies between the two contracting parties of a financial investment is obtained only with the *CCI* regime, which makes the accumulation with continuous conversion of interest that is accrued during the use of the capital[13]. The mathematical calculations have the difficulty of considering infinitesimal times, and it is necessary to use infinitesimal calculus. We will keep the hypothesis of per period rates and intensities constant in time.

We can consider two different ways to undertake the calculation:

1) the first is to assume the continuous conversion as the limit case of the fractional conversion when the frequency diverges (i.e. $m \to +\infty$);

2) the second, having general validity and also being suitable to describe the eventuality of time variable returns, consists of a direct approach to the formation of interest and their conversion, described with differential calculus. This is spontaneously related, in the case of constant in time returns, to the *exponential regime* described in section 2.6. We showed that the laws for such a regime, and only these, satisfy the properties of decomposability (and of strong decomposability, if we consider the couple of conjugate interest and discount laws) and uniformity in time.

The *first way* brings us to consider the limit of (3.30) and (3.31) with diverging m. By the limit of (3.30), given the instantaneous intensity (constant over time) of return, denoted by δ, we obtain the equivalent annual rate i, which is also the upper bound for $m = 1$, of the intensities $j(m)$ referred to the fractional conversion (according to the convexity of $e^{\delta t}$). By the limit of (3.31), given the annual rate i, we obtain the equivalent instantaneous intensity δ, lower bound for $m \to +\infty$ of the intensities $j(m)$. Using formulae

$$\begin{cases} i = \lim_{m \to \infty} f(\delta, m) = \lim_{m \to \infty} \left\{ \left[1 + \frac{\delta}{m} \right]^m - 1 \right\} = e^{\delta} - 1 \\ \delta = \lim_{m \to \infty} g(i, m) = \lim_{m \to \infty} \frac{(1+i)^{1/m} - 1}{1/m} = ln(1+i) \end{cases} \tag{3.30'}$$

EXAMPLE 3.12.– By using the data in Table 3.2, given the constant intensity $j = 0.12$ and taking the limit $m \to +\infty$, it is calculated that in continuous accumulation $i = f(0.12, +\infty) = 0.1274969$ holds. Instead, using the effective annual rate $i = 0.12$, it is calculated that in continuous accumulation the instantaneous intensity $\delta = g(0.12, +\infty) = 0.1133287$ holds. By comparing these results with the last

13 In the hydraulic analogy of footnote 10, in continuous accumulation the second tank always has the input and output taps open, so that the “drops” of interest just formed go to the first tank and the second tank is almost always empty.

row of Table 3.2, it can be seen that the daily values (m = 360) are a good approximation of the continuous conversion's values.

An annual time horizon is not needed to define fractional and continuous accumulation. More generally, accumulating in the interval $[0,T]$ using the intensity j, the amount in T corresponding to the principal C invested in 0 with equally spaced conversions in $[0,T]$ is:

$$M(T) = C(1 + j\frac{T}{m})^m \tag{3.32}$$

and taking the limit for $m \to +\infty$, if the intensity $j(i,m)$ (varying with m so that i remains unchanged) converges to a real value indicated with δ, in CCI the following is obtained[14]

$$M(T) = \lim_{m\to\infty} \left\{ C(1 + j(i,m)\frac{T}{m})^{m} \right\} = \tag{3.33}$$

$$= C \lim_{m\to\infty} \left\{ (1 + \frac{j(i,m)T}{m})^{\frac{m}{j(i,m)T}} \right\}^{j(i,m)T} = C\, e^{j(i,+\infty)\,T} = C\, e^{\delta T}$$

The *second way* formalizes the continuous conversion with constant rate. It follows from the following postulates:

– the *linearity*, that is, the proportionality between interests flow and the principal that generates them;

– the *circularity*, that is, the immediate and continuous transferring of earned interests to the profitable fund that generates them.

14 This formulation of continuous accumulation, intended as a limit of the fractional accumulation, introduces the restriction of all equal subdivision intervals. Furthermore, in this limit we need the convergence of the intensity as a function of the fractioning. This last property exists in both cases examined in the table with varying m:

– when in the different fractioning situations the annual rate i (or a given per period rate $i_{1/m}$) is kept unchanged because in such a hypothesis the intensity, being decreasing, converges to $\delta = \ln(1+i) = \ln(1+i_{1/m})^m\}$;

– when the intensity does not change.

From these follows the equality between the amount's increment between t and $t+dt$, approximated by $dM(t)=M'(t)dt$, and the infinitesimal interest $\delta \cdot M(t) \cdot dt$. Then the simple differential equation (which is linear homogenous of the 1st order and with separable variables) is derived

$$M'(t) = \delta\, M(t) \tag{3.34}$$

for which the particular solution, relative to the condition $M(0) = C$, is

$$M(t) = C\, e^{\delta t},\ \forall t \in [0,T]\ ^{15} \tag{3.35}$$

Equation (3.34) can be obtained with more details from the following considerations. Given the constant intensity $\delta > 0$, investing C at time 0 and without any interest conversion, at time T the interest is δC and if at that time the interest is added to the principal, the amount $M(t)$ becomes $C(1+\delta T)$. This SDI scheme satisfies linearity but not circularity, in the time interval $[0,T]$, where circularity instead implies that in the infinitesimal interval dt between times t and $t+dt$ in $[0,T]$, the amount is increased by the earned interest, expressed by $\delta \cdot M(t)dt + o(dt)$, where $o(dt)$ represents an infinitesimal error of order greater than dt. The following differential relation holds, $\forall t \in [0,T]$

$$M(t+dt) = M(t) + \delta\, M(t)dt + o(dt) \tag{3.34'}$$

which gives the amount, originated by the principal C invested in $t=0$ and without any other financial flow, as a function of t that is continuous and differentiable $\forall t > 0$. Taking the limit for $dt \to 0$ and taking into account that $\lim o(dt)/dt = 0$, (3.34) is obtained.

It is obvious that such a financial mechanism, based on linearity and circularity, realizes the *CCI regime with constant rate,* that, taking into account (2.50), is

15 Considering, for the sake of simplicity, the unitary capital invested at time 0, the accumulated amount in t without previous conversion (then only due to linearity) is: $M_1(t) = 1+\delta t$, while with previous continuous conversion (then also due to circularity) it is

$$M_\infty(t) = e^{\delta t} = 1 + \delta t + \frac{\delta^2}{2} t^2 + \frac{\delta^3}{3!} t^3 + \ldots + \frac{\delta^n}{n!} t^n + \ldots > 1 + \delta t = M_1(t)$$

which shows that $M_\infty(t) > M_1(t)$ and also $M_1(t)$ is the linear approximation in $t=0$ of $M_\infty(t)$.

equivalent to the accumulation with the *exponential regime*[16]. A law of such a regime, and which is called an *exponential law*, applicable in the interval $[0,T]$, given the annual rate i or the per period rate $i_{1/m}$, is obtained from (3.35) using (see footnote 15)

$$\delta = \ln (1+i) = m \ln (1+i_{1/m}). \tag{3.31'}$$

If the intensity δ is given, the following inverse formulae hold

$$i = e^{\delta} - 1 \quad ; \quad i_{1/m} = e^{\delta / m} - 1. \tag{3.31''}$$

Then, given the annual rate i, (3.35) can be written as

$$M(t) = C\,(1+i)^t,\ t \geq 0 \tag{3.35'}$$

It follows from (3.35) and (3.35') that for the accumulation laws in the CCI regime[17]:

– the *accumulation factor* for the length $t = y\text{-}x > 0$ is

$$u_t = (1+i)^t = e^{\delta t} \tag{3.36}$$

– the *per period interest rate* for the length t is

$$i_t = (1+i)^t - 1 = e^{\delta t} - 1 \tag{3.37}$$

– the *per period interest intensity* for the length t is

$$j_t = i_t / t = [(1+i)^t - 1]/\, t = (e^{\delta t} - 1)/t \tag{3.38}$$

Exercise 3.12

Let us consider an investment of €4,550 in the CCI regime at the annual rate of 6.78% for 5 months and 18 days. Calculate the accumulation factor, the per period rate and the corresponding intensity, the instantaneous intensity and the earned interest at time t.

---

16 Let us consider here the exponential regime in the formation of interests for problems of accumulation with constant rate (or with “flat structure”). For the analogous discount regime see section 3.8.

17 For these and other questions of financial mathematics, see S. Kellison (1991), Irwin; Poncet, Portait, Hayart (1993).

A. Adopting the bank year, this results in:

$t = 5/12+18/360 = 0.46667$; then:

$u_t = 1.0678^{0.466667} = 1.031087$; $i_t = 1.067^{0.466667} - 1 = 0.031087$;

$j_t = 0.031087/0.466667 = 0.066615$/year;

$\delta = \ln 1.0678 = 0.0656$/year; $I_t = 4550 \cdot 0.031087 =$ €141.45.

Exercise 3.13

Let us consider the investment of the previous exercise with the same interest rate but for a length of 2 years, 3 months and 7 days. Calculate the accumulation factor, the per period rate, the corresponding intensity and the interest earned at time t.

A. $t = 2+3/12+7/360 = 2.269444$ holds, and then:

$u_t = 1.0678^{2.269444} = 1.161023$; $i_t = 1.0678^{2.269444} - 1 = 0.161023$;

$j_t = 1.161023/2.269444 = 0.070953$; $I_t = 4550 \cdot 0.160123 =$ €732.65.

The problem of equivalent rate and intensities in the CCI regime is resolved by a generalization of (3.28) and (3.29), which is useable only if we consider natural numbers >1, since now we have to assume $t \in \Re^+$. Two *per period rates* for different periods t' and t'' are *equivalent* if, expressed as annual rates in the aforementioned regime, they give the same return in terms of rate i or instantaneous intensity δ. Two *per period intensities* are equivalent if they correspond to equivalent rates. In formulae, to have equivalence, the rates $i_{t'}$ and $i_{t''}$ must satisfy

$$(1+i_{t'})^{1/t'} = (1+i_{t''})^{1/t''} \; (= 1+i = e^{\delta}) \tag{3.39}$$

and the intensities $j_{t'}$ and $j_{t''}$ must satisfy

$$(1+ j_{t'}\, t')^{1/t'} = (1+j_{t''}\, t'')^{1/t''} \; (= 1+i = e^{\delta}) \tag{3.40}$$

Exercise 3.14

Let us consider an investment of €156,000 in the CCI regime for the length t' = (7m+24d) = 0.651620 year at the per period rate 0.0371. Calculate: 1) the corresponding intensity; 2) the rates and intensities equivalent to the previous ones, extending the investment for the length t'' = (1y+4m+17d)=1.380556 year; 3) the interest earned after one year of investment.

A. Using (3.39) and (3.40) the following is obtained:

1) $j_{t'} = 0.0371/0.651620 = 0.056935$/year;

2) $i_{t''} = (1+i_{t'})^{t''/t'} - 1 = 1.0371^{2.118652} - 1 = 0.080235$; the intensity $j(t'')$ follows from (3.40) or (3.38), which leads to

$$j_{t''} = [\{1+t' j_{t'}\}^{t''/t'}-1]/t'' = [\{1 + 0.056935\cdot 0.651620\}^{2.118652}-1]/1.380556$$
$$= 0.058118; \text{ or } j_{t''} = i_{t''}/t'' = 0.080235 / 1.380556 = 0.058118;$$

3) by inverting (3.37) the equivalent annual rate is obtained $i = 1.0371^{1/0.651620} - 1 = 0.057496$, and then the interest for one year of investment is

$$I_1 = 156{,}000\cdot 0.057496 = €8{,}969.38.$$

3.8. The regime of continuously compound discount (CCD)

We now consider the compound discount, only with regard to the *continuously compound discount (CCD)* (or *exponential*) regime which gives rise to a family of discount laws conjugated to those of CCI that can be specified by the instantaneous discount intensity θ. The function *C(t) = discount value of M for effect of an anticipation of length t* verifies the differential relation:

$$C(t+dt) = C(t) - \theta C(t)dt - o(dt) \qquad (3.41)$$

(where $\theta C(t)dt$ is the *elementary discount* between t and $t+dt$) under the initial condition $C(0) = M$. Then $C(t)$ is expressed by

$$C(t) = M\, e^{-\theta t} \qquad (3.42)$$

Recalling (3.35), it is obvious that the law of exponential discount in (3.38) with parameter θ is conjugated to the law of exponential accumulation with parameter δ if and only if θ = δ.

Working with a CCD law characterized by the intensity θ on *annual interval* (t =1) or *fraction of year* (t =1/m), it follows from (3.42) that the *annual discount factor* v, the *per period discount factor* $v_{1/m}$, the *annual discount rate* d and the *per period discount rate* $d_{1/m}$ for time 1/m are given respectively by:

$$v = C(1)/M = e^{-\theta} \;;\; v_{1/m} = C(1/m)/M = e^{-\theta/m} \qquad (3.43)$$

$$d = \{M - C(1)\}/M = 1 - v = 1 - e^{-\theta} \qquad (3.44)$$

$$d_{1/m} = \{M - C(1/m)\}/M = 1 - e^{-\theta/m} \tag{3.45}$$

resulting in the following equivalence relations on discount rates for different frequencies

$$(1 - d_{1/m'})^{m'} = (1 - d_{1/m})^{m} = 1 - d \tag{3.46}$$

In addition, the definition of *per period discount intensity* relative to the frequency *m* (also called *nominal discount rate convertible m times a year*) is expressed by

$$\rho(m) = m\, d_{1/m} = m\,(1 - e^{-\theta/m}) \tag{3.47}$$

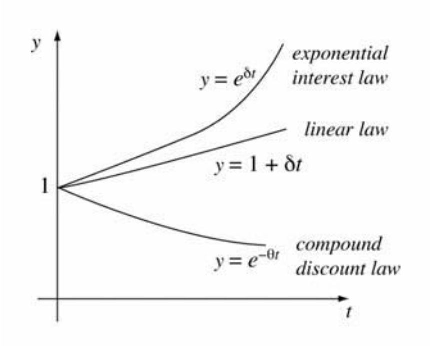

Figure 3.4. *Interest and discount exponential law*

EXAMPLE 3.13.– Considering the CCD law with $\theta = 0.0689$, we obtain for the factor and the rate of annual discount the values $v = e^{-0.0689} = 0.93342$ and $d = 1 - v = 0.06658$, while for the quarterly discount length (m=4) we obtain for the per period factor, the per period rate and the corresponding intensities the following values:

$v_{1/4} = e^{-0.0689/4} = 0.982923$; $d_{1/4} = 1 - v_{1/4} = 0.017077$; $\rho(4) = 4\, d_{1/4} = 0.06831$.

If $m' = 6$ (= bimonthly period), for (3.46) the equivalent discount rate is $d_{1/6} = 1 - (1 - d_{1/4})^{4/6} = 0.011418$.

Comparing (3.45) with (3.26), the rates $i_{1/m}$ and $d_{1/m}$ come from m-fractional and conjugated compound laws if the following relation holds:

$$(1 + i_{1/m})\,(1 - d_{1/m}) = 1 \tag{3.48}$$

and, taking into account (3.26') and (3.47), the intensities for conjugated laws satisfy the relation

$$(1 + j(m)/m)(1 - \rho(m)/m) = 1 \qquad (3.48')$$

Exercise 3.15

Using the CCD law with instantaneous intensity $\theta = 0.0523$, calculate the rates and the per period intensities, equivalent to each other, of such a law for the usual frequencies. Also calculate the rates and the per period intensities of interest for the same frequencies, based on the instantaneous intensity $\delta = \theta$ or $\delta = 0.0473 \neq \theta$.

A. Using Excel, the rates $d_{1/m}$ and the intensities $\rho(m)$ for changing m are obtained using (3.45) and (3.47). Furthermore, if $\delta = \theta = 0.0523$, the CCI law is conjugated to the CCD law; so the rates $i_{1/m}$ and the intensities $j(m)$ are obtained using (3.48) and (3.49). The following table is obtained.

CALCULATION OF EQUIVALENT PER PERIOD RATES AND INTENSITIES
with conjugated CCD and CCI laws

	intensity θ =	intensity δ =	0.0523	
m	$D_{1/m}$	. $\theta(m)$	$i_{1/m}$	*J(m)*
1	5.096%	0.05096	5.369%	0.05369
2	2.581%	0.05162	2.649%	0.05299
3	1. 728%	0.05185	1.759%	0.05276
4	1.299%	0.05196	1.316%	0.05264
6	0.868%	0.05207	0.875%	0.05253
12	0.435%	0.05219	0.437%	0.05241
52	0.101%	0.05227	0.101%	0.05233
360	0.015%	0.05230	0.015%	0.05230

Table 3.3. *Equivalent per period rates and intensities*

The Excel instructions are as follows. Rows 1, 2, 4, 5 are for data and titles; F4: 0.0523. The 4th row is empty. From the 6th to 13th rows:

– column A (frequency): insert the given frequencies;

– column B: empty;

– column C (per period disc. rate): C6:= 1-EXP(-$F$4/A6);

– column D (per period disc. intensity): D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (per period interest rate): F6:= 1/(1-C6)-1; copy F6, then paste on F7 to F13;

– column G (per period interest intensity): G6:= A6*(1/(1-D6/A6)-1); copy G6, then paste on G7 to G13.

The convergence of the per period intensities to $\delta = \theta = 0.0523$ is verified.

If instead $\delta = 0.0473$, the laws are not conjugated and the calculation of the interest rates and intensities proceeds autonomously on the basis of (3.31') and (3.38) with $t = 1/m$. We then obtain the following table.

CALCULATION OF EQUIVALENT PER PERIOD RATES AND INTENSITIES
with unconjugated CCD and CCI laws

	intensity θ =	0.0523	intensity δ =	0.0473
m	*d.1/m*	Q*(m)*	*i.1/m*	*J(m)*
1	5.096%	0.05096	4.844%	0.04844
2	2.581%	0.05162	2.393%	0.04786
3	1.728%	0.05185	1.589%	0.04767
4	1.299%	0.05196	1.190%	0.04758
6	0.868%	0.05207	0.791%	0.04749
12	0.435%	0.05219	0.395%	0.04739
52	0.101%	0.05227	0.091%	0.04732
360	0. 015%	0.05230	0.013%	0.04730

Table 3.4. *Equivalent per period rates and intensities*

The Excel instructions are as follows. Rows 1, 2, 4, 5 are for data and titles; D4: 0.0523; G4: 0.0473; the 4^{th} row is empty; from the 6^{th} to 13^{th} rows:

– column A (frequency): insert the given frequencies;

– column B: empty;

– column C (per period disc. rate): C6:= 1-EXP(-$D$4/A6);

– column D (per period disc. intensity): D6:= A6*C6; copy D6, then paste on D7 to D13;

– column E: empty;

– column F (per period interest rate): F6:= EXP($G$4/A6)-1; copy F6, then paste on F7 to F13;

– column G (per period interest intensity): G6:= A6*F6; copy G6, then paste on G7 to G13.

This verifies the convergence of the per period intensities to the respective instantaneous intensities with frequency divergence.

In general, with diverging *m*, (3.47) converges to the instantaneous intensity θ. Indeed, using $h = -1/m$

$$\lim_{m \to \infty} \rho(m) = \lim_{h \to 0} (e^{\theta h} - 1)/h = \theta \tag{3.49}$$

Working with a CCD law whose instantaneous intensity is θ, on *any discount length t* >0 due to (3.42) the *discount factor*, the *per period discount rate* and the *per period discount intensity* for the length *t* are given respectively by

$$v_t = e^{-\theta t} \tag{3.50}$$

$$d_t = 1 - e^{-\theta t} \tag{3.51}$$

$$\rho_t = d_t \,/\, t = (1 - e^{-\theta t})/t \tag{3.52}$$

EXAMPLE 3.14.– Using the same discount law as in Example 3.13 and applying t = (2y+7m+21d) = 2.641667, the following values for (3.50), (3.51) and (3.52) are obtained:

$$v_t = 0.83359229;\ d_t = 0.16640771;\ \rho_t = 0.06299345$$

The comparison between (3.50) and (3.36) shows that if $\theta = \delta = \ln(1+i)$, u_t and v_t are reciprocal, where the corresponding CCI and CCD laws are conjugated.

3.9. Complements and exercises on compound regimes

Complement 1: graphical interpretation

Let us recall that the exponential functions differ from their reciprocal functions only by the sign of the exponent: $1/e^{\delta\tau} = e^{-\delta\tau} = e^{\delta(-\tau)}$. Therefore, it can be concluded that the same function $e^{\delta\tau}$ represents, depending on the sign of τ, compound accumulation or discount, if we consider the following durations:

– in the first case a positive duration $\tau > 0$ between the beginning and the end of accumulation;

– in the second case a negative duration proceeding backwards from the maturity, taken as origin, until time $\tau < 0$ where the discount is carried out.

This enables us to represent in only one graph $f(t)= e^{\delta t}$, $\forall$t, shown in Figure 3.5, the typical quantities of the exponential regime, choosing an intensity $\delta>0$ which represents the interest intensity for the accumulation law and the discount intensity for the discounting law.

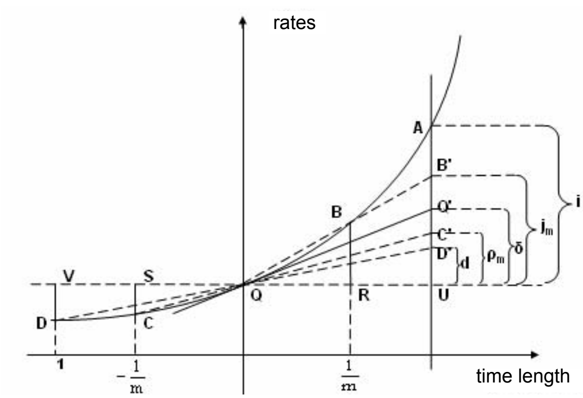

Figure 3.5. *Rates and intensities in the exponential law*

Interpretation of Figure 3.5

Let us consider in Figure 3.5 the following typical points of the graph of the function $f(t) = e^{\delta t}$ identified by the Cartesian coordinates on the plain O*tf*:

$$A = (1,e^{\delta});\ B = (1/m,e^{\delta/m});\ Q = (0,1);\ C = (-1/m,e^{-\delta/m});\ D = (-1,e^{-\delta})$$

where m is the conversion frequency. The points B', C', D' are intersections with t=1 of the secants of the exponential $f = e^{\delta t}$ respectively for the fixed point Q and the varying points B, C, D; furthermore, point Q' is the intersection in t=1 of the tangent of the curve in Q (= limit line of the secants). The point U, R, S, V on the horizontal $f=1$ have the same abscissa as A, B, C, D.

Let us observe that because of the proportionality between catheti of similar triangles QRB and QUB': $\overline{UB'} = \overline{RB}/\overline{QR}$ = slope of the secant QB. Using a similar argument: $\overline{UC'} = \overline{SC}/\overline{QS}$ = slope of the secant QC. Because $\overline{QU}=1$, then UQ' = slope of the line QQ' tangent in Q to $e^{\delta t}$ as well.

Because $e^{\delta} = 1+i$, $e^{\delta/m} = 1+i_{1/}$, $e^{-\delta/m} = 1-d_{1/m}$, $e^{-\delta} = 1-d$ and also $j(m) = m \cdot i_{1/m}$, $\rho(m) = m \cdot d_{1/m}$, the following graphical interpretation can be obtained:

– ordinate of D = v = discount factor for one year;

– ordinate of C = $v^{1/m}$ = discount factor for $1/m$ of one year;

– ordinate of B = $u^{1/m}$ = accumulation factor for $1/m$ of one year;

– ordinate of A = u = accumulation factor for one year;

– $\overline{VD} = \overline{UD'} = d = \rho(1)$ = annual discount rate = discount intensity on one year;

– $\overline{RB} = d_{1/m}$ = discount rate per period for $1/m$ of one year;

– $\overline{SC} = i_{1/m}$ = interest rate per period for $1/m$ of one year;

– $\overline{UA} = i = j(1)$ = annual interest rate = interest intensity on one year base;

– $\overline{UB'} = j(m)$ = interest intensity per period on $1/m$ of one year;

– $\overline{UQ'} = \delta$ = instantaneous interest intensity;

– $\overline{UC'} = \rho(m)$ = discount intensity per period on $1/m$ of one year.

It is clear that rates and intensities relative to different periods, taken from the same function $e^{\delta t}$, are equivalent.

Using the same graph as in Figure 3.5, the relations between the fundamental quantities *u, v, i, d* (which, referring to one year, are valid for all uniform regimes considered in this chapter) can be considered, as well as the relations between the fundamental quantities and the instantaneous intensity δ. Such relations are summarized in the following table, where each quantity given in the 1st column is expressed as a function of the quantities given in the 1st row.

	u	**v**	**i**	**d**	**δ**
u	u	$\frac{1}{v}$	$1+i$	$\frac{1}{1-d}$	e^{δ}
v	$\frac{1}{u}$	v	$\frac{1}{1+i}$	$1-d$	$e^{-\delta}$
i	$u-1$	$\frac{1-v}{v}$	i	$\frac{d}{1-d}$	$e^{\delta}-1$
d	$\frac{u-1}{u}$	$1-v$	$\frac{i}{1+i}$	d	$1-e^{-\delta}$
δ	$\ln u$	$-\ln v$	$\ln(1+i)$	$-\ln(1-d)$	δ

(3.53)

Table 3.5. *Transformation formulae between rates or intensities*

Complement 2: average length and average rate in the compound regime

By applying the same considerations as in sections 2.5.3 and 2.5.4 to the compound regime, it can be easily verified that:

– using a CCI law and the accumulation factor $u_t = (1+i)^t$, the *average length* $\hat{t}$ (equal to the average term if the investment starts at 0) is given by the exponential mean with base $(1+i)$ of the length t_h of the investment on the principal C_h. Then:

$$(1+i)^{\hat{t}} = \frac{\sum_{h=1}^{n} C_h (1+i)^{t_h}}{\sum_{h=1}^{n} C_h} \tag{3.54}$$

In the same way, using a CCD law and the discount factor $v_t = (1+i)^{-t}$, the *average length* $\hat{t}$ is given by the exponential mean with base $(1+i)^{-1}$ of the length t_h of the discount on the terminal amount M_h. Then:

$$(1+i)^{-\hat{t}} = \frac{\sum_{h=1}^{n} M_h (1+i)^{-t_h}}{\sum_{h=1}^{n} M_h} \tag{3.55}$$

– using a CCI law, the *average rate* $\hat{i}$ relative to the investment of principal C_h for the same length t made with rate i_h is the mean of powers with exponent t defined by

$$\left(1+\hat{i}\right)^t = \frac{\sum_{h=1}^{n} C_h \left(1+i_h\right)^t}{\sum_{h=1}^{n} C_h} \tag{3.56}$$

In the same way, using a CCD law, the *average rate* $\hat{i}$ relative to the discount on terminal value M_h for the same length t made at rate i_h is the mean of powers with exponent -t defined by

$$\left(1+\hat{i}\right)^{-t} = \frac{\sum_{h=1}^{n} M_h \left(1+i_h\right)^{-t}}{\sum_{h=1}^{n} M_h} \tag{3.57}$$

Complement 3: plurality of accounts and problems of averaging

Let us consider the following application which implies an averaging problem. A company has to operate financially through a plurality of accounts, all ruled by a compound regime, which is decomposable, but with different rates. Let u_h be the annual accumulation factor on the principal $C_h > 0$ invested at time 0 in the h^{th} account (h = 1, ..., n).

We are interested in valuing the characteristics of this accumulation regime connected to the *total financial management* of the n account, considering only the effect of such initial investments. So the accumulation factor for the 1st year is $m(0,1) = \sum_h C_h u_h / \sum_h C_h = \hat{u}$ = weighted arithmetic mean of the single factors (= *first moment* of the distribution $\{u_{h,} C_h\}$); for two years of consecutive investment the accumulation factor is: $m(0,2) = \sum_h C_h u_h^2 / \sum_h C_h$ = mean of squared u_h (= *second moment* of the distribution $\{u_{h,} C_h\}$).

The decomposability valid on each account is maintained at a global level as long as, supposing for example an interruption after one year, further accumulation for the 2nd year of the obtained amounts $C_h u_h$ is made $\forall h$ with the same factor u_h valid in the 1st year. The following is obtained indeed:

$$m(1,2) = \frac{\sum_h (C_h u_h) u_h}{\sum_h C_h u_h} = \frac{\sum_h C_h u_h^2}{\sum_h C_h u_h} = \frac{m(0,2)}{m(0,1)} \tag{3.58}$$

However, it can be observed that:

– the values in (3.58) are given by the anti-harmonic mean of the factors u_h, which is not associative[18];

– the total amount = $\sum_h C_h u_h$ at time 1 of the principal $\sum_h C_h$ can be obtained by also applying the mean annual rate $\hat{u}$ to each account, but if the rate $\hat{u}$ is also applied to each account in the 2nd year, after the interruption, we would obtain a lower total amount and the global process would not be separable. The result of such a hypothesis is that:

$$\mu_1 = m(0,2) / m(0,1) = \left(\sum_h C_h u_h^2\right) / \left(\hat{u} \sum_h C_h\right)$$

$$\mu_2 = m(1,2) = \sum_h \left(C_h u_h\right) \hat{u} / \sum_h C_h u_h = \hat{u}$$

Putting $\mu_3 = \left(\sum_h C_h u_h\right) \sum_h C_h > 0$, we obtain

$$(\mu_1 - \mu_2)\, \mu_3 = \sum_{h<k} C_h C_k (u_h - u_k)^2 > 0$$

and then $\mu_1 > \mu_2$. Thus, the statement is proved.

The problem is more complicated if some of the amounts are credits and some are debits, without the possibility of compensation.

Such simple observations should make the financial operator consider the delicacy of such problems and the attention needed in choices when averaged values are used.

Exercises on equivalent rates and intensities

It is convenient to stress that the consideration of a rate per period for $1/m$ of a year does not have meaning in annual conversion; it only has meaning in m-fractional conversion or m'-fractional, with m' multiple of m, or in an exponential

18 Generalizing this conclusion, we can observe that a feature of the compound regime is the fact that the continuing annual accumulation factor for the k^{th} year is the anti-harmonic mean of order k given by $\sum_h C_h u_h^h / \sum_h C_h u_h^{h-1}$; see: Caliri (1981).

regime ($m \to +\infty$). In the latter case the interest rate can be considered for any period t, expressed by $e^{\delta t}$-1.

Exercise 3.16

Firm Y receives from Bank X a short-term loan with 7.60% nominal annual rate with quarterly conversion and uses it in an operation with monthly income. Calculate the minimum monthly rate of return necessary to assure a positive spread of 2% on the cost rate in terms of effective annual rates.

A. The parameter 0.076 = nominal annual rate 4-convertible = $j(4)$, is an intensity, referred to the quarterly conversion. It corresponds to effective annual rate $i = (1+j(4))^4 - 1 = 0.078194$. Therefore, the minimum annual rate of return is: $i' = 0.098194$, to which corresponds the monthly rate $(1+i')^{1/12} - 1 = 0.007836 = 0.7836\%$.

Exercise 3.17

For the loan of the principal C = €250,000, there will be delayed bimonthly interest payments of €2,900, until the time of repayment in one transaction.

Calculate the amount of per period equivalent interest payments:

a) in the case of monthly advance payments;

b) in the case of semi-annual delayed payments;

c) in the case of quarterly advance payments.

A. Having established the final repayment of the total loan, the installments paid by the debtor are pure interest. Furthermore, the equivalent installments have to be calculated using the same DCI law with the monthly conversion (monthly because 12 is the least common multiple of the frequencies considered here).

So, because of the data, the accumulation law used here gives rise to a value for the bimonthly interest rate equal to

$$i_{1/6} = 2{,}900/250{,}000 = 0.0116 = 1.16\%$$

Recalling (3.28), (3.46) and (3.48):

$$\text{a) } d_{1/12} = \frac{i_{1/12}}{1+i_{1/12}} = \frac{(1+i_{1/6})^{0.5}-1}{(1+i_{1/6})^{0.5}} = 0.005750$$

therefore the equivalent advance monthly installment is: $C\, d_{1/12}$ = €1,437.

b) $i_{1/2} = (1 + i_{1/6})^3 - 1 = 0.035205$

therefore the equivalent delayed semi-annual installment is: $C\, i_{1/2}$ = €8,801.

$$\text{c) } d_{1/4} = \frac{i_{1/4}}{1+i_{1/4}} = \frac{(1+i_{1/6})^{1.5}-1}{(1+i_{1/6})^{1.5}} = 0.017151$$

therefore the equivalent advance quarterly installment is: $C\, d_{1/4}$= €4,288.

Exercise 3.18

1) For an investment of €10,000 in compound regime at the annual effective rate of 5%, let us compare the amount after 5 years and 7 months in the three following options:

a) with CCI law;

b) with mixed law with quarterly conversion;

c) with mixed law with annual conversion.

For b) and c) use the assumption that the investment is made at one prefixed time of conversion (for example on 1 January).

A. In case a), apply (3.35'), use C = 10,000; i = 0.05; t = 5+7/12 = 5.583333; then:

$$M_a = 10{,}000\cdot(1.05)^{5.583333} = €13{,}131.27.$$

In case b), apply (3.27) with $f_1 = 0$; $f_2 = 1/12$; $m = 4$; $k = 22$; $j(4) = 4\,(1.10^{1/4} - 1) = 0.040989$; C = 10,000; then:

$$M_b = 10{,}000\cdot(1+0.040989/4)^{22}\cdot(1+0.040989\cdot 0.083333) = €13{,}131.50.$$

In case c), apply (3.25) with $f_1 = 0$; $n = 5$; $f_2 = 7/12$; C = 10,000; then:

$$M_c = 10{,}000\cdot(1.05)^5\cdot(1+0.05\cdot 0.583333) = €13{,}135.06.$$

The amounts are in increasing order, given that they follow from the same effective rates. In addition, M_b is very close to M_a.

2) Make the comparison for the amounts made in 1 but for a length of 5 years.

A. As will be shown in section 3.10, for integer length the three amounts are the same. For 5 years this gives: $M_a = M_b = M_c = 12762.82$.

3) Make the comparison as in 1, but calculating for 5 years and 7 months with a common intensity $j = 0.05$ for any frequency of conversion.

A. In such a case, introducing j both in the compound law for integer year and in the linear law for fractions of a year, we obtain:

$M_a = 10{,}000\ e^{0.055 \cdot 583333} = €13{,}220.27$

$M_b = 10{,}000\ (1{+}0.05/4)^{22}\ (1{+}0.05 \cdot 0.083333) = €13{,}197.64$

$M_c = 10{,}000\ (1{+}0.05)^5\ (1{+}0.05 \cdot 0.583333) = €13{,}135.06$

The value M_c coincides with that in 1 because numerically $i = j(1)$. The amounts are now in decreasing order with the decreasing number of conversions ($M_a > M_c$ because $e^{\delta\tau} > 1{+}\delta\ t$).

4) Make the comparison as in 1), but for 5 years as in 2).

A. Obviously the equality between the amounts is lost and then:

$$M_a = €12{,}840.25;\ M_b = €12{,}820.37;\ M_c = €13{,}762.82.$$

Exercise 3.19

Consider the same problem as in Exercise 3.18, 1), using the same data, but removing the assumption that the investment starts at the conversion dates, but instead starts 12 days in advance.

A. Using the bank year (= 12 months of 30 days each), results in:

– case a), no changes because the exponential law depends only on the total length, which has not changed; therefore, $M_a = €13{,}131.27$;

– cases b) and c) concern mixed law, then not a uniform law, and the result changes.

In case b), using in (3.27): $f_1 = 0.03333 = (12\ d)$; $k = 22$; $f_2 = 0.05\ (= 18\ d)$; $j(4) = 0.049089$; $C = €10{,}000$, the following is obtained:

$$M_b = 10{,}000\ (1{+}1.049089 \cdot 0.03333) \cdot (1{+}0.049089/4)^{22}\ (1{+}0.049086 \cdot 0.05) = €13{,}131.55.$$

In case c), putting in (3.25): $f_1 = 0.033333$; $f_2 = 7/12 - 12/360 = 0.55$; $n = 5$; $i = 0.05$; $C = 10{,}000$, the following is obtained

$$M_c = 10{,}000\ (1+1.05\cdot 0.03333)\cdot(1+0.05)^5 \cdot(1+0.05\cdot 0.55) = €13{,}135.65.$$

If with the law assumed in case b), used in banks on current accounts, the fractions f_1 and f_2 are calculated relating the effective numbers of day to the bank year, i.e. 360, and can assume values greater than 1/4 (so that from 1 July to 29 September inclusive, there are 91 days, resulting in $91/360 = 0.252778>1/4$).

Exercise 3.20

In Exercise 3.19 we verified that, with the same interest and length, in mixed accumulation the result changes according to the placement of the investment interval with respect to the conversion interval. Calculate the values that, using the same data, maximize the amount.

A. Considering for the sake of simplicity case c), we have to work on variables f_1 and f_2 such that $f_1+f_2 = t\text{-}n$ = constant = H and maintaining the number $n+2$ of conversions. Using $f_1 = x$, $f_2 = H\text{-}x$, with the data of Exercise 3.17 it is necessary to maximize the accumulation factor

$$g(x) = M(t)/C = (1 + 0.05x)\cdot 1.05^5\cdot[1 + 0.05(H\text{-}x)];$$

its graph is a concave downward parabola, thus having only one maximum point where the first derivative is zero. It is $g'(x) = 0$ for $x = H/2$, i.e. when $f_1 = f_2$.

In conclusion, if the length and frequency (annual, but this also holds for the fractional case, as it is easy to verify) are given, it is convenient for the creditor that the interval of investment is positioned symmetrically with respect to the conversion intervals.

EXAMPLE 3.15.– Given an investment for 3 years and 6 months between 2005 and 2009 at an annual rate of 5.50%, with conversion at the end of the calendar year, taking into account that the beginning cannot be before 1 July 2005 and the term cannot be after 3 June 2009, we obtain the maximum accumulation factor, equal to 1.206755, when the investment begins on 1 October 2005 and ends on 31 March 2009. Indicating by x the number of months in 2005 and by $y = 6\text{-}x$ the number of months in 2009, by varying x with the respect of the given constraints, we obtain the following results which gives the order of magnitude of the variations.

Investment intervals	*x*	*y*	*Accumulation factor g(x)*
01/07/05 – 31/12/08	6	0	1.206533
01/08/05 – 31/01/09	5	1	1.206656
01/09/05 – 28/02/09	4	2	1.206730
01/10/05 – 31/03/09	3	3	1.206755
01/11/05 – 30/04/09	2	4	1.206730
01/12/05 – 30/05/09	1	5	1.206656
01/01/06 – 30/06/09	0	6	1.206533

Table 3.6. *Comparison among accumulation factors*

3.10. Comparison of laws of different regimes

After collecting the results of previous section we can make a comparison between the amounts obtainable with different uniform accumulation regimes already considered or between the present values connected with different uniform discount regimes.

We will consider in this section:

a) *in accumulation*, the comparison among simple, delayed or advance, and continuously compound interest laws;

b) *in discount*, the comparison among rational, simple and continuously compound discount laws.

The result of such a comparison depends on the functional form of the exchange factors but also on the return parameters (rates or intensities) used for the single laws.

When referring to the different *accumulation regimes*, if we only consider a comparison in the assumption of equal i, i.e. among:

– an SDI law with annual rate i;

– a CCI law with the same annual rate i;

– an SAI law with annual rate $d = i/(1+i)$;

we can conclude straight away that:

1) the three SDI, CCI and SAI laws give rise to the same return of interest after one year of investment, i.e. the *indifference length* is 1;

2) indicating here with $\succ$ the preference among laws

$$(SDI) \succ (CCI) \succ (SAI), \text{ if } t < 1,$$

$$(SAI) \succ (CCI) \quad (SDI), \text{ if } t > 1.$$

Regarding comparison among *discount regimes*, it is enough to observe that the RD, CCD and SD regimes give rise to conjugated laws, respectively, to SDI, CCI and SAI. Then it is enough to consider the reciprocal factors and repeat all reasoning, to conclude, when comparing the following:

– an RD law with annual rate i;

– a CCD law with the same annual rate;

– an SD law with annual rate $d = i/(1+i)$;

that

1) the three RD, CCD and SD laws give rise to the same discount return after one year of anticipation, i.e. the *indifference length* is 1;

2) the preference among laws, indicated here by ƭ , is

$$(SD) \succ (CCD) \succ (RD), \text{ if } t < 1,$$

$$(RD) \succ (CCD) \succ (SD), \text{ if } t > 1.$$

Graphical interpretation

Figure 3.6 shows the comparison among interest laws: (SDI)→ line (a), (CCI)→ line (b), (SAI)→ line (c), when the delayed interest rates coincide in the different law and the indifferent length is 1. The comparison among discount conjugate law (RD) →line(a'), (CCD)→line(b'), (SD)→line(c'), with the same conditions and indifferent length, is also shown.

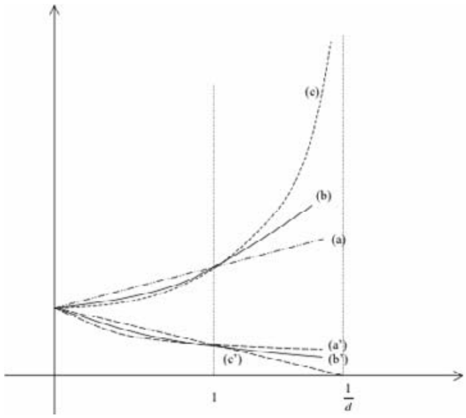

Figure 3.6. *Comparisons among interest and discount laws*

Let us now solve the *problem of comparing* the various regimes two by two, when different rates are applied to the laws of accumulation or discount. In this way we can also find the *indifference lengths* which depend on the couples of the chosen rates.

With reference to *interest laws*, the following results are obtained.

A1) *Comparison between SDI and CCI laws*

Let i_0 be the annual rate of an SDI law and i the annual effective rates for a CCI law. With reference to the accumulation factors, the principal and the amount being proportional, the returns coincide in both laws if the length t satisfies the relation

$$1 + i_0 t = (1 + i)^t \tag{3.59}$$

We will not consider the solution $t = 0$, because we are interested only in a positive solution t':

– if $i_0 > \delta = \ln(1+i)$, such a solution exists and is unique, given the upward concavity of $(1+i)^t$. The calculation of indifference length t' must be done numerically. If i and i_0 satisfy (3.59), $1+i_0 t > (1+i)^t$ if $t<t'$ holds, while $1+i_0 t < (1+i)^t$ if $t>t'$. Therefore, the compound law is preferable for the investor only for a length greater than the indifference length, which is

$$t' = 1/m \text{ if } i_0 = j(m);\ t' = 1 \text{ if } i_0 = i.$$

– if $i_0 \leq \delta$, there is no indifference length and the compound law is always preferable.

Exercise 3.21

Given the (SDI) law with an annual rate of $i_0 = 0.061$ and the (CCI) law with an annual rate of $i = 0.062$, calculate the indifference length using the methods described in this section.

A. Given the annual rate $i_0 = 0.061 > \ln(1+i) = 0.060154$, there exists the indifference length $t' > 0$. We have $t' = 1$ if $i_0 = i$; but being $i_0 < i$, $t' < 1$ follows. Finally: $0 < t' < 1$ and the compound factor prevails if $t > t'$.

Indicating with $\zeta(t) = (1+i)^t - (1+i_0 t)$ the spread between the factors (where by definition $\zeta(t')=0$) is $\zeta(1) = i\text{-}i_0$ and with the given rates: $\zeta(1) = 0.001$. Let us calculate in the interval (0,1) a time t such that $\zeta(t) < 0$. With decreasing t we have for example: $\zeta(0,4) = -0.000047$. Proceeding initially with the *dichotomic method* (see section 4.5.3) between t=1 and t=0.40, we obtain: $\zeta(0.70) = 0.000307$; $\zeta(0.55) = 0.000088$; etc. The convergence is slow.

Let us proceed with the *secant method* (see section 4.5.4), with upper bound t = 0.55 fixed and increasing lower bound from t = 0.40.

1st step: linear interpolation between t = 0.40 and t = 0.55:

$$\frac{t-0.40}{0.55-0.40} = \frac{0-\zeta(0.40)}{\zeta(0.55)-\zeta(0.40)} = \frac{47}{88+47} = 0.348148$$

then $t = 0.40 + 0.15 \cdot 0.348148 = 0.452222$; $\zeta(t) = -0.000009$.

2nd step: linear interpolation between t = 0.452222 and t = 0.55:

$$\frac{t-0.452222}{0.55-0.452222} = \frac{0-\zeta(0.452222)}{\zeta(0.55)-\zeta(0.452222)} = \frac{9}{88+9} = 0.092784$$

then $t = 0.452222 + 0.097778 \cdot 0{,}092784 = 0.461294$; $\zeta(t) = -0.000002$.

3rd step: linear interpolation between t = 0.461294 and t = 0.55:

$$\frac{t-0.461294}{0.55-0.461294} = \frac{0-\zeta(0.461294)}{\zeta(0.55)-\zeta(0.461294)} = \frac{2}{88+2} = 0.022222$$

then $t = 0.461294 + 0.088706 \cdot 0.022222 = 0.463265$; $\zeta(t) = -0.000000035$.

Let us stop the iterative process, because time 0.463265 = (5m+17d) is a good estimation (approximated by defect) of the indifference length t', implying a spread ζ of almost zero.

A2) *Comparison between SDI and SAI laws*

Let i be the annual rate of an SDI law and d the annual discount rate of an SAI law. We have coincidence of returns (for length $t < 1/d$) if

$$1 + it = (1 - d\,t)^{-1} \tag{3.60}$$

and we have the only positive solution $t' = (i_0 - d)/i_0 d$ if and only if $i_0 > d$. In particular $t' = 1$ if $i_0 = d/(1-d)$.

Due to the sign of concavity $(1 - dt)^{-1}$, the SDI law is convenient for the investor if $t<t'$, but the SAI law is convenient if $t>t'$.

EXAMPLE 3.16.– Comparing an SDI law with an annual rate i_0 = 4.70% with an SAI law with an annual advance rate d = 4.30%, the indifference length is given by:

$$t' = (0.047\text{-}0.043)/(0.047\cdot 0.043) = 1.979218 = 1\text{y}+11\text{m}+23\text{d}.$$

Using instead the corresponding rate d = 0.047/1.047 = 4.489% we obtain t' = 1.

A3) *Comparison between SAI and CCI laws*

Let d be the annual discount rate of a SAI law and i the effective annual rate of a CCI law. The returns are the same if the length satisfies the relation

$$(1 - d\,t)^{-1} = (1+i)^t \quad , \quad t < 1/d \tag{3.61}$$

For this comparison the calculation of indifference length t' must be performed numerically. We have a solution $t'>0$ (which can be shown to be unique) to the problem of equivalent length if and only if $d<\delta$.[19] In such a case, if $t<t'$ the CCI law is convenient for the investor; if $t>t'$, then the SAI laws are convenient. If instead $d>\delta$, the SAI law is always convenient for the investor.

19 This is because the curves $(1-dt)^{-1}$ and $(1+i)^t$ are both convex and have right derivatives in t=0 equal respectively to d and δ.

Exercise 3.22

Given the law (CCI) at the annual rate $i = 0.062$ and the law (SAI) at the annual delayed rate $d = 0.059$, calculate the indifference length using the method described in section 4.5.

A. Given that $d = 0.059 < \delta = \ln(1+i) = 0.060154$, there exists the indifference length $t' > 0$. To calculate this, we proceed as in Exercise 3.21, where the CCI and SDI laws are compared. Furthermore, with length t=1 the SAI law is convenient, because the following is obtained for the accumulation factors: $1/(1\text{-}d) = 1.062699 > 1.062000 = 1+i$. Then: $0<t'<1$ and the simple advance factor prevails if $t > t'$.

Indicating with $\xi(t) = (1\text{-}d\ t)^{-1}\text{-}(1+i)^t$ the spread between the factors (where by definition $\xi(t')$=0), with the given rates we obtain: $\xi(1) = 0.000699$. In addition, $\xi(0.5) = -0.000137$. Starting with the *dichotomic method* between t=1 and t=0.50, we obtain: $\zeta(0.750) = 0.000150$; $\zeta(0.625) = -0.000025$; .....

To speed up the convergence, we proceed with the *secant method*, using the upper bound $t = 0.750$ fixed and the increasing lower bound from $t = 0.625$.

1st step: linear interpolation between $t = 0.625$ and $t = 0.750$:

$$\frac{t-0.625}{0.750-0.625}=\frac{0-\xi(0.625)}{\xi(0.750)-\xi(0.625)}=\frac{25}{150+25}=0.142857$$

from which $t = 0.625 + 0.125\cdot 0.142857 = 0.642857$; $\xi(t) = -0.000004$.

2nd step: linear interpolation between $t = 0.642857$ and $t = 0.750$:

$$\frac{t-0.642857}{0.750-0.642857}=\frac{0-\xi(0.642857)}{\xi(0.750)-\xi(0.642857)}=\frac{4}{150+4}=0{,}025974$$

from which: $t = 0.642857 + 0.107143\cdot 0.025974 = 0.645640$; $\xi(t) = -0.000001$.

We stop here: time 0.645640 = (7m+22d) is a good estimation (approximated by default) of the indifference length t', because the spread ξ is close to zero.

With reference to *discount laws*, for the problem of

B1) *comparison between RD and CCD laws*;

B2) *comparison between RD and SD laws*;

B3) *comparison between SD and CCD laws*.

we obtain the same indifference length t' valid for the interest conjugate laws, as it is simple to prove by observing that the equations giving the solutions concern the reciprocals of the terms which appear in equations (3.59), (3.60), (3.61) and then coincide with the aforementioned relations.

Furthermore, for length $t \neq t'$, going from interest laws to their conjugated discount laws, the preference relations are inverted[20].

20 In fact the discount laws give rise to factors reciprocal to those of the interest laws conjugated with the previous discount laws. Therefore, the inequalities and the sign of concavities of the corresponding graphs are inverted. In addition, considering discount, in the right derivatives in t=0 only the sign changes, i.e. there are $-i_0$, -d, $-d$. This is in agreement with the generally valid property, that the differentiable functions $f(x)$ and their reciprocal function have in the intersection points opposite derivatives. Indeed, if $f(x_0) = 1/f(x_0)$, we obtain: $[f(x_0)]^2 = 1$ and then $1/f(x)_{x=x_0} = -f'(x_0)$.

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Chapter 4

Financial Operations and their Evaluation: Decisional Criteria

4.1. Calculation of capital values: fairness

In financial practice there is often the problem of evaluation, at a given time and based on a given exchange law, of a finite set of financial supplies, i.e. of incomes or payments to be made at a fixed time. It is easy to generalize about an infinite number of supplies or to a continuous flow of payments as a theoretical model which approximates a sequence of financial transactions of small amount with close maturity.

Such a set is called a *financial operation* because it is the financial reflex of economic acts regarding flows or funds (like transferring assets, payments for services, loans with a unique or periodic repayment schedule, installation or management of industrial equipment, etc.).

Referring to the concept of *financial supply (T,S)* as well as to the *equivalence principle* based on a given exchange law, a *financial operation O*, which we will firstly consider as discrete payments, can then be defined as *union of supplies*, using

$$O = \bigcup_{h=1}^{n} (T_h, S_h) \tag{4.1}$$

If $n \to \infty$, it is necessary to introduce some conditions. Without loss of generality we will consider $\{T_h\}$ increasing with h, i.e. in chronological order; in addition, S_h>0 are incomes for the agent "A" whereas S_h<0 are payments. The operation O is also called a *financial project* if it is referred to dated amounts that are expected by a feasible project.

The operation O can be alternatively expressed in transposed form with respect to (4.1), using a pair of n-dimensional vectors ("maturities $\{T_h\}$, cash flow[1] $\{S_h\}$", h=1,...,n), instead of a n-tuple of two-dimensional vectors which identify the supplies "time T_h, amount S_h". Therefore, we can write

$$O = (T_1, T_2, ..., T_n)\&(S_1, S_2, ..., S_n) = \{T_h\}\&\{S_h\} \qquad (4.1')$$

where & = *correspondence between vector components* and where corresponding pairs (T_h,S_h) with the same h identify the supply.

It is usual to distinguish in (4.1) between the cases n=2, which give rise to a *simple operation*, and n>2, which give rise to a *complex operation*. A simple operation, if S_1 and S_2 have opposite sign, is just an exchange between two amounts with different maturities.

Let us consider an economic agent "A" who wants to value O at time T, based on an indifference relation ≈ which gives rise to the exchange factors $z(X,Y)$, given by (2,5'), which express the used financial law. We then define as *capital value* (or just *value*) of the operation O at time T (from the point of view of agent "A") the amount $V(T;O,z)$ so that "A" considers O to be fairly exchangeable with the supply $(T, V(T;O,z))$. In other words, from the point of view of "A" there is indifference between obtaining the supplies O and acquiring the amount $V(T;O,z)$ in T.

We can apply what we have said above to calculate the selling value of a company. If O, expressed by (4.1), concerns all financial transactions related to its management and expected by one party (for example, the seller) in $T<T_1$ (and then $T<T_h,\forall h$), then $V(T;O,z)$ is the value in T given to the company based on the law z, which has to be compared with the offered price to judge whether it is convenient to sell[2].

To measure $V(T;O,z)$ we can consider that, because of results in section 2.2, the amount exchangeable in T with S_h in T_h is $S'_h = S_h{:}z(T_h,T)$, and this is the value in T of (T_h,S_h); furthermore, we will usually assume the *additive property*, by which the

1 Correctly speaking, flow should be used for the continuous case, but it is also frequently used in finance for the discrete case when there is a sequence of payments.
2 The comparison between values (subjective, as a consequence of the choice of z) and prices (objective, because they are fixed by the market) is the basis of the decisions and choices theory between "financial projects" that we will consider later.

value in T of the union of supplies $O = \bigcup_{h=1}^{n}(T_h, S_h)$ is the sum of the values at the same time T of each supply. Therefore

$$V(T;O,z) = \sum_{h=1}^{n} S'_h = \sum_{h=1}^{n} S_h z(T_h, T) . \tag{4.2}$$

When there is no ambiguity on O and on the law z, we will write $V(T)$ instead of $V(T;O,z)$.

We will say that "A" considers *fair* (or *well-balanced*) the operation O in T based on his choice of law z or, briefly, that *the operation O is fair in T if and only if* $V(T;O,z) = 0$.

Adopting exchange laws $z(X,Y)$ which are always positive with any X and Y, the *fairness of O* implies that there is no concordance in algebraic sign of all amounts of O.

A simple fair operation is a pure exchange (i.e. a *repurchase agreement*) balanced according to z, resulting in $S_2 = -S_1 \cdot z(T_1, T_2)$.

Complements on fair operations

We will say that an exchange law identified by ≈, according to which the fairness of an operation O is valued, verifies the *invariance property* if an operation considered fair in T_0 is also fair in all other times T. Furthermore, given that fairness implies the zero value of O, the additive property implies that the union of two or more operations (defined as the union of the sets of their supplies), all judged fair in T_0, is fair in T_0.

As invariance does not generally hold, then the value $V(T;O,z)$ if it is zero in $T=T_0$ can become different from zero in a different T; it is then necessary to specify the evaluation time. However, given that, as can be proved, the strong decomposability implies additivity and invariance together, if z satisfies such a property, the judgment of fairness of the operations does not depend on the evaluation time T.

It is important to observe here that – given that an exchange law implies the payment of interest for the deferring of the availability of a principal amount – saying that "A" considers the operation O to be fair having assumed the law $z(X,Y)$ means that O gives exactly the return expressed by z. In other words, i.e. with inflows and outflows at the times T_h on a profitable account ruled by such a law, if $V(T_0;O,z) = 0$, this means that in T_0 the evaluations of S_h, taking into account the interest, are balanced. If instead $V(T_0;O,z) > 0$ (< 0), the operation O gives rise to a spread of positive (negative) returns added to the return implied by the law z. This is the starting point of the theory of comparisons and choices between financial operations on the basis of the returns.

If the law z identified by $\approx$ is *uniform, not decomposable*, and therefore the exchange factor has the form $g(\tau)$ (see (2.40)), it is enough to replace in (4.2) $z(T_h,T) = g(\tau_h)$, where $\tau_h = T\text{-}T_h$, and then

$$V(T;O,z) = \sum_{h=1}^{n} S_h \; g(T - T_h) \tag{4.3}$$

Particular cases

a) Simple delayed interest (SDI) law at rate i and its conjugate rational discount (rd)

$$\text{SDI: } g(\tau) = 1 + i\,\tau, \text{ if } \tau > 0; \text{ rd: } g(\tau) = 1/(1 + i\,|\tau|), \text{ if } \tau < 0$$

b) Simple advance interest (SAI) law at rate d and its conjugate simple discount (sd)

$$\text{SAI: } g(\tau) = 1/(1 - d\,\tau), \text{ if } \tau > 0; \text{ sd: } g(\tau) = 1 - d\,|\tau|, \text{ if } \tau < 0$$

If the law z identified by $\approx$ is *strongly decomposable (s.dec), non-uniform,* characterized by an intensity $\delta(\lambda)$ as a function of current time λ (see section 2.4), the exchange factor in (4.2) is written[3]: $z(T_h,T) = \exp\left(\int_{T_h}^{T} \delta(\lambda)\, d\lambda\right)$.

If the law z identified by $\approx$ is *uniform and also s.dec*, it falls within (as shown in Chapter 2) the *exponential regime* where $\approx$ is a relation of uniform equivalence. Therefore, it gives rise to the following case.

c) Continuously compound interest (CCI) law with intensity δ *and its conjugate continuously compound discount (CCD)*

$$\text{CCI: } g(\tau) = e^{\delta\tau}, (\tau > 0); \text{ CCD: } g(\tau) = e^{\delta\tau}, (\tau < 0)$$

and then

3 It is well known that $\exp\left(\int_a^b \delta(\lambda)\, d\lambda\right)$ is often used, for simplicity, instead of $e^{\int_a^b \delta(\lambda)\, d\lambda}$.

$$V(T;O,z) = \sum_{h=1}^{n} S_h \; e^{\delta(T\text{-}T_h)} = \sum_{h=1}^{n} S_h \; (1+i)^{T\text{-}T_h} \tag{4.4}$$

Exercise 4.1

Let us consider the following operation

O = {(0, -1,500)∪(2.5, -1,850)∪(3.5, 520∪(5, 4,500)}

where time is measured in months, and let us calculate its value in T=4 using the SDI law with an annual rate i = 5.5% and its conjugate rd.

A. By applying (4.3), we obtain

$$V(4) = -1{,}500\ (1+0.055\frac{4}{12}) - 1{,}850\ (1+V(4) = 0.055\frac{1.5}{12}) + 520\ (1+0.055\frac{0.5}{12})$$
$$+\ 4{,}500\ \frac{1}{1+0.055\frac{1}{12}} = 1{,}610.44$$

Exercise 4.2

Let us consider the same operation as in Exercise 4.1, i.e.

O = {(0, -1,500)∪(2.5, -1,850)∪(3.5, 520)∪(5, 4,500)}

where time is measured in months, and let us calculate its value at time T=4 using the SAI law with an annual discount rate d equivalent to i =5.5% and its conjugate sd.

A. The equivalent rate d is 0.052133. Applying (4.3)

$$V(4) = -1{,}500 \frac{1}{1\text{-}0.052133\frac{4}{12}} - 1{,}850 \frac{1}{1\text{-}0.052133\frac{1.5}{12}} +$$
$$520 \frac{1}{1\text{-}0.052133\frac{0.5}{12}} + 4{,}500\,(1-0.052133\frac{1}{12}) = 1{,}612.920$$

Exercise 4.3

Let us consider the same operation as in Exercise 4.1, i.e.

$$O = \{(0, -1{,}500)\cup(2.5, -1{,}850)\cup(3.5, 520)\cup(5, 4{,}500)\}$$

where time is measured in years, and let us calculate its value at time T=4 using the exponential exchange law with an annual interest i = 6%.

A. The equivalent instantaneous intensity δ is 0.058269. By applying (4.4)

$$g(\tau) = e^{0.058269\,\tau} = 1.06^{\tau}$$

and then

$$V(4) = -1500 \cdot 1.06^{4} - 1850 \cdot 1.06^{1,5} + 520 \cdot 1.06^{0,5} + 4500 \cdot 1.06^{-1} = 867.97$$

The value can also be found with an Excel spreadsheet where in the 1st row we put the rate values and evaluation time, in the 2nd row the column's titles and from the 3rd to 6th rows the needed values: terms and amounts of supplies; exchange factors from the terms to 4; amount valued at time 4, then the sum gives $V(4)$ = 867.97. The following table is obtained.

Rate = 0.06		Time = 4	
Term	*Amount*	*Exchange factor at 4*	*Value at 4*
0.0	–1,500.00	1.2624770	–1,893.715
2.5	–1,850.00	1.0913368	–2,018.973
3.5	520.00	1.0295630	535.373
5.0	4,500.00	0.9433962	4,245.283
		V(4) =	867.967

Table 4.1. *Calculation of values*

The Excel instructions are as follows. B1: 0.06; D1: 4. The first two rows are for data and column titles; from the 3rd to 6th rows:

– column A (maturity): A3: 0; A4: 2.5; A5: 3.5; A6: 5;

– column B (amounts): B3: –1,500; B4: –1,850; B5: 520; B6: 4,500;

– column C (exchange factors in 4): C3: = (1+B$1)^(D$1-A3); copy C3, then paste on C4 to C6;

– column D (evaluation in 4): D3: = B3*C3; copy D3, then paste on D4 to D6; (value at time 4): D7: = SUM(D3;D6).

Financial operation with continuous flow

Let us consider briefly the calculation of the value of operations with continuous flow. In the continuous case the elementary supply is expressed by [t, $\varphi(t)dt$], with $\varphi(t)$ defined in $t' \le t \le t''$ and therein continuous (however, to calculate the values the integrability is enough). Then (4.2), (4.3), (4.4) become respectively

$$V(T;O,z) = \int_{t'}^{t''} \varphi(t) z(t,T) dt \tag{4.2'}$$

$$V(T;O) = \int_{t'}^{t''} \varphi(t) g(T-t) dt \tag{4.3'}$$

$$V(T;O) = \int_{t'}^{t''} \varphi(t) e^{\delta(T-t)} dt \tag{4.4'}$$

Using an s.dec and non-uniform exchange law, the exponential in (4.4') becomes $\exp\left(\int_t^T \delta(\lambda) d\lambda\right)$.

Naturally, there can be a *mixed operation*, which puts together continuous and discrete operations, and due to additivity the value will be given by the sum of the values (4.2) and (4.2') (or of the values for the other particular cases).

4.2. Retrospective and prospective reserve

With reference to an operation O and an exchange law $z(X,Y)$, let us assume T in the interval $[T_1,T_n]$ is *logically distinct*[4] from each T_h. Let r be the number of supplies before T and n-r those after T ($0 \le r \le n$).

We then define as *retrospective reserve* (briefly: *retro-reserve*) of O at time T according to z the amount $M(T;O,z)$ given by the opposite of the value in T of the sub-operation O' consisting of the set of all supplies of O *before* T. If there is no ambiguity, we write $M(T)$ instead of $M(T;O,z)$.

We define as *prospective reserve* (briefly: *pro-reserve*), or *residual value*, of O at time T according to z the amount $W(T;O,z)$ given by the value in T of the sub-operation O'' consisting of the set of all supplies of O *after* T. If there is no ambiguity, we write $W(T)$ instead of $W(T;O,z)$.

---

4 By *logically distinct* we mean that T is different from the time of payment T_h or, if they coincide, that there is a method to establish if the corresponding S_h has to be added to the payments before or after T (for example, with a rule of "delayed" or "advance" payment per period).

Because of (4.1) we have:

$$M(T;O,z) = -\sum_{h=1}^{r} S_h \, z(T_h, T) \tag{4.5}$$

$$W(T;O,z) = \sum_{h=r+1}^{n} S_h \, z(T_h, T) \tag{4.6}$$

Notice that in (4.5) we have only accumulation processes and in (4.6) only discount processes. Therefore, due to (2.5'), instead of $z(T_h,T)$ we can use $m(T_h,T)$ in (4.5) and $a(T_h,T)$ in (4.6).

If the exchange law is *uniform (non-decomposable)*, (4.5) and (4.6) can be written, giving $t_h = |\tau_h|$ and recalling (2.42)

$$M(T) = -\sum_{h=1}^{r} S_h \, u(t_h) \tag{4.7}$$

$$W(T) = \sum_{h=r+1}^{n} S_h \, v(t_h) \tag{4.8}$$

If the exchange law is *s.dec (non-uniform)* with intensity $\delta(\lambda)>0$, the exchange factor in (4.5) and (4.6) is written as: $z\left(T_h, T\right) = \exp\left(\int_{T_h}^{T} \delta(\lambda) \, d\lambda\right)$.

In particular, if the exchange law is *exponential* with rate i, the expressions for the reserves are

$$M(T) = -\sum_{h=1}^{r} S_h \, (1+i)^{(T-T_h)} \; ; \; W(T) = \sum_{h=r+1}^{n} S_h \, (1+i)^{-(T_h-T)} \tag{4.9}$$

From the definitions, whichever exchange law is used for the operation O, it follows that

$$V(T) = W(T) - M(T), \quad \forall T \tag{4.10}$$

Therefore, if the exchange law implies fairness for O at time T_0 *only for this time* it follows that

$$M(T_0) = W(T_0) \tag{4.11}$$

However, if for such a law the invariance property holds (in particular if the s.dec holds) (4.11) implies that

$$M(T) = W(T), \quad \forall T \tag{4.12}$$

In the *continuous case*, with operations spread in *(t',t'')*, the reserves are obtained by adopting the previous formulae. Therefore:

– in the general case of two variables law *z*:

$$M(T;O,z) = -\int_{t'}^{T} \varphi(t) z(t,T) dt \;\; ; \;\; W(T;O,z) = \int_{T}^{t''} \varphi(t) z(t,T) dt \tag{4.5'}$$

– with the non-decomposable uniform law:

$$M(T) = -\int_{t'}^{T} \varphi(t) g(T-t) dt \;\; ; \;\; W(T) = \int_{T}^{t''} \varphi(t) g(T-t) dt \tag{4.7'}$$

– with the exponential law:

$$M(T) = -\int_{t'}^{T} \varphi(t) e^{\delta(T-t)} dt \;\; ; \;\; W(T) = \int_{T}^{t''} \varphi(t) e^{-\delta(t-T)} dt \tag{4.9'}$$

while with any s.dec law with intensity $\delta(\lambda)$ the exponentials in (4.9') must be replaced by $\exp(\int_t^T \delta(\lambda)\, d\lambda)$.

With a *mixed operation*, valued with a law of two variables, the retro-reserve is obtained by adding *M(T)* written in (4.5) and (4.5') and the pro-reserve by adding *W(T)* written in (4.6) and (4.5'); when particular regimes are used, the aforementioned corresponding expressions for *M(T)* and *W(T)* must be added.

Observation

The previous definitions need some interpretation. As mentioned in footnote 1 of Chapter 1, a financial transaction is usually coupled with a real transaction of opposite side. In particular, an operation *O* can be concerned with financial transactions, managed by Mr A, connected with the management of a company for the production or the trade of assets or services, which we call *project O*. According to the accounting principle of "double entry", such transactions are registered by Mr A on an account giving interest, assigned to *O*; each payment implies a charging of the account and then the creating of a credit of Mr A (or the settlement of a debt) while each income implies a crediting on the account and then the creating of a debt of Mr A (or the settlement of a credit).

Given that according to the definitions the retrospective reserve *M(T;O,z)* represents the financial statement of Mr A at time *T* following the transactions with the sub-operation *O'*; it is positive or negative (i.e. a credit or a debit for Mr A) depending on the fact that before *T* is greater the number of payments or else incomes for Mr A, valued financially in T through the exchange law z. Consequently, $M(T;O,z)$ is the amount that, if the supplies of O subsequent to T would cancel, Mr A should cash (algebraically) in T such that the resulting operation, consisting of $\{O' \cup [T, M(T;O,z)]\}$, would be fair in T.

Vice versa, the prospective reserve $W(T)$ is the capital value in T of the supplies of the sub-operation O'', i.e. the amount that, if the supplies of O before T would cancel, Mr A should pay (algebraically) in T such that the resulting operation, consisting of $\{O'' \cup [T, -W(T;O,z)]\}$, is fair in T.

The names *retrospective reserve* and *prospective reserve* are also used (referring to expected values) in the stochastic financial insurance operations.

Exercise 4.4

Let us consider again Exercise 4.3 and observe that O is not fair, given that $V(4)$=867.97. It is enough to add the supply (4, -867.97) to obtain, using a rate of 6%, a fair operation (at each time, given that the adopted law is decomposable)

$$\hat{O} = \{(0, -1{,}500) \cup (2.5, -1{,}850) \cup (3.5, 520) \cup ((4, -867.97) \cup (5; 4{,}500)\}$$

Calculate the reserves of $\hat{O}$ in T=3 verifying the validity of (4.12).

A. By applying (4.9) we obtain for $\hat{O}$:

$$M(3) = 1{,}500 \cdot 1.06^3 + 1{,}850 \cdot 1.06^{0.5} = 3{,}691.22$$

$$W(3) = -867.966 \cdot 1.06^{-1} + 520 \cdot 1.06^{-0.5} + 4{,}500 \cdot 1.06^{-2} = 3{,}691.22$$

Using an Excel spreadsheet for the same calculation, we have to proceed as follows. For the calculation of the retro-reserve it is necessary to take into account only the supplies before T=3; therefore, expanding along the columns, the supplies below are not considered; vice versa for the calculation of the pro-reserve it is necessary to take into account only the supplies after T=3; therefore, expanding along the columns, the supplies above are not considered. To do this (if the Excel macros are not applied), given that

$$a' = (a + |a|)/2a = 1 \text{ if } a > 0, = 0 \text{ if } a < 0; \; a'' = (a - |a|)/2a = 0 \text{ if } a > 0, = 1 \text{ if } a < 0$$

then for the calculation of $M(T)$ in the 1st of (4.9) $-S_h(1+i)^{(T-T_h)}$ is preserved if $T - T_h > 0$ and we use 0 if $T - T_h < 0$; on the contrary for the calculation of $W(T)$ in the 2nd formula of (4.9) we use 0 if $T - T_h > 0$ and $S_h(1+i)^{-(T_h - T)}$ is preserved if $T - T_h < 0$. Therefore, setting such values by columns, we obtain the retro-reserve adding the amounts valued at time 3 multiplied by a', while the pro-reserve is obtained adding the amounts valued at time 3 multiplied by a''.

i = 0.06				T = 3			
T_h	S_h	T-T_h	$S_h(T)$	a'	a''	$Am.(T>T_h)$	$Am.(T<T_h)$
0.0	-1,500.000	3.00	-1,786.52	1.0	0.0	1,786.524	0.000
2.5	-1,850.000	0.50	-1,1904.69	1.0	0.0	1,904.692	0.000
3.5	520.000	-0.50	505.07	0.0	1.0	0.000	505.069
4.0	-867.967	-1.00	-818.84	0.0	1.0	0.000	-818.837
5.0	4,500.000	-2.00	4,004.98	0.0	1.0	0.000	4,004.984
						3,691.216	3,691.216

Table 4.2. *Calculation of retro-reserves and pro-reserves*

The Excel instructions are as follows. The first two rows are for data and titles; C1: 0.06; F1: 3. From the 3rd to 7th rows:

– column A (maturity): A3: 0; A4: 2.5; A5: 3.5; A6: 4; A7: 5;

– column B (amounts): B3: –1,500; B4: –1,850; B5: 520; B6: 867.967; B7: 4,500;

– column C (maturity): C3: = F$1-A3; copy C3, then paste on C4 to C7;

– column D (amounts valued in 3): D3:= B3*(1-C$1)^(F$1-A3); copy D3, then paste on D4 to D7;

– column E (a' = indicates *M(T)*): E3: = (C3+ABS(C3))/2/C3; copy E3, then paste on E4 to E7;

– column F (a" = indicates *W(T)*): F3: = (C3-ABS(C3))/2/C3; copy F3, then paste on F4 to F7;

– column G (amounts for *M(T)*): G3: = -D3*E3; copy G3, then paste on G4 to G7;

– column H (amounts for *W(T)*): H3: = D3*F3; copy H3, then paste on H4 to H7; (retro-reserve in 3): G8: = SUM(G3;G7); (pro-reserve in 3): H8: = SUM(H3;H7). Then: G8 = H8.

Exercise 4.5

From the data of the operation considered in Exercise 4.1, we observe that

$$\hat{O}=\{(0, -1{,}500)\cup(\tfrac{2.5}{12}, -1{,}850)\cup(\tfrac{3.5}{12}, 520)\cup(\tfrac{4}{12}, -1{,}610.44)\cup(\tfrac{5}{12}, 4{,}500)\}$$

is fair in $T_0 = \frac{4}{12}$. Verify the validity of (4.11) if $T_0 = \frac{4}{12}$ and its non-validity (i.e. unfairness of $\hat{O}$) if $T_0 = \frac{1}{12}$, due to the non-decomposability of the adopted laws.

A. We add $(\frac{4}{12}, -1{,}610.44)$ to the payments after $\frac{4}{12}$. By calculating the accumulation with an SDI law at the annual rate of 5.5% we obtain $M(\frac{4}{12})$ = 3,390.18; by calculating the discount with an RD law with the same rate we obtain $V(\frac{4}{12})$ = 3,390.18 = $M(\frac{4}{12})$. By evaluating at the time $\frac{1}{12}$, we obtain $M(\frac{1}{12})$ = 1,506.87; instead $V(\frac{1}{12})$ = 1,507.12 $\neq M(\frac{1}{12})$, then $\hat{O}$ is not fair if valued at such time.

The "differential equation of accumulated value" with principal flow

A particular mixed O in the interval $(0,T)$, which at the same time allows a generalization of (3.35), is obtained by considering an initial supply $(0,S_0)$ and other later supplies with infinitesimal amounts $(t,\varphi(t)dt)$, $(0 \leq t \leq T)$, following a continuous principal flow $\varphi(t)$. This is useful to schematize the management of a small firm, by considering an initial cost for establishment and then small financial transactions as inflows and outflows.

The accumulation of interest always proceeds according to the cci law with instantaneous intensity δ. With such a hypothesis the retro-reserve $M(t)$ varies for effect of the financial transactions due to the flow $\varphi(t)$, as well as for the continuous accumulation of interest, due in S_0 and in $\varphi(\tau)d\tau$, $(\tau \leq t)$ according to the flow $\delta \cdot M(t)$. This process can then be obtained by solving the following linear and non-homogenous differential equation:

$$M'(t) = \delta\, M(t) - \varphi(t) \tag{4.13}$$

In fact, generalizing (3.34'), $\forall t \in (0,T)$ in the given hypothesis the dynamics of the retro-reserve are described by

$$M(t+dt) = M(t) + \delta\, M(t)\, dt - \varphi(t)dt + o(dt) \tag{4.13'}$$

Dividing by dt and taking the limit $dt \to 0$, given that $o(dt)/dt \to 0$, (4.13) follows. This equation is also called the *differential equation of the accumulated value*. It is indeed easy to see that the retro-reserve in T coincides with the amount, valued in T, of the invested principal due to the outflows, subtracting the inflows, before T.

The analytical solution of (4.13) is immediate. In fact, by multiplying both members by $e^{-\delta t}$, it is soon found that the general integral is

$$M(t) = e^{\delta t}\left\{-\int \varphi(t)e^{-\delta t}dt + \text{constant}\right\} \tag{4.14}$$

and, due to the continuity of *M(t)* after 0, the particular solution of (4.13), where $M(0) = -S_0$, is

$$M(t) = e^{\delta t}\left\{M(0) - \int_0^t \varphi(\tau)e^{-\delta\tau}d\tau\right\} = M(0)e^{\delta t} - \int_0^t \varphi(\tau)e^{\delta(t-\tau)}d\tau \tag{4.14'}$$

Value (4.14') can be financially interpreted observing that, due to the s.dec of the exponential exchange law, the retro-reserve in *t* can be obtained by accumulating in *t* the property evaluations in 0 connected to the financial transactions occurring between 0 and *t* (and then of opposite sign).

If an s.dec law is used with intensity $\delta(\lambda)$ instead of the cci, solution (4.14') is generalized as

$$M(t) = M(0)\exp\left(\int_0^t \delta(\lambda)d\lambda\right) - \int_0^t \varphi(\tau)\exp\left(\int_\tau^t \delta(\lambda)d\lambda\right)d\tau \tag{4.15}$$

EXAMPLE 4.1.– Mr. B opens in a financial institution a c/a both for deposit (when Mr B is in credit) and for lending (when Mr B is in debt), ruled by an exponential exchange regime and with *reciprocal rate*, i.e. with the same instantaneous intensity, both the earned interest on the credits and the passive interest on the debts are obtained and converted time by time. Assuming the monetary unit MU = €1,000 euros, let us suppose that the transaction in the c/a in the interval (0,*t*) is given by a deposit in 0 of MU 25.48 followed by deposits and withdrawals based on a continuous flow which is assumed with a parabolic shape $\varphi(\tau) = a+b\tau+c\tau^2$, $(0\le\tau\le t)$. Let us use $t=2$, finding the function $\varphi(\tau)$ by interpolation on the basis of the values at times 0, 1, 2, that are respectively: $\varphi(0) = -4$ (= infinitesimal payment $-4dt$), $\varphi(1) = +5$ (= infinitesimal income $5dt$); $\varphi(2) = +12$ (= infinitesimal income $12dt$). Mr B wants to estimate the retro-reserve at time 2, i.e. his position $M(2)$ (of credit if $M(2)>0$, of debit if $M(2)<0$), with a CCI law at the reciprocal annual rate of 5%.

To do this we firstly need to calculate the flow function imposing its passage for the points $P_i = (i, \varphi(i))$, i=1,2,3, and deducing the parameters *a, b, c*. Then we have to solve the linear system

$$a = -4\ ;\ a+b+c = 5;\ a+2b+4c = 12$$

which has as unique solution: $a = -4$, $b = +10$, $c = -1$. Then: $\varphi(\tau) = -4+10\tau-\tau^2$. In addition, $\delta = \ln(1.05) = 0.04879$. Therefore, due to (4.14'), $M(2)$ is obtained from the following expression

$$M(2) = e^{0.09758}\left\{25.48-\int_0^2[-4+10\tau-\tau^2]e^{-0.04879\tau}d\tau\right\}.$$

Integrating *by parts*: $\int[-4+10\tau-\tau^2]e^{-0.04879\tau}d\tau =$

$$= \frac{-e^{-0.04879\tau}}{0.04879}\left[\left\{-4+\frac{1}{0.04879}\left(10-\frac{2}{0.04879}\right)\right\}+\left(10-\frac{2}{0.04879}\right)\tau-\tau^2\right]+cost.$$

and then the integral: $\int_0^2[-4+10\tau-\tau^2]e^{-0.09758}d\tau =$

$$=\frac{1-e^{-0.09758}}{0.04879}\left\{-4+\frac{1}{0.04879}\left(10-\frac{2}{0.04879}\right)\right\}-\frac{e^{-0.09758}}{0.04879}\left\{\left(10-\frac{2}{0.04879}\right)2-4\right\} =$$
$$= 8.64434$$

Therefore: $M(2) = e^{0.09758}\{25.48-8.64434\} = 18.56131$ MU

It follows that, withdrawing from $\varphi(\tau)$ the accumulated incomes on the accumulated payments, the result in t=2 is negative, i.e. there is a decrement of credit, equal to €9,530.38, compared to 25480·1.1025 = €28,091.69 = credit in the c/a that Mr B would have in absence of the flow $\varphi(\tau)$ in the interval [0,2].

4.3. Usufruct and bare ownership in "discrete" and "continuous" cases

Assuming an s.dec exchange law z and supposing that the operation O is fair at a given time T_0 and then, due to the invariance of z, $\forall T$ (if this is not true, to make O fair it is enough to add to the original supplies $(T_0,-V(T_0))$), it is important, under the practical point of view the *decomposition into two parts of the pro-reserve W(T)* at time T (where inside the symbol in $W()$ are implicit the symbols O and z):

a) the first, called *usufruct* and indicated by *U(T)*, is the evaluation in T of the financial transactions due only to the interest settled after T;

b) the second, called *bare ownership* and indicated by *P(T)*, is the evaluation in T of the remaining transactions, i.e. the supplies of O'' without interest.

Then by definition

$$W(T) = U(T) + P(T),\ \forall T \tag{4.16}$$

and, once *W(T)* and *U(T)* are calculated, the bare ownership is given by the difference. The distinction made in (4.16) is important, because in a financial operation there can be difference in the owners of the rights to the two supplies.

The calculation of the usufruct and the bare ownership is usually done according to the *discrete* scheme that approximates the scheme of the interest continuous formation. Taking as an example a lending operation with periodic payments, at such times the *interest shares* are also calculated based on the current debt position and the time that has passed since the last payment. The *usufruct in discrete case U(T)* is then the present value in *T* of the interest shares after *T* and the *bare ownership in discrete case P(T)* is the present value in *T* of the principal share after *T*. On this point we will talk about amortizations and their evaluation in the following chapters.

However, usufruct and bare ownership can be calculated rigorously according to the *continuous* scheme, assuming the *continuous payment of interest* as obtained. Indicating by $\tilde{U}(t)$ and $\tilde{P}(t)$ the *usufruct and bare ownership in continuous case*, we have

$$W(t) = \tilde{U}(t) + \tilde{P}(t),\ \forall t \tag{4.16'}$$

For $\tilde{U}(t)$ and $\tilde{P}(t)$ clearly additivity holds, i.e. they are obtained as the sum of the usufructs and the bare ownerships, calculated by the continuous scheme, of each supply of *O"*.

Let us find the expression for $\tilde{U}(t)$ and $\tilde{P}(t)$ in the hypothesis that *O* consists only of transactions (T_h,S_h). Let $\delta(\lambda)$ be the instantaneous intensity connected to an s.dec law and then dependent only on the current time λ). Also let T_h $(h = r+1,\dots,n)$ be the maturity times later than *T*. We obtain

$$\tilde{U}(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\} \int_T^{T_h} \delta(\lambda)d\lambda \tag{4.17}$$

$$\tilde{P}(t) = W(t) - \tilde{U}(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}\left[1 - \int_T^{T_h} \delta(\lambda)d\lambda\right] \tag{4.18}$$

Proof

Given the additivity, we first consider one supply (T_h,S_h) with *h* fixed between $r+1$ and *n*. The earned interest between time *X* and $X+dX$ is (not considering errors that vanish with the following integration) $M(X)\delta(X)dX$, but because *z* is s.dec and *O* is fair, we have $M(X) = W(X)$, $\forall X$. To calculate $\tilde{U}(t)$ we must integrate from *T* onwards the interest $M(X)\delta(X)dX = W(X)\delta(X)dX$ discounted in T, multiplying by $\exp\left\{-\int_X^T \delta(\lambda)d\lambda\right\}$. The prospective reserve on (T_h, S_h) is: $W_h(X) = S_h \exp\left\{-\int_X^{T_h} \delta(\lambda)d\lambda\right\}$ if $X \le T_h$; otherwise $W_h(X) = 0$ and then the integration of interest for $X > T_h$ gives no contribution. After some calculations we obtain:

$$\tilde{U}(t) = \sum_{h=r+1}^{n} W_h(X)\delta(X)\exp\left\{-\int_T^X \delta(\lambda)d\lambda\right\}dX =$$

$$\sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}\int_T^{T_h} \delta(\lambda)d\lambda$$

i.e. (4.17) holds. Expression (4.18) for $P(t)$ is obtained subtracting the 2nd member of (4.17) from $W(t) = \sum_{h=r+1}^{n} S_h \exp\left\{-\int_T^{T_h} \delta(\lambda)d\lambda\right\}$.

Let us find the expression for $\tilde{U}(t)$ and $\tilde{P}(t)$ maintaining the hypothesis of fair O, which we assume to be mixed, extended to the time interval (t', t'') with discrete and continuous supplies, and assuming an exponential exchange law with intensity δ. Because of uniformity of the law, we can assume the beginning of the operation in the origin of time, i.e. $t' = 0$. Using (4.17) and (4.18) for the discrete component and considering that for the continuous component it is enough that the sums of the amounts S_h, with $t_h \in (t, t'')$, are replaced by integral on the time of the flows $\varphi(\lambda)$ with $\lambda \in (t, t'')$, the following formulae are easily obtained

$$\tilde{U}(t) = \delta \sum_{h=r+1}^{n} S_h e^{-\delta(t_h - t)}(t_h - t) + \delta \int_t^{t''} \varphi(\lambda) e^{-\delta(\lambda - t)}(\lambda - t)d\lambda \tag{4.17'}$$

$$\tilde{P}(t) = \sum_{h=r+1}^{n} S_h e^{-\delta(t_h - t)}[1 - \delta(t_h - t)] + \int_t^{t''} \varphi(\lambda) e^{-\delta(\lambda - t)}[1 - \delta(\lambda - t)]d\lambda \tag{4.18'}$$

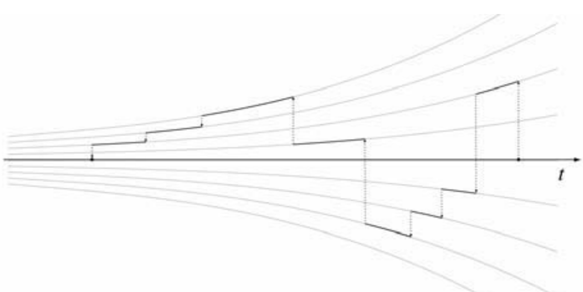

Figure 4.1. *Plot of the values M(t) =W(t) if O is fair*

4.4. Methods and models for financial decisions and choices

4.4.1. *Internal rate as return index*

We will now discuss the parameters of implicit return in a financial operation, for which we have already considered the evaluation of the whole or of some of the parts (reserve, usufruct or bare ownership), as well as the decisional criteria for financial operations (discrete) $O = \{T_h\}\&\{S_h\}$ that, considering the set of economic and technical facts below the set of supplies (S_h, T_h), we will call *financial projects* (of whatever type: realized in agriculture, industry, commerce, services, etc.). It is fundamental to give a general definition of *internal rate* (of return for the investor, of costs for the borrower), relative to O.

Let us recall the advantage of using a uniform financial law, according to which all evaluations can be performed at time zero, and then the exchange factors are all discount factors. If the law is also decomposable, which we will suppose to be true from now on, then it is the exponential law, characterized by a constant intensity. With reference to the payments, we will still use a positive sign for the incomes (or receipts, or cash inflows) and a negative sign for the outcomes (or outlays, or cash outflows) from the point of view of the subject who evaluates. We should also recall that an operation at a given rate is fair if its balance at a given time (and then at all times if the law is decomposable) is zero. However, we have seen in section 4.1 that the fairness of an operation depends not only on its supplies, but also on the used exchange law z. If we adopt an exponential exchange law, this is identified by the annual rate i.

The rate i^* of the CCI (or exponential) exchange law that makes the given operation O fair, i.e. which makes (4.4) zero, is called the *internal rate of return* (IRR)[5]. It summarizes the return of the project made in O and, as we will see, is normally used as the basis of a decisional criterion on financial projects.

Let us analyze the concept of IRR to better clarify its meaning and its limits as a return measure. We have already shown that in a simple financial operation, of pure exchange, made of the supplies $(T_1,-C)$, $(T_2,+M)$, the percentage variation $(M-C)/C$, i.e. the related rate per period, is a measure of the return of the operation. More generally, for a complex operation O, defined in (4.1) or (4.1') with $n>2$, the IRR of the operation $O = \{T_h\}\&\{S_h\}$ is generalized as mentioned, as the interest rate of the exponential operation that makes O fair.

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5 We note that with respect to O an operator can be both the investor and the borrower. We also observe that $V(T;O,z) = 0$ with an exponential law at rate i^* implies, due to the decomposability of the law, a compensation between payments and incomes of O taking into account the interest at the rate i^* and the current balances.

The meaning given to the internal rate, as a parameter whose use assures the fairness to the operation, implies that:

– such a parameter can be used to measure the investment return (or the financing cost) in the sense that summarizes the instantaneous returns, also a variable in time, in their evolution in the time interval of O;

– therefore, it corresponds to a constant instantaneous intensity in such a time interval, which becomes an exponential financial law.

Indeed we have seen in section 4.1 that if O is fair in cci at the rate i^*, clearly *the discount factor i*=IRR is also the return rate* inherent in the supplies of O together with the interest on the current reserves. We observe that it is not necessary that the retro-reserves (coinciding with the pro-reserves if O is fair) keep their sign in the time (i.e. that O is a *pure project*). The property also holds in the case of sign alternation (O = *mixed project*), as long as i^* is a reciprocal rate, i.e. it is valid both for earned and passive interest[6]. Using (4.4) with $i=i^*$ we obtain

$$- S_1 = \sum_{h=2}^{n} S_h \,(1+i^*)^{-(T_1-T_h)} \tag{4.4''}$$

i.e. O can be interpreted, considering the case $S_1>0$, as the investment of S_1 at time T_1 that gives rights to the supplies (T_h,S_h), $h=2,...,n$, and i^* is the return rate of the operation.

If the payments are *periodic*, it is not restrictive to assume a *unitary* period (changing the unit measure and using the equivalent rate in the compound regime). In such a case, using $v = (1+i)^{-1}$, owing to (4.4) the calculation of IRR starts from the equation

$$V(0;O,i) = \sum_{h=1}^{n} S_h \, v^h = 0 \tag{4.19}$$

i.e. an algebraic equation of degree n in the unknown v. From the solution v^* the IRR $i^* = 1/v^* - 1$ is obtained. Well known theorems give information on the solutions of (4.19) in relation to the coefficients S_h.

6 For the general *mixed project* ruled by a non-reciprocal rate, it is necessary to use more general methods, which will be discussed in section 4.4.6. The interpretation of IRR as a return index in the mixed projects also implies a more rigorous widening of its meaning. This would lead to giving a return meaning to the IRR only when it is an index of totally detached or totally incorporated return, and only in the case of uniqueness of IRR for a given operation (see footnote 7).

The existence and uniqueness of the IRR for the operation *O* is not always verified. However, in order that the problem of IRR is mathematically *well posed* and financially meaningful as a return index, it is necessary that the solution v^* (and then i^*) exists and is unique. In this case we will say that i^* is IRR *operative*. This eventuality is verified in some special type of investments (or also of financings, obtainable reversing the algebraic sign of investment amounts), in which there is only one sign inversion in the sequence of amounts S_h. We will consider this later.

The value V(0;*O*,*i*) is called the *discounted cash-flow* (DCF) and i^* is the rate that makes the DCF become zero.

4.4.2. *Outline on GDCF and "internal financial law"*

The concepts of DCF and IRR can be generalized going from a flat structure of interest rates to any structure, as long as it follows an s.dec financial law.

If we consider a financial operation *O* made of the amounts s_0, s_1,...,s_n paid at increasing time t_0, t_1,...,t_n (= intervals from a given time origin) then $O = \bigcup_{h=1}^{n}(t_h, s_h)$. We know that according to the signs of s_h, *O* can be an investment or a financing. Let us evaluate according to a exchange law $z(\xi,\eta)$ that, using $x<y$, becomes $z(y,x) = a(y,x)$ for discount from *y* to *x*, while $z(x,y) = m(x,y)$ for accumulation from *x* to *y*, being $m(x,y) = 1/a(y,x)$ in the symmetric hypothesis. Let us only consider discounting. Then the functional $G(\mathbf{a}) = \sum_{h=0}^{n} s_h\, a(t_h, t_0)$ depends on the function *a* and is called the *generalized discounted cash-flow* (GDCF) of *O* in t_0.

If $G(\hat{\mathbf{a}}) = \sum_{h=0}^{n} s_h\, \hat{a}(t_h, t_0) = 0$ results, the discount law is called the *internal financial law* (IFL) for *O* and is identified with $\hat{a}(t_h, t_0)$, defined in the payment times. Using IFL we obtain the fairness of *O*, i.e. the balancing between income and payments valued in t_0.

If $\hat{a}(t_h, t_0) = (1+i^*)^{-(t_h - t_0)}$, the GDCF gives rise to DCF and the IFL gives rise to the only parameter i^* (=IRR), that is the *internal rate of return* for *O*.

If the law $a(y,x)$ is s.dec and symmetric, we have $a(y,x) = 1/m(x,y)$ and the fairness does not depend on the evaluation time. Given the payment times $\{t_0, t_1,...,t_n\}$, we put

$$a_h = a(t_{h\text{-}1}, t_h)\ ;\ m_h = m(t_{h\text{-}1}, t_h),\ h=1,....,n \tag{4.20}$$

where $m_h = 1/a_h$.

Let us consider again the retro-reserve for *O*, which coincides with the pro-reserve because of the fairness. Let us indicate by c_h the retro-reserve calculated at time t_h *just after* the transaction s_h, which represents the credit obtained in t_h due to the previous transactions. The extreme values of the sequence $\{c_h\}$ are constrained by:

– $c_0 = -s_0$, having no earned interest yet;

– $c_n = 0$, due to the fairness of *O* valued with the IFL.

For the other values there exists a wide flexibility, connected to the choice of the accumulation factors $\{m_h\}$.

The following fundamental theorem holds.

THEOREM.– *For all operations O consisting of the cash-flow* $\bigcup_{h=1}^{n}(t_h, s_h)$, *each sequence* $\{c_h\}$ *of retro-reserves, with* $c_0 = -s_0$, $c_n = 0$ *under the condition* $c_h \neq 0, \forall h<n$, *gives rise biuniquely to a sequence of per period accumulated factors* $\{\hat{m}_h\} = \{\hat{m}(t_{h\text{-}1}, t_h)\}$ *that form an IFL for O, owing to the recurrent system*

$$c_h = c_{h\text{-}1}\, m_h - s_h\,,\ h = 1,....,n. \tag{4.21}$$

Proof

The evidence of the biunique correspondence is based on the fact that the constraints give rise to a system of determined equations, which, given the supplies (t_h, s_h), identifies $\{m_h\}$ as a function of $\{c_h\}$ and vice versa. We can prove that $\{m_h\}$ is an IFL for *O* proceeding by induction on *h*. In fact

$$c_h = c_0 \prod_{z=1}^{h} m_z - \sum_{u=1}^{h} s_u \prod_{z=u+1}^{h} m_z = c_0\, m(t_0, t_h) - \sum_{u=1}^{h} s_u\, m(t_u, t_h) \tag{4.22}$$

due to the decomposability of the financial law. Using $h=n$ and taking into account the constraints on the extreme of $\{c_h\}$, we obtain the fairness condition

$$\sum_{u=0}^{n} s_u\, m(t_u, t_n) = 0 \tag{4.23}$$

satisfied by the sequence $\{\hat{m}(t_{h\text{-}1}, t_h)\}$, which is IFL for *O*. Going backwards, we can prove the opposite. From the factors $\hat{m}_h$ forming an IFL, we can obtain the intensities for the interval $(t_{h\text{-}1}, t_h)$

$$j_h = (\hat{m}_h)^{1/(t_h - t_{h-1})} - 1 \tag{4.24}$$

that give rise to the corresponding annual return.

4.4.3. *Classifications and property of financial projects*

If the retro-reserve of a project at the initial time 0 is zero, the value V at this time obviously coincides with the pro-reserve W.

If $V(i)$ is a decreasing function of i in the interval $(0,+\infty)$ and if (from the point of view of the investor)

$$\begin{cases} \lim_{i \to 0^+} V(i) = \sum_h S_h > 0 \\ \lim_{i \to +\infty} V(i) = S_0 < 0 \end{cases} \tag{4.25}$$

this is a sufficient condition for existence and uniqueness of a positive solution $i=i^*$ of (4.19) that gives the internal rate (sometimes called *implicit rate*), while its definition has no operative meaning without uniqueness.

The previous sufficient condition implies that for the investor the initial supply is a payment and that the algebraic sum of the amounts of O is positive, i.e. there exists a reward given by the surplus between incomes and payments. Starting the project with a payment (or an income) is in fact a characteristic of *investment operations* (or *financing operations*). The following definitions hold.

A project is called:

– *investment in the strict sense*, if all payments come before all incomes;

– *investment in the broad sense*, if the average maturity (defined in section 2.5.2) of payments comes before that of incomes at any evaluation rate.

We have a *financing (in strict or broad sense)* if the aforementioned definitions hold after the inversion of the sign of the amounts.

A project can be characterized by input and output amounts paid only one time or spread over more times, giving rise to four possibilities:

1) PIPO (= point, input, point output);

2) CIPO (= continuous input, point output);

3) PICO (= point input, continuous output);

4) CICO (= continuous input, continuous output).

For PIPO and PICO an investment project is *simple* if it is formed by one payment followed only by incomes; symmetrically a financing project is *simple* if it is formed by one income followed only by payments. For the simple projects of investment (or financing), the decreasing monotonic (or, respectively, increasing monotonic) *V(i)* is assured, and if

$$\left[\sum_h S_h + S_0 > 0\right] \cap \left[S_0 < 0\right] \text{ or } \left[\sum_h S_h + S_0 < 0\right] \cap \left[S_0 > 0\right]$$

the operative internal rate exists.

Considering investment projects, it is interesting to give more general conditions for the existence for the operative IRR. It can be proved that in the hypothesis of $V(0) > 0$ of the project, besides the simple investment PIPO and PICO projects (which are investment in strict sense), there is existence and uniqueness of the solutions of (4.19) also in the following cases:

– in the other investment projects in the strict CIPO and CICO sense;

– in the investment project in the broad sense, of type CICO, when the condition (also sufficient because the project is an investment in the broad sense) that the arithmetic mean of the time of payments (= their average maturity when $i \to 0$) comes before the time for the first income (= average maturity of incomes when $i \to +\infty$) is satisfied.

Indeed, in both cases *V(i)* is decreasing until it remains positive, and approaches $S_0 < 0$ when *i* diverges, which gives the existence and uniqueness of its roots.

The IRR has the following properties:

– it does not change with a proportional change of the amounts;

– the project sum of two projects with internal rate has an internal rate with value between the rates of the two projects, then, if the two rates coincide, the rate of the project sum coincides with them.

If the aforementioned conditions for the average maturities are not satisfied, we can lose the existence or uniqueness of the roots of (4.19), no longer having the possibility of defining an operative IRR. Reversing the sign of the amounts in the cash-flow, the investment becomes a financing. Then the decreasing of *V(i)* changes to increasing and the return rate is a cost rate for the financing, maintaining the previous property.

Summarizing the operative interpretation of the IRR of an investment project, this rate is (if it exists) a return index, because it is just the interest rate of a profitable account fed only from the financial transactions connected with the project, such that, also considering the interest, the balance is zero just after the last transaction[7].

EXAMPLE 4.2.– We give here two examples, regarding cash-flow of investment projects, for which the properties defined above are satisfied. To consider financing projects of the same type, it is enough to reverse the sign of the monetary amounts:

– a project of *investment in the strict sense* of type CICO, called $\mathcal{A}$, is as follows. In an industrial plant the following costs and revenues apply: for the first 2 semesters only costs apply; so €70,000 in the 1st semester for buying the plant, and €40,000 in the 2nd semester for installations in the plant; in each of the following 10 semesters we have operating costs for €6,000 and income for €19,000; in the 13th semester the divestment of the plant occurs with a net return of €15,000. The algebraic sum of these transactions is €35,000, thus it is a profitable investment. Valuing the amounts at the end of each semester, the cash-flow of $\mathcal{A}$ is shown by the following graph:

-70,000	-40,000	13,000	13,000	....	13,000	13,000	15,000
1	2	3	4		11	12	13

$\mathcal{A}$ is an investment project in the strict sense, with a unique IRR, because when balancing in the year incomes and payments, the time of the last (net) payment comes before that of the first (net);

– an *investment project in the broad sense* assumes an average maturity of payments preceding that of incomes (in net terms) at whatever evaluation rate is used; for this it is sufficient condition that the arithmetic mean maturity of payments (= their average maturity when $i \to 0$) comes before the time of first income (= average maturity of incomes when $i \to +\infty$).

7 Here it is assumed that, if the balance does not remain constant in sign due to the dynamic of the financial supplies, the allowed interest rate is the same as the charged rate, i.e. the c/a bears reciprocal rate. In some cases, this is a strong limitation, the overcoming of which requires a more general approach (see section 4.4.6).

An example of such a type of project can be obtained by modifying the project $\mathcal{A}$ in $\mathcal{B}$ adding a payment of €15,000 after 6 semesters. Compensating with a net operating income in the period for €13,000 we obtain for the 6th semester a net payment of €2,000 and then the cash-flow of $\mathcal{B}$ can be described by the following graph:

-70,000	-40,000	13,000	13,000	13,000	- 2,000	3,000	....	13,000	15,000
1	2	3	4	5	6	7		12	13

Measuring in semesters, the arithmetic mean maturity of $\mathcal{B}$ payments is:

$$(-70{,}000 - 40{,}000 \cdot 2 - 2{,}000 \cdot 6)/(-112{,}000) = 1.4469$$

whereas the time for the first income is 3 and then the project $\mathcal{B}$ is an investment in the broad sense with unique IRR; it is profitable because the algebraic sum of the amounts is +5,000.

4.4.4. *Decisional criteria for financial projects*

It is fundamental in mathematical finance to give a criterion to decide if it is convenient or not for an economic subject to realize a project identified by the financial operation O[8].

Not considering a particular criterion based on parameters and particular points of view, we will focus our attention on the two more important criteria as they are better justified in general in the light of financial equivalencies and are universally used in business practice.

The first criterion is based on the *value* V of O in a given time of evaluation[9]. This has been defined in this chapter and its meaning is clear in quantifying O. The second criterion is based on the *internal rate of return* of O, already defined, if it is operative.

Usually the first criterion is considered *subjective*, because V depends on the evaluation rate x that is subjectively fixed by the decision maker, while the second criterion is considered *objective*, because the internal rate depends only on objective elements, such as the fixed supplies of O. However, looking carefully at both

8 For further discussion on the internal rate and, in general, on decision and choice in financial projects, see in Italian literature: Levi (1967); Trovato (1972).

9 If the valuation time is not after the beginning of the operation or if the supplies before such time do not matter, the value V can be changed with the prospective reserve of O.

criteria, as we will see later, the decision follows from a comparison between an *external* value (due to the *market*, which implies a *subjective* evaluation by the decision maker) and an *internal* value (connected with the *objective* features of the *project*, i.e. its supplies or internal rate).

It is clear that the evaluation depends on the choice of the financial law, but if, as is usually the case, an exponential regime is used, a feature of which is a flat structure of rates, it is sufficient to fix the annual rate to identify the exponential law. In this way the decision is not influenced by a changing of the evaluation time and not even by a uniform deferment of the financial supplies.

We can therefore enunciate the *first decisional criterion* in the following way.

CRITERION OF THE PRESENT VALUE (PV): *the project identified by a financial operation O^* of investment or financing is convenient for the economic subject "A", and then the decision on it is positive, if and only if, at the evaluation rate x^* chosen by "A", the value V of O^* at the evaluation time results in: $V(x^*) > 0$. The project identified by O^* is not convenient, and then the decision on it is negative, if and only if in the same conditions the result is: $V(x^*) < 0$. Finally, if and only if $V = 0$ the project is indifferent*[10].

This is reasonable according to the profit criterion: if and only if the cash-flow of the projects implies withdrawals and deposits on a profitable account at the (reciprocal) rate x^* (that – taking into account the received and allowed interest on the balance, initially zero, that is accruing from time to time – give rise to a final positive spread, for which the discounted evaluation $V(x^*)$ remains positive), the project is convenient. Otherwise it is unacceptable or, at most, indifferent.

From the above the criterion to choose the evaluation rate immediately follows: it is necessary to choose the *market rate* of the financial operations, which are alternatives to the examined project.

We could not add anything on the projects decision, given the overall validity of the PV criterion and its dependence on the fundamental principles of financial

10 The decisional criterion can be extended to many projects, if the decision maker has available funds and he is interested in sustaining more than one, in the following way: *given $n \geq 2$ investment projects $O_1,\dots,O_n$, each being convenient according to the evaluation rate x^* (concerning the opportunities of financing the projects, or by vanishing profit or by rising cost), let us put them in decreasing order of their value, such that $V_1(x^*) > \dots > V_n(x^*)$. For the decision maker it is convenient to carry on the first $r \leq n$ projects with values $V_1(x^*),\dots, V_r(x^*)$ for which he has enough funds. He can also add one of the subsequent projects, in convenience order, if it can be split (as, for example, is done with stocks or company's share), thus buying a part of the project.*

equivalency and on the arbitrariness of the choice of the evaluation law. Furthermore, we need to take into account the common wish of economic operators, especially in the business world, to fix a criterion on an objective basis. This explains the wide spread of the *second decisional criterion*, which can be given as follows.

CRITERION OF THE INTERNAL RATE OF RETURN (IRR): *if an investment project, identified by an operation O^*, with internal rate i^* (to be considered like return rate), is convenient, then the decision is positive, if and only if $i^* > x^*$, where x^* is the evaluation rate, in particular the market rate (to be considered like external rate of the financing costs needed for the investment). If O^* is not convenient, then the decision is negative, if and only if $i^* < x^*$.*

For the financing projects it is enough to change signs and the inequalities side, and the following formulation holds: if a financing project O^*, with internal rate i^* (to be considered likecost rate) is convenient, then the decision is positive, if and only if $i^* < x^*$, where x^* is the evaluation rate, in particular the market rate (to be considered like external rate of return of the investments following the financing). If O^* is not convenient, then the decision is negative, if and only if $i^* > x^*$.

Regarding the criterion IRR to decide on a single project, we observe that:

– it does not have an overall validity, because it assumes the existence of an operative internal rate for O^*;

– the arbitrariness is not eliminated because in any case it is necessary to choose the external rate x^*, in order to compare it with i^*;

– the foundation of the criterion, when it is enforceable, comes from that of the present value criterion, as follows by the proof here schematized for investment or financing projects with a positive internal rate.

If $V(i)$, which is strictly decreasing or increasing depending on the project being an investment or a financing, has only one root i^*, the following inequality couples are equivalent (in the sense that one is necessary and a sufficient condition for the other):

– *investment projects:*

$V(x^*) > 0 \Leftrightarrow i^* > x^*$: O^* = convenient investment

$V(x^*) < 0 \Leftrightarrow i^* < x^*$: O^* = non-convenient investment

– *financing projects:*

$V(x^*) > 0 \Leftrightarrow i^* < x^*$: O^* = convenient financing

$V(x^*) < 0 \Leftrightarrow i^* > x^*$: O^* = non-convenient financing

Comment

The positive decision on the project identified by O^* is equivalent to the choice of O^* instead of the "no project" (that is, the project of doing nothing), featured by the absence of cash-flow and then by the maintenance of the "status quo ante", according to which the wealth of the economic subject was profitably invested, for example at the evaluation rate x^*. The criterion IRR in the previous formulation then follows, because "doing O^*" means the "transferring financial funds", from the market to the project if O^* is an investment or from the project to the market if O^* is a financing.

EXAMPLE 4.3.– *Evaluation of investment projects. Calculation of IRR. Decisions.*

We will consider three investment projects, all having an operative IRR.

A) The simple investment project $\mathcal{A}$ of the PICO type is the buying with cash at time 0 of a real estate equipped with an industrial plant that is producing a detached return, with a sale after 5 year that implies an incorporated return. Let the purchase price be € 47,500, the semiannual return balanced at the end of term, after tax and operating expenses, is €3,000, the selling price at the end of the 5th year is €50,000. It is then a financial project featured by the following supplies:

-47,500	+3,000	+3,000	.........+3,000	+53,000	(amounts)
0	1/2	1	9/2	5	(time)

Using "semester" as a measure of time and indicating with x the semiannual evaluation rate, the PV (or DCF) of the project $\mathcal{A}$, given by the initial value of its supplies, is expressed by

$$V(x) = -47{,}500 + 3{,}000\,[(1+x)^{-1}+\ldots+(1+x)^{-10}] + 50{,}000\,(1+x)^{-10}$$

which results in:

$$V(0.062) = 1{,}770.626;\ V(0.07) = -\,1{,}011.791$$

Decisions according to PV criterion. By using money borrowed at a semiannual rate of 6.2% (for self-financing by *vanishing profit* or external financing by *rising cost*) the investment is convenient; however, by using money borrowed at a semiannual rate of 7% the investment is not convenient.

Calculation of semiannual IRR. The semiannual IRR, unique and certainly between 0.062 and 0.07, will be calculated first of all, obtaining an estimation through linear interpolation in the interval (0.062, 0.07) and then, to obtain the solution (with 9 exact decimals required), proceeding with a classical iteration (for more detail on these methods, see section 4.5) starting from the approximate solution, using Excel spreadsheets[11].

The linear interpolation leads to solving the equation

$$\frac{x-0.062}{0.07-0.062}=\frac{0-1,770.92}{-1,011.79-1,770.92}$$

with solution $\bar{x}$ = 0.067091. To obtain the exact solution $\hat{x}$ with 9 decimals, we apply to the classical iteration the transformation "$f(x) = xg(x)/g_0$" to go from equation "$f(x) = x$" to the equivalent equation "$g(x) = g_0$", which is more useful here, thus using

$g(x) = V(x) + 47{,}500$ (= present value of incomes); $g_0 = 47{,}500$ (= initial payment)

With an Excel spreadsheet we build the following five columns. The resulting Excel table is as follows.

11 We notice that Excel has included a function for the calculation of IRR, starting from a given cash-flow with periodic payments. Furthermore, for calculating the IRR by means of the Excel method we need a starting approximated evaluation, which can be obtained with the formula in footnote 22 of this chapter regarding the linear interpolation method. However, for the calculation of IRR, it is useful to give an illustration of classical numerical methods.

0	0.067091000	47,475.956	0.999494	0.067057039
1	0.067057039	47,487.647	0.999740	0.067039600
2	0.067039600	47,493.652	0.999866	0.067030641
3	0.067030641	47,496.737	0.999931	0.067026037
4	0.067026037	47,498.323	0.999965	0.067023670
5	0.067023670	47,499.138	0.999982	0.067022454
6	0.067022454	47,499.557	0.999991	0.067021829
7	0.067021829	47,499.772	0.999995	0.067021508
8	0.067021508	47,499.883	0.999998	0.067021342
9	0.067021342	47,499.940	0.999999	0.067021257
10	0.067021257	47,499.969	0.999999	0.067021214
11	0.067021214	47,499.984	1.000000	0.067021191
12	0.067021191	47,499.992	1.000000	0.067021180
13	0.067021180	47,499.996	1.000000	0.067021174
14	0.067021174	47,499.998	1.000000	0.067021171
15	0.067021171	47,499.999	1.000000	0.067021169
16	0.067021169	47,499.999	1.000000	0.067021168
17	0.067021168	47,500.000	1.000000	0.067021168
18	0.067021168	47,500.000	1.000000	0.067021168

Table 4.3. *Intermediate calculations for the case of A*

The first column is built with natural numbers (i = number of steps) and the 1st row (i=0) has components: 0; $x_0=\overline{x}=0.067091$; $g(x_0)$; $g(x_0)/g_0$; $f(x_0)$. The 2nd row (i=1) starts with: 1; $x_1 = f(x_0)$; the remaining part is built, using the "copy and paste" function, by columns. In the 3rd, 4th and 5th columns the values $g(x_i)$; $g(x_i)/g_0$; $f(x_i)$ for $i \geq 1$ are obtained; and then in the 2nd column we obtain the values x_i for $i \geq 2$, that are the sequence converging to the solution $\hat{x}= 0.067021168$. We get such value by the 17th iteration, because $g(x_{17}) = 47{,}500$ and at the 18th iteration we can see that with 9 decimals $x_{18} = x_{17}$ and then such a value is $\hat{x}$. The compound annual IRR is $(1+\hat{x})^2 - 1 = 0.138534$.

The Excel instructions are as follows: A1: 0; B1: 0.067091; C1: = 3000*(1-(1+B1)^-10)/B1+50,000*(1+B1)^-10; D1: = C1/47500; E1: = B1*D1.

A2: = A1+1; B2: = E1; copy B1,C1,D1,E1,A2, then paste on the subsequent elements of the same column.

Decisions according to IRR criterion. If the annual evaluation rate (in practice the market rate by *vanishing profit* or by *rising cost*) is less than 13.8534%, the decision on the project is positive; otherwise it is negative.

B) The pure investment project $\mathcal{B}$ in the strict sense of CICO type, and then with unique IRR, is as follows. Let us consider an industrial plant which gives: for the first 2 years only payments, so a purchase cost of €7,000 in the 1st year and an installation cost of €4,000 in the 2nd year; for the next 5 years in operative phase one has annual managing costs of €1,200 and annual incomes of €3,800; for the next year the plant is sold for a net return of €1,500. The algebraic sum of the transactions is €3,500 and then we have a profitable investment. Valuing the amounts at the end of each year, the cash-flow of $\mathcal{B}$ is expressed by

-7,000	-4,000	+2,600	+2,600	+2,600	+2,600	+260	+1,500	(amounts)
1	2	3	4	5	6	7	8	(time)

Using $v = (1+x)^{-1}$, the PV is expressed by

$$V(x) = -7{,}000\, v + [-4{,}000 + 2{,}600(v+v^2+v^3+v^4+v^5)]v^2 + 1{,}500\, v^8$$

and then

$$V(0.07) = +148.51 \quad ; \quad V(0.08) = -200.36$$

Decisions according to PV criterion. If money is borrowed at the annual rate of 7%, the investment is convenient, but if money is borrowed at the annual rate of 8% (both rates by *vanishing profit* or by *rising cost*) the investment is inconvenient.

Calculation of IRR. The IRR, which is unique and certainly between 0.07 and 0.08, will be calculated starting with an approximation through linear interpolation in the interval (0.07; 0.08) and then, to obtain the solution (with 9 decimals required), with a classical iteration starting from the approximate solution. The linear interpolation leads to the solution of the following equation:

$$\frac{x-0.07}{0.08-0.07} = \frac{0-148.51}{-200.36-148.51}$$

This (approximate by excess) solution is $\bar{x}$ = 0.074257. In order to obtain the exact solution $\hat{x}$, we apply the classical iteration method. Furthermore, we cannot start from the equation $x = f(x)$, where $f(x) = x + V(x)$, because in a neighborhood of $\hat{x}$ is $|f'(x)| > 1$. We need to start from the equation $x = h(x)$ (equivalent to $x = f(x)$) where $h(x) = [f(x)-mx]/[1-m]$. We should use $m = f'(\hat{x})$ to obtain $h'(\hat{x})=0$ and an immediate convergence, but this is not possible because $\hat{x}$ is unknown. Furthermore, in order to obtain the convergence with the sequence $\{x_i\}$ obtained by $x_{i+1} = h(x_i)$, it is enough that $|h'(\hat{x})| <1$, for which m must be a well approximated value of $f'(\hat{x})$. We can use: $m = \Delta f/\Delta x = 1+\Delta V/\Delta x$ in a neighborhood of $\hat{x}$. Thus, we obtain the following function, suitable for the iteration

$$h(x) = \frac{V(x) - x - \left[1 + \frac{\Delta V}{\Delta x}\right]x}{1 - 1 - \frac{\Delta V}{\Delta x}} = \frac{V(x) - x\frac{\Delta V}{\Delta x}}{-\frac{\Delta V}{\Delta x}}$$

We assume $\Delta V/\Delta x$ on the interval (0.07; 0.08) containing $\hat{x}$, thus using some previous calculations: $\Delta V/\Delta x$ = [-200.36 -148.51]/ [0.08-0.07] = -34887 results, which is to be substituted in the previous expression for $h(x)$. Given that, let us use Excel and proceed as in section A of this example. We obtain the results in Table 4.4.

0	0.074257000	-3.22391125	0.074164590
1	0.074164590	0.01840163	0.074165117
2	0.074165117	-0.00011144	0.074165114
3	0.074165114	0.00000067	0.074165114
4	0.074165114	0.00000000	0.074165114

Table 4.4. *Intermediate calculations for the case of B*

The 1st row is the vector with components: 0, $\bar{x}$, $V(\bar{x})$, $[V(\bar{x})+34{,}887\,\bar{x}]/34{,}887$; the 2nd row starts with the values 1, $[V(\bar{x})+34{,}887\,\bar{x}]/34{,}887$ and the remaining part of the table is completed using the copy and paste function. The 2nd column shows the rate sequence converging to the solution; already in the 4th iteration we obtain $\hat{x}$= 0.074165114 with 9 exact decimals and $V(\hat{x}) = 0$.

The Excel instructions are as follows:

– A1: 0; B1: 0.074257;

– C1: = -7,000*(1+B1)^-1+(1+B1)^-2*(-4,000+2,600*(1-(1+B1)^-5)/B1+1,500*(1+B1)^-8;

– D1:= (C1+3,4887*B1)/34,887;

– A2: = A1+1; B2: = D1; copy B1, D1, A2, then paste on the following column cells.

Decisions according to IRR criterion. If the annual evaluation rate is less than 7.4165114%, the decision on the project is positive; otherwise it is negative.

C) Let us consider the project obtained modifying $\mathcal{B}$ with the addition of repairing costs after 6 months for €3,000. Compensating with the net operating income for €2,600, at the 6^{th} year we have a net payment of €400 and the cash-flow is given by

-7,000	-4,000	+2,600	+2,600	+2,600	-400	+2,600	+1,500	*(amount)*
1	2	3	4	5	6	7	8	*(time)*

The new project is not an investment in the strict sense, but it verifies the sufficient condition in order that it is an investment in the broad sense and the IRR exists and is unique. In fact, the arithmetic averaged maturity of payments is (-7,000 -4,000·2 - 400·6)/(-11,400) = 1.53, and the time of first income is 3. Furthermore, the project is profitable because the algebraic sum of the monetary transactions is +500. Using $v = (1+x)^{-1}$, the PV is given by

$$V(x) = -7{,}000\, v + (-4{,}000 + 2{,}600\,(v+v^2+v^3+v^5)]\, v^2 - 400\, v^6 + 1{,}500\, v^8$$

and

$$V(0.01) = +77.49\ ;\ \ V(0.07) = -311.98.$$

Decisions according to the PV criterion. If money is borrowed at the annual rate of 1% (by *vanishing profit* or by *rising cost*) the investment is convenient, but if money is borrowed at the annual rate of 2% the investment is inconvenient.

Calculation of the IRR. We proceed in the same way as for $\mathcal{B}$, having the same conditions. The IRR, which is unique and between 0.01 and 0.02, will be calculated starting from an approximated value estimated using linear interpolation in the interval (0.01; 0.02) and then, to obtain the solution (with 9 exact decimals required), proceeding with classical iteration starting from the approximated solution. The linear interpolation leads to the following solution of the equation:

$$\frac{x-0.01}{0.02-0.01} = \frac{0-77.49}{-311.98-77.49}$$

and then the solution $\bar{x}$ = 0.011989. To obtain the exact solution $\hat{x}$, we apply the classical iteration method, with the same transformation and procedures applied in section B of this example. The function to use for the iteration on the equation $x = h(x)$ is

$$h(x) = \frac{V(x) - x\dfrac{\Delta V}{\Delta x}}{-\dfrac{\Delta V}{\Delta x}}$$

We assume $\Delta V/\Delta x$ in the interval (0.01;0.02) containing $\hat{x}$: we obtain

$$\Delta V/\Delta x = (-311.98 - 77.49)/(0.02 - 0.01) = -38947$$

and such a value is to be substituted in the previous expression for *h(x)*. As a result, using Excel as in section A, the following table is obtained.

0	0.011989000	1,426.860635	-1,429.377990	-2.517356	0.011924365
1	0.011924365	1,429.813919	-1,429.751608	0.062310	0.011925964
2	0.011925964	1,429.740801	-1,429.742360	-0.001559	0.011925924
3	0.011925924	1,429.742630	-1,429.742591	0.000039	0.011925925
4	0.011925925	1,429.742584	-1,429.742585	-0.000001	0.011925925
5	0.011925925	1,429.742585	-1,429.742585	0.000000	0.011925925

Table 4.5. *Intermediate calculations for the case C*

The 1st row is the 6 component vectors: 0, $\bar{x}$, two addends of $V(\bar{x})$ needed for the calculation, $V(\bar{x})$, $[V(\bar{x})+38{,}947\,\bar{x}]/38{,}947$; the 2nd row starts with the values 1, $[V(\bar{x})+38{,}947\,\bar{x}]/38{,}947$ and the remainder of the table is completed with the copy and paste function. The 2nd column shows the sequence of rates converging till the solution; at the 5th iteration we obtain $\hat{x}$= 0.011925925 with 9 exact decimals and $V(\hat{x}) = 0$.

The Excel instructions are as follows.

– A1: 0;

– B1: 0.011989;

– C1: = -7,000*(1+B1)^-1+(1+B1)^-2*(-4,000+2,600*(1-(1+B1)^-5)/B1);

– D1: = -3,000*(1+B1)^-6+1,500*(1+B1)^-8;

– E1: = C1*D1;

– A2: = A1+1;

– B2: = F1; copy C1,D1,E1,A2,B2, then paste on the subsequent column cells.

Decisions according to the IRR criterion. If the annual evaluation rate (really the market rate by *vanishing profit* or by *rising cost*) is less than 1.1925925%, the decision on the project is positive; otherwise it is negative.

Comparison between Example 4.3B and Example 4.3C. The input in $\mathcal{B}$ of the payments of €3,000 at the 6^{th} year reduces the convenience threshold from 7.42% to 1.19% in terms of the highest acceptable rate, making the investment C almost inconvenient at the current rates.

4.4.5. *Choice criteria for mutually exclusive financial projects*

In section 4.4.3 we considered a project as the only alternative to *no project*. However, it can be necessary to make a *choice between two projects*, which are each convenient on their own, but not together[12].

For this particular problem we can use a criterion based on the present value or otherwise on the internal rate. However, we first need to observe that a coherent choice implies *comparability* between the cash-flows of the mutually exclusive project in homogenous conditions; we use in this case the expression *complete alternative*, which, in the case of PIPO or PICO investment projects with only one initial payment, means:

– same initial payment;

– same time length.

12 We stress that the external comparison rate to judge the convenience of two projects could not be the same, with a choice based on the circumstances. Then if the first project can be realized with self-financing, disinvesting a previous investment, the internal rate must be compared with its return rate (by vanishing profit), whereas if the second project can be realized only by borrowing money, its convenience must be evaluated by comparing the internal rate with the external rate of the financing (by rising cost), which is usually greater than the return rate.

If there is not a *complete alternative* between the two projects, we need to consider *additional operations* of shorter initial cost or length for the projects, which allow a comparison between homogenous elements. In short – considering two projects – if O_1 and O_2 do not give rise to *complete alternatives*, in order to make the comparison it is necessary to go back to *complete alternatives*, considering two additional projects Q_1 and Q_2 so that the unions $O_1 \cup Q_1$ and $O_2 \cup Q_2$ (where $\cup$ is a union between projects) are a *complete alternative*[13].

Therefore, in order to choose between two projects, we can apply the following criterion.

PRESENT VALUE CRITERION. *Given two investment projects characterized by cash-flows with values* $V_1(x^*)$, $V_2(x^*)$ *positive according to an evaluation rate* x^* *(concerning the projects' financing opportunities, by vanishing profit or by rising cost), the decision maker who, in a complete alternative condition, can carry on only one of them chooses the project that gives rise to a higher value*[14]. *The same decisional criterion holds for mutually exclusive financing projects.*

The consideration of criteria based on the internal rate points out delicate questions that need to be clarified. Indeed, for the financial valuations only in the initial part of the company's life we can neglect the pre-existing conditions; when considering the alternative projects, we need to think in terms of substitutive projects and then consider the difference of cash-flows.

Indeed, if $\mathcal{P}$ represents the projects already realized by the considered company (assuming a past activity) and if the owner has to decide between two new projects $\mathcal{A}$ and $\mathcal{B}$ (both acceptable, if individually considered), the owner does not have to compare $\mathcal{A}$ and $\mathcal{B}$ but $\mathcal{P} \cup \mathcal{A}$ and $\mathcal{P} \cup \mathcal{B}$ and then the owner must choose, assuming a favorable preliminary decision on $\mathcal{B}$, whether or not to substitute $\mathcal{B}$ for $\mathcal{A}$ and then must decide on $\mathcal{A} - \mathcal{B} = (\mathcal{P} \cup \mathcal{A}) - (\mathcal{P} \cup \mathcal{B})$. In such a case the owner, after having decided to add $\mathcal{B}$ and $\mathcal{P}$, considers it more advantageous to withdraw such a decision (i.e. subtract $\mathcal{B}$ to go back from $\mathcal{P} \cup \mathcal{B}$ to $\mathcal{P}$, not considering costs) and then to add $\mathcal{A}$ to $\mathcal{P}$.

In summary, a choice in alternative between $\mathcal{A}$ and $\mathcal{B}$, both investment or both financing projects, is equivalent to a decision on $\mathcal{B} - \mathcal{A}$. Thus, the general extension of the IRR criterion to alternative choices with reference to the rates of the projects to be compared is not justified. Because of the overall validity of the evaluation, it is

---

13 It is not necessary to take into account Q_1 (or Q_2) if its internal rate equals the evaluation rate, because then the value increment due to the integration is zero.

14 Notice that a such criterion can be obtained as a particular case of the criterion described in footnote 10 using $n = 2$, $r = 1$.

indeed necessary to value the *difference project* of two alternative projects (considering that $\mathcal{A} - \mathcal{B}$ and $\mathcal{B} - \mathcal{A}$ have the same internal rate, if it exists operative, as defined in section 4.4.1) and to apply to such *difference project* the decisional criterion IRR (see Figures 4.2a and 4.2c).

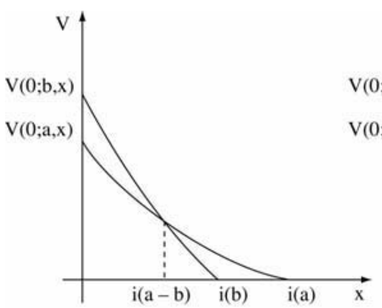
Furthermore, in the particular case of *dominance* between projects, we will say that the project ***b*** is *dominant* over the project ***a*** if

$$V(0;\boldsymbol{b},x^*) > V(0;\boldsymbol{a},x^*) \,, \quad \forall x^* \in X^* \tag{4.26}$$

results (where X^* is the set of variation of all possible evaluation rates), there is no IRR for the difference operations. In addition

– with reference to *investment,* the project with higher value for each rate also has higher IRR. In particular, if (4.26) holds, because of the decrease of *V*, the inequality $i^*(\boldsymbol{b}) > i^*(\boldsymbol{a})$ concerning the IRR of **a** and **b** (see Figure 4.2b);

– with reference to *financing,* the project with higher value for each rate also has lower IRR. In particular, if (4.26) holds, because of the increase of *V*, the inequality $i^*(\boldsymbol{b}) > i^*(\boldsymbol{a})$ concerning the IRR of **a** and **b** (see Figure 4.2d).



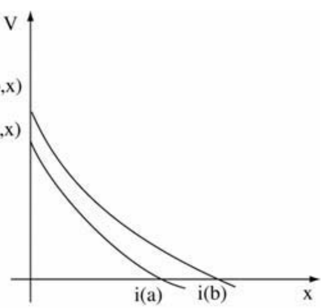
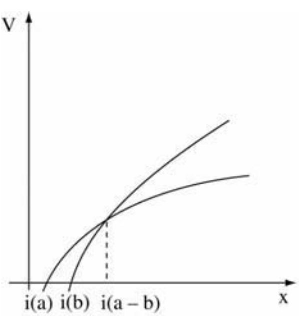


Figure 4.2a. *Investment projects* **Figure 4.2b**



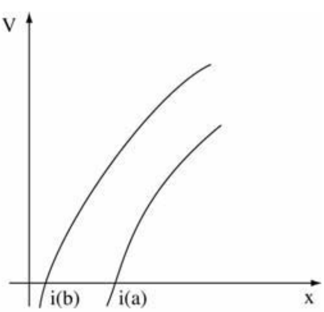


Figure 4.2c. *Financing projects* **Figure 4.2d**

In conclusion, for the alternative choices we can enunciate the following.

INTERNAL RATE CRITERION

1) Case of no dominance. Given two convenient investment projects **a** and **b** without dominance in the rates set X*, if there exists operative IRR i* of the difference operation **a-b** (and then of **b-a**) and if at the rate 0 is $V(0;\boldsymbol{b},0) > V(0;\boldsymbol{a},0)$, then, indicating by x* the external evaluation rate chosen by the decision maker, if x* < i*, **b** is preferred to **a**; if x* > i*, **a** is preferred to **b**.

The opposite inequalities hold if $V(0;\boldsymbol{b},0) < V(0;\boldsymbol{a},0)$.

2) Case of dominance. Given two convenient investment projects **a** and **b** in a dominance relation in the rates set X* and having operative IRR i*(**a**) and i*(**b**), the decision maker prefers the project with the higher IRR.

If the two projects to be compared are financing, their cash-flows are obtained from that of the investment projects changing the amounts' sign. So, the previous criterion holds except for an inversion of the inequalities between the rates considered in case 1), whereas the project with lower IRR is chosen in case 2).

4.4.6. *Mixed projects: the TRM method*

In section 4.4.5 we clarified the relation connecting the choice between two alternative projects and the so-called *substitutive financial operations*, which are obtained formally as *difference operations* between two operations (of investment or financing). Such operations are used when, among other things, we want to cancel a project already chosen to substitute it with another project corresponding better to the new company's aim[15]. We saw that the choice between two projects that are both acceptable can lead back to a difference operation.

The next problem is moreover that a difference operation $\mathcal{A} - \mathcal{B}$ between two investment (or financing) projects having operative IRR is not always a project with operative IRR. This leads to careful discussions about the non-existence or plurality of solutions to the IRR problem for a substitutive project. However, it is an added basic issue which, when resolved, leads to canceling the discussed problems and some formulation defects. We wish to consider this problem.

The present value criterion described in sections 4.4.4 and 4.4.5 uses the evaluation rate referable to the received rates in the investment markets or the allowed rates in the financing markets. If the project is profitable for the

15 See Volpe di Prignano and Sica (1981).

entrepreneur, he uses his profits on the investment market at a received rate of return; however, if the project receives money from the entrepreneur, he gets it from the financing market at an allowed cost rate. Yet because the project to which the criterion is applied changes, as usual, between investment and financing periods, the received and allowed rates in the respective markets are usually different and it does not correspond to reality to use the same rate, i.e. operate at a *reciprocal rate*, as implicitly postulated by the criterion on decisions and choices problems. We then have to leave, for this type of project, the restrictive view followed so far and to introduce new definitions and formulations, also increasing the dimension of the variability space of the examined quantities.

With that aim we will now follow a formalized approach that has given rise to the TRM[16] method. The generalization in the approach will be clarified later.

In such an approach the entrepreneur is seen as an operator in intermediate position between *market* and *project*, which can be realized in an industrial (or commercial or financial or other) venture, which:

a) obtains money, as input to investments for the entrepreneur – who puts his own means, which were profitably invested, or is financed by the external money market, for the most the bank system – with a cost that in the first case is *lost profit*, and in the second case is *emerging cost*;

b) gives subsequently (but the cycle could be reversed, as in the insurance sector) output as money that the entrepreneur then invests in a profitable manner.

In the process described here, there are, in general, four different rates that are considered:

– a pair of rates, usually different[17], for the cost and profit of money, which are the received and allowed rates (for the entrepreneur) of *external* type on the money market, i.e. the cost rate r^* of financing or self-financing needed for investment in the project and the return rate k^* of the investments made with the profit from the project;

– a pair of *internal* rates, $\hat{r}$ to debit and $\hat{k}$ to credit of the project, i.e. that relate to the objective characteristics of the project. As we will see, this second pair is enlarged on an infinite set of equivalent pairs.

Let us assume a *discrete* approach, i.e. considering the supplies of the entrepreneur relative to the project and the following balances only periodically (in

16 See Teichroew, Robichek, Montalbano (1965a) and (1965b).
17 We will come back to this later.

particular, annually). In addition, let us indicate by $\{a_i\}$, ($i = 0, 1,..., n$) the cash-flow of the project (i.e. $a_i < 0$ for payments, $a_i > 0$ for income of the entrepreneur).

In the TRM method the following classification is introduced, relative to the dynamic of the cumulated current balance $S_t = S_t\ (k,r)$, (t= 0,1,..., n), of a c/a devoted to the examined project. From this c/a the inputs come out and to this c/a the outputs come in, both connected to the project[18].

A project is said to be *pure (at a given rate)* when accounting the financial transactions connected to a project in a profitable c/a at such a rate, the balances S_h, h=0,...,n-1 before the last transaction have the same sign, while $S_n = S_n(k,r)$ can be >=< 0, constituting the *final result* of the project (from the viewpoint of the entrepreneur).

Obviously:

– we have a *project of pure investment (at the investment rate r)* if the first financial transaction is a payment and, accounting for the interest charged to the project at the rate r, the balances S_h, h=0,...,n-1, remain ≤ 0. In formulae this is:

$$\begin{cases} S_t = S_{t-1}(1+r)+a_t\ , \quad t=1,...,n \\ S_0 = a_0 < 0 \quad ; \quad S_h \le 0\ \ , \ \ h=1,...,n\text{-}1 \end{cases} \tag{4.27}$$

– we have a *project of pure financing (at the financing rate k)* if the first financial transaction is an income and, accounting for the interest charged to the project at the rate k, the balances S_h, h=0,...,n-1, remain ≥ 0[19]. In formulae this is:

$$\begin{cases} S_t = S_{t-1}(1+k)+a_t\ , \quad t=1,...,n \\ S_0 = a_0 > 0 \quad ; \quad S_h \ge 0\ \ , \ \ h=1,...,n\text{-}1 \end{cases} \tag{4.28}$$

A project is said to be *mixed (at the investment rate r and at the financing rate k)* if it is neither a pure investment at the rate r, nor a pure financing at the rate k. Thus, accounting the interest charged or favorable to the project at such rates, the balances S_h, h=0,...,n-1, do not remain of constant sign, and it can result in >=< 0, with alternating phases of investment and financing. In formulae

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18 Observe that the balance from the viewpoint of the entrepreneur is equal to the retro-reserve from the viewpoint of the project, identified as a counterpart of the entrepreneur.

19 An example of a project of pure investment is given, if $S(t_n) = 0$, by the management of a loan at the rate r from the viewpoint of the lender. An example of a pure financing project is given, if $S(t_n) = 0$, by managing a loan at the rate k from the viewpoint of the borrower. For the management of loan with amortization, see Chapter 6.

$$\begin{cases} S_0 = a_0 \\ S_t = S_{t-1}(1+r) + a_t \quad if \;\; S_{t\text{-}1} < 0 \\ S_t = S_{t-1}(1+k) + a_t \quad if \;\; S_{t\text{-}1} \geq 0 \\ t = 1,\ldots,n \end{cases} \tag{4.29}$$

Both for mixed and pure projects the value

$$S(k,r) = S_n = S_n(k,r) \tag{4.30}$$

can be >=< 0, constituting the final result of the project (from the viewpoint of the entrepreneur), given by the returns, which are also financial, the net of costs, which are also financial, at the final time instead of at the initial time (as seen for the present value), and on the basis of a complex and non-decomposable financial law, identified by (4.29), if the project is mixed and a non-reciprocal rate is applied, i.e. if r and k are both used for calculation and they are different.

We can now see the substantial limit of the present value criterion (see section 4.4.4) which is not useful for projects that interchange in time the role of investment taking money (if $S_t < 0$) and that of loan giving money (if $S_t > 0$) because we use only one evaluation rate whereas the rules of the money and financial market usually give rise to different return rates in the two cases.

The specification of the rate, when the pure or mixed feature of the project is specified, is needed because such a feature depends on the level of the investment or financing rate. In fact, every *mixed* project with $a_0 < 0$, when r increases and exceeds a given r-min, becomes a project of *pure investment* and the result does not depend on k; in the same way every mixed project with $a_0 > 0$, when k increases beyond a given k-min, becomes a project *of pure financing* and the result does not depend on r.

The property can be proved immediately observing that, if $a_0 < 0$ and thus the interest is initially accrued at the rate r, there exists a proper r-min, such that, $\forall h$ = 1,...,n-1, the absolute value of the decrement $S_h - S_{h\text{-}1}$ exceeds every $a_h > 0$. Thus the balances, initially negative, remain negative for the whole time-length before the last transaction that gives the final result and the project results of a pure investment. Analogously, if $a_0 > 0$ and then the interest is initially accrued at the rate k, there exists a proper k-min such that $\forall h$ = 1,...,n-1 the increment $S_h - S_{h\text{-}1}$ exceeds the absolute value of each $a_h \leq 0$. Thus the balances, initially positive, remain positive for the whole time length before the last transaction that gives the final result and the project results of a pure financing.

Let us consider the geometric approach on the Cartesian plane. Restricting oneself to the *mixed projects* on the basis of the given rates (then with r<r-min if a_0 < 0, with k<k-min if a_0 > 0), while (4.30) generalizes the concept of present value of a project, the generalization of the internal rate is obtained choosing among the level curves of the surface (4.30), defined on the quadrant $k \geq 0,\ r \geq 0$ of the plane *Okr*, the one corresponding to the parameter = 0 and then to a nil final result. Analogously the IRR is solution $\hat{i}$ of the equation $V(\hat{i}) = 0$. Therefore the equation

$$S(k,r) = 0 \tag{4.31}$$

implicitly defines on the aforementioned quadrant a continuous and increasing curve, that is called *final fairness curve* (or more briefly, *fairness curve*, or also *curve of equilibrium*) of the project, generally asymptotic to the straight line r = r-min if a_0 < 0 or to the straight line k = k-min if a_0 > 0. The explicit forms of (4.31), to be considered alternatively, can be written as

$$\begin{cases} r = r_0(k) \\ k = k_0(r) \end{cases} \tag{4.32}$$

where r_0 and k_0 are the functional operators, one inverse of the other, which realize the final fairness of the project. Any pair $(\hat{k}, \hat{r})$ satisfying (4.31) and then, because of (4.32), such that

$$\hat{r} = r_0(\hat{k}) \text{ or } \hat{k} = k_0(\hat{r}) \tag{4.33}$$

identifies a point on the fairness curve and then has the following meaning: given the cash-flow of the mixed project, if in the investment stages of the project, interests (positive for the entrepreneur) are debited to it at the rate $\hat{r}$, such that the project is in equilibrium (i.e. gives rise to a null final balance) we have to credit interest (negative for the entrepreneur) at the rate $\hat{k}$ during the financing stages of the project. And the inverse holds. The points on curve (4.31), with coordinates of type $(\hat{k}, \hat{r})$, are financially equivalent in the sense that they all assure a zero final result

THEOREM.– The generalization of IRR implies a connection between *internal* allowed and charged interest rates, i.e. needed to maintain the equilibrium of the project, with necessarily concordant variations rate.

Proof

Given that it is not restrictive to suppose that the function *S(k,r)* has continuous first order partial derivatives, we have

$$\frac{\partial S}{\partial k} > 0 \quad ; \quad \frac{\partial S}{\partial r} < 0 \tag{4.34}$$

This is because according to (4.29), on curve (4.31) if k increases, fixing r, at the end of the process, S increases; while if r increases, fixing k, S decreases. Given that the derivative of the function $r = r_0(k)$ exists and is continuous, explicit equation of the curve S=0. Moreover, $\frac{dr_0(k)}{dk} = \frac{dr}{dk} = -\frac{\partial S}{\partial k} / \frac{\partial S}{\partial r} > 0$ holds. Analogous conclusion for the inverse function $k = k_0(r)$ with derivative $\frac{dk_0(r)}{dr} = \frac{dk}{dr} = -\frac{\partial S}{\partial r} / \frac{\partial S}{\partial k} > 0$.

On the contrary, if $a_0 < 0$ and $r = r$-min or $a_0 > 0$ and $k = k$-min, the project becomes *pure* and the *fairness curve* becomes a parallel to the $r = 0$ or $k = 0$ axis respectively. In such a case it is enough to consider only one of the *external rates,* r^* of cost or k^* of return; then the problem is led back to the one-dimensional case and to the criteria as given in sections 4.4.4 and 4.4.5.

Summarizing the geometric point of view, on varying the supplies, the fairness curves (4.31), locus of the points (k,r), cover: a) if $a_0 < 0$, a *mixed* region defined by (k>0, 0<r<r-min) and in the top a region as pure investment defined by (k>0, $r \geq r$-min); b) if $a_0 > 0$, a *mixed* region defined by (0<k<k-min, r>0) and in the right side a region as *pure financing* defined by ($k \geq k$-min, r>0) (see Figure 4.3).

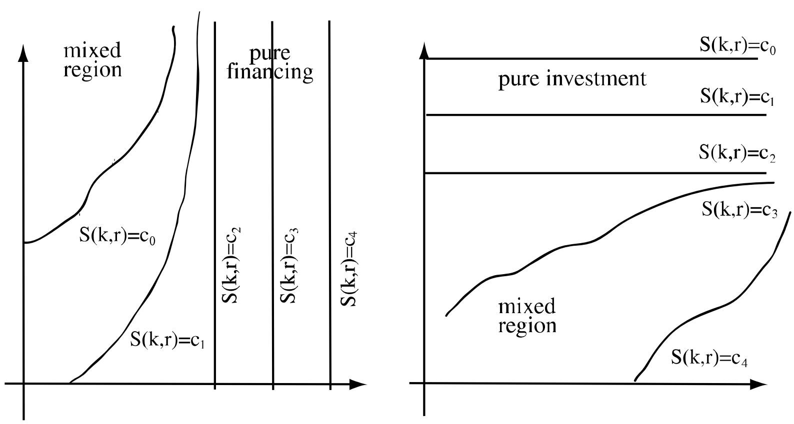

Figure 4.3a and b. *Fairness curves*

4.4.7. *Decisional criteria on mixed projects*

Let us now extend the decisional and choice criteria for mixed project as seen with the one rate approach in sections 4.4.4 and 4.4.5. In addition, we observe that it is enough to consider here decisional criteria: indeed, substitutive operations can be led back to a mixed project and we have shown in section 4.4.5 the way in which decision on them is equivalent to choices in alternative between projects of investment or financing. The decisional criterion discussed later has a geometric interpretation in Figure 4.4 with reference to the plane *Okr*.

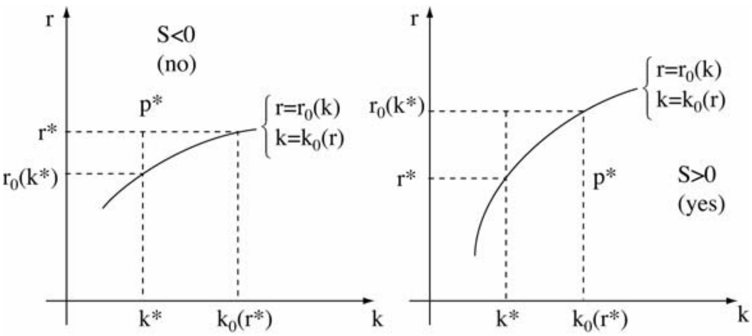

Figure 4.4a and b. *Plot of the decisional criterion*

When we are interested in rates that generate a mixed project (i.e. when there is plurality in the sign of the periodic balances sequence), from the viewpoint of the entrepreneur, the broker between the project and the financial market, the following decisional criteria apply.

a) *The final result (FR) criterion, which extends the present values (PV) criterion*

The decision depends on the sign of the final balance evaluated at the external cost rates r^* and returns k^*, already defined, on the capital market. Therefore, the criterion can be formulated as follows.

If

$$S(k^*,r^*) > 0 \tag{4.35}$$

the mixed project is convenient *for the firm; if*

$$S(k^*,r^*) < 0 \tag{4.35'}$$

the mixed project is inconvenient *for the firm; if*

$$S(k^*,r^*) = 0 \qquad (4.35'')$$

the mixed project is indifferent *for the firm.*

Proof

It is obvious that (4.35), (4.35') and (4.35”) generalize the criterion of present value described in section 4.4.4, which uses the relations $V(x^*)>0$, $V(x^*)<0$ and $V(x^*)=0$. To prove the FR criterion with a direct argument, it is enough to observe that, from the viewpoint of the acting firm, the amounts of the cash-flow towards the market regarding the examined project are exactly the opposite of those of the cash-flow towards the project. In other words, the firm takes the amounts from the market with charged interest rate k^* and invests them in the project at the allowed rate $\hat{r}$, and, in addition, takes the amounts from the project with charged interest rate $\hat{k}$ and invests them in the market at the allowed rate r^*. If the final effect of such a transaction is a spread $S(k^*,r^*)>0$, then the mixed project is convenient, otherwise it is not.

b) *With the return and cost rates (RCR) criterion, which extends the internal rate (IRR) criterion*

In such a criterion (which, in contrast to the IRR criterion, can always be applied) it is necessary to consider four types of rates already indicated, i.e. the pair (k^*,r^*) of return and cost rates on the external market and the infinite pairs $(k_0(r),r_0(k))$, coordinates of points on the well established *fairness curve* which replaces the IRR, with varying $\hat{r}$ = *investment rate in the project* and $\hat{k}$ = *financing rate from the project*, from the viewpoint of the firm. The *RCR criterion* can then be formulated as follows.

Given a mixed project $\mathcal{P}$ *(at the considered rate) with a fairness curve of equation* (4.31) *with explicit form:* $r = r_0(k)$ *(i.e.:* $k = k_0(r)$*), using (from the viewpoint of the acting firm):*

– k^* = *external allowed return rate on the market;*

– r^* = *external charged cost rate on the market*[20]*;*

20 The original formulation of the TRM criterion, more limited than the one shown here, introduces only one external rate called *cost rate of the capital*, then assuming $k^*=r^*$ and considering the point P^* bounded on the bisector $r=k$. Thus, only the intersection points are meaningful of the fairness curve with the bisector $r=k$, which have coordinates equal to each other and to the IRR of the project, which are non-operative in the case of absence or plurality of solutions, leading back to the inconveniences of the IRR criterion. We consider more advantageous a more general schematization to complete the innovative contribution of this

– *$k_0(r^*)$ = charged internal financing rate from the project, corresponding to the internal investment rate $\hat{r} = r^*$;*

– *$r_0(k^*)$ = allowed internal investment rate in the project, corresponding to the internal financing rate $\hat{k} = k^*$; in the hypothesis that the firm has access without limitation to the financing market at the rate r* and to the investment market at the rate k* (constant rates for the whole length of the mixed project), thus;*

– *if and only if the point $P^* = (k^*,r^*)$ is below the fairness curve, where*

$$r_0(k^*) > r^* ; \quad k_0(r^*) < k^* \tag{4.36}$$

(the inequalities are either both true or both false due to the increasing behavior of the fairness curve), the project $\mathcal{P}$ is convenient *for the firm;*

– *with the same hypothesis and positions, if and only if the point $P^* = (k^*,r^*)$ is above the fairness curve (4.31), where*

$$r_0(k^*) < r^* ; \quad k_0(r^*) > k^* \tag{4.36'}$$

(the inequalities are either both true or both false), the project $\mathcal{P}$ is inconvenient *for the firm;*

– *with the same hypothesis and positions, if and only if the point $P^* = (k^*, r^*)$ is on the fairness curve (4.31), where*

$$r_0(k^*) = r^*; \quad k_0(r^*) = k^* \tag{4.36''}$$

(the equalities are either both true or both false), the project $\mathcal{P}$ is indifferent *for the firm.*

Proof

(4.36") follows from the definition of fairness curve. In addition, we can verify the equivalence between (4.35) and (4.36), between (4.35') and (4.36'), and between (4.35") and (4.36"), from which the *RCR criterion* follows.

To prove the validity of the RCR criterion with a direct argument, we firstly observe that a project is convenient or inconvenient if it gives rise to a property variation that increases the return rate and/or decreases the cost rate or it gives rise to

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scheme, also due to the fact that in the capital market it is used to work with non-reciprocal rates.

a property variation with opposite effects. Besides, let us indicate by S_t the balance with accumulated interest until time t, as defined in (4.29).

So if, when $S_t < 0$, the firm has invested in the mixed project $\mathcal{P}$ at the rate r^* equal to the cost rate on the market (then without spread), then in equilibrium condition the financing from $\mathcal{P}$, when $S_t > 0$, is ruled by the rate $k_0(r^*)$. Therefore:

– if the second inequality of (4.36) holds, investing the profit from such financing on the market at the rate $k^* > k_0(r^*)$, the firm has a positive spread and $\mathcal{P}$ is *convenient*;

– if the second inequality of (4.36’) holds, investing the profit from such financing on the market at the rate $k^* < k_0(r^*)$, the firm has a negative spread and $\mathcal{P}$ is *inconvenient*;

– if the second equality of (4.36”) holds, investing the profit from such financing on the market at the rate $k^* = k_0(r^*)$, the firm has zero spread and $\mathcal{P}$ is *indifferent.*

Otherwise if, when $S_t > 0$, the firm is financed from the mixed project $\mathcal{P}$ at the rate k^* equal to the return rate on the market (then without spread), then in equilibrium condition the investment in $\mathcal{P}$, when $S_t < 0$, is ruled by the rate $r_0(k^*)$. Therefore

– if the first inequality of (4.36) holds, taking money on the market at the rate $r^* < r_0(k^*)$, the firm has a positive spread and $\mathcal{P}$ is *convenient*;

– if the first inequality of (4.36’) holds, taking money on the market at the rate $r^* > r_0(k^*)$, the firm has a negative spread and $\mathcal{P}$ is *inconvenient*;

– if the first inequality of (4.36”) holds, taking money on the market at the rate $r^* = r_0(k^*)$, the firm has zero spread and $\mathcal{P}$ is *indifferent.*

EXAMPLE 4.4.– Regarding decisions on mixed projects, we use the classical example of the *oil pump project* shown by *Lorie and Savage*[21] to be a typical substitutive operation in the industrial field. Let us suppose that from an oil well, containing crude valued US$20,000 (we are using low numbers for sake of brevity: it would be enough to assume as unit a suitable power of 10), oil is being extracting at time 0 with a pump system that enables the completion of the extraction in 2 years, and there will be a gross profit of US$10,000 at the end of the 1st year and the same at the end of the 2nd year. It is then necessary to evaluate at time 0 the convenience of the installation of a more efficient pump, with a substitution cost of US$1,600, which enables the extraction to be completed within one year, with a profit of US$20,000 before the end of the 1st year.

21 See Lorie and Savage (1955) (see footnote 16).

Not considering previous flows, which have no influence here, the project $\mathcal{A}$ (= old pump) implies the following supply sequence:

$$\mathcal{A} : (1; +10{,}000)\cup(2; +10{,}000)$$

while the project $\mathcal{B}$ (= new pump) implies the following supplies:

$$\mathcal{B} : (0; -1{,}600)\cup(1; +20{,}000).$$

Note that $\mathcal{A}$ does not have IRR and it is convenient with any evaluation rate, while $\mathcal{B}$ is a simple project with IRR = 1,150%. However, here, being a substitutive operation, we are interested in the difference project given by

$$\mathcal{A} - \mathcal{B}: (0, -1{,}600)\cup(1, +10{,}000)\cup(2, -10{,}000)$$

Now $\mathcal{B} - \mathcal{A}$ is a mixed project that starts with a payment and becomes a pure investment project only if $r \geq r\text{-min} = 525\%$.

The final fairness curve $r = r_0(k)$ has equation

$$r = 5.25 - 6.25/(1 + k) \tag{4.37}$$

an explicit form of $S = 0$, which in this case is written as follows:

$$[-1{,}600\ (1 + r) +10{,}000]\ (1 + k) -10{,}000 = 0$$

On the basis of the original criterion TRM (see footnote 20), the intersections of (4.37) with the bisector $r=k$ correspond to the following IRR values:

$$k^* = r^* = 0.25 = 25\%\ ; \quad k^* = r^* = 4 = 400\%$$

(non-operative IRR because we have obtained more solutions). Therefore, the substitution of the pump, i.e. the change from $\mathcal{A}$ to $\mathcal{B}$ is convenient if and only if the market rate, reciprocal for investments and financings, chosen for the evaluation is between 25% and 400%. Indeed, the fairness curve (4.37) has the concavity downwards and, intersecting the 1st bisector in points P_1 = (0.25;0.25) and P_2 = (4;4), all points P^* of such a bisector between P_1 and P_2, with $0.25 < k^* = r^* < 4$, are such that $S(P^*) > 0$. So, with such external rates the substitution is convenient. Instead, if $k^* = r^*$ is > 4 or < 0.25, the substitution is inconvenient.

On the basis of the version of TRM introduced here, which considers market rates k^* and r^* to be different, it is necessary to evaluate the pair (k^*,r^*) to adopt and accept the substitution if and only if the point $P^* = (k^*,r^*)$ is below the curve (4.37).

Observe that the oil company, if it performs the substitution, will have to obtain at time 0 a loan of US$1,600 at the cost rate r^* and will have to invest at time 1 the higher profit of US$10,000 at the return rate k^*.

4.5. Appendix: outline on numerical methods for the solution of equations

4.5.1. *General aspects*

As seen in this chapter and as will be seen in Chapter 5 in the particular case of *annuity flows*, the congruity relations between the flows of a financial operation and their capital values at a given time are often considered in financial mathematics under different hypotheses on the adopted financial laws. However, such a relation can be thought of as equations where the unknown is the length or the rate, and all other quantities are given. The duration is seldom considered unknown indeed, while the rate is often considered like this. Then there is the classic problem of the *calculation of the IRR of a financial operation O*. We saw the importance of this in the previous sections, but we did not consider its calculation.

The solution of such a problem is not simple and sometimes it is impossible from the algebraic viewpoint, when the equation on the rate is not simple enough to give a solution in closed form. It is then necessary to apply numerical methods that give approximate solutions, unless iterative methods are applied to obtain the numerically exact solution in the desired number of decimals. The field of application of such methods is much greater than the calculation of IRR. We then consider it opportune to give a brief insight into the theoretical and applicative aspects of more suitable numerical methods even if the software available on PCs enables an easy evaluation of equation roots, in particular the IRR of an operation *O*.

On the choice of the calculation methods for IRR, it is necessary to consider that due to the versatility and popularity of PCs, and also of the pocket calculator, many of the methods used in the past are now obsolete. We will consider few classic methods, favoring the iterative ones.

4.5.2. *The linear interpolation method*

Given a function $f(x)$, which is continuous and monotonic in an assigned interval, and given a value k, it is necessary to find the value $\tilde{x}$ such that $f(\tilde{x})=k$, i.e. the root of the equation:

$$g(x) := f(x) - k = 0 \tag{4.38}$$

The linear interpolation is done starting from the values x_1 and x_2, which are close enough between them and to the root[22], and such that $f(x_1) > k$, $f(x_2) < k$ [23] as well. Then an approximate estimation of $\tilde{x}$, which we will indicate by x_0, is obtained from the abscissa of the intersection with $y = k$ of the secant line to the graph of $y=f(x)$ in the points with abscissas x_1 and x_2. Indeed, x_0 is also the root of the secant the graph of $g(x)$ in the points with abscissas x_1 and x_2. This easily results in

$$x_0 = x_1 + \frac{k - f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1) \tag{4.39}$$

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22 The search for x_1 and x_2 "by chance" is not easy and not very scientific. In the financial problems in which we are interested, a starting approximate value $\hat{x}$ of the solution $\tilde{x}$, in order to find by small steps such values x_1 and x_2 , can be easily obtained on the basis of linear hypothesis, using the arithmetic mean and the SDI law. In particular we consider an O satisfying the condition that the arithmetic mean maturity of the payments precedes the time of first income; we know that it is enough for O to be an investment in the broad sense and then to have an operative IRR. Let us denote by τ_i the arithmetic mean of incomes, with τ_e that of payments, with E the sum of payments and with I the sum of incomes. Thus, being $\tau_e < \tau_i$, the IRR approximated $\hat{x}$ is found from: $E[1+(\tau_i-\tau_e)\hat{x}] = I$. In the particular case of simple investment with payment V_0 in 0 and incomes R_h in h, the arithmetic mean maturity of payments is then $\tau_e = 0$, that of incomes is $\tau_i = \sum_h hR_h / \sum_h R_h$ and $\hat{x}$ follows from: $V_0(1 + \tau_i\hat{x}) = \sum_h R_h$. We have such a situation in the amortization of the debt V_0 with installments R_h (see Chapters 5 and 6).

23 Then $x_1 > x_2$ if f increases in the interval, and $x_1 < x_2$ if f decreases. If $f(x)$ is the initial value of an annuity at rate x, then f is decreasing and convex, so $\tilde{x} < x_0$ (see Chapter 5).

Proof

The secant defined before is the straight line through $P_1 \equiv [x_1, f(x_1)]$ and $P_2 \equiv [x_2, f(x_2)]$, with equation

$$\frac{y - f(x_1)}{f(x_2) - f(x_1)} = \frac{x - x_1}{x_2 - x_1} \tag{4.39'}$$

and making a system with $y=k$, the solution $x=x_0$ expressed by (4.39) is obtained.

The solution x_0 is an approximation *by excess*, i.e. greater than the exact value, if *f(x)* (and then *g(x)*) is also decreasing and convex (i.e. with upwards concavity)[24] or increasing and concave (i.e. with downwards concavity); instead x_0 is an approximation *by defect*, i.e. smaller than the exact value, if *f(x)* (and then *g(x)*) is decreasing and concave or increasing and convex.

The linear interpolation procedure can be iterated using in the procedure the found approximated root, i.e., using the initial positions, acting analogously on the interval (x_0, x_1) or alternatively (x_0, x_2), thus satisfying the condition of sign discordance between $f(x_i)-k$, $i = 1$ or 2, and $f(x_0)-k$. The estimation $x_0^{(2)}$ of the root is then obtained. Proceeding analogously again we obtain a sequence $x_0^{(i)}$, $i = 2, 3,...$ converging to the root $\tilde{x}$, i.e. such that $f(x_0^{(i)})$ converges to k; this procedure gives the *secant method* (see below)

Exercise 4.6

To buy a shed to be used in an industrial company for the price of €170,000, the entrepreneur sells stocks earning €22,000 and for the remaining part he enters into a loan to repay (for the amortization procedure see Chapter 6) with 10 annual delayed installments $R_h = R+hD$, ($h = 1,...,10$), where $R = 17{,}030$, $D = 0.04R$; then $R_1 = 17{,}711.20$; $R_2 = 18{,}392.40$;.....; $R_{10} = 23{,}842.00$. Find the loan rate.

A. The loaned amount is $S = 170{,}000 - 22{,}000 =$ €148,000; the length is $n = 10$; the installments are given. To find the rate we apply the linear interpolation method on the function *f(x)*, related to the equivalence (6.2) specified in section 6.2

$$S = f(x) \coloneqq \sum_{h=1}^{10} R_h (1+x)^{-h}$$

We start from a rough rate $\hat{x}$ according to footnote 22. This results in

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24 We find this case in the search for IRR in an amortization operation for a capital given as loan, with given cash-flow and maturity of installments (see Chapter 6), solving an equation of type $V(i) = 0$, with $V(i)$ decreasing and with upwards concavity.

$$\tau_i = \sum_h h(R+hD)/\sum_h (R+hD) = (55+0.04\cdot 385)/(10+0.04\cdot 55) = 5.77;\ V_0$$
$$(1+\tau_i\,\hat{x}) = \sum_h R_h \text{ becomes: } 148{,}000\ (1+5.77\,\hat{x}) = 207.766$$

then: $\hat{x} = 0.06999$. We obtain: g(0.06999) = -4,724.11 and with *g(x)* decreasing the root is a value < 7%. Acting on *f(x)*, with decreasing *x* we easily obtain using Excel

$$f(0.0675) = 145{,}037.74$$
$$f(0.0650) = 146{,}832.06$$
$$f(0.0625) = 148{,}659.58$$

The solution $\tilde{x}$ is evidently between 6.50% and 6.25%. Applying (4.39) we obtain:

$$x_0 = 0.0625 + \frac{148{,}000.00 - 148{,}659.58}{146{,}832.06 - 148{,}659.58}0.025 \ = 0.06340$$

In accordance with the fact that x_0 is approximation by excess, f(0.0634) = 147,996.13 < 148,000 follows. Interpolating on the interval (0.0625; 0.0634) (2nd step of the "secant method", of which the starting interpolation is the 1st) we obtain:

$$x_{00} = 0.0625 + \frac{148{,}000.00 - 148{,}659.58}{147{,}996.13 - 148{,}659.58}(0.0634 - 0.062) \ = 0.063395$$

The process can be iterated again if we require many exact decimal digits; otherwise the rate 6.3395% is a satisfactory estimation of the loan rate, giving rise to a relative spread of less than 10^{-4}.

Financial application with numerical solution using the linear interpolation methods has been developed in section 4.4.4, Example 4.3.

4.5.3. *Dichotomic method (or for successive divisions)*

This is a procedure with slow convergence, which can be applied, due to its simplicity, if we have a calculator.

In order to solve equation (4.38), assuming *f(x)* to be continuous and monotonic, by assumption we know that the searched root is in an interval with known extreme *a, b*, with *a<b,* considering the case: *f(a)* > *k*, *f(b)* < *k*.

Let us define $x_0 = (a+b)/2$, $d = |a-b|$ and then, with recursive process,

$$x_h = x_{h-1} \pm \frac{d}{2^h} \quad ; \quad h = 1,\ 2,\ ... \tag{4.40}$$

using in (4.40) the sign + or - according to, respectively, $f(x_{h-1}) > k$ or $f(x_{h-1}) < k$. In such a case we obtain a sequence $x_0, x_1,\ldots,x_h,\ldots$ approximating a root $\tilde{x}$ of (4.38), in the sense that the pairs of consecutive values x_{h-1}, x_h are extremes of numeric interval containing $\tilde{x}$ and of amplitude $d/2^h$ geometrically decreasing with ratio 1/2.

4.5.4. *Secants and tangents method*

Given equation (4.38) with $f(x)$ continuous and differentiable, considering the case that $f(x)$ is decreasing and has upwards concavity (the changes are obvious for the other cases), it is necessary to find the abscissa $\tilde{x}$ of the point X intersection of the graphs $y = f(x)$ and $y = k$ (see Figure 4.5). The procedure on the secants being considered here is a particular case of that shown in section 4.5.2.

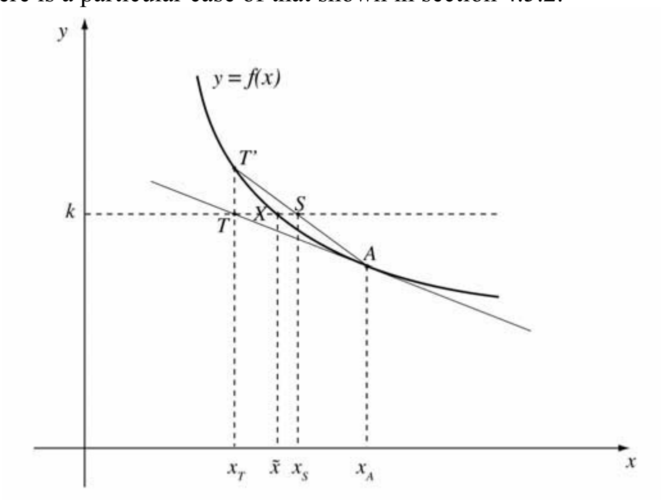

Figure 4.5. *Secants and tangents method*

Initial step

Starting from point A of the $f(x)$ graph with abscissa $x_A > \tilde{x}$ (where $f(x_A) < k$), possibly already approximated to $\tilde{x}$ on the basis of preliminary information (see footnote 22), the *tangent* to the curve in A is analytically found, whose equation is

$$y - f(x_A) = f'(x_A) - (x - x_A) \tag{4.41}$$

and by using $y=k$ we obtain the abscissa x_T of the point T intersection of the tangent with the line $y=k$:

$$x_T = x_A + (k - f(x_A))/f'(x_A) \tag{4.42}$$

Due to a well known property of the upwards concave function, the inequalities $x_T < \tilde{x} < x_A$ hold. Then (4.42) is an estimate of $\hat{x}$ approximated by defect.

Furthermore, we obtain the equation of the *secant* to the curve through A and the point T' of the graph with abscissa of T, obtaining

$$\frac{y - f(x_A)}{f(x_T) - f(x_A)} = \frac{x - x_A}{x_T - x_A} \tag{4.43}$$

and using $y=k$ we find from (4.43) that x_S is an approximation that is better than x_A. We then find the numeric interval, with extremes x_T , x_S, which includes the searched solution $\tilde{x}$.

Next step

The procedure can be iterated, starting from x_S, instead of x_A, obtaining an approximated interval, which is contained in the previous intervals and has decreasing amplitude converging to 0, thus obtaining a very good estimate of $\tilde{x}$.

4.5.5. *Classical iteration method*

Let us give a brief insight on a widely-applied method – *classical iteration* – which requires the availability of a personal computer (or a programmable calculator). Let us consider an equation, written in the form $x = f(x)$, where $f(x)$ exists continuously in an interval containing the root α, for which the approximate value x_0 is known. If the equation is given in the form: $g(x) = g_0$, it is enough to use: $f(x) = x+g(x) - g_0$ or, if $g_0 \neq 0$, to use: $f(x) = x\, g(x)/g_0$. The method consists of finding the following sequence, starting from x_0:

$$x_h = f(x_{h-1}); \quad (h = 1, 2, 3,...) \tag{4.44}$$

because if $\{x_h\}$ converges, its limit is necessarily given by the root of the given equation.

The validity of the procedure arises from the following theorem.

THEOREM.– *If there exists a root α of the equation x = f(x) – where f(x) is defined, continuous and differentiable in the interval J containing* α *and* x_0 *is a value of first approximation of* α *– and moreover if, for a given prefixed constant H satisfying: 0 <H <*1*, we obtain* $\forall x \in J$*:* $|f'(x)| \leq H$*, then the sequence* $\{x_h\}$ *converges and its limit is the root* α*. If, instead, in J we obtain:* $|f'(x)| > 1$*, then the sequence does not converge to* α[25].

Figures 4.6a and b geometrically show the convergence and divergence cases of the iteration method.

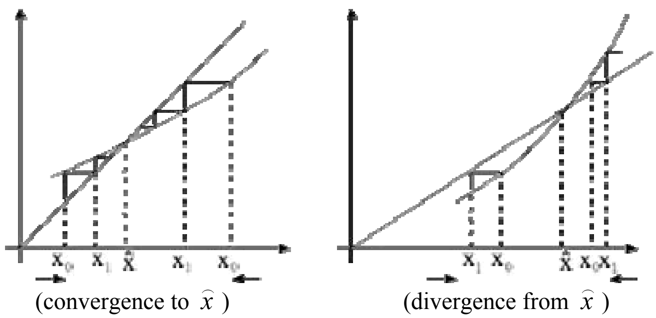

Figure 4.6a and b. *Convergence and divergence cases of the iteration method*

Referring to the graphs of $y = x$ and $y = f(x)$, it is geometrically obvious that the convergence of the procedure is faster the closer the value of $|f'(\alpha)|$ is to zero. Therefore, the following transformation is used to accelerate the convergence or to make it possible when it is not on f, i.e. if $|f'(\alpha)| > 1$. Using

25 Let us give a brief proof of the theorem. If $|f'(x)| \leq H$ with $0<H<1$, due to the Cavalieri-Lagrange theorem we can write, given that by definition $\tilde{x} = f(\tilde{x})$, recalling (4.44) and introducing $\bar{x}_0$ between $\tilde{x}$ and x_0):

$$|\tilde{x} - x_1| = |f(\tilde{x}) - f(x_0)| = |f'(\bar{x}_0)|\,|\tilde{x} - x_0| \leq \alpha|\tilde{x} - x_0| < |\tilde{x} - x_0|$$

and analogously, for $h = 2,3,...$

$$|\tilde{x} - x_h| = |f(\tilde{x}) - f(x_{h-1})| = |f'(\bar{x}_{h-1})|\,|\tilde{x} - x_{h-1}| \leq \alpha|\tilde{x} - x_{h-1}| \leq \alpha^h |\tilde{x} - x_0|$$

Because of the formula: $\lim_{h \to \infty} \alpha^h = 0$, and of a comparison theorem,

$$0 \leq \lim_{h \to \infty} |\tilde{x} - x_h| = |\tilde{x} - x_0| \lim_{h \to \infty} \alpha^h = 0 \qquad \text{that is} \qquad \lim_{h \to \infty} x_h = \tilde{x}$$

follows, and then the thesis. Again for a comparison theorem, if $|f'(x)|>1$ holds in a neighborhood of $\tilde{x}$, a diverging geometric sequence is minorant of $\{x_h - x_0\}$, therefore this sequence is also diverging and its values deviate from the solution $\tilde{x}$.

$$m = f'(\alpha) \quad ; \quad g(x) = [f(x) - m\,x]/(1 - m) \tag{4.45}$$

$g'(\alpha) = 0$ follows, and it is easy to verify that the equation $x = g(x)$, to which the iteration method can be applied in the best conditions, is equivalent to $x = f(x)$, so it has the same roots. Obviously m cannot at first be found in an exact way because it is unknown α; however, an approximate value almost satisfies the condition and then the method can be applied in order to obtain a quick convergence.

Financial applications using the classical iteration method for numerical solutions have been developed in section 4.4.4, Example 4.3.

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Chapter 5

Annuities-Certain and their Value at Fixed Rate

5.1. General aspects

From now on we will consider problems that more frequently come into the financial practice, solving them in light of given theoretical formulation and on the basis of the *financial equivalence principle* following a prefixed exchange law.

The aforementioned principle was applied in Chapter 4 where, referring mostly to complex financial operations evaluated with exchange laws at *fixed rate* (i.e. constant in time; we can thus talk – as already mentioned – of *flat structure* rates, as in the regimes described in Chapter 3), their *values* $V(t)$ and also *reserves* $M(t)$ and $W(t)$ are found at a generic time t. We stressed there the importance of fair operations, such that, if the exchange law is strongly decomposable, $\forall t$, we obtain $V(t) = W(t)-M(t) = 0$.

In this chapter we will consider the application of the correspondences in both sides between *flows* given by the operation and *funds* given by their capital values, $V(t)$ at a given time t, all in a specific case: that of operation $\hat{O}$ constituted by a finite or infinite sequence of dated amounts *with the same sign*, that for one of the contracting parts is positive (and then they are incomes). Assuming, as it is used, an exchange law such that the equivalent amounts at different times always keep the same sign in the given temporal interval, the capital value $V(t)$ of $\hat{O}$ at whichever time t has the same sign as the concordant transactions and therefore $\hat{O}$ can never be a fair operation, whichever exchange law parameters are used. However, a fair operation O^* is obtained by adding to $\hat{O}$ the supply made by the opposite of such capital value paid at the evaluation time.

We will usually call an *annuity* the particular unfair operation $\hat{O}$, formed by a sequence of dated amounts with the same sign and made at equal intervals in time[1]. We will use the following definitions when referring to an annuity:

– *period* = constant temporal distance between two consecutive payments, usually of one year, or a multiple or sub-multiple of this;

– *frequency* = inverse of period, i.e. the number of payments per year;

– *interval* = time separating the beginning of the first period and the end of the last;

– *term* = length of the interval;

– *installment* = payment amount, constant or varying.

In addition, we will distinguish the annuities in the following ways:

– *annual*, when the period is one year, standard unit measure of time, or *fractional* or *pluriannual,* if the period is a submultiple or a multiple of one year;

– *annuity-due*, when the payment is made at the beginning of each period, or *annuity-immediate,* when it is made at the end of each period; a case of theoretical interest is that of *continuous annuity*, when the period tends towards zero and we have a continuous flow of payments;

– *certain*, if we assume that the established payments will be made with certainty, or *contingent*, if we assume that the payment of each installment is made only if a given event occurs[2]; in this part we will not consider *contingent* annuities and thus "certain" annuities will always be implied.

– *constant or varying*, referring to the installment sequence;

– *temporary or perpetuity*, if the term is finite or not.

EXAMPLE 5.1

1) The monthly payments for the rent of real estate can be considered as a certain annuity-due which is constant (or varying), monthly and temporary.

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1 Originally the meaning of the word "annuity" was restricted to annual payments, but it has been extended to include payments made at other regular intervals as well.

2 For a discussion on *contingent* annuities, which we consider in the field of "actuarial mathematics", it is necessary to have knowledge of basic probability calculus.

2) The future wages of a worker is a contingent annuity (considering the possibility of leaving the job due to death, invalidity, resignation, etc.), varying (due to the variation of wage), weekly- or monthly-immediate and temporary.

3) The "landed rent" of a cultivated field is, from an objective viewpoint (i.e. not considering the change of owners), a certain annuity, varying for perpetuity.

4) Another example of annuities is the payment of bills, accommodation expenses, etc.

In the problems of evaluation and negotiation we are interested in the *annuity capital value* even more than the annuity itself. This is, on the basis of what we have stated above, the amount that, associated with the evaluation time, gives rise to the indifferent supply[3] to the sequence of concordant supplies that form the annuity. The value thus depends on the evaluation time and the financial exchange law.

Usually this time is at the end of the annuity interval or at its initial time but it can even be before this. In the first case the capital value is called *final value or accumulated value*; in the second case *initial value* or *present value of a prompt annuity*; in the third case *present value of a delayed annuity*[4].

As concerns the exchange law, if the annuities are multi-year, a compound law, with a given interest conversion period and the corresponding rate per period, is usually used. For a short-term annuity, we usually use a simple interest law for the evaluation of the final value and a simple discount law for that of the initial value. Such financial laws are uniform, thus the annuity interval can always be translated, without changing the results[5].

3 Or "equivalent" in the sense specified in Chapter 2, if the exchange law is strongly decomposable (s.dec.).

4 We usually distinguish between "annuity" and "delayed annuity" according to the comparison of their initial times and their evaluation times, something that does not take the payment characteristics into consideration. We observe that if we limit ourselves to the preceding choices, the evaluation time is never inside the annuity interval, so that in order to calculate the capital value, consideration of a complete exchange law, union of accumulation and, possibly conjugate, discount laws is not needed. In fact, it is enough to use a discount law for the present value and an accumulation law for the final value. Therefore, the weak decomposability of such laws is enough to obtain the equivalence between $\hat{O}$ and its final or present value at the evaluation time.

5 Sometimes a distinction between *simple annuity*, when the conversion and payment period coincide, and *general annuity*, when such periods do not coincide, is introduced; but they are usually commensurable (i.e. the ratio of their length is a rational number). Furthermore, a general annuity can always be led back to a simple annuity using equivalent rates to obtain a conversion with the same period as the annuity rates (see Hummel, Seebeck (1969)).

In Chapter 6, when discussing *amortization* and *accumulation*, we will consider in applicative terms the step, briefly considered above, from a unfair operation $\hat{O}$ of an *annuity* to a fair operation O^* associated with it. It will be enough to add to $\hat{O}$ a supply given by the couple of numbers: [a time extreme of the annuity interval; the opposite of the value at such time]. Due to the financial equivalence between the whole of the supplies of an annuity between T_1 and T_2 and its initial value in T_1 or final value in T_2, we can conclude that:

a) the annuity payments are installments of a debt *amortization* equal to their initial value $V(T_1)$, in the sense that if a loan of amount $V(T_1)$ has been made, the annuity supplies *amortize* the debt i.e. pay it back both for the principal and for the charged interest, if the discount law applied to the annuity corresponds to the law that rules the loan;

b) the annuity payments are installments for the *accumulation* of a capital (i.e. *funding*) equal to their final value $V(T_2)$, in the sense that, depositing the dated amounts of the annuity into a profitable account according to the applied accumulation law, such an account accumulates (considering also the accrued allowed interests) a credit that will reach in T_2 the value $V(T_2)$.

5.2. Evaluation of constant installment annuities in the compound regime

5.2.1. *Temporary annual annuity*

For simplicity, choosing as t=0 the beginning of the interval, let us now calculate the *initial value* (IV) at the annual rate i of a *temporary annual* annuity – thus featured in the interval $[0,n]$ by payments at the beginning or end of each year, according to the annuity being *due* or *immediate*, which is defined as the sum of the present values of each payment, and indicated with the symbol V_0 or $\ddot{V}_0$. Let n be the length of the annuity and thus the number of payments.

In the specific case of a *unitary* annual annuity (i.e. with unitary installments) which is *temporary*, or respectively *immediate* or *due*, for the IV we use the symbols $a_{\overline{n}|i}$ or $\ddot{a}_{\overline{n}|i}$ [6], referring to annual periods and rates, and by definition

6 Such symbols, separately for immediate and due case, depend on the duration (or number of periods) n and the per period equivalent rate i. The diaeresis denotes annuity-due. The results of suitable calculations of these values for the immediate case (those for the due case can be calculated using the previous case: see e.g. (5.2)) and of other quantities are scheduled in specific "financial tables", depending on the most important parameters. However, the increasing availability of very good pocket scientific calculators enables exact calculations of the value of any parameters, thus making tables obsolete.

$$a_{\overline{n}|i} := [(1+i)^{-1} + (1+i)^{-2} + ... + (1+i)^{-n}] \tag{5.1}$$

$$\ddot{a}_{\overline{n}|i} := [1 + (1+i)^{-1} + (1+i)^{-2} + ... + (1+i)^{-(n-1)}] = (1+i)\ a_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

According to the equivalence principle, it is immediately verified that, as $v = (1+i)^{-1}$ = annual discount factor, $d = 1-v = iv$ = annual discount rate, we obtain[7]:

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n}{i};\ \ddot{a}_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{d} = \frac{1-v^n}{d} \tag{5.2}$$

and thus, for the annuity-immediate with installment R and annuity-due with installment $\ddot{R}$, the IV are respectively:

$$V_0 = R\ a_{\overline{n}|i};\ \ddot{V}_0 = \ddot{R}\ \ddot{a}_{\overline{n}|i} \tag{5.3}$$

Due to the observation at the end of section 5.1, in (5.3), R is the *constant installment of delayed amortization* in n years at the annual rate i of the debt V_0, ($\ddot{R}$ is for the *advance* amortization of the debt $\ddot{V}_0$).

Exercise 5.1

Calculate the amount to be paid today as an alternative to 5 payments of €1,000 with a deadline at the end of each year, with the annuity starting today, and adopting a compound annual exchange law at the annual delayed interest rate of 8.25%

A. We apply (5.3) using: $R = 1{,}000$; $i = 0.0825$; $n = 5$.

The following is obtained

$$V_0 = 1{,}000\ (1 - 1.0825^{-5})/0.0825 = 3{,}966.54$$

---

7 This and the following formulations can be proved algebraically, but we prefer to use financial arguments, as we consider them to be more appropriate here. Thus, to obtain (5.2), recalling that it is indifferent to defer an income if in the meantime the interest is accrued according to the prefixed accumulation law, using the compound regime and valuing at time 0, it is indifferent to receive the amount S at time 0 (present value = S) or receiving it at time n (present value = $S\ v^n$) with the addition of the delayed annual interests, forming an annual annuity-immediate for n years of installment Si (present value = $Si\ a_{\overline{n}|i}$) or advance, forming an annual annuity-due for n years of installment Sd (present value = $Sd\ \ddot{a}_{\overline{n}|i}$). Thus, $\forall S$, the financial equivalences: $S = S\ v^n + Si\ a_{\overline{n}|i}$; $S = S\ v^n + Sd\ \ddot{a}_{\overline{n}|i}$, and thus (5.2) can be obtained.

If we are interested in the *final value* (*FV*) V_n (or $\ddot{V}_n$) of the *annual temporary annuity-immediate or -due*, defined as the sum of the accumulated values in n of each payment, due to the decomposability of the compound law, it is equivalent to accumulating each payment until time n and adding the results or discount each payment until time 0 and accumulating for n years the sum of the obtained values. Therefore,

$$V_n = (1+i)^n V_0 \quad ; \quad \ddot{V}_n = (1+i)^n \ddot{V}_0 \tag{5.4}$$

Therefore, indicating with $s_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$ the *final value* of *unitary temporary annual annuity*, respectively *-immediate* and *-due*, and using the same argument as for the IV, by definition the following is the result:

$$s_{\overline{n}|i} := 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1} \tag{5.5}$$

$$\ddot{s}_{\overline{n}|i} := (1+i) + (1+i)^2 + \ldots + (1+i)^n$$

and the following is easily obtained[8]

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \tag{5.6}$$

$$\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1 = \ddot{a}_{\overline{n}|i} \, (1+i)^n = \frac{(1+i)^n - 1}{d} \, s_{\overline{n}|i}$$

while for annuity-immediate with installment R or -due with installment $\ddot{R}$ it results in

$$V_n = R \, s_{\overline{n}|i} \quad ; \quad \ddot{V}_n = \ddot{R} \, \ddot{s}_{\overline{n}|i} \tag{5.7}$$

8 The last terms of (5.6) can be obtained from financial equivalence valuing at time n the amount S paid in 0 or the same amount paid in n plus the annual delayed or advance interest, between 0 and n and obtaining the equalities: $S\,(1+i)^n = S + Si\, s_{\overline{n}|i}$; $S\,(1+i)^n = S + Sd\, \ddot{s}_{\overline{n}|i}$.

In (5.3) R is the *constant delayed installment of the accumulation* of V_n in n years at the annual rate i whereas $\ddot{R}$ is the *constant advance installment of the accumulation* of $\ddot{V}_n$.

The following symbols are frequently used and can be found on tables for the most common values of n and i:

$$\alpha_{\overline{n}|i} = 1/a_{\overline{n}|i} \; ; \; \ddot{\alpha}_{\overline{n}|i} = 1/\ddot{a}_{\overline{n}|i} \; ; \; \sigma_{\overline{n}|i} = 1/s_{\overline{n}|i} \; ; \; \ddot{\sigma}_{\overline{n}|i} = 1/\ddot{s}_{\overline{n}|i} \, . \qquad (5.8)$$

These form the coefficient to be applied to the IV or FV of an annuity-immediate or annuity-due to obtain the constant installments. In fact, obtaining R and $\ddot{R}$ from (5.3) and (5.7), it follows that

$$R = V_0 \, \alpha_{\overline{n}|i} \; ; \; \ddot{R} = \ddot{V}_0 \, \ddot{\alpha}_{\overline{n}|i} \; ; \; R = V_n \, \sigma_{\overline{n}|i} \; ; \; \ddot{R} = \ddot{V}_n \, \ddot{\sigma}_{\overline{n}|i} \qquad (5.8')$$

The values in (5.8') thus give the amortization installment of the debt V_0 and the delayed or advance funding installment of the capital V_n[9].

Calculation of rate and length

Considering only the annuity-immediate case, (5.3) is a constraint between the quantities V_0, R, n, i, which enables expression one to be dependent on the other three. The first parts of (5.3) and (5.8) explain V_0 and R. The calculation of i is reduced to that of the IRR (see Chapter 4) of the operation $O^* = \hat{O} \cup (0,-V_0)$ defined in section 5.1. Sometimes more needs to be said about the calculation of the *implicit length n*.

From the 1st part of (5.3) we obtain, recalling (5.2):

$$\frac{V_0}{R} = \frac{1-(1+i)^{-n}}{i} \quad \text{i.e.} \quad 1 - \frac{iV_0}{R} = (1+i)^{-n} = e^{-\delta n}$$

Considering the natural logarithm and (3.30') we obtain the implicit length:

9 The comparison between (5.8) and (5.8') makes it possible to give a financial meaning to the values $\alpha_{\overline{n}|i}$, $\ddot{\alpha}_{\overline{n}|i}$, $\sigma_{\overline{n}|i}$, $\ddot{\sigma}_{\overline{n}|i}$. They are, in order, the constant delayed and advance installments of amortization of the unitary debt and of funding of the unitary capital.

$$n = \frac{-\ln\left(1 - \frac{iV_0}{R}\right)}{\delta} \tag{5.8''}$$

Solution (5.8") is positive, because $0 < 1 - iV_0/R < 1$, but usually it is not a natural number. We can consider the natural n_0 that better approximates the solution, again calculating (if desired) the IV as a function of n_0.

Between the quantities (5.8) there are the following relations, which have a relevant financial meaning[10]:

$$\alpha_{\overline{n}|i} = \sigma_{\overline{n}|i} + i \; ; \; \ddot{\alpha}_{\overline{n}|i} = \ddot{\sigma}_{\overline{n}|i} + d \tag{5.9}$$

It is useful to consider that (as can be deduced from their algebraic values and financial meaning):

– $a_{\overline{n}|i}$ is an increasing function of n and decreasing of i;

– $\alpha_{\overline{n}|i}$ is a decreasing function of n and increasing of i;

– $s_{\overline{n}|i}$ is an increasing function of n and increasing of i;

– $\sigma_{\overline{n}|i}$ is a decreasing function of n and decreasing of i.

The same dynamics apply to the annuity-due values.

We have examined, so far, the evaluation of annuities carried out at the beginning of the interval (and thus, as already specified in section 5.1, we talk about IV and prompt annuities). Furthermore, we obtain *present values of delayed annuity* (PVDA) if the evaluation time precedes the beginning of the interval. Putting it in $-r$ we have a deferment, and then an increment, of the discount times of all payments, of r years (r can also be not integer). Therefore, indicating with ${}_{r/}a_{\overline{n}|i}$ or ${}_{r/}\ddot{a}_{\overline{n}|i}$ the PVDA in the case of unitary temporary installments, annuity-immediate or annuity-due, it is obvious that

10 Equation (5.9) can be easily deduced algebraically but it can be justified financially with an equivalence between amortizations: for the debtor it is equivalent to paying, at the end or beginning of the year, to the creditor the constant amortization installment referred to as the unitary debt (for n years at rate i) or to paying only the interest amount, i or d, and accumulating in n years the unitary capital to pay back to the creditor in only one payment at the end. In the alternative, the annual constant payments must be equal. Thus, equation (5.9) is justified. We will come back to this in Chapter 6 when considering "American" amortization.

$$ {}_{r/}a_{\overline{n}|i} = v^r a_{\overline{n}|i} = \frac{v^r - v^{r+n}}{i}; \; {}_{r/}\ddot{a}_{\overline{n}|i} = v^r \ddot{a}_{\overline{n}|i} = \frac{v^r - v^{r+n}}{d} \tag{5.10} $$

and in the case of constant installment R of an annuity-immediate or $\ddot{R}$ of an annuity-due, we obtain:

$$ {}_{r/}V_0 = R\, {}_{r/}a_{\overline{n}|i} \; ; \; {}_{r/}\ddot{V}_0 = \ddot{R}\, {}_{r/}\ddot{a}_{\overline{n}|i} \tag{5.3'} $$

5.2.2. *Annual perpetuity*

Let us now consider *annual perpetuity*, observing that, in the case of constant installments, the FV is not considered because it goes to infinity[11]. However, the IV are finite and are obtained using $n \to +\infty$ in the previous formulae from (5.1) to (5.3). It follows that[12]

$$ a_{\overline{\infty}|i} = 1/i, \; \ddot{a}_{\overline{\infty}|i} = 1/d \; ; \; V_0 = R/i, \; \ddot{V}_0 = \ddot{R}/d \tag{5.11} $$

and, for the PVDA:

$$ {}_{r/}a_{\overline{\infty}|i} = v^r/i; \; {}_{r/}\ddot{a}_{\overline{\infty}|i} = v^r/d; \; {}_{r/}V_0 = R\, v^r/i; \; {}_{r/}\ddot{V}_0 = \ddot{R}\, v^r/d \tag{5.11'} $$

Exercise 5.2

1) Calculate the initial and final value of an annual annuity-due with constant installment $\ddot{R}$ = €150, annual interest rate i = 8.55% = 0.0855, length n = 17.

A. Using (5.3) we obtain

$$ \ddot{V}_0 = 150 + 150\,(1 - 1.0855^{-16})/0.0855 = 150\,(1 + 8.5484723) = 1{,}432.27 $$

In addition, we obtain d = 0.0855/1.0855 = 0.0787656 and, applying (5.6),

$$ \ddot{V}_n = \ddot{R}\, \ddot{s}_{\overline{n}|i} = (1.0855^{17} - 1)/0.07876555 = (1.0855^{18} - 1)/0.0855 = 38.515980 $$

and thus

---

11 It should be calculated in the compound regime, at whatever non-negative rate, as the sum of the elements of a geometric sequence with ratio ≥ 1, which is positively diverging.

12 The expressions in (5.11), with their formal simplicity, are of fundamental importance in the accumulation problems of perpetual incomes of lasting assets.

$$\ddot{V}_{17} = 150 \cdot 38.5159796 = 5{,}777.40$$

(5.4) is soon verified, resulting in

$$(1.0855)^{17} 1432.27 = 4.033732164\ 1432.27 = 5{,}777.40$$

2) An estate can be bought with an advance of €5,600 and a loan that involves 15 annual delayed installments of €850 each. Calculate the equivalent price in cash, if the annual loan rate is 6%.

A. Let P be such a price, and applying (5.2) the result is:

$$P = 5600 + 850\ (1\text{-}1.06^{-15})/0.06 = 5600 + 8255.41 = 13{,}855.41.$$

3) A reserve fund of a company at the closing balance is €156,500. If we want to increase it in 5 years to the level of €420,000 through constant earmarking at the end of each following year in a savings account at the compound annual interest of 6%, calculate the amount of each annual earmarking, assuming they are constant.

A. Denoting the earmarking by C, it is given by

$$C = (420{,}000 - 156{,}500)\,\sigma_{5|0.06} = 263{,}500\ 0.06/(1.06^5 - 1) = 263{,}500 \cdot 0.17739640$$
$$= €41{,}743$$

4) Verify (5.9) for the values $n = 15$, $i = 0.09$

A. The first formula gives rise to the equality: 0.12405888 = 0.03405888 + 0.09 and the second to: 0.11381549 = 0.03124668 + 0.08256881, obtained from the previous one multiplying by v=1/(1 + i) = 1/1.09 = 0.91743119.

5.2.3. *Fractional and pluriannual annuities*

In sections 5.2.1 and 5.2.2 we considered the evaluation in the compound regime, with annual conversion of interest, of annuities with annual installments. The same formulae can be used for *m-fractional annuities* i.e. with installments of frequency m (usually m = 2, 3, 4, 6, 12, 52, 360 for the usual fraction of a year, even if only the constraint $m\text{-}1 \in \mathcal{N}$ follows from the definition) for the evaluation of which we use the m-fractional conversion of interest and then it is given the delayed per period rate $i_{1/m}$ for $1/m$ of a year or the intensity $j(m) = m\ i_{1/m}$. It is sufficient to use in such formulae $i_{1/m}$ instead of i and as a temporal parameter the number of payments, then changing the unit measure of time. For the *pluriannual annuities* with a payment every p years it is sufficient to use $m = 1/p$ and instead of i the p-annual equivalent rate, given by $i_p = (1+i)^p - 1$ [13].

In fact, we recall that in a compound accumulation process we obtain the same return with an annual conversion at rate i or with the m-fractional conversion at the rate per period $i_{1/m}$ if the two rates are equivalent, i.e. linked by (3.26).

Temporary fractional annuities

The IV at the delayed rate i of an *m-fractional annuity with length* $n \leq +\infty$ (i.e. an annuity such that the annual amount R is fractionated into the annual interval in m equally spaced installments, the amount of which is $R_{1/m} = R/m$, so the annuity has period $1/m$) can be evaluated, distinguishing the annuity-immediate from the annuity-due case, through formulae analogous to (5.3), obtaining

$$V_0^{(m)} = R_{1/m}\, a_{m\overline{n}|i_{1/m}} \;;\; \ddot{V}_0^{(m)} = R_{1/m}\, \ddot{a}_{m\overline{n}|i_{1/m}} \quad [14] \tag{5.12}$$

In the specific case of an annuity *unitary m-fractional* (i.e. with installments of amount $1/m$ so as to have a unitary annual amount) *temporary*, respectively *annuity-immediate* or *annuity-due,* for the IV we use the symbols $a^{(m)}_{\overline{n}|i}$ or $\ddot{a}_{\overline{n}|i}$ and, analogously to (5.2), we obtain the following formulae

$$a^{(m)}_{\overline{n}|i} = \frac{1-v^n}{j(m)};\; \ddot{a}^{(m)}_{\overline{n}|i} = (1+i)^{1/m}\, a^{(m)}_{\overline{n}|i} = \frac{1-v^n}{\rho(m)} \quad [15] \tag{5.13}$$

In general, with an annual total R, (5.12) can be rewritten, using (5.13), as

$$V_0^{(m)} = R_{1/m}\, a_{m\overline{n}|i_{1/m}} \;;\; \ddot{V}_0^{(m)} = R_{1/m}\, \ddot{a}_{m\overline{n}|i_{1/m}} \tag{5.12'}$$

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13 Given that these transformations leave the period of the annuity and the conversion unchanged, we are still in the case of basic annuities, specified in footnote 5.

14 In the fractional annuity the number of payments nm can be large and, if we use financial tables, it can be higher than the maximum in the table. In such cases, the following decomposition can be useful: $a_{\overline{n+p}|i} = a_{\overline{n}|i} + (1+i)^{-n} a_{\overline{p}|i}$, which enables calculation of the 1st member when the length $n+p$ goes beyond the limit of the table, provided that $(n.p)$ is whichever duration included in the table.

15 Proceeding analogously to footnote 7, (5.13) can be obtained taking into account the financial equivalence between the amount S in 0 or the same amount in n adding the delayed or advance interest paid with frequency m. Their annual total is thus, respectively $mSi_{1/m}$ or $mSd_{1/m}$. Therefore, the equivalences give rise to the equations: $S = S\, j(m)\, a^{(m)}_{\overline{n}|i} + S\,(1+i)^{-n}$, $S = S\, \rho(m)\, a^{(m)}_{\overline{n}|i} + S\,(1+i)^{-n}$, and thus (5.13).

The step from (5.12) to (5.12') is justified observing that

$$V_0^{(m)} = \frac{R}{m}\frac{1-(1+i_{1/m})^{-mn}}{i_{1/m}} = R\frac{1-(1+i)^{-n}}{j(m)}$$

$$\ddot{V}_0^{(m)} = \frac{R}{m}\ \frac{1-(1+i_{1/m})^{-mn}}{i_{1/m}}(1+i_{1/m}) = R\ \frac{1-(1+i)^{-n}}{\rho(m)}$$

To obtain the FV of a temporary m-fractional annuity for n years, annuity-immediate or annuity-due, given the decomposability of the compound laws it is enough to accumulate the IV during the interval of the annuity. If the annual total is unitary, the FV, indicated in the two cases with $s_{\overline{n}|i}^{(m)}$ and $\ddot{s}_{\overline{n}|i}^{(m)}$, are

$$s_{\overline{n}|i}^{(m)} = (1+i)^n\ a_{\overline{n}|i}^{(m)}\ ;\ \ddot{s}_{\overline{n}|i}^{(m)} = (1+i)^n\ \ddot{a}_{\overline{n}|i}^{(m)} \qquad (5.6')$$

In general, with annual total R, analogously to (5.4'), the FV are given by

$$V_n^{(m)} = R\ s_{\overline{n}|i}^{(m)} = R\,(1+i)^n\, a_{\overline{n}|i}^{(m)} = (1+i)^n\ V_0^{(m)} \qquad (5.4')$$

$$\ddot{V}_n^{(m)} = R\ \ddot{s}_{\overline{n}|i}^{(m)} = R\,(1+i)^n\ \ddot{a}_{\overline{n}|i}^{(m)} = (1+i)^n\,\ddot{V}_0^{(m)}$$

Exercise 5.3

Calculate the IV and FV of the annuity formed by the income flow with monthly delayed installment of €650 for 10 years at the nominal annual rate 12-convertible of 9%.

A. We have $i_{1/12} = 0.0075$, $i = 0.0938069$, $R = 7{,}800$, and then

$$V_0^{(12)} = 650\ \frac{1-1.0075^{-120}}{0.0075} = 650\cdot 78.9416927 = 51{,}312.10\text{, or}$$

$$V_0^{(12)} = 7{,}800\ a_{\overline{10}|i}^{(12)} = 7{,}800\frac{1-1.0938069^{-10}}{0.09} = 7{,}800\cdot 6.5784744 = 51{,}312.10$$

In addition: $V_n^{(12)} = 5{,}1312.10\cdot 1.0075^{120} = 125{,}784.28$

Applying footnote 14 and assuming there is a financial table with a maximum length of 100, choosing the length 70 and 50, with 70+50=120, this results in

$$a_{\overline{120}|0.0075} = a_{\overline{70}|0.0075} + 1.0075^{-70} a_{\overline{50}|0.0075} = 54.3046221 + 0.59271533 \cdot 41.5664471 = 78.9416925,$$

and thus $V_0^{(12)} = 51312.10$, i.e. the same value as previously.

If, with the same data, the installments are in advance, this results in: $d_{1/12} = 0.0074442$; $\rho(12) = 0.08933$ and

$$\ddot{V}_0^{(12)} = 7,800\ \ddot{a}^{(12)}_{\overline{10}|0.0938069} = 7,800\ \frac{1-1.0938069^{-10}}{0.08933} = 7,800 \cdot 6.6278135 = 51,696.96$$

$$\ddot{V}_{10}^{(12)} = 51696.96 \cdot 1.0075^{120} = 126,727.71$$

For completeness, let us mention briefly annuities *m-fractional delayed* for r years, for which the PVDA are obtained multiplying by v^r the corresponding IV With a unitary annual total we have, with the obvious meaning of the symbols

$$_{r/}a^{(m)}_{n|i} = v^r a^{(m)}_{\overline{n}|i} = \frac{v^r - v^{n+r}}{j(m)};\ _{r/}\ddot{a}^{(m)}_{n|i} = (1+i)^{1/m}\ _{r/}a^{(m)}_{n|i} = \frac{v^r - v^{n+r}}{\rho(m)} \quad (5.13')$$

while in general, with installment R/m it is enough to multiply by R the values (5.13').

Exercise 5.4

Using the data in exercise 5.3, calculate the PVDA with 4 years deferment.

A. The following is obtained:

$$_{4/}a^{(12)}_{\overline{10}|9.38069\%} = 1.0938069^{-4}\, a^{(12)}_{\overline{10}|9.38069\%} = 0.6986141 \cdot 6.5784744 = 4.5958166$$

$$_{4/}\ddot{a}^{(12)}_{\overline{10}|9.38069\%} = 1.0938069^{-4}\, \ddot{a}^{(12)}_{\overline{10}|9.38069\%} = 0.6986141 \cdot 6.6278135 = 4.6302842$$

Fractional perpetuity

The IV of the *fractional perpetuity* are obtained from those for temporary values putting $n \to +\infty$ and taking into account that in such a case $v^n \to 0$. If the annual total is unitary, we obtain, analogously to (5.11),

$$a^{(m)}_{\overline{\infty}|i} = 1/j(m) \;\; ; \;\; \ddot{a}^{(m)}_{\overline{\infty}|i} = (1+i)^{1/m} \; a^{(m)}_{\overline{\infty}|i} = 1/\rho(m) \tag{5.14}$$

while with installment *R/m*

$$V_0^{(m)} = R \; a^{(m)}_{\overline{\infty}|i} = R/j(m) \;\; ; \;\; \ddot{V}_0^{(m)} = R \; \ddot{a}^{(m)}_{\overline{\infty}|i} = R/\rho(m) \tag{5.14'}$$

and, for the PVDA the result is[16]

$$_{r/}a^{(m)}_{\overline{\infty}|i} = v^r/j(m) \;\; ; \;\; _{r/}\ddot{a}^{(m)}_{\overline{\infty}|i} = v^r/\rho(m) \tag{5.15}$$

$$_{r/}V_0^{(m)} = R\, v^r/\,j(m) \;\; ; \;\; _{r/}\ddot{V}_0^{(m)} = \ddot{R} \;\; v^r/\rho(m)$$

Exercise 5.5

Using the data from Exercise 5.4, calculate the IV and the PVDA of an immediate or due perpetuity with flow equal to €5,600/year.

A. Applying (5.14') for the IV we obtain

$$V_0^{(m)} = 5{,}600\, a^{(12)}_{\overline{\infty}|9.38069\%} = \frac{5{,}600}{0.09} = 62{,}222.22$$

$$\ddot{V}_0^{(m)} = 5{,}600\, \ddot{a}^{(12)}_{\overline{\infty}|9.38069\%} = \frac{5{,}600}{0.08933} = 62{,}688.91;$$

and for the PVDA discounting we obtain

16 A comparison between the formulae shows the intuitive fact that both for annual annuity, a fractional annuity and (as we will see) a continuous annuity, the following decomposition holds: the *IV* of a perpetuity is the sum of the *IV* of a corresponding temporary annuity and of the *PVDA* of the corresponding delayed annuity at the end of the previous one. In the simplest case, of a unitary annual annuity-immediate, the result is: $a_{\overline{\infty}|i} = a_{\overline{r}|i} + {}_{r/}a_{\overline{\infty}|i}$, following the identity: $1/i = (1-v^r)/i + v^r/i$. This splitting up is similar to the juridical splitting with usufruct and bare ownership (but in a different meaning as used in Chapter 4). The usufruct is like the temporary annuity, whereas the bare ownership is like the delayed annuity, which starts after the end of the temporary one. However – unlike what occurs in annuities-certain – the splitting up with usufruct and bare ownership leads to uncertain values, since it is linked to a random usufructuary lifetime. Therefore, in a random case we calculate mean values according to expected lifetime.

$$ {}_{4/}V_0^{(12)} = 0.698614 \cdot 62222.22 = 43{,}469.32; $$

$$ {}_{4/}\ddot{V}_0^{(12)} = 0.698614 \cdot 62688.91 = 43{,}795.35 $$

or applying (5.15)

$$ {}_{4/}V_0^{(12)} = 5{,}600 \frac{0.698614}{0.09} = 43469.32; \; {}_{4/}V_0^{(12)} = 5{,}600 \frac{0.698614}{0.08933} = 43{,}795.35 $$

Continuous annuities

Let us briefly consider *continuous annuities (temporary or perpetuities, prompt or delayed)*, characterized by a continuous flow of payments, which we assume here to be constant. They can be considered as a specific case of fractional annuities, for $m \to +\infty$. For uniformity of symbols and easier comparison we assume the flow of R per year, where R is also the amount paid in one year. Since the period goes to 0, the distinction between annuity-immediate and annuity-due does not make sense.

Indicating with $a_{\overline{n}|i}^{(\infty)}$ the IV, with $s_{\overline{n}|i}^{(\infty)}$ the FV (only if $n<\infty$), with ${}_{r/}a_{\overline{n}|i}^{(\infty)}$ the PVDA of the *unitary* annuity (R=1), taking into account (5.13), (5.13'), (5.14), (5.15) and the convergences $\rho(m) \to \delta \leftarrow j(m)$ when $m \to +\infty$, using these limits the following is easily obtained:

$$ a_{\overline{n}|i}^{(\infty)} = \frac{1-v^n}{\delta}; \; s_{\overline{n}|i}^{(\infty)} = \frac{(1+i)^n - 1}{\delta}; \; {}_{r/}a_{\overline{n}|i}^{(\infty)} = \frac{v^r - v^{r+n}}{\delta}; \tag{5.16} $$

$$ a_{\overline{\infty}|i}^{(\infty)} = \frac{1}{\delta}; \; {}_{r/}a_{\overline{\infty}|i}^{(\infty)} = \frac{v^r}{\delta} $$

Exercise 5.6

Using the data in exercise 5.4, calculate the values in (5.16) of the unitary perpetuities.

A. We have: i = 0.0938069 and δ = ln 1.0938069 = 0.0896642; thus the following is obtained:

$$ a_{\overline{10}|i}^{(\infty)} = 6.603113; \; s_{\overline{10}|i}^{(\infty)} = 16.186588; \; {}_{4/}a_{\overline{10}|i}^{(\infty)} = 4.613027; $$

$$ a_{\overline{\infty}|i}^{(\infty)} = 11.152723; \; {}_{4/}a_{\overline{\infty}|i}^{(\infty)} = 7.791450. $$

If the annuity has flow *R,* it is enough to multiply by *R* the values of (5.16)[17].

Pluriannual annuity, temporary or perpetuities

We will now comment briefly on *pluriannual annuities*, characterized by constant installments, annuity-immediate or annuity-due, equally spaced over *p* years, therefore with frequency $1/p$. They find application, for example, in the evaluation of the charges due to industrial equipment renewal.

If such annuities are temporary, it is necessary that $n = kp$ (where $k \in \mathcal{N}$ is the number of installments). Thus, indicating with $i_p = (1+i)^p - 1 = i\, s_{\overline{p}|i}$ the *p*-annual rate equivalent to *i*, to obtain the capital value it is enough to assume an interval of *p* years as the new unit measure and apply the formulae for the annual annuities using $i_{1/p}$ as the rate and *k* as the length.

In more detail, considering as *unitary* (referring to the annual amount) *p-annual annuity* that with installment $R_p = p$, we indicate with $a^{(1/p)}_{\overline{n}|i}$ or $\ddot{a}^{(1/p)}_{\overline{n}|i}$ the IV of the *temporary* one with length *n*, *-immediate* or *-due*. It thus follows that:

$$a^{(1/p)}_{\overline{n}|i} = p\,\frac{1-(1+i_p)^{-k}}{i_p} = p\,\frac{1-(1+i)^{-n}}{(1+i)^p - 1} = p\,\sigma_{\overline{p}|i} < a_{\overline{n}|i} \tag{5.17}$$

$$\ddot{a}^{(1/p)}_{\overline{n}|i} = (1+i)^p\, a^{(1/p)}_{\overline{n}|i} = p\,\frac{1-(1+i)^{-n}}{1-(1+i)^{-p}} = p\, a_{\overline{n}|i}\,\alpha_{\overline{p}|i} > a_{\overline{n}|i}$$

17 The value of continuous annuity can be calculated analytically in the compound regime using the integrals of continuous flow, which are discounted or accumulated. If the flow is the constant *R*, the results are:

$$R\, a^{(\infty)}_{\overline{n}|i} = R\frac{1-v^n}{\delta}\,; \quad R\, s^{(\infty)}_{\overline{n}|i} = \int_0^n R\, e^{\delta(n-t)}\,dt = R\frac{e^{\delta n}-1}{\delta}\,;$$

$$R_{r/}a^{(\infty)}_{\overline{n}|i} = \int_0^n R\, e^{-\delta t}\,dt = \int_r^{r+n} R\, e^{-\delta t}\,dt = R\frac{e^{-\delta r} - e^{-\delta(r+n)}}{\delta}\,;$$

$$R\, a^{(\infty)}_{\overline{\infty}|i} = \int_0^{+\infty} R\, e^{-\delta t}\,dt = \frac{R}{\delta}\,; \quad R_{r/}a^{(\infty)}_{\overline{\infty}|i} = \int_r^{+\infty} R\, e^{-\delta t}\,dt = R\frac{e^{-\delta r}}{\delta}\,,$$

i.e. the values obtained from (5.16). With varying flow $\varphi(t)$, the *IV* of a temporary annuity is given by $\int_0^n \varphi(t)\, e^{-\delta t}\,dt$, with obvious modification for the other cases.

and, in general, the IV of the analogous annuity with installment R_p-*immediate* or R_p-*due* are

$$V_0^{(1/p)} = R_p \frac{1-(1+i)^{-n}}{(1+i)^p - 1} = R_p \; a_{\overline{n}|i} \, \sigma_{\overline{p}|i}; \tag{5.17'}$$

$$\ddot{V}_0^{(1/p)} = R_p \frac{1-(1+i)^{-n}}{1-(1+i)^{-p}} = R_p \; a_{\overline{n}|i} \, \alpha_{\overline{p}|i}$$

Multiplying (5.17') by $(1+i)^n$ the FV of such annuities are obtained. Multiplying them instead by $(1+i)^{-r}$ the PVDA of the analogous annuity *p-annual temporary delayed* for r years are obtained. Using instead $n \to +\infty$, the IV of the analogous *p-annual perpetuity* are obtained. For the unitary IV it is found that

$$a_{\overline{\infty}|i}^{(1/p)} = \frac{p}{(1+i)^p - 1} = \frac{1}{j(1/p)} \tag{5.18}$$

$$\ddot{a}_{\overline{\infty}|i}^{(1/p)} = \frac{p}{1-(1+i)^p} = \frac{1}{\rho(1/p)} = a_{\overline{\infty}|i}^{(1/p)} + p$$ [18]

and it is generally sufficient to substitute R_p to p, obtaining:

$$V_0^{(1/p)} = \frac{R_p}{(1+i)^p - 1} = R_p \; a_{\overline{\infty}|i} \, \sigma_{\overline{p}|i} \tag{5.18'}$$

$$\ddot{V}_0^{(1/p)} = \frac{R_p}{1-(1+i)^{-p}} = R_p \; a_{\overline{\infty}|i} \, \alpha_{\overline{p}|i}$$

18 Observe that, as for (5.14), the values in (5.18) represent the reciprocal of the intensity per period on a length of p years. In the second part of (5.18), the last term is justified noting that the annuity-due value is obtained from the annuity-immediate value adding the initial $R_{1/p} = p$ and subtracting nothing due to perpetuity.

and also: $\ddot{V}_0^{(1/p)} = V_0^{(1/p)} + R$.

Exercise 5.7

1) Calculate the IV of a five-yearly annuity at the annual rate of 7%, -immediate or -due, with a length of 20 years or perpetuity, with delayed installments of €58,500.

A. By applying formulae from (5.17) to (5.18') the following is obtained:

– in the temporary case:

$$a_{\overline{20}|0.07}^{(1/5)} = 5\,\frac{1-(1.07)^{-20}}{(1.07)^5-1} = 9.2110032$$

$$V_0^{(1/5)} = 58{,}500\,\frac{1-(1.07)^{-20}}{(1.07)^5-1} = 107{,}768.74$$

$$\ddot{a}_{\overline{20}|0.07}^{(1/5)} = 5\,\frac{1-(1.07)^{-20}}{1-(1.07)^{-5}} = 12.9189075$$

$$\ddot{V}_0^{(1/5)} = 58{,}500\,\frac{1-(1.07)^{-20}}{1-(1.07)^{-5}} = 151{,}151.22$$

– in the perpetual case:

$$a_{\overline{\infty}|0,07}^{(1/5)} = \frac{5}{(1,07)^5-1} = 12.4207639 \quad ; \quad V_0^{(1/5)} = \frac{58500}{(1.07)^5-1} = 145{,}322.92$$

$$\ddot{a}_{\overline{\infty}|0,07}^{(1/5)} = \frac{5}{1-(1,07)^{-5}} = 17.9189075 \quad ; \quad \ddot{V}_0^{(1/5)} = \frac{58500}{(1.07)^5-1} = 203{,}822.92$$

2) For the functioning of a company the owner buys equipment that must be replaced, due to wear and obsolescence, every 5 years. A horizon of 20 years is established for the company's activity, for which the return rate is 7.45%. The mean cost of the equipment is evaluated in €255,000. On the basis of such estimations, calculate the IV for the purchase expenses of such equipment for the whole time horizon.

A. The IV asked for is that of a 5-year temporary annuity-due for 20 years, and according to (5.17') it is

$$\ddot{V}_0^{(1/5)} = \ddot{a}_{\overline{20}|0.0745}^{(1/5)} = 255{,}000\,\frac{1-(1.0745)^{-20}}{1-(1.0745)^{-5}} = 255{,}000 \cdot 2.5259699 = 644{,}122.34$$

3) A forestry company wants to buy a wood, the income of which follows the periodic cutting down of the trees and use of the wood, with spontaneous reforestation, at a price which, according to the principle of "capitalization of income", is given by the present value, at a rate corresponding to cost-opportunity, of future profits. Supposing that:

– the trees are cut down every 12 years;

– costs and returns between the periods are compensated;

– the profit due to each harvest is €55,000;

– the evaluation rate is 6.20%;

calculate the price offered by the company, in the alternative hypotheses that

a) the trees have only just been cut down;

b) all the trees have an age of 7 years;

c) all the trees have an age of 12 years.

A. In the given problem the return can be considered perpetual[19]. Therefore, the price P following the accumulation of the profit (then the IV of the annuity of future returns) is obtained as follows:

a) $P = P_a$ is the IV of the 12-year annuity-immediate with constant installment $R_p = 55{,}000$:

$$P_a = \frac{R_{12}}{i_{12}} = \frac{55000}{1.062^{12} - 1} = 51{,}973.51$$

b) $P = P_b$ is the PVDA of a 12-year annuity-due delayed for 5 years, with constant installment $R_p = 55{,}000$:

$$P_b = (1+i)^{-5} \frac{R_{12}\,(1+i_{12})}{i_{12}} = \frac{55000 \cdot 1.062^{7}}{1.062^{12} - 1} = 70.210{,}91$$

c) $P = P_c$ is the IV of a 12-year annuity-due with constant installment $R_p = 55{,}000$:

19 A strong limitation for the meaning of this calculus, and all those concerning perpetuities with constant installment, follows the unrealistic hypothesis of periodic constant profits in an unlimited time. Furthermore, if we suppose profit changing every p years in geometric progression, i.e. varying with constant rate, then – as we will see in the case of annuity with varying rates – Fisher's equation permits exact calculation by means of constant annuities, which are equivalent to those varying in geometric progression, if one assumes a new evaluation rate as a function of the given one and of the one in progression.

$$P_c = \frac{R_{12}\, i_{12}}{1+i_{12}} = \frac{55000 \cdot 1.062^{12}}{1.062^{12}-1} = 106{,}973.51$$

4) A field with poplars is bought by a private person just after the harvest and following reforestation. He leases the field to a forestry company for 4 productive cycles; if the cut is every 8 years, the length is 32 years. The rent on the basis of market prices is €43,500 to pay after each cut, with the tenant bearing the cost of reforestation. Calculate the IV of such a contract at the annual evaluation rate of 4.5%. In addition, in the hypothesis that the owner is able to obtain from the tenant the same total amount, but divided into annual advance installments, calculate the percentage increments of the contract value.

A. The IV of the standard contract is that of an 8-year temporary annuity-immediate for 32 years; thus, according to (5.17'),

$$V_0^{(1/8)} = R_8\, a^{(1/8)}_{\overline{32}|0.045} = 43{,}500\, \frac{1\text{-}1.045^{-32}}{1.045^{8}-1} = 77{,}858.22$$

The IV $\hat{V}_0^{(1/8)}$ of the contract is that of an annual annuity-due, temporary for 32 years, with installment $R_8/8$. Therefore, using $d = 0.0430622$, we obtain

$$\frac{R_8}{8}\ddot{a}^{(1/8)}_{\overline{32}|0.045} = 5{,}437.50\,\frac{1-1.045^{-32}}{0.0430622} = 5{,}437.50\cdot 17.5443913 = 95{,}397.63$$

and thus the percentage increment is

$$100\ \frac{\hat{V}_0^{(1/8)} - V_0^{(1/8)}}{V_0^{(1/8)}} = 22.527\ \%$$

5.2.4. *Inequalities between annuity values with different frequency: correction factors*

We have seen that, in relation to all the unitary annuities considered so far, by changing frequency their IV are in inverse relation to the corresponding per period interest or discount intensities. Therefore, denoting by $j(1/p)$ and $\rho(1/p)$ the p-annual interest and discount intensities and by m the frequency of the fractional annuity, given using the compound regime we obtain: $\rho(1/p) < d < \rho(m) < \delta < j(m) < i < j(1/p)$, the following inequalities hold:

$$\ddot{a}^{(1/p)}_{\overline{n}|i} > \ddot{a}_{\overline{n}|i} > \ddot{a}^{(m)}_{\overline{n}|i} > a^{(\infty)}_{\overline{n}|i} > a^{(m)}_{\overline{n}|i} > a_{\overline{n}|i} > a^{(1/p)}_{\overline{n}|i};\ (n \le \infty) \qquad (5.19)$$

Analogous inequalities hold for the PVDA with same deferment r and for the FV of temporary unitary annuities.

Formulae shown in section 5.2.3 enable direct calculation of the capital value of non-annual annuities. Furthermore, it can be convenient to use *correction factors* to apply to the IV or to the installments of annual annuities, if these elements are easily available, to obtain, by multiplying, the IV or the FV or the *equivalent* installments of fractional or pluriannual annuities.

The correction factor to go from the IV to the FV of a temporary annuity for n years of whichever type is, in all cases, $(1+i)^n$ and it is the reciprocal for the inverse transformation.

More complex are the factors to go from annual annuities to fractional or pluriannual ones, and vice versa.

In light of this, we distinguish between two problems on such transformations, from frequency 1 (*annual* case) to frequency m with $m\text{-}1 \in \mathcal{N}$ (*fractional* case) or $m=1/p$ (*pluriannual* case with payments every p years).

PROBLEM A.– *Transformation of the capital value (at a given time) of the annual annuity in that of the annuity with the same length and annual rate but with frequency m (or, more generally, with a frequency changing from m' to m") which leaves unchanged the total annual payment*[20].

PROBLEM B.– *Transformation of the installment of the annual annuity in that of the annuity with the same length and annual rate but with frequency m (or, more generally, with a frequency changing from m' to m") which leaves unchanged the capital value (at a given time).*

Problem A is solved by applying to the capital value the correction factor f_A given by the reciprocal of the ratio between the corresponding per period intensities, i.e.:

– for annuity-immediate, $f_A = i/j(m) > 1$ (being $i = j(1)$), applying it to the value of the annual annuity, and $f_A = j(m')/j(m'')$ in general;

– for annuity-due, $f_A = d/\rho(m) < 1$ (being $d = \rho(1)$), applying it to the value of the annual annuity[21], and $f_A = \rho(m')/\rho(m'')$ in general.

20 Observe that corresponding annuities in the sense of Problem A have installments proportional to the periods.

21 In practice this factor is seldom used, it is preferred to apply the factor $(1+i)^{1/m}\, i/j_{(m)}$ to the value of the annual annuity-immediate.

Problem B is solved by applying to the capital value the correction factor f_B given by the ratio between the corresponding per period rates, i.e.:

– for annuity-immediate, $f_B = i_{1/m}/i < 1/m$, applying it to the annual rate R, and $f_B = i_{1/m''}/i_{1/m'}$ in general;

– for annuity-due, $f_B = d_{1/m}/d > 1/m$, applying it to the annual rate $\ddot{R}$, and $f_B = d_{1/m''}/d_{1/m'}$ in general.

The installments for different frequencies, obtained solving Problem B, can be said to be *equivalent* because they are obtained by proportionality at different rates. The argument still holds if a regime different from the compound regime is used.

Reciprocal factors are applied for inverse transformations.

The factors for Problem A are directly justified, on the basis of the expressions for the values of the annuities considered in sections 5.2.2 and 5.2.3, observing that inverse proportionality exists between such values and the per period delayed or advance intensities with corresponding fractioning. Limiting ourselves to the IV of a temporary annuity (given that in all other cases the development is the same, changing only the numerator of the ratios), by indicating with R the annual total of the payments that remains unchanged, we have:

– for annuity-immediate: $R\, a^{(m)}_{\overline{n}|i} = R\, a_{\overline{n}|i}\; i/j(m)$; $R\, a^{(m'')}_{\overline{n}|i} = R\, a^{(m')}_{\overline{n}|i}\; j(m')/j(m'')$;

– for annuity-due: $R\, \ddot{a}^{(m)}_{\overline{n}|i} = R\, \ddot{a}_{\overline{n}|i}\; d/\rho(m) = R\, a^{(m)}_{\overline{n}|i}\; (1+i)^{1/m} = R\, a_{\overline{n}|i}\; (i/j(m))(1+i)^{1/m} = R\, a_{\overline{n}|i}\; i/\rho(m)$ (i.e. $i/\rho(m)$ is correction factor from $a_{\overline{n}|i}$ to $\ddot{a}^{(m)}_{\overline{n}|i}$); $R\, \ddot{a}^{(m'')}_{\overline{n}|i} = R\, \ddot{a}^{(m')}_{\overline{n}|i}\; \rho(m)'/\rho(m'')$

In the case of *pluriannual annuities*, it is obvious that the correction factor to go from the IV of the annual annuity-immediate to the p-annual one, if -immediate is $i/j(1/p)$, if -due is $i/\rho(1/p)$.

As concerns the factors from problem B, it is obvious that $R_{1/m}$ is the installment of an annuity m-fractional -immediate equivalent (in the sense of the equality of capital values) to an annual annuity-immediate with installment R if and only if it is the installment of accumulation in one year of the amount R. Thus, the result is: $R\,\sigma_{\overline{m}|i_{1/m}}\, 1/m = R \dfrac{i_{1/m}}{(1+i_{1/m})^m} = R \dfrac{i_{1/m}}{i}$ [22]. We obtain an analogous result in general.

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22 Note that, $i_{1/m}/i$ being the installment to be paid at the end of each m^{th} year to capitalize at the end of the year the unitary capital, it is also the correction factor to be applied to the

With annuity-due, $\ddot{R}_{1/m}$ is the installment of a m-fractional annuity-due equivalent to the annual annuity-due with installment $\ddot{R}$ if and only if it is the advance installment of amortization in one year of the amount $\ddot{R}$. Therefore we have:

$$\ddot{R}_{1/m} = \ddot{R}\ \ddot{\alpha}_{\overline{m}|i_{1/m}} = R\ \frac{d_{1/m}(1+i_{1/m})^m}{(1+i_{1/m})^m - 1} = R\ \frac{d_{1/m}}{i}(1+i) = R\ \frac{d_{1/m}}{d}$$

We obtain an analogous result in general.

It is obvious that if we transform the annual delayed installment (see footnote 20), $\ddot{R}_{1/m}$ is the corresponding accumulation advance installment, thus the correction factor is $\dfrac{i_{1/m}}{i(1+i_{1/m})}$.

In the case of *pluriannual annuity*, the correction factor to go from the annual delayed installment to the p-annual one, if -immediate, is $s_{\overline{p}|i}$ and, if -due, is $a_{\overline{p}|i}$[23].

EXAMPLE 5.2.– Using the data in exercise 5.3, as i = 0.0938069 and thus d = 0.085768, ρ_{12} = 0.08933, the unitary annual and monthly annuity are

$$a_{\overline{10}|i} = 6.3116048;\ a^{(12)}_{\overline{10}|i} = 6.5784744;\ \ddot{a}^{(12)}_{\overline{10}|i} = 6.6278135;\ \ddot{a}_{\overline{10}|i} = 6.9035675$$

and thus the transformations using the correction factors are easily verified:

$$a^{(12)}_{\overline{10}|i} = a_{\overline{10}|i} 0.0938069/0.09 \quad ; \quad \ddot{a}^{(12)}_{\overline{10}|i} = \ddot{a}_{\overline{10}|i}\ 0.08933/0.0857618$$

---

delayed annual installment of amortization or accumulation in a prefixed number of years to obtain the equivalent delayed m-fractional installment of amortization or accumulation in the same number of years. We obtain an analogous result for advance payments, considering the fractional installment $d_{1/m}/d$.

23 If the annual installment is in advance, it is enough to use respectively $\ddot{s}_{\overline{p}|i}$ or $\ddot{a}_{\overline{p}|i}$. This can be verified directly using m=1/p or simply observing that the annual installment is an installment of accumulation in p years of the amount given by the p-annual delayed installment or else installment of amortization in p years of the amount given by the p-annual advance installment.

The unitary continuous annuity is $a^{(\infty)}_{\overline{10}|i}$ = 6.6032175 and the inequalities are verified as

$$a_{\overline{10}|i} < a^{(12)}_{\overline{10}|i} < a^{(\infty)}_{\overline{10}|i} < \ddot{a}^{(12)}_{\overline{10}|i} < \ddot{a}_{\overline{10}|i}$$

Exercise 5.8

1) Calculate the IV of the 10-year 4-fractional unitary annuity, both -immediate and -due, at the annual rate of 5%, knowing that the annual annuity-immediate value is 7.1217349.

A. By applying the correction factors with given rates and times, we obtain

$$a^{(12)}_{\overline{10}|0.05} = a_{\overline{10}|0.05} \frac{0.05}{0.0490889} = 7.8650458$$
$$\ddot{a}^{(12)}_{\overline{10}|0.05} = \ddot{a}_{\overline{10}|0.05} \frac{0.05\ (1.05)^{1/4}}{0.0490889} = 7.9615675$$

2) Solve Problem A with data from exercise 5.3, maintaining the annual total of €7,800 and calculating the IV of the quarterly annuity-immediate through the correction factor on the IV of the monthly one (see Example 5.1).

A. We have i = 0.0938069 and thus $j(4)$ = 0.0906767; the following is obtained:

$$a^{(4)}_{\overline{10}|i} = a^{(12)}_{\overline{10}|i} \frac{j(12)}{j(4)} = 6.5784744 \frac{0.09}{0.0906767} = 6.5293807$$

3) Solve Problem A of question 2 but referring to annuity-due.

A. We have ρ(4) = 0.0886667 and thus ρ(12) = 0.08933; the following is obtained:

$$\ddot{a}^{(4)}_{\overline{10}|i} = \ddot{a}^{(12)}_{\overline{10}|i} \frac{\rho(12)}{\rho(4)} = 6.627813 \frac{0.0893300}{0.0886667} = 6.6773945$$

4) Solve Problem B with data from exercise 5.3, calculating the monthly installment equivalent to the annual one of €7,800, both in the -immediate and -due cases.

A. With delayed installments, using R = 7,800, the result is

$$R_{1/12} = R\frac{i_{1/12}}{i} = 7{,}800 \frac{0.0075}{0.0938069} = 623.62.$$

With advance installments, given $\ddot{R}$ = 7,800 and being $d_{1/12}$ = 0.0074442 from which d = 0.0857618, we obtain

$$\ddot{R}_{1/12} = \ddot{R}\,\frac{d_{1/12}}{d} = 7{,}800\,\frac{0.0074442}{0.0857618} = 677.04$$

5) We have to amortize (in the sense specified in section 5.1) the debt V_0 = €50,000 over 5 years at the annual rate of 8.75% with delayed annual installments R. Calculate the constant installments. To amortize with six-monthly delayed or advance installments, calculate the value using correction factors.

A. The annual installment is $R = V_0\ \alpha_{\overline{5}|0.0875}$ = 12,771.35, and thus the six-monthly equivalent delayed and advance installments are:

$$R_{1/2} = R\frac{i_{1/2}}{i} = 12771.35 \cdot 0.4895164 = 6{,}251.79$$

$$\ddot{R}_{1/2} = R\,\frac{i_{1/2}}{i(1+i_{1/2})} = 6251.79 \cdot 0.9589266 = 5{,}995.00$$

6) We have to accumulate (in the sense specified in section 5.1) a capital sum of €37,500 in 8 years at the annual rate of 6.15% with constant installments, annual delayed or quarterly. Calculate these installments.

A. The annual installment is $R = V_8\ \sigma_{\overline{8}|0.0615}$ = 3,768.50 and then the quarterly equivalent delayed or advance installments are

$$R_{1/4} = R\frac{i_{1/4}}{i} = 3768.50\ \ 0.244329 = €921.15$$

$$\ddot{R}_{1/4} = R\,\frac{i_{1/4}}{i(1+i_{1/4})} = 3768.50 \cdot 0.2408129 = €907.50$$

Exercise 5.9

1) We have to amortize, at the annual rate of 6.60%, a debt of 1,450,000 monetary units (MU) with constant annual delayed installments over 20 years. To evaluate the convenience of an amortization with four-yearly installments, solve Problem B calculating the equivalent delayed and advance installment.

A. The annual installment is $R = V_0\ \alpha_{\overline{20}|0.066}$ = 1450000·0.0914786 = 132643.95. The 4-yearly equivalent delayed and advance installments are

$$R_4 = R\ \sigma_{\overline{4}|0.066} = 132643.95 \cdot 4.4137115 = 585{,}452.13$$

$$\ddot{R}_4 = R\ a_{\overline{4}|0.066} = 132643.95 \cdot 3.4180241 = 453{,}380.22$$

2) Solve Problem B of question 1) in circumstances where the annual constant amortization installment is advance.

A. The annual installment is $\ddot{R} = V_0\,\ddot{\alpha}_{\overline{20}|0.066} = 1450000 \cdot 0.0858148 =$ 124,431.48. The 4-annual equivalent delayed and advance installments are

$$R_4 = \ddot{R}\ \ddot{\sigma}_{\overline{4}|0.066} = 124431.48 \cdot 4.7050165 = 585{,}452.16$$

$$\ddot{R}_4 = \ddot{R}\ \ddot{\alpha}_{\overline{4}|0.066} = 124431.48 \cdot 3.6436137 = 453{,}380.25$$

Obviously the values R_4 and $\ddot{R}_4$ are the same in the results of 1) and 2) (except for rounding-off errors), due to the decomposability of the financial law used.

5.3. Evaluation of constant installment annuities according to linear laws

5.3.1. *The direct problem*

We have already mentioned that uniform financial laws, different from the compound laws, are sometimes used to evaluate annuities. It is worth studying the problem in detail.

As seen in Chapter 2, the need for simplicity leads us to use, for short lengths of time, the simple interest law in an accumulation process and the simple discount law in a discounting process[24]. Thus for some applications the following questions are relevant:

– the initial evaluation of an annuity on the basis of the simple discount law;

– the final evaluation of an annuity on the basis of the simple interest law.

Although the reader is referred to section 5.5 for the case of general installments, we give here the most important formulae for the case of *m-fractional annuities* (owing to short times) *with constant installments* and, always fixing 0 as the beginning of the annuity's interval, let us give the following definitions:

– $m > 1$ = annual frequency of payments;

24 For them the exchange factor is linear, and they are called *linear laws*. Their conjugate, with hyperbolic factors, are usually used for indirect problems, e.g. offsetting, etc.

– s = total number of payments;

– i = interest intensity = annual interest rate;

– d = discount intensity = annual discount rate;

– R = delayed constant installment;

– $\ddot{R}$ = advance constant installment.

If the payments are *delayed*, the h^{th} amount R is paid at time h/m, and the IV on the basis of the simple discount (SD) law at the rate d and the FV in s on the basis of the simple delayed interest (SDI) at rate i are given, respectively, by

$$V_0 = s\ R\ \left(1 - d(s+1)/2m\right)\ ;\ V_s = s\ \mathrm{R}\ \left(1 + \mathrm{i}(s-1)/2m\right) \tag{5.20}$$

Instead, if payments are in *advance*, the h^{th} amount $\ddot{R}$ is paid at time $(h\text{-}1)/2m$, and the IV on the basis of the SD law at rate d and the FV in s on the basis of the SDI law at rate i are given, respectively, by

$$\ddot{V}_0 = s\ \ddot{\mathrm{R}}\ \left(1\text{-d(s-1)}/2\mathrm{m}\right)\ ;\ \ddot{V}_s = s\ \ddot{R}\ \left(1 + i(s+1)/2m\right) \tag{5.21}$$

Equations (5.20) and (5.21) are obtained from the sum of terms in arithmetic progression. More simply, observing that t' = $(s\text{-}1)/2m$ is the average length of accumulation of the s delayed installments and the average length of the discounting of the s advance installments, while t'' = $(s+1)/2m$ is the average length of accumulation of the s advance installments and the average length of the discounting of the s delayed installments, we obtain:

$$V_0 = s\ R\ (1 - d\ t'')\quad ;\ V_s = s\ R\ (1 + i\ t') \tag{5.20'}$$

$$V_0 = s\ \ddot{R}\ (1 - d\ t')\ \ ;\ V_s = s\ \ddot{R}\ (1 + i\ t'') \tag{5.21'}$$

equivalent to (5.20) and (5.21).

Exercise 5.10

1) We have to build up a fund of €12,000 with 10 constant delayed or advance monthly payments in a saving account at 6% annual in the SDI regime: calculate the value of each installment.

A. In the case of delayed payments, from the 2nd expression of (5.20) the following is obtained

$$R = 12,000 / \left[10 \ (1 + 0.06 \frac{9}{24}) \right] = €1,173.59$$

The difference of 264.10 between the accumulated amount of €12,000 and the total payments, which amount to €11,735.90, is due to the interest accrued in the fund. In the case of advance payments, due to the 2nd expression of (5.21) the installment is

$$\ddot{R} = 12,000 / \left[10 \ (1 + 0.06 \frac{11}{24}) \right] = €1,167.88$$

2) In a hire purchase the client accepts 10 quarterly delayed payments of €400 each. If the seller is able to discount the payments at the annual rate of 8% in a simple discount regime, calculate the amount obtainable by the seller.

A. The obtainable amount is equal to the initial value V_0 given by the 1st expression of (5.20). This results in

$$V_0 = 4,000 \left(1 - 0.08 \frac{11}{8} \right) = 3,560$$

The spread of €440 with respect to the total payments of €4,000 is the amount of discount, as reward for the advance availability.

5.3.2. *Use of correction factors*

If the SDI law is used with factor $u(t) = 1+it$ and the m-fractional annuity is considered with accumulation of interest only at the end of the year, the correction factors to be applied to the annual delayed installment R to have the equivalent m-fractional delayed installment $R_{1/m}$ or advance installment $\ddot{R}_{1/m}$, are obviously

$$\textit{delayed case}: \ f_p = \frac{1}{m + \frac{m-1}{2} i}; \ \textit{advance case}: \ f_a = \frac{1}{m + \frac{m+1}{2} i} \tag{5.22}$$

The correction factor is also the periodic installment for the accumulation of unit capital in one year. In fact, considering the temporal interval between 0 and 1, at time 1 the FV V_1 of the annual payment is *R,* while the FV $V_1^{(m)}$ of the delayed m-fractional payments is $R_{1/m}m$ (for the principal) + $R_{1/m}\dfrac{m(m-1)}{2}i$ (for the interest); therefore, under the constraint we obtain $R_{1/m}/R = f_p$ given by the first formula in (5.22). With advance m-fractional payments we have $\ddot{V}_1^{(m)} = \ddot{R}_{1/m}m(1+\dfrac{m+1}{2}i)$ and the constraint $V_1 = V_1^{(m)}$ implies $\ddot{R}_{1/m}/R = f_a$ given by the second formula in (5.22).

5.3.3. *Inverse problem*

Equations (5.20) and (5.21) have been presented for the solution of the *direct problem*, consisting of the evaluation of the *initial value* and *final value* of an annuity given according to linear law. However, the same formulae solve univocally the *inverse problem*, consisting of:

– the calculation of the constant delayed (or advance) installment of amortization of the debt V_0 (or $\ddot{V}_0$) with a simple discount law;

– the calculation of the constant delayed (or advance) installment of capital funding V_s (or $\ddot{V}_s$) with a simple interest law.

Amortization and accumulation are usually carried out with such laws for short durations.

Exercise 5.11

We have to extinguish a debt of €5,000 at 9% annually in 3 years with delayed annual installments in the annual compound regime. There is the choice to amortize the debt with constant monthly delayed or advance installments with the assumption that the payments during the year produce *simple* interest, which only at the end of the year are accumulated and used for the amortization. Calculate the installments.

A. The amount for the annual installment is $R = 5{,}000\ \alpha_{\overline{3}|0.09} = 1{,}975.27$.

Using the correction factors (5.22) on *R*, the following values for the other are obtained:

– monthly delayed: $R_{1/12} = R\,f_p = 1{,}975.27/(12+\dfrac{11}{2}0.09) = 1{,}975.27\cdot 0.08003 =$ 158.08;

– monthly advance: $\ddot{R}_{1/12} = \ddot{R}\ f_a = 1{,}975.27/\left(12+\frac{13}{2}0.09\right) = 1{,}975.27 \cdot 0.07946$ $= 156.95$

In the monthly compound regime it would be:

$$i_{1/m}\ /\ i = 0.0800814;\ \text{monthly delayed installment} = 158.18$$

$$i_{1/m}\,(1+i)^{-1/m}\ /\ i = 0.0795083;\ \text{monthly advance installment} = 157.05$$

5.4. Evaluation of varying installments annuities in the compound regime

5.4.1. *General case*

For the evaluation in the compound regime of annuities with varying installments, with the same signs, we can follow the classification shown in section 5.2. Here we will deal with such an argument, showing the similarities, but taking into account that the schemes based on the regularity of installments are not conserved.

Thus, we will limit ourselves to developing the calculus for the IV V_0 of annual temporary annuities-immediate (for n years), as for the other annuity schemes it is enough to take into account that, starting from the previous case, we can apply the following changes, valid with varying installments as well as constant installments:

– in the *fractional* (or *pluriannual*) case it is enough to use in the formulae, instead of years and annual rate, the number of payments and the equivalent per period rate;

– in the *-due* case each amount is paid one year before, thus the IV is $V_0(1+i)$;

– in the *delayed*[25] case each amount is paid after r years, thus the PVDA are $V_0(1+i)^{-r}$;

– the FV V_n is given by $V_0(1+i)^n$;

– in the *perpetuities* case it is enough to use $n \to +\infty$, but such a calculation can be carried out only if a rule on the formation of installments in an unlimited time is given.

By putting together the previous five rules we can obtain all the results of the classification seen in section 5.2 if the aforementioned value V_0 has been calculated.

25 It is almost unnecessary to observe that, in the case of varying installments, the annuity-immediate value coincides with that of the corresponding annuity-due delayed by one period.

Denoting by R_h or $\ddot{R}_h$ (in chronological order) the different delayed or advance installments the annuity is the union of the concordant dated amounts $\bigcup_{h=1}^{n}(h,R_h)$ or $\bigcup_{h=1}^{n}(h-1,\ddot{R}_h)$. Considering from now on the case of non-negative installments (and at least one positive one), due to (4.3") the IV of the annuity-immediate or -due are:

$$V_0 = \sum_{h=1}^{n} R_h\ (1+i)^{-h} \quad ; \quad \ddot{V}_0 = \sum_{h=1}^{n} \ddot{R}_h\ (1+i)^{-(h-1)} \tag{5.23}$$

Proceeding analogously, the FV of the annuity-immediate or -due are

$$V_n = \sum_{h=1}^{n} R_h (1+i)^{n-h} \quad ; \quad \ddot{V}_n = \sum_{h=1}^{n} \ddot{R}_h (1+i)^{n-h+1} \tag{5.24}$$

EXAMPLE 5.3.– Applying (5.23) and (5.24), calculate the IV and FV of annuities-immediate or -due. An Excel spreadsheet can be used and the installments put directly into columns, applying recurrent formulae, such as

– $V_{h-1} = (R_h+V_h)(1+i)^{-1}$ from V_n=0, to calculate pro-reserves and IV in the -immediate case;

– $V_{h-1} = \ddot{R}_{h-1} + V_h(1+i)^{-1}$ from V_n=0, to calculate pro-reserves and IV in the -due case;

– $M_h = M_{h-1}(1+i)+R_h$ from M_0=0 to calculate retro-reserves and FV in the -immediate case;

– $M_h = (M_{h-1}+ \ddot{R}_{h-1})(1+i)$ from M_0=0, to calculate retro-reserves and FV in the -due case.

The following table is obtained where, both in the -immediate and the -due case, the IV is given by the pro-reserve in 0 and the values below in the column give the pro-reserve for the following years, while the FV is given by the retro-reserve in n as credit for the counterpart which pays the installments, and the values above in the column give the retro-reserve for the preceding years. Obviously, given that an annuity operation is unfair, the retro-reserves and pro-reserves will never coincide in the various years.

	Rate =	0.042		*Length* =	10	
Year	delayed installment	advance installment	delayed pro-reserve	advance pro-reserve	delayed retro-reserve	advance retro-reserve
0	0.00	521.44	5,230.48	5,450.16	0.00	0.00
1	521.44	412.36	4,928.72	5,135.73	521.44	543.34
2	412.36	125.61	4,723.37	4,921.75	955.70	995.84
3	125.61	1,544.98	4,796.14	4,997.58	1,121.45	1,168.55
4	1,544.98	897.33	3,452.60	3,597.61	2,713.53	2,827.50
5	897.33	69.55	2,700.28	2,813.69	3,724.83	3,881.27
6	69.55	587.11	2,744.14	2,859.39	3,950.82	4,116.76
7	587.11	897.54	2,272.28	2,367.72	4,703.87	4,901.43
8	897.54	1,258.32	1,470.18	1,531.93	5,798.97	6,042.53
9	1,258.32	285.10	273.61	285.10	7,300.85	7,607.48
10	285.10	0.00	0.00	0.00	7,892.58	8,224.07
	IV annuity-immediate =	5,230.48				
	IV annuity-due =			5,450.16		
	FV annuity immediate =				7,892.58	
	FV annuity due. =					8,224.07

Table 5.1. *Pro-reserves and retro-reserves in the immediate and due case*

The Excel instructions are as follows. C1: 0.042; F1: 10; use the first two rows for data and column titles, the annual values from 0 to 10 are in rows 3-13:

– column A (year): A3: 0; A4:= A3+1; copy A4, then paste on A5-A13;

– column B (installments in the -immediate case): B3: 0; from B4 to B13: (insert data: delayed installments);

– column C (installments in the -due case): copy from B4 to B13, then paste on C3 to C12 (insert data: advance installments); C13: 0;

– column D (pro-reserve in the -immediate case): D13: 0; D12: = (B13+D13)*(1+C$1)^-1; copy D12, then paste backwards on D11 to D3;

– column E (pro-reserve in the -due case): E13: 0; E12: = C12+E13*(1+C$1)^-1; copy E12, then paste backwards on E11 to E3;

– column F (opposite of the retro-reserve in the -immediate case): F3: 0; F4: = F3*(1+C$1)+B4; copy F4, then paste on F5 to F13;

– column G (opposite of the retro-reserve in the -due case): G3: 0; G4: = (G3+C3)*(1+C$1); copy G4, then paste on G5 to G13; (initial and final value of annuities-immediate and -due): D15: = D3; E16: = E3; F17: = F13; G18: = G13.

Continuous flow

In case of continuous flow $\varphi(t)$ of annuity from 0 to n, the IV and the FV are expressed respectively by

$$\overline{V}_0 = \int_0^n \varphi(t) e^{-\delta t} dt \tag{5.23'}$$

$$\overline{V}_n = \int_0^n \varphi(t) e^{\delta(n-t)} dt \tag{5.24'}$$

With the previous formulae the *direct problem* is solved by finding the IV or the FV of an annuity with varying installments. The same formulae form a constraint for the *inverse problem*, by finding an annuity, i.e. a sequence of dated amounts with the same sign, which has a given IV or FV. Thus, as already seen regarding annuities with constant installments, if the IV is given, we have a *problem of gradual amortization* of an initial debt, while if the FV is given, we have a *problem of gradual funding* at the end of the time interval. In the case of constant installments we obtain a unique solution, owing to n-1 equality constraints between the installments. Instead, in general the solution of the inverse problem is not unique, having n-1 degrees of freedom. Furthermore, in the amortization, due to technical and juridical reasons, inequality constraints are introduced so that the amortization installments cover at least the accrued interests.

5.4.2. *Specific cases: annual annuities in arithmetic progression*

Let us here consider some relevant models, which refer to specific cases of annual annuities with varying installments. Among them, we can consider the installment evolution in arithmetic progression (AP). We obtain such a feature when all the subsequent installments vary according to a constant rate γ (positive or negative) of the first installment R. Thus, the subsequent differences are constant, and are given by the *ratio* D. Therefore, $D = \gamma R$ and

$$R_h = R + (h\text{-}1)D > 0, (h = 1, \ldots, n) \tag{5.25}$$

Let us focus first on the *normalized* or *unitary* annuity, also called an *increasing annuity,* where the first installment and the ratio are unitary; therefore $R_h = h$. In the *temporary case* the IV is indicated with the symbol $(\mathrm{I}a)_{\overline{n}|i}$; its value is

$$(\mathrm{I}a)_{\overline{n}|i} = \sum_{h=1}^{n} h\,(1+i)^{-h} = \frac{1}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right] \tag{5.26}$$

For *perpetuities* ($n = \infty$), prompt and delayed, given that $\lim_{n \to +\infty} n(1+i)^{-n} = 0$, we obtain the IV of an *increasing perpetuity*

$$(\mathrm{I}a)_{\overline{\infty}|i} = \sum_{h=1}^{+\infty} h\,(1+i)^{-h} = \frac{1}{i\,d}\ ;\quad {}_{r/}(\mathrm{I}a)_{\overline{\infty}|i} = \frac{v^r}{i\,d} \tag{5.26'}$$

Denoting by $(\mathrm{I}s)_{\overline{n}|i} = (1+i)^n\,(\mathrm{I}a)_{\overline{n}|i}$ the FV in the delayed case, in the other cases the symbols for the values of the *increasing annuities* are easily extended as in section 5.2.

To deduce the closed form given in (5.26) and then in (5.26'), some algebraic developments are needed. However, it is also possible to use financial equivalences, which we will use starting from perpetuities. Let us observe, first, that the relation $a_{\overline{n}|i} = a_{\overline{\infty}|i} - {}_{n/}a_{\overline{\infty}|i}$ (see footnote 16) can be generalized as follows

$$(\mathrm{I}a)_{\overline{n}|i} = (\mathrm{I}a)_{\overline{\infty}|i} - {}_{n/}(\mathrm{I}a)_{\overline{\infty}|i} - n\;{}_{n/}a_{\overline{\infty}|i} \tag{5.27}$$

(because $(\mathrm{I}a)_{\overline{\infty}|i} - {}_{n/}(\mathrm{I}a)_{\overline{\infty}|i}$ is the IV of a perpetuity, increasing until time n but with constant installments after n; thus to obtain the IV of a temporary annuity we still need to subtract $n\;{}_{n/}a_{\overline{\infty}|i}$). We have an analogous conclusion for the -due case.

The 1st part of (5.26') is justified for the transitivity property of the equalities. We can observe, in fact, that using the delayed evaluation rate i (equivalent to the advance rate $d = i/(1+i)$), the supply $(0,S)$ is equivalent to the perpetuity of its advance interests, i.e. $\bigcup_{h=0}^{+\infty}(h, Sd)$ with graph

$$\begin{array}{ccccccccc} 0 & & Sd & & Sd & & & & Sd \\ & -- & & -- & & --\ldots-- & & & \quad -- \\ 0 & & 1 & & 2 & & & & h \end{array}$$

Furthermore, each supply (h,Sd) is equivalent to the perpetuity, delayed by h years, of its delayed interests Sid, i.e. $\bigcup_{k=h+1}^{+\infty}(k, Sid)$ with graph

$$\begin{array}{cccccc} Sid & & Sid & & Sid & \\ & -- & & --\ldots-- & & -- \\ h+1 & & h+2 & & k & \end{array}$$

and adding to h, for all supplies, the triangular development $\bigcup_{h=0}^{+\infty}\bigcup_{k=h+1}^{+\infty}(k,Sid)$ = $\bigcup_{k=1}^{+\infty}(k,kSid)$ is obtained, with graph

$$[\overset{0}{\underset{0}{}}--\overset{Sid}{\underset{1}{}}--\overset{2Sid}{\underset{2}{}}--...--\overset{kSid}{\underset{k}{}}--].$$

This last annuity, for which the IV is Sid $(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}}$ [26], is equivalent to $(0,S)$ for which the IV is S. Therefore, Sid $(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}} = S$, i.e.: $(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}} = 1/id$. This proves the 1st part of (5.26'). The 2nd part is obvious. Developing (5.27), we obtain

$$(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}} - {}_{n/}(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}} - n\,{}_{n/}a_{\overline{\infty}|\mathrm{i}} =$$

$$= \frac{1}{i\,d} - \frac{(1+i)^{-n}}{i\,d} - n\frac{1-(1+i)^{-n}}{i} = \frac{1}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right]$$

i.e. (5.26).

After what has been said about the relationship between the different cases, this is an exercise to give the expressions for the other values of the annual *increasing annuity*. Starting from (5.26) and (5.26'), it is found that

$$(\mathrm{I}s)_{\overline{n}|\mathrm{i}} = \frac{(1+i)^n}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right] \quad (\mathrm{I}\ddot{s})_{\overline{n}|\mathrm{i}} = \frac{(1+i)^n}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right];$$

26 Considering that, by definition, $(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}}$ is the IV of $\bigcup_{k=1}^{+\infty}(k,k)$, by multiplying the amounts by Sid the IV of the annuity $\bigcup_{k=1}^{+\infty}(k,kSid)$ is obtained. We ascertain here the strength of the compound discount: at whichever rate, the present value of an annuity with diverging installment and infinite length is finite, i.e. it is in no case diverging! This is due to the fact that an increasing exponential function becomes infinite faster than a linear one, and also a polynomial one. Therefore, the result also holds true for *increasing* perpetuities of the higher order, which we can define for subsequent sums, in this manner: the h^{th} installment of the perpetuity-due with IV $1/d^m$ ($m>2$) is the sum of the first h installments of the annuity for which the IV is $1/d^{m-1}$. With $m=3$, $1/d^3$ is the IV of the perpetuity with advance installments which are the partial sums of the installments' sequence corresponding to the IV $1/d^2$, i.e.: 1; 1+2=3; 1+2+3=6, ..., $1+...+n = n(n+1)/2$, ...

$$(I\ddot{a})_{\overline{n}|i} = \frac{1}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right];$$

$$(I\ddot{a})_{\overline{\infty}|i} = \frac{1}{d^2};$$

$$_{r/}(I\ddot{a})_{\overline{n}|i} = \frac{v^r}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right]$$

$$_{r/}(I\ddot{a})_{\overline{\infty}|i} = \frac{v^r}{d^2}$$

EXAMPLE 5.4.– To have an order of magnitude, let us give in Table 5.2 the IV, PVDA, FV for two parametric scenarios.

Type of annuity	*Symbol*	(i = 4.20%; r = 5) $n = 20$	$n = \infty$	(i = 11.35%; r = 2) $n = 9$	$n = \infty$
-immediate IV	$(Ia)_{\overline{n}\|i}$	122.141386	590.702948	23.457999	86.436774
-due IV	$(I\ddot{a})_{\overline{n}\|i}$	127.271304	615.512472	26.120482	96.247340
-immediate PVDA	$_{r/}(Ia)_{\overline{n}\|i}$	99.431559	480.87316	18.919538	69.713696
-due PVDA	$_{r/}(I\ddot{a})_{\overline{n}\|i}$	103.607684	50.069840	21.066905	77.626194
-immediate FV	$(Is)_{\overline{n}\|i}$	278.110396	∞	61.730949	∞
-due FV	$(I\ddot{s})_{\overline{n}\|i}$	289.791032	∞	68.737412	∞

Table 5.2. *IV, PVDA, FV calculation*

In addition, for the two scenario perpetuities with n and i, we obtain

$_{n/}(Ia)_{\overline{\infty}\|i}$	259.426752	32.846309
$_{n/}(I\ddot{a})_{\overline{\infty}\|i}$	270.322676	36.574365
$_{n/}a_{\overline{\infty}\|i}$	10.456740	3.348052
$_{n/}\ddot{a}_{\overline{\infty}\|i}$	10.895924	3.728056

(5.27) is verified with these parameters in the two scenarios, distinguishing between annuity-immediate and annuity-due:

1st scenario, annuity-immediate:	590.702948 – 259.426752 – 20·10.456740 = 122.141386
1st scenario, annuity-due:	615.512472 – 270.322676 – 20·10.895924 = 127.271324
2nd scenario, annuity-immediate:	86.436774 – 32.846309 – 9·3.348052 = 23.457999
2nd scenario, annuity-due:	96.574365 – 36.574365 – 9·3.72805 = 26.120482

We can now give the formulae for an annual annuity in AP which is characterized by the couple (*R,D*). These formulae generalize those of the *increasing annuity.* For the reasons already mentioned, we can consider only the IV and FV of an annuity-immediate, temporary for *n* years. The following is obtained:

$$V_0 = R\ a_{\overline{n}|i} + D\ {}_{1/}(\mathrm{I}a)_{\overline{n-1}|i} = (R-D)\ a_{\overline{n}|i} + D(\mathrm{I}a)_{\overline{n}|i} \tag{5.28}$$

$$V_n = V_0\,(1+i)^n = (R-D)\ s_{\overline{n}|i} + D(\mathrm{I}s)_{\overline{n}|i} \tag{5.28'}$$

In fact, by definition, particularizing (5.23) with R_h given by (5.25):

$$V_0 = \sum_{h=1}^{n} R_h\ (1+i)^{-h} = \sum_{h=1}^{n} \left[R+(h-1)D\right]\ (1+i)^{-h} = R\,a_{\overline{n}|i} + D\ {}_{1/}(\mathrm{I}a)_{\overline{n-1}|i}$$

The last side of (5.28) follows from the simple identity: $R+(h-1)D = (R-D)+ hD$.

Exercise 5.12

A lease of a company has been agreed between the parties for an annual rent, delayed for 12 years, of €285,000 for the first year, with an annual increment of 3% of the initial rent. At the compound annual rate of 6.20%, calculate the IV of such an annuity.

A. By applying (5.28) where: i=0.062; d=0.058380; v=0.941620; R=285000; D=8,550, the following is the result

$$V_0 = R\,a_{\overline{12}|0.062} + D\,(1.062)^{-1}\,(\mathrm{I}a)_{\overline{11}|0.062} =$$

$$= 285{,}000\cdot 8.292677 + 8{,}550\cdot 0.941620\cdot 33.856815 = 2{,}635{,}989.12$$

5.4.3. *Specific cases: fractional and pluriannual annuities in arithmetic progression*

The linear variability of the installments of an annuity in AP is in practice more frequently applied using fractional installment.

Let us observe, first, that the fractioning can concern both the *frequency of payments k, and the frequency of variations h*, where $h \in \mathcal{N}$, $k=wh$ being $w \in \mathcal{N}$ the number of consecutive unchanged payments. Considering that formulae to generalize (5.28) are needed to obtain the IV of annuities in AP[27], it is sufficient here to extend to the *fractional* case the *increasing annuity* that will enter into the calculations, through an appropriate *normalization* that is convenient to apply *so as to maintain the increments in the annual total of payments as unitary.*

Given the above, it is easy to verify that:

– the *increasing* fractional annuity-immediate (-due) with *annual* increment of the installment (proportionally to a natural number), i.e. with h=1, k=w>1, is formed by installments payable at the end (beginning) of each k^{th} of year and of amount 1/k in the 1st year, 2/k in the 2nd year, etc. Generalizing (5.26), the IV of annuities-immediate is

$$(\mathrm{I}a)^{\mathrm{k|1}}_{\overline{\mathrm{n}}|\mathrm{i}} = \frac{1}{j(k)\, d}\left[1-(1+n\, d)(1+i)^{-n}\right];\ (\mathrm{I}a)^{\mathrm{k|1}}_{\overline{\infty}|\mathrm{i}} = \frac{1}{j(k)\, d} \tag{5.29}$$

which is obtainable by applying to the value of the annual annuity the same correction factor *i/j(k)* (= ratio between intensities) already used for constant annuity. The same factor has to be applied also for FV and PVDA;

– the *increasing* fractional annuity-immediate (-due) with h>1, k=wh>1, is formed, due to the aforementioned normalization, by installments payable at the end (beginning) of each k^{th} of year, so that, for the -immediate case, the first w payments of the 1st year are of amount 1/hk, the second w payments of the 1st year are of amount 2/hk,... the last w payments of the 1st year are of amount 1/k,.... the first w payments of the n^{th} year are of amount $[(n\text{-}1)h+1]/hk$, the second w payments of the n^{th} year are of amount $[(n\text{-}1)h+2]/hk$, …. the last w payments of the n^{th} year (in the case of a temporary annuity for n years) are of amount n/k. The IV of this annuity-immediate is

$$(\mathrm{I}a)^{\mathrm{k|h}}_{\overline{\mathrm{n}}|\mathrm{i}} = \frac{1}{j(k)\rho(h)}\left[1-(1+n\ \rho(h))(1+i)^{-n}\right];(\mathrm{I}a)^{\mathrm{k|h}}_{\overline{\infty}|\mathrm{i}} = \frac{1}{j(k)\rho(h)} \tag{5.30}$$

and generalizes (5.26) in the sense that the annual intensities i and d are substituted in (5.30) for those relative to frequency k and h. The same thing holds true for FV and PVDA;

– if all the payments of an *increasing* fractional annuity-immediate, with h=1 or h>1, are backdated for 1/k of a year, we obtain an *increasing* fractional annuity-due, the IV of which follows from that of the annuity-immediate on multiplying by

27 In generalizing (5.28) for the fractional case it is convenient to consider its last term, at least when h>1, which implies installment variations during the year, to avoid the complication of deferment for a fraction of years.

$(1+i_k)$; therefore it is sufficient to substitute $j(k)$ with $\rho(k)$ into the formulae in (5.29) and (5.30).

Proof

Let us first observe that (5.27) is generalized in

$$(\mathrm{I}a)^{k|h}_{\overline{n}|i} = (\mathrm{I}a)^{k|h}_{\overline{\infty}|i} - {}_{n/}(\mathrm{I}a)^{k|h}_{\overline{\infty}|i} - n\ {}_{n/}a^{(k)}_{\overline{\infty}|i} \tag{5.27'}$$

Therefore, to prove (5.29) and (5.30) it is sufficient to consider the perpetuities, because using their value we obtain that of the temporary annuities. (5.29) is proved observing that, analogously to what was seen for the annual annuity, an amount S in 0 is equivalent to the annual perpetuity, starting with 0, of advance interest Sd and that each installment Sd is equivalent to the subsequent k-fractional perpetuity of delayed interest $Sdi_{1/k}$. The total of the payments is, therefore:

– $S\,d\,i_{1/k}$ at the end of each period with duration $1/k$ of the 1st year;

– $2\,S\,d\,i_{1/k}$ at the end of each period with duration $1/k$ of the 1st year;

– etc.

Therefore it is sufficient to use $S=1/j(k)d$ in order to obtain (5.29) as IV of the annuity with fractional payments $1/k$ in the 1st year, $2/k$ in the 2nd year, etc., and thus unitary increments in the annual total of payments, which is 1 in the 1st year.

Equation (5.30), which generalizes (5.29), is proved observing that the supply $(0,S)$ is equivalent to the subsequent h-fractional perpetuity-due with constant installments $Sd_{1/h}$, each of which is equivalent to the following k-fractional annuity-immediate of constant installments $Sd_{1/h}i_{1/k}$. The amount S is thus the IV of the perpetuity with payments:

– $Sd_{1/h}i_{1/k}$ at the end of each of the first k/h periods with duration $1/k$ of the 1st year;

– $2Sd_{1/h}i_{1/k}$ at the end of each of the second k/h periods with duration $1/k$ of the 1st year;

– $hSd_{1/h}i_{1/k}$ at the end of each of the last k/h periods with duration $1/k$ of the 1st year;

– $(h+1)Sd_{1/h}i_{1/k}$ at the end of each of the first k/h periods with duration $1/k$ of the 2nd year;

– $(h+2)Sd_{1/h}i_{1/k}$ at the end of each of the second k/h periods with duration $1/k$ of the 2nd year;

– $2hSd_{1/h}i_{1/k}$ at the end of each of the last *k/h* periods with duration 1/*k* of the 2nd year;

– etc.

It is sufficient to use $S = 1/[j(k)\rho(h)]$ in order to obtain the perpetuity that starts with the fractional payment 1/(*hk*), reaching the level *n/k* after *n* years, for which the IV is given by the 2nd of (5.30).

It is obvious that the IV of the annuity, temporary (or perpetuity), which has installments proportional to those of an *increasing* fractional annuity-immediate (with $h>1$, $k=wh>1$) and first payment *H*, is obtained from the first (or second) value in (5.30) multiplying by *Hhk*. More generally, the IV of whichever fractional annuity in AP is obtained with an appropriate linear combination of the unitary IV $a^{(k)}_{\overline{n}|i}$ and $(Ia)^{k|h}_{\overline{n}|i}$.

Observation

When $h>1$, the value of payments of the1st year is no longer unitary; its value is $T_1 = (h+1)/2h$. In general the total payment of the year s+1 is

$$T_{s+1} = \frac{k}{h}\left(\frac{sh+1}{hk} + ... + \frac{(s+1)h}{hk}\right) = \frac{h+1}{2h} + s,\ s = 0,1,2,... \tag{5.31}$$

thus $T_{s+1} = T_s + 1$, $\forall(s,h)$ and the unitary normalization of the annual increments is verified. The total of payments in the first *n* years is

$$T^{(n)} = \sum_{s=0}^{n-1} T_{s+1} = \frac{n}{2}\left(n + \frac{1}{h}\right) \tag{5.31'}$$

Continuous increasing annuity

The values in the continuous case are obtained, as usual, on diverging the frequency. However, in this case we have two frequencies: the frequency of payments and the frequency of increments.

Recalling that $\delta = \lim_{k\to\infty} j(k) = \lim_{h\to\infty} \rho(h)$, we observe that there is no distinction between -due and -immediate in the case of varying installments as well. Let us give the results, that can be easily proved, starting from (5.30), in both cases:

– if only the payment frequency *k* diverges, the IV is

$$(Ia)^{\infty|h}_{\overline{n}|i} = \frac{1}{\delta\ \rho(h)}\left[1 - (1 + n\ \rho(h))(1+i)^{-n}\right];\ (Ia)^{\infty|h}_{\overline{\infty}|i} = \frac{1}{\delta\ \rho(h)} \tag{5.32}$$

– if both the payment frequency k and the increment frequency h diverge, the IV is

$$(\mathrm{I}a)_{\overline{n}|\mathrm{i}}^{\infty|\infty} = \frac{1}{\delta^2}\left[1-(1+n\,\delta)(1+i)^{-n}\right]\ ;\ (\mathrm{I}a)_{\overline{\infty}|\mathrm{i}}^{\infty|\infty} = \int_0^{+\infty} t\, e^{-\delta\, t} dt = \frac{1}{\delta^2} \qquad (5.32')$$

and the total of payments in the first n years is $n^2/2$.

EXAMPLE 5.5.– Let us make some numerical comparisons on the fractional *increasing* annuity, changing the fractioning with the constraint $h \leq k$, verifying the increasing behavior if using the same h from higher deferment to higher anticipation, decreasing if h increases, fixing the other parameters. Let us assume i=0.07; n=10. Considering the frequency 1, 4, 12, the equivalent values are: d=0.0654206, $j(4)$=0.0682341, $j(12)$=0.0678497, $\rho(4)$=0.0670897, $\rho(12)$=0.0674683, δ=0.0676586 and the following table is obtained, where $T^{(10)}$ is the maximum value obtainable (h,k) at zero rate.

k	h	$T^{(10)}$	$(\mathrm{I}a)_{\overline{\infty}|\mathrm{i}}^{\mathrm{k}|\mathrm{h}}$	$(\mathrm{I}\ddot{a})_{\overline{\infty}|\mathrm{i}}^{\mathrm{k}|\mathrm{h}}$
1	1	55.000	34.7390688	37.1707813
4	1	55.000	35.6381167	36.2460231
12	1	55.000	35.8400231	36.0426277
∞	1	55.000	35.9412524	35.9412524
4	4	51.250	32.8980119	33.4591782
12	4	51.250	33.0843943	33.2714212
∞	4	51.250	33.1778403	33.1778403
12	12	50.417	32.4783090	32.6619097
∞	12	50.417	32.5700432	32.5700432
∞	∞	50.000	32.2671080	32.2671080

Table 5.3. *Comparisons on the fractional increasing annuities*

Exercise 5.13

1) An industrial company has to pay a monthly delayed rent for leasing (equipment, etc., see section 6.5) equal to the amortization installment at the interest rate of 9.50% for 7 years proportional to the initial value of the plant of €48,500, net of 5% of the value initially paid as an advance, and without any other clause except for an annual increment of 12% on the initial rent. Calculate the rents for the 7 years and the initial value of the borrowed amount at the evaluation rate for the supposed income of 12% per year.

A.

a) *calculation of the installments*: Borrowed amount = M = 48,500 (1–0.05) = €46,075; for (5.13), being j_{12} = 0.0910984, initial rent = C = $M/(12\, a^{(12)}_{\overline{7}|0.095})$ = 46,075/(12·5.1615977) = €743.87.

Given the adjustment clause there is an annual increment of the monthly rent of €18.60 resulting in:

– Monthly rent in the 1st year = €743.87

– Monthly rent in the 2nd year = €762.47

– Monthly rent in the 3rd year = €781.07

– Monthly rent in the 4th year = €799.67

– Monthly rent in the 5th year = €818.27

– Monthly rent in the 6th year = €836.87

– Monthly rent in the 7th year = €855.47

A.

b) *calculation of the* IV: using (5.13), (5.29) and: n=7; i=0.12; k=12; h=1; R = 743.874 · 12 = 8926.488; D = 18.597·2 = 223.164 (in terms of annual flows) and generalizing (5.28) in

$$V_0 = R\, a^{(k)}_{\overline{n}|i} + D\; {}_{1/}(\mathrm{I}a)^{k|h}_{\overline{n}|i} = (R\text{-}D)\, a^{(k)}_{\overline{n}|i} + D\,(\mathrm{I}a)^{k|h}_{\overline{n}|i},$$

the following is obtained:

$$V_0 = 8926.488 \cdot 4.8096288 + 223.164 \cdot 13.7441973 = €46,000.30$$

2) A three-year work contract starting on the 1st January has an annual wage bill of €13,390 to be paid in 12 delayed monthly salaries + 13th salary for Christmas, and also an increasing benefit to add to the 12 monthly, initially equal to 5% of the initial salary but with quarterly increments all equal to it, not affecting the 13th one. To calculate the end of job indemnity, let us value the FV of such a contract at an evaluation rate of 6.60%.

A. It is convenient to first calculate the IV, afterwards accumulating it for 3 years, and keep separate the 13th salary from the ordinary monthly salaries, including the increasing benefit.

The payments for the 13th salary give rise to an annual annuity-immediate with 3 installments of 13,390/13 = €1,030, with IV of: $V'_0 = 1030\, a_{\overline{3}|0.066}$ = €2,722.92.

The ordinary monthly salaries give rise to a 12-fractional *increasing* temporary annuity with quarterly increments, formed by 3 delayed monthly installments of €1,081.50 followed by another 3 of €1,133.00, etc. The parameters are: n=3; h=4; k=12; i=0.066. Thus applying the last term of (5.28), its IV is

$$V''_0 = 1{,}030 \cdot 12 \cdot a^{(12)}_{\overline{3}|0.066} + 1{,}030 \cdot 0.05 \cdot 4 \cdot 12 \cdot (\mathrm{I}a)^{12|4}_{\overline{3}|0.066} = 31{,}784.29 + 10{,}027.48 =$$

$$= €41{,}811.77$$

Thus, the IV of the contract is, at the rate of 6.6%, $V_0 = V'_0 + V''_0$ = €44,534.69 and the FV is $V_3 = 1066^3\, V_0$ = €53,947.34.

Pluriannual increasing annuities

Let us briefly mention the pluriannual *increasing* annuities, from which can be deduced with linear combination the IV of the pluriannual annuities in AP, which find practical application generalizing the annuities with constant installments. Let us consider only the case of $h=k=1/p$ (i.e. p-annual *increasing* annuity with period $1/k$ and increment after each payment)[28]. The following normalization implies that the s^{th} installment is sp^2; thus the r^{th}, at the end of n years, if $n=rp$, is $rp^2=np$. Therefore, the IV are obtained using (5.30), where the expressions already used in (5.18) for the p-annual intensities $j(1/p) = [(1+i)^p-1]/p$; $\rho(1/p) = [1-(1+i)^{-p}]/p$ are taken into account. For the perpetuities we obtain the following results

$$(\mathrm{I}a)^{\frac{1}{p}|\frac{1}{p}}_{\overline{\infty}|i} = \frac{1}{j(1/p)\rho(1/p)} = \frac{p^2}{(1+i)^p + (1+i)^{-p} - 2} \tag{5.30'}$$

$$(\mathrm{I}\ddot{a})^{\frac{1}{p}|\frac{1}{p}}_{\overline{\infty}|i} = \frac{1}{\rho^2(1/p)} = \frac{p^2}{\left[1-(1+i)^{-p}\right]^2}$$

28 The conclusions for this specific case can be easily obtained from those of the annual *increasing* annuity (see (5.26) and (5.26')) assuming as the new unit measure the p-year and thus adopting proper measures for time and rates.

while for the temporary case it is enough to multiply (5.30') by [1-(1+$n\rho(1/p)$)(1+i)^n]. To have the values of the annuity, proportional to the previous one, for which the first installment is H, it is sufficient to multiply by H/p^2.

Exercise 5.14

Consider again the problem of Exercise 5.7 assuming that the cost of the plants due to the five-yearly replacements increases by 26% in respect to the initial cost.

A. The parameters are: n=20; p=5; i=7.45% thus $\rho_{(1/5)} = (1\text{-}1.0745^{-5})/5$ = 0.0603638. The cost for the plant at time 0 is €255,000; the increment of cost for each replacement is: 255,000·0.26 = €66,300 starting at time 5 for 3 times. To avoid deferments let us divide the five-yearly varying cost by the sum of a five-yearly advance cost of 255,000-66,300 = €188,700 and by an increasing cost proportional to an *increasing* five-yearly annuity-due with a first installment of €66,300. Thus, applying (5.17) and (5.30') modified for the temporary case, the IV of such an operation is

$$V_0 = \frac{188,700}{5} \ddot{a}^{(1/5)}_{\overline{20}|0.0745} + \frac{66,300}{25} (\mathrm{I}\ddot{a})^{\frac{1}{5}|\frac{1}{5}}_{\overline{20}|0.0745} = 37,740 \cdot 5 \cdot \frac{1-1.0745^{-20}}{1-1.0745^{-5}} +$$

$$+ \frac{66,300}{25} \frac{25}{\left[1-1.0745^{-5}\right]^2} [1-(1+20 \cdot 0.0603638)(1.0745)^{-20}] = 37,740 \cdot 12.6298497 +$$

$$+ 2,652 \cdot 130.5014439 = 476,650.53 + 346,089.83 = €822,740.36$$

5.4.4. *Specific cases: annual annuity in geometric progression*

Temporary annuities

Often in annuities the installment variation is proportional to a fixed ratio of the previous installment instead of the initial one. It follows that the behavior of installments is in geometric progression (GP), the ratio of which we will indicate with q. Typical are those phenomenon of adjustment with constant rate: if a salary increases at the rate of 5% the following index numbers are obtained

100, 105, 110.75, 115.7625, 121.5506, etc.

in GP with ratio q=1.05.

Let us define a *temporary annual unitary annuity-immediate in geometric progression* with ratio q>0 with the following operation:

$$\bigcup_{h=1}^{n}(h, q^{h-1}) \tag{5.33}$$

with graph: $\begin{bmatrix} 1 & -- & q & --\ldots-- & q^{n-1} \\ 1 & & 2 & & n \end{bmatrix}$.

If the period is not annual, it can always be assumed to be a unit measure of time (and in such case i is the corresponding per period rate), thus unifying the treatment.

The IV of the unitary annuity (5.33) in a compound regime is given by

$$(\mathrm{G}a)_{\overline{n}|i}^{[q]} = \sum_{h=1}^{n} q^{h-1} v^{h} = \begin{cases} nv \ , \text{ if } q = 1+i \\ v\dfrac{1-(qv)^n}{1-qv}, \text{ if } q \neq 1+i \end{cases} \tag{5.34}$$

If we have an unitary annuity-due, its IV $(\mathrm{G}\ddot{a})_{\overline{n}|i}^{[q]}$ is obtained multiplying the values in (5.34) by (1+i) and

$$(\mathrm{G}\ddot{a})_{\overline{n}|i}^{[q]} = \sum_{h=1}^{n} q^{h-1} v^{h-1} = \begin{cases} n \ , \text{ if } q = 1+i \\ \dfrac{1-(qv)^n}{1-qv}, \text{ if } q \neq 1+i \end{cases} \tag{5.34'}$$

More generally, the IV of annuities in GP, -immediate or -due, with a first installment equal to R are given by

$$V_0 = R(\mathrm{G}a)_{\overline{n}|i}^{[q]}; \qquad \ddot{V}_0 = R(\mathrm{G}\ddot{a})_{\overline{n}|i}^{[q]} \tag{5.34''}$$

Using $\eta = q-1$ (=algebraic rate of variation of the GP), if $q<1+i$ i.e. $\eta<i$, qv is the discount factor at the rate $\lambda = (1/qv)-1 > 0$ such that the IV (5.34') is also that of a constant annuity-due at the rate λ (see (5.2)). If instead $q>1+i$ i.e. $\eta>i$, qv is the accumulation factor at the rate $\mu = qv-1 > 0$ such that the IV (5.34') is also the FV of a unitary constant annuity-immediate at the rate μ (see (5.6)). In formulae[29]:

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29 See the observation in footnote 19. We obtain a formula analogous to the 1st expression of (5.35) for annuity-immediate if it is normalized assuming the first installment equal to q, coherently with the following viewpoint: considering the annuity in g.p. $\bigcup_{k=0}^{n}(k, q^k)$, the IV of the annuity-due is calculated on the first n supplies; the IV of the annuity-immediate takes account of the following n supplies after the first one. For all choices of q and i, the two

$$q<1+i:\ (\mathrm{G}n)^{[q]}_{\overline{n}|i} = \ddot{a}_{\overline{n}|\lambda}\ ;\ q>1+i:\ (\mathrm{G}\ddot{a})^{[q]}_{\overline{n}|i} = s_{\overline{n}|\mu} \tag{5.35}$$

where

$$q = 1+\eta = \frac{1+i}{1+\lambda} = (1+i)(1+\mu) \tag{5.36}$$

Equation (5.35) speeds up the calculation of (5.34) and (5.34') leading it back to that of the values of constant annuities.

Due to the decomposability, the FV $(\mathrm{G}s)$, $(\mathrm{G}\ddot{s})$ and the *p.v.d.a* . $_{r/}(\mathrm{G}a)$, $_{r/}(\mathrm{G}\ddot{a})$ of unitary annuities in GP, -immediate and -due, are obtained from IV with the usual factors:

$$(\mathrm{G}s)^{[q]}_{\overline{n}|i} = (1+i)^n(\mathrm{G}a)^{[q]}_{\overline{n}|i}\ ;\ (\mathrm{G}\ddot{s})^{[q]}_{\overline{n}|i} = (1+i)^n(\mathrm{G}\ddot{a})^{[q]}_{\overline{n}|i} \tag{5.37}$$

$$_{r/}(\mathrm{G}a)^{[q]}_{\overline{n}|i} = (1+i)^{-r}(\mathrm{G}a)^{[q]}_{\overline{n}|i}\ ;\ _{r/}(\mathrm{G}\ddot{a})^{[q]}_{\overline{n}|i} = (1+i)^{-r}(\mathrm{G}\ddot{a})^{[q]}_{\overline{n}|i} \tag{5.38}$$

From a general point of view, let us consider annuities with installments that are sum of two addends: the former is constant, the latter is varying in GP Considering an annual temporary annuity-immediate (or with another period to assume as unitary) with installment $R_h = H + Kq^{h-1}$, its IV and FV are

$$\begin{cases} V_0 = H\, a_{\overline{n}|i} + K(\mathrm{G}a)^{[q]}_{\overline{n}|i} \\ V_n = H\, s_{\overline{n}|i} + K(\mathrm{G}s)^{[q]}_{\overline{n}|i} \end{cases} \tag{5.39}$$

Analogous formulae hold for other types of annuities in GP.

Real and monetary variations

The formulation that leads to (5.36) is a specific case – which considers rates that are constant in time – of the problem of financial evaluation with rates that vary in time and with variation of the purchasing power of money. Such a problem, which has an important application in macroeconomics and finance, can be shown with a simple argument. Let m_t and c_t be the interest rate for the year (t-1,t), on the

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theoretical rates λ and μ introduced in (5.35), because of (5.36) are linked by the relation $(1+\lambda)(1+\mu) = 1$, thus they have opposite signs. Also, we have to consider the case q=1+i, in which $\lambda = \mu = 0$.

monetary market and on the commodity market (such as wheat, for example) respectively, in the sense that:

– for M euros loaned in t-1, we pay back $M(1+m_t)$ euros in t;

– for C kilograms of wheat loaned in t-1, we give back $C(1+c_t)$ kilograms of wheat in t.

In addition, let r_t be the variation rate of the wheat price in euros, i.e. C kilograms are traded today for M euros and after one year for $M(1+r_t)$ euros. It is obvious that the three rates m_t, c_t, r_t are bound by an equation, which is deduced as follows. If at time t-1 the C kilograms of wheat are traded on the market for M euros, two equivalent loans of C and M lead in t to the equivalent return of $C(1+c_t)$ kilos and $M(1+m_t)$ euros; but in such time C kilos are traded with $M(1+r_t)$ euros, and thus $C(1+c_t)$ kilograms are traded with $M(1+r_t)(1+c_t)$ euros. For comparison the multiplicative relation is found, which is also called *Fisher's equation*[30],

$$1 + m_t = (1 + r_t)(1 + c_t) \tag{5.40}$$

Supposing a market economy with only one commodity (wheat), m_t is the monetary interest rate (or *rate in value*), c_t is the real interest rate (or *rate in volume*), r_t is the variation rate of the commodity price. (5.40) thus expresses the market constraint in terms of exchange annual factors. If r_t and c_t are small, the product $r_t\, c_t$ in the development of $(1+r_t)(1+c_t)$ can be ignored and (5.40) can be approximated using the simple relation

$$m_t = r_t + c_t \tag{5.40'}$$

usually used (and sometimes abused) in the description of macroeconomic phenomena.

In the specific case of constant rates, putting $m=i$, $c=\lambda$, $r=\eta$, (5.40) is reduced to (5.36) and adding the effects for n years, the 1st expression of (5.35) is found, which expresses the equality between: a) the IV at rate i (which acts as the monetary rate m) of the annuity in GP with ratio q, i.e. with variation rate η, and: b) the IV at rate λ, which act as real rate c, of the constant annuity[31]. This also holds in cases in

30 See Fisher (1907).

31 If there is a devaluation of the commodity compared to the money, then $0<q<1$, while in the case of appreciation it is $q>1$. If and only if the real rate λ is positive, then $\eta<i$, while $\lambda=0$ implies $\eta=i$. It can happen that the rate of price increment is higher than the monetary interest rate, so we obtain a real rate $\lambda<0$. In this last case, to avoid the use of a negative rate in the formulae, it is enough to introduce the value μ linked to λ by (5.36) and to apply (5.35).

which all the installments are multiplied by the constant R. Therefore, we have to consider equivalent to a discount:

– at a real rate λ, a constant annuity with installments given by the constant values of fixed quantities at the constant price of the initial year; or

– at the corresponding monetary rate i, the annuity of the varying values of the same quantities at the current prices that vary at the rate η[32].

Perpetuities

If the annuity in GP is a perpetuity – differently from what happens for perpetuity linearly increasing[33] or according to powers with integer exponent greater than 1 – of time, its IV assumes a finite value only if $q<1+i$. We obtain in such a case: $\lim_{n\to\infty}[q^n(1+i)^{-n}]=0$ and thus

$$(\mathrm{G}a)_{\infty|\mathrm{i}}^{[\mathrm{q}]} = \frac{v}{1-qv} \; ; \; (\mathrm{G}\ddot{a})_{\infty|\mathrm{i}}^{[\mathrm{q}]} = \frac{1}{1-qv} \tag{5.41}$$

In general, with an initial installment R_1, the IV V_0 is obtained from (5.41) and multiplying by R_1. The other values are obtained simply by applying the corresponding factors.

Exercise 5.15

1) Let a loan have paid back delayed installments indexed at 3%, the first of which coincides with the constant amortization installment of the debt of €140,000 over 10 years at the rate of 6.3%. Calculate the sequence of installments and the IV of the temporary annuity and the IV of the corresponding perpetuity, if it is finite.

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32 To generalize the 1st expression of (5.35) in the case of installments R_h and varying rates m_t, r_t, c_t it is enough to replicate Fisher's equation (5.40) using t=1,2,..,n, and we obtain with simple developments the equality

$$\sum_{h=1}^{n} R_h \prod_{t=1}^{h} \left(\frac{1+r_t}{1+m_t}\right)^h = \sum_{h=1}^{n} R_h \prod_{t=1}^{h} \left(\frac{1}{1+c_t}\right)^h$$

between the IV at the monetary rates m_t of the varying amounts R_h indexed at the rate r_t, i.e. evaluating the commodity at the current prices, and the IV at the real rates c_t of the amounts R_h which are not indexed, i.e. evaluating the commodity at a constant price. If the rates are linked by (5.40'), the equality is an approximation.

33 We can observe that the arithmetic progression behavior is a discretization of the linear behavior while the geometric progression behavior is a discretization of the exponential behavior. It is important to analyze this comment thoroughly, from the viewpoint of the mathematical analysis, and the problems that come up when we consider perpetuities.

A. We obtain qv = 0.9689558 < 1, R_1 = 140,000 $\alpha_{\overline{10}|0.063}$, R_h = 1.03 $R_{h\text{-}1}$ (h=2,...,10) and the following values in euros are found for the installments:

R_1 = 19,292.79; R_2 = 19,871.57; R_3 = 20,467.72; R_4 = 21,081.75; R_5 = 21,714.20
R_6 = 22,365.63 ; R_7 = 23,036.60; R_8 = 23,727.70;R_9 = 24,439.53;R_{10} = 25,172.71

Due to (5.34"), the IV is

$$V_0 = R_1 (\mathrm{G}a)^{[1.03]}_{10|0.063} = 19,292.79 \frac{1-\left(\frac{1.03}{1.063}\right)^{10}}{1.063\left(1-\frac{1.03}{1.063}\right)} = 19,292.79 \cdot 8.1962415 = €158,128.37$$

which has to compare with the value 140,000 of the annuity with constant installment.

If we consider a perpetuity, using q = 1.03 < 1+i = 1.063, (5.41) is applied and the IV of the perpetuity is bounded and is

$$V_0 = R_1 (\mathrm{G}a)^{[1.03]}_{\infty|0.063} = \frac{19,292.79}{1.063\left(1-\frac{1.03}{1.063}\right)} = 19,292.79 \cdot 34.2414848 = €660,613.77$$

2) Assuming the compound regime and using the annual rate of 6%, let us consider a sequence of advance annual rent indexed at 9% for 10 years, the first of which coincides with the funding annual constant installment of the amount of €100,000 in 10 years. Calculate the IV and the FV of the aforementioned annuity, and also the rate of the equivalent constant annuity. Also consider the perpetuity.

A. The first advance installment is

$$R_1 = 100,000 \ \ddot{\sigma}_{\overline{10}|0.06} = 100,000 \cdot 0.0715735 = 7,157.35$$

and the following installments are

$$R_2 = 1.09\ R_1 = 7,586.80; \ ....; R_{10} = 1.09^9\ R_1 = 12,092.20$$

By applying (5.34") and (5.37), the following is the result:

$$\ddot{V}_0 = R_1 (\mathrm{G}\ddot{a})^{[1.09]}_{10|6\%} = 7157.35 \cdot 11.3746307 = €81,412.21$$

$$\ddot{V}_{10} = R_1 (\mathrm{G}\ddot{s})^{[1.09]}_{10|6\%} = 7157.35 \cdot 20.3702312 = €145,796.87$$

The value €145,796.87, if compared with the capital of €100,000 accumulated with 10 constant installments of €7,157.35, shows the effect of the compound index at 9%.

The rate μ applied in (5.35) results here as (1.09/1.06)-1 = 0.0283019 independently from n and forming an active rate of accumulation. The constraint $(1+\lambda)(1+\mu) = 1$ holds true (see footnote 29), and thus if we exchange the two rates i and $\eta=q-1$, using the index at 6% and the interest at 9%, the following is obtained: $\lambda = (1.12/1.09)^{-1} - 1 = 0.0283019$, coinciding with μ but to be interpreted as the allowed amortization rate.

As $q>1+i$, not only the values of the temporary annuities, but also the single discounted values, increase with n, and thus the perpetuity has unlimited value.

5.4.5. *Specific cases: fractional and pluriannual annuity in geometric progression*

Proceeding analogously as for the annuities in AP, let us briefly examine the changes connected with the fractioning of annuities in GP.

This is useful because sequences of payments and variations subdivided during the year are widely used (see section 5.4.5). With the positions already used, let $h \in \mathcal{N}$ be the variation frequency and $k=wh$ the payment frequency, with $w \in \mathcal{N}$ being the number of consecutive unchanged payments.

To simplify the discussion without loss of generality, it is convenient to use h=1 and then k=w. This is obtained assuming as a *new unit measure of time, the period between two consecutive installment variations*, which we will call the *invariance period*, having fractioned the installment in k equal parts[34], with delayed and advance payments at each k^{th} of the period; rate i will be the equivalent rate.

The normalization that leads to the *unitary fractional prompt* annuity is that in which, fractioning the payment of each period into k equal parts, the total of payments of the first period is unitary and those for the following periods proceeds in GP of ratio q, as shown in the following graph for an -immediate with temporary n

$$\left[\frac{\frac{1}{k}}{\frac{1}{k}},\frac{\frac{1}{k}}{\frac{2}{k}},\ldots,\frac{\frac{1}{k}}{\frac{1}{k}};\frac{\frac{q}{k}}{\frac{k+1}{k}},\ldots,\frac{\frac{q}{k}}{\frac{2k}{k}};\frac{\frac{q^2}{k}}{\frac{2k+1}{k}},\ldots,\frac{\frac{q^2}{k}}{\frac{3k}{k}};\ldots\ldots;\frac{\frac{q^{n-1}}{k}}{\frac{(n-1)k+1}{k}},\ldots,\frac{\frac{q^{n-1}}{k}}{\frac{nk}{k}}\right]$$

34 With the symbols used in section 5.4.3, this is the case of h=1, k=r.

while for an -due the same amounts are backdated by $1/k$. Their IV are obtained from (5.34) and (5.41) by applying for each invariance period, and thus for the whole n years, the same correction factors f_A obtained in section 5.2.4 for *Problem A*, respectively $i/j(k)$ and $i/\rho(k)$[35].

With the symbols taking their obvious meanings, we obtain:

$$(Ga)^{[q;k]}_{\overline{n}|i} = \begin{cases} d\,n/j(k)\ , & \text{if } q = 1+i \\ \dfrac{d[1-(qv)^n]}{j(k)[1-qv]}, & \text{if } q \neq 1+i \end{cases} \tag{5.42}$$

$$(Ga)^{[q;k]}_{\overline{\infty}|i} = \frac{d}{j(k)[1-qv]},\ \text{if } q < 1+i \tag{5.42'}$$

$$(G\ddot{a})^{[q;k]}_{\overline{n}|i} = \begin{cases} d\,n/\rho(k)\ , & \text{if } q=1+i \\ \dfrac{d[1-(qv)^n]}{\rho(k)[1-qv]}, & \text{if } q \neq 1+i \end{cases} \tag{5.42''}$$

$$(G\ddot{a})^{[q;k]}_{\overline{\infty}|i} = \frac{d}{\rho(k)[1-qv]},\ \text{if } q < 1+i \tag{5.42'''}$$

If all of the payments of the first invariance period are R, to obtain V_0, it is enough to multiply R by (5.42) or (5.42') in the immediate case, or else to multiply R by (5.42”) or (5.42”’). To obtain the FV and the PVDA of r periods, multiply by $(1+i)^n$ and v^r. For the IV in perpetuity, use $(qv)^n = 0$.

For Problem B the correction factors are those already used, i.e. those needed to obtain the k-fractional installments with an addendum in GP from one invariance period to the next. Therefore, in the -immediate case we can write the installments

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35 A proof based on equivalences is the following. For the $(h+1)$-th period we want the financial equivalence in $h+1$ (which is the instant of the end of the period and also the instant of payment of the per period non-fractional installment q^h) between such an installment and all of the fractional installments of the period, for which the valuation is

$$\Sigma_{r=1}^{k}\frac{1}{k}q^h(1+i_{1/k})^{k-r} = \frac{1}{k}q^h(1+i_{1/k})^k\Sigma_{r=1}^{k}(1+i_{1/k})^{-r} =$$

$$= \frac{1}{k}q^h(1+i_{1/k})^k(1+i_{1/k})^{-k}\frac{(1+i_{1/k})^k-1}{1+i_{1/k}-1} = q^h\frac{i}{j_k}$$

Therefore, also in this case: $f_A = \dfrac{i}{j_k}$.

as $R_h^{(k)} = H^{(k)} + K^{(k)} q^{h-1}$, so to not modify the values (5.39). In the -due case the results are analogous, obtaining $\ddot{H}^{(k)}$ and $\ddot{K}^{(k)}$ from $\ddot{H}$ and $\ddot{K}$. With the latter we have[36]

$$H^{(k)} = H\frac{i_{1/k}}{i}, K^{(k)} = K\frac{i_{1/k}}{i}, \ddot{H}^{(k)} = \ddot{H}\frac{i_{1/k}}{i(1+i_{1/k})}, \ddot{K}^{(k)} = \ddot{K}\frac{i_{1/k}}{i(1+i_{1/k})} \qquad (5.43)$$

For the *pluriannual case*, going back to the annual unit of measure, we only consider the case h=k=1/p, i.e. of a normalized p-annual annuity with variation at each payment; it is not restrictive to assume, to be consistent with the parameters of the annual annuity, the ratio q^p (that would be obtained by annually applying the ratio q). To calculate the IV of a temporary annuity it is sufficient to apply (5.34) and (5.34'), assuming as the unit of measure the interval of p years. Therefore, in terms of annual parameters, the following is easily obtained, in the -immediate case:

$$(\mathrm{G}a)_{\mathrm{n}|\mathrm{i}}^{[\mathrm{q}^{\mathrm{p}};\frac{1}{\mathrm{p}}]} = \begin{cases} n(1+i)^{-p}/p \ , & \text{if } q = 1+i \\ \dfrac{1-(qv)^n}{(1+i)^p - q^p} \ , & \text{if } q \neq 1+i \end{cases} \qquad (5.44)$$

and in the -due case:

$$(\mathrm{G}\ddot{a})_{\mathrm{n}|\mathrm{i}}^{[\mathrm{q}^{\mathrm{p}};\frac{1}{\mathrm{p}}]} = \begin{cases} n/p \ , & \text{if } q=1+i \\ \dfrac{1-(qv)^n}{1-(qv)^p} \ , & \text{if } q \neq 1+i \end{cases} \qquad (5.44')$$

The IV of a perpetuity assumes a finite value only if q<1+i, resulting, in such a case, in: $\lim_{n\to\infty} (qv)^n = 0$, and thus

$$(\mathrm{G}a)_{\infty|\mathrm{i}}^{[\mathrm{q};\frac{1}{\mathrm{p}}]} = \frac{1}{(1+i)^p - q} \quad ; \quad (\mathrm{G}\ddot{a})_{\infty|\mathrm{i}}^{[\mathrm{q};\frac{1}{\mathrm{p}}]} = \frac{1}{1-(qv)^p} \qquad (5.45)$$

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36 It is enough to observe that also here H, K , $\ddot{H}, \ddot{K}$ are the *FV* of the annuities in the invariance period with installments respectively $H^{(k)}, K^{(k)}, \ddot{H}^{(k)}, \ddot{K}^{(k)}$.

By using q=1, we go back to the values of constant annuities. The usual factors are used to obtain the FV and the PVDA In order to obtain the corresponding values of the effective annuities, multiply the values in (5.44), (5.44') or (5.45) for the first installment.

Continuous annuities in geometric progression

With continuous flows of payments, it is enough to consider two possibilities which, acting on the amplitude of the invariance periods, cover all cases:

a) *continuous constant flow* in each year (or more generally in each invariance period to which the parameters refer) and variations in GP with ratio q from one period to the next;

b) *varying continuous flow* in exponential way.

The normalized values for case a) are obtained from those of the k-fractional annuities with $k \to \infty$ (the non-normalized values are obtained by multiplying for the total of the first period). By applying the correction factor d/δ from (5.34) and (5.41) the IV are obtained.

$$(\mathrm{G}a)^{[q;\infty]}_{n|i} = \begin{cases} dn/\delta & \text{if } q = 1+i \\ \dfrac{d[1-(qv)^n]}{\delta(1-qv)} & \text{if } q \neq 1+i \end{cases} ; \quad (\mathrm{G}a)^{[q;\infty]}_{\infty|i} = \frac{d}{\delta(1-qv)} \text{if } q < 1+i \quad (5.46)$$

To obtain the normalized values for case b), which give the highest continuity degree with $k \to \infty$, $h \to \infty$, let us first define the *variation intensity of continuous flow* (constant, because the payment flow evolves in an exponential way) given by $\psi = \ln q$ which is consistent with the annual ratio $q = e^{\psi}$. Thus, the evolution of the discounted flows is given by $e^{(\psi-\delta)t}$ and the IV can be obtained using the integral calculus (analogously to the case of constant annuities: see footnote 17) obtaining[37]

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37 We can write: $V_0 = \int_0^n e^{(\psi-\delta)t}dt = \dfrac{e^{(\psi-\delta)n}-1}{\psi-\delta} = \dfrac{(qv)^n-1}{\ln(qv)}$, if $q \neq 1+i$; $V_0 = n$ if $q = 1+i$, that implies $\psi = \delta$. Recalling the Taylor series of $\ln(1+x)$, it is seen that qv-1 is the linear approximation of *ln(qv)*, which is very precise when $qv \cong 1$. In such cases, the difference between the normalized *IV* of annual annuity and of continuous annuity are negligible; in fact the change of deadlines does not bring practical effects because increasing the flow compensates the discount.

$$(G\overline{a})^{[q]}_{n|i} = \begin{cases} n\ , & \text{if } q=1+i \\ \dfrac{(qv)^n - 1}{ln(qv)}, & \text{if } q \neq 1+i \end{cases} \quad ; \ (G\overline{a})^{[q]}_{\infty|i} = -1/\ln(qv), \ \text{if } q < 1+i \qquad (5.46')$$

The IV of a continuous perpetuity in GP in cases a) and b) assume a finite value only if $q<1+i$.

Exercise 5.16

Calculate the IV and FV at the 4-convertible annual rate = 0.056 on the three-yearly interval of validity of the contract, of the annuity given by a delayed monthly wage of a worker. This wage is set up by a fixed part of €1,700 and by a benefit initially at €400 and then increasing at the quarterly ratio of 0.8%. Compare the results with those of a continuous annuity with the same financial parameters.

A. It is convenient to assume the quarter period to be unitary and to use: fixed part = 5,100; initial benefit =1,200; ratio q=1,008; rate i=0.056/4=0.014; frequency of payments k=3; length n=12; thus: v=0.9861933; $d/j(3)$=0.9907814; qv=0.9940828; $(qv)^{12}$=0.9312595. Using (5.12) and (5.42) the IV is

$$V_0 = 5{,}100\, a^{(3)}_{\overline{12}|0.014} + 1{,}200\, (Ga)^{[1.008;3]}_{\overline{12}|0.014} = 5{,}100\cdot 11.0267981 + 1{,}200\cdot 11.5099724 =$$
$$56{,}236.67 + 13{,}811.97 = 70{,}048.64$$

The FV is

$$V_n = 1.014^{12}\ V_0 = 1.1815591\cdot 70{,}048.64 = 82{,}766.61$$

For comparison, let us calculate, using the same parameters of amount and rate, the values in the case of continuous flow with continuous increments. Leaving the quarter as the unit measure of time and using (5.16) and (5.46'), the following is obtained

$$V_0 = 5{,}100\, \overline{a}^{(\infty)}_{\overline{12}|0.014} + 1{,}200\ (G\overline{a})^{[1.008]}_{\overline{12}|0.014} = 5{,}100\cdot 11.0524121 + 1{,}200\cdot 11.5826613$$
$$= 56{,}367.30 + 13{,}899.19 = 70{,}266.50$$

$$V_n = 1.014^{12}\ V_0 = 1.1815591\cdot 70{,}266.50 = 83{,}024.02$$

The values of the constant continuous unitary annuity are only different by a small amount from those of the analogous monthly annuity and this also holds true for the varying annuity in GP given that $qv = 0.9940828 \cong 1$ (see footnote 37).

Exercise 5.17

An industrial company works at a plant for which the initial cost of €280,000 is already covered, but – expecting an average economic length of 5 years, without break-up value and with cost increments for periodic renewal at the annual compound rate of 5% – wants to cover the renewal cost in 20 years through semiannual delayed payment in a profitable bank fund at the compound rate of 6%. Calculate the semiannual payments:

a) using the hypothesis of constant payments in the 20 years;

b) using the hypothesis of payments increasing every 5 years in progression corresponding to the variation of the annual compound rate of 5%.

A. *Computation of cost of five-yearly renewals:*

– after 5 years: $C_1 = 280{,}000(1.05)^5 =$ €357,358.84;

– after 10 years: $C_2 = C_1(1.05)^5 =$ €456,090.50;

– after 15 years: $C_3 = C_2\,(1.05)^5 =$ €582,099.89;

– after 20 years: $C_4 = C_3\,(1.05)^5 =$ €742,923.36.

The outflows for such costs give rise to a pluriannual annuity-immediate in GP with $p = 5$; $n = 20$; $i = 0.06$; ratio $q^p = 1.05^5 = 1.2762816$, and the IV, due to (5.44), is

$$V_0 = 357{,}358.84\ \ (\mathrm{G}a)_{20|0.06}^{[1.2762816,\frac{1}{5}]} = 357{,}358.84\ \frac{1-0.990566^{20}}{1.06^5-1.05^5} =$$
$$= 357{,}358.84 \cdot 2.7878276 = €996{,}254.84$$

We now have to find the installments of the annuity to accumulate in the fund what is needed for the periodic renewal in the two cases a) and b) specified above.

Hypothesis a)

Assuming the year is the unit measure, using i=0.06; n=20; m=2 and using the correction factor $i_{1/2}/i$ specified in section 5.2.4, the semiannual delayed installment $R_{1/2}$ to deposit in a fund that provides the payments for the costs $C_1,\ldots,C_4$ calculated above, is obtained. The following result holds true:

$$R_{1/2} = V_0\,\alpha_{\overline{n}|i}\ i_{1/2}/i = 996{,}254.84 \cdot 0.0871846 \cdot 0.4927169 = €42{,}796.45$$

The values $R_{1/2}$ form constant outflows so as to balance over the 20 years the deposits in a fund with increasing costs; therefore during the 20 years a reserve is formed, equal to the current balance and always to the credit of the depositing person and in debt of the institution that is managing the fund (so 6% is always a debit rate for this institution). This fund dies away after 20 years, as is confirmed by the following scheme of balances at the end of each five-years of the 20 years, where the FV of the five-yearly annuity at 6% annual of payments $R_{1/2}$ is 489,627.13. In the following table each row refers to a five-year period.

NG.	Existing balance (1)	FV of 5-year payments (2)	Updated balance (3) = (1)+(2)	Withdraw for renewal (4)	Residual of 5 - year period (5) = (3) – (4)	Fund after 5 years (6)=(5)1.06^5
1	0.00	489,627.13	489,627.13	357,358.84	132,268.30	177,004.82
2	177,004.82	489,627.13	666,631.96	456,090.50	210,541.46	281,751.97
3	281,751.97	489,627.13	771,379.11	582,099.89	189,279.22	253,298.29
4	253,298.29	489,627.13	742,925.43	742,923.36	(°) 2.07	

(°) Apparent final balance = 2.07 instead of 0, due to rounding-off.

Table 5.4. *Dynamics of a fund in hypothesis a)*

Hypothesis b)

In order to form the amount of five-yearly costs for renewal, we now have delayed constant semi-annual payments inside each five-year period, increasing when passing from one five-year period to the next with the same annual ratio of 5% with which the renewal costs increase. This implies, as we will verify, the balancing between the FV of the payments and the absorption of substitutions, already calculated, with consequent lack of residuals and thus zeroing of the reserve at the end of each five-year period. We can develop the calculation assuming the five-year period as a unit measure of time and solving in respect to K the first equation in (5.39), using: H=0; i = $(1.06)^5$-1 = 0.3382256; n=4; V_0 = 996254.84. This equation becomes

$$V_0 = K(\mathrm{G}a)^{[1.2762816;\frac{1}{5}]}_{\overline{20}|0.06}$$

from which: K=357,358.84. This value is the equivalent FV of the semiannual payments $K^{(10)}$ of the first five-year period, which are obtainable by applying the correction factor $i_{1/10}/i$=0.0973983; therefore, we have $K^{(10)}$=0.0973983·357,358.84 = 31,235.38. For the following five-year periods the semiannual payments and their five-year period FV increase in GP with ratio 1.2762816 every 5 years. The evolution of such payments in the four five-year periods and the verification of the zeroing of residuals are shown in the following table, where each row is referred to a five-year period.

N.	Existing balance (1)	Semiannual payment (2)	FV of 5-year payments (3)	Withdrawal for renewal (4)	Residual of 5-year period (5)=(3) – (4)	Fund after 5 years (6)=(5)1,06
1	0	31,235.38	357,358.84	357,358.84	0	0
2	0	39,865.14	456,090.50	456,090.50	0	0
3	0	50,879.15	582,099.89	582,099.89	0	0
4	0	64,936.12	742,923.36	742,923.36	0	0

Table 5.5. *Dynamics of a fund in hypothesis b)*

Also in hypothesis b), the fund is never in credit because, due to the semiannual payments, it remains in debt inside each five-year period, but becomes 0 at its end and remains 0 during the first following half-year.

Exercise 5.18

1) Recall the second problem of exercise 5.15, which considers a temporary annual annuity-due in GP; using the same data let us consider the following variations:

a) annually varying continuous flow with the same progression;

b) continuous flow with continuous increments, given by $e^{\psi t}$, with $\psi = \ln 1.09$, discounted according to the intensity $\delta = \ln 1.06$.

A. *Case a*) The annual installment is substituted for the constant annual flow, equal to €7,157.35 during the 1st year, afterward in GP at 9% for 10 years, all evaluated in the compound regime with $i = 6\%$. Due to (5.46), the IV is obtained by applying the correction factor $d/\delta = 0.9714233$ to that of the annual case. We obtain: $V_0 = 7157.35 \cdot 11.0495810 = 79{,}085.72$. The FV is given by: $V_{10} = V_0(1.06)^{10} = 79085.72 \cdot 1.7908477 = 141{,}630.48$.

A. *Case b*) By applying (5.46'), IV=$V_0 = 7{,}157.35 \dfrac{(1.09/1.06)^{10} - 1}{\ln(1.09/1.06)} =$ $7{,}157.35 \cdot 11.5348438 = 82{,}558.91$ is obtained, and also FV = $V_{10} = V_0(1.06)^{10} = 82{,}558.91 \cdot 1.7908477 = 147{,}850.44$.

Compare the results obtained here with those from the second part of Exercise 5.15.

2) Calculate the IV for the normalized annuities as in part 2 of exercise 5.15 and part 1 of exercise 5.18, but assuming $q = 1.03$.

A. Using the formulae already discussed, the normalized IV are summarized in the following table.

Type of payment of the annuity	*temporary (10 years)*	*perpetuity*
Advance annual payments	8.8179309	35.3332994
Continuous flow with annual increments	8.5659496	34.3236102
Continuous flow with continuous increments	8.6925517	34.8309069

Table 5.6. *Calculation of the normalized IV*

5.5. Evaluation of varying installment annuities according to linear laws

5.5.1. *General case*

Also varying installments are often used for short periods linear exchange laws. Let us find here the IV and FV at time s of a *m-fractional annuity with varying installments*, considering again the symbols and assumptions of section 5.3, but indicating with R_h the h^{th} delayed installment and with $\ddot{R}_h$ the h^{th} advance installment. Such annuities are the operations $\hat{O}$ for which the supplies are (h/m, R_h) in the delayed case and ((h-1)/m, R_h) in the advance case.

Therefore, if the payments are *delayed*, the IV according to the SD law at rate d and the FV in s according to the SDI law at rate i are given, respectively, by

$$V_0 = \sum_{h=1}^{s} R_h \left(1 - d\,\frac{h}{m}\right);\; V_s = \sum_{h=1}^{s} R_h \left(1 + i\,\frac{s-h}{m}\right) \tag{5.47}$$

However, if the payments are in *advance*, the IV, according to the SD law at rate d and the FV in s according to the SDI law at rate i, are given, respectively, by

$$V_0 = \sum_{h=1}^{s} R_h \left(1 - d\,\frac{h}{m}\right);\quad V_s = \sum_{h=1}^{s} R_h \left(1 + i\,\frac{s-h}{m}\right) \tag{5.47'}$$

Equations (5.47) and (5.47') solve the already-mentioned *direct problem* (installment→value)[38], but we also have to consider here the *inverse problem* (value→installment) to solve amortization and accumulation calculus with varying installments according to linear laws. We have to consider that, as opposed to what we have seen in section 5.3, with constant installments where there is always a unique solution for the installment, here the variability of the installments usually leads to infinite solutions. The problem becomes determinate, and thus has a unique solution due to the linearity of the installment in (5.47) and (5.47'), only if the number of constraints between the installments at different maturities is enough to cancel out that of the degrees of freedom; i.e. s-1 further constraints in addition to (5.47) or (5.47'). This is obtained, in particular, imposing that the installments evolve in AP or in GP

5.5.2. *Specific cases: annuities in arithmetic progression*

Let us consider annuities in AP using

$$R_h = \ddot{R}_h = H+D\,h \qquad (5.48)$$

Under the hypothesis of $R_h=\ddot{R}_h=h$ (i.e. H=0, D=1 in (5.48)) the IV of the unitary annuity in AP -immediate or -due, i.e. of the i*ncreasing annuity* with SD law, are obtained[39] and expressed respectively by

$$I_s^{(m)} = \sum_{h=1}^{n} h\left(1-d\frac{h}{m}\right) = \frac{s(s+1)}{2}\left(1-d\frac{2s+1}{3m}\right) \qquad (5.49)$$

$$\ddot{I}_s^{(m)} = \sum_{h=1}^{n} h\left(1-d\frac{h-1}{m}\right) = \frac{s(s+1)}{2}\left(1-d\frac{s^2-1}{3m}\right) \qquad (5.49')$$

Thus for the IV of annuities with delayed or advance installments given in (5.48) the following is easily obtained, from the 1st part of (5.47) and (5.47'),

38 We only consider the IV and FV of a temporary annuity because: 1) due to the short application interval of the linear law it is not relevant to consider perpetuities; 2) for the same reason the PVDA are not important; furthermore, given the decomposability of the SD law, the PVDA is not obtained from the IV applying the discount; a direct calculation is needed.

39 We will use here: $\sum_{h=1}^{n} h^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{h=1}^{n} h(h-1) = \frac{n(n^2-1)}{3}$.

$$V_0 = H\sum_{h=1}^{s}\left[1-d\frac{h}{m}\right]+D\,I_s^{(m)} = s\left\{H\left[1-d\frac{s+1}{2m}\right]+D\frac{s+1}{2}\left[1-d\frac{2s+1}{3m}\right]\right\} \quad (5.50)$$

$$\ddot{V}_0 = H\sum_{h=1}^{s}\left[1-\frac{d(h\text{-}1)}{m}\right]+D\,\ddot{I}_s^{(m)} = s\left\{H\left[1-\frac{d(s+1)}{2m}\right]+D\left[\frac{(s+1)}{2}-\frac{d}{3m}(s^2-1)\right]\right\} \quad (5.50')$$

For the FV of the aforementioned annuities, from the 2nd part of (5.47) and (5.47') the following is obtained

$$V_s = \sum_{h=1}^{s}(H+Dh)\left\{1+i\frac{s-h}{m}\right\} = sH\left[1+\frac{i}{m}\left(s-\frac{s+1}{2}\right)\right]+D\frac{s(s+1)}{2}\left[1+i\frac{s-1}{3m}\right]$$

(5.51)

$$\ddot{V}_S = \sum_{h=1}^{s}(H+D\,h)\left\{1+i\frac{s+1-h}{m}\right\} = sH\left[1+i\frac{s+1}{2m}\right]+D\frac{s(s+1)}{2}\left[1+i\frac{s+2}{3m}\right] \quad (5.51')$$

Exercise 5.19

Calculate the IV and FV in the -immediate and -due case, of an annuity formed by 15 monthly payments, the first one of €6,500 and the following payments varying in arithmetic progression with a ratio of €150, evaluating with linear laws and equivalent rates at the annual discount rate of 6.4%. Consider the *inverse problem* for amortization and accumulation.

A. The IV of the annuity-immediate is obtained by applying (5.50) with: H=6,350; D=150; m=12; s=15, d=0.064.

$$V_0 = 15\left\{6,350\left[1-0.064\frac{16}{24}\right]+150\frac{16}{2}\left[1-0.064\frac{31}{36}\right]\right\} = 91,186 + 17,008 = 108,194.00$$

The IV of the annuity-due is obtained by applying (5.50') with: H=6,350; D=150; m=12; s=15; d=0.064. The result is:

$$\ddot{V}_0 = 15\left\{6,350\left[1-0.064\frac{16}{24}\right]+150\left[\frac{16}{2}-0.064\frac{224}{36}\right]\right\} = 91,694 + 17,104 = 108,798.00$$

The FV of the annuity-immediate is obtained by applying (5.51) with: H=7,100; D=160; m=12; s=15; i = 0.064/0.936 = 0.0683761. The result is:

$$V_{15} = 15\left\{6,350\left[1+\frac{0.0683761}{12}\left(15-\frac{16}{2}\right)\right]+150\frac{16}{2}\left[1+0.0683761\frac{14}{36}\right]\right\} = 117{,}527.78$$

The FV of the annuity-due is obtained by applying (5.51') with: H=7,100; D=160; m=12; s=15; i = 0.064/0.936 = 0.068376. The result is:

$$\ddot{V}_{15} = 15\left\{6,350\left[1+0.0683761\frac{16}{24}\right]+150\frac{16}{2}\left[1+0.0683761\frac{17}{36}\right]\right\} = 118{,}173.08$$

For the inverse problem, let us observe that the percentage ratio in the first installment is: $\gamma = D/(H+D)$ = 0.0230769. Therefore, if we want to amortize, using an SD law, the debt of €108,194 by 15 increasing delayed monthly installments in AP at 2.30769% of the first installment, this and thus all payments are found to solve the system formed by two equations: (5.50), with the given parameters and V_0 = 108194, and D = 0.0230769(H+D), in the two unknowns H and D. The result is: H = 6350, D = 150, from which the first installment is 6,500 and the other increase by 150 per month. The same installments, if paid at the beginning instead of the end of each month, are consistent to amortize a debt of €108,798, as we see using (5.50') with $\ddot{V}_0$ = 108,798.

Proceeding analogously using (5.51) and (5.51'), we can see that, with SDI law at the rate of 6.83761%, the same 15 monthly installments form a final capital of €117,527.78 if delayed, or of €118,173.08 if advance.

5.5.3. *Specific cases: annuities in geometric progression*

Let us consider the problems of section 5.5.2 using fractional annuities in GP, writing the installments in the form

$$R_h = \ddot{R}_h = R\, q^{h-1} \tag{5.52}$$

where R is the first installment, m is the frequency, s is the total number of installments and q is the ratio of the GP. Using:

$$G_s = \sum_{k=0}^{s-1} k\, q^k = \frac{(s-1)q^s}{q-1} - \frac{q^s-1}{(q-1)^2} \tag{5.53}$$

the IV of the annuity-due with SD law at the rate d and with installments written in (5.52) is given, according to (5.53), by

$$\ddot{V}_0 = \sum_{k=0}^{S-1} R\, q^k \left(1 - d\,\frac{k}{m}\right) = R\left(\frac{q^S - 1}{q - 1} - \frac{d}{m} G_S\right) \qquad (5.54)$$

For the IV of the annuity-immediate with installments in (5.52), for comparison with (5.54), we obtain:

$$V_0 = \sum_{h=1}^{S} R\, q^{h-1} \left(1 - d\,\frac{h}{m}\right) = \ddot{V}_0 - R\,\frac{d}{m}\frac{q^S - 1}{q - 1}$$

and therefore[40]

$$V_0 = R\left\{\left(1 - \frac{d}{m}\right)\frac{q^S - 1}{q - 1} - \frac{d}{m} G_S\right\} \qquad (5.54')$$

With similar development as above, the FV of the annuity-due with installments (5.52), according to the SDI law with rate i, is obtained. It follows that

$$\ddot{V}_S = \sum_{k=0}^{S-1} R\, q^k \left[1 + \frac{i}{m}\,(s - k)\right] = R\left\{\left(1 + \frac{is}{m}\right)\frac{q^S - 1}{q - 1} - \frac{i}{m} G_S\right\} \qquad (5.55)$$

Comparing with (5.55), we obtain

$$V_S = \sum_{h=1}^{S} R\, q^{h-1} \left(1 + i\,\frac{s\text{-}h}{m}\right) = \ddot{V}_0 - R\,\frac{i}{m}\frac{q^S - 1}{q - 1}$$

and thus

40 Observe that V_0 is the arithmetic mean of $R\dfrac{q^S - 1}{q - 1}$ and $-RG_S$.

$$V_s = R\left\{\left(1+\frac{i(s-1)}{m}\right)\frac{q^s-1}{q-1}-\frac{i}{m}G_s\right\} \tag{5.55'}$$

Exercise 5.20

A small loan is amortized over a short period according to a SD law with monthly advance installments in GP. Let us assume the following parameters:

– initial installment R = €650;

– variation monthly rate = 1.2%;

– annual discount rate for the amortization = 5.60%;

– number of monthly rate s = 10.

Calculate the debt to amortize and, also, the debt in the case of delayed installments.

A. Due to (5.53), (5.54) and (5.54') we have

$$G_{10} = \frac{9\cdot 1{,}012^{10}}{0{,}012} - \frac{1{,}012^{10}-1}{(0{,}012)^2} = 47.9724838 \ ;$$

$$\ddot{V}_0 = 650\ (9.4443164 - 0.2238716) = 5{,}993.29$$

$$V_0 = 650\ (0.9953333\cdot 9.4443164 - 0.2238716) = 5{,}964.64$$

Obviously, if we assign, with the rate, time and ratio given above,

a) the debt of 5,993.29 to amortize with monthly advance payments;

b) the debt of 5,964.64 to amortize with monthly delayed installments,

the given installments would be found as a solution.

Exercise 5.21

An industrial company, with increasing turnover, has to replace an old plant over a short period of time. To partially finance the replacement they are able to deposit, at the beginning of every quarter, amounts increasing at 2.5%, the first of which is €6,900, into a savings account with SDI law at 6% per year, for 9 months. Calculate the final balance of the account. Also, calculate in the case of delayed payments.

A. $s = 3$; $q = 1.025$. Due to (5.53), (5.55) and (5.55')

$$G_3 = \frac{2 \cdot 1.025^{\,3}}{0.025} - \frac{1.025^{\,3} - 1}{0.025^{\,2}} = -36.87375$$

$$V_3 = 6900\ (1.03 \cdot 3.0756250 + 0.5531062) = 25{,}674.90$$

Obviously if we assign, with the rate, time and ratio given above, the capital of 25,993.23 to accumulate with quarterly advance installments, or the capital of 25674.90 to accumulate with quarterly delayed installments, the given installments would be found as solution.

Chapter 6

Loan Amortization and Funding Methods

6.1. General features of loan amortization

We have already seen in Chapter 5 that, given a discount law, the inverse problem of the computation of initial value of an annuity is an amortization, in the sense that *the annuity's installments are the amortization's installment of a debt equal to its initial value*.

Having then clarified the general concept of *loan amortization*, we will consider in this chapter the classification and description of the most common amortization methods of a debt contracted at a given time. We will consider in sections 6.2, 6.3, and 6.4 the *amortization of unshared loans* (i.e. with only one lender, which is the creditor, and only one borrower, which is the debtor) *at fixed rates and at varying rates*. In the loan mortgage contract the borrower guarantees payment to the lender.

The exchange law used will be that of discrete compound interest (*DCI*), because in this context we usually consider operations having pluriennial length and the calculation of interest is performed periodically when the borrower pays.

In sections 6.8 and 6.9 we will consider *shared loans* (i.e. among a number of creditors, in the presence of quite a large amount of debt) at fixed rate. The need for the same conditions among many creditors leads to technical complications and problems in financial evaluations, that we have to consider.

In an unshared loan, assuming a discrete scheme for repayment[1], we can distinguish between the following:

a) only one lump-sum repayment of the principal at the end of the term:

a_1) with only one interest payment at the end of the term,

a_2) with periodic interest payment;

b) periodic repayments of the principal together with the accrued interests.

The formulation will consider the scheme of annual payments: for different cases, it is enough to assume the used period as the unit measure, and introduce the equivalent rate. We will then indicate with *i* the rate per period and with *n* the number of periods.

In case a_1), operation $\hat{O}$ is simple, consisting of the exchange between the principal *C* given by the lender in 0 and the amount *M* paid by the borrower in *n* as repayment of the debt and payment of all the accrued interests. *M* is obviously the accumulated value of *C* after *n* periods, obtainable using equation (3.24) for a fixed-rate loan and equation (3.23) for loans with varying rates and given times. Therefore, $\hat{O} = (0,-C) \cup (n,+M)$ from the viewpoint of the lender.

In case a_2), still considering only one final repayment, the interest, which is always calculated on the initial debt, is paid at the end (or beginning) of each period and calculated at rate *i* (or respectively at the rate *d*). In the two cases the operations are written as

$$(0,-C) \cup (1,Ci) \cup ... \cup (n-1,Ci) \cup (n,C(1+i)) \tag{6.1}$$

$$(0,-C(1-d)) \cup (1,Cd) \cup ... \cup (n-1,Cd) \cup (n,C) \tag{6.1'}$$

Example 6.1

Referring to a debt of €255,000 to pay back with the scheme in a_1), after 5 years at the fixed rate of 6.50%, the final debt amount, together with interest, is

$$D = 255{,}000 \cdot 1.065^5 = 255{,}000 \cdot 1.3700867 = €349{,}372.10.$$

Using scheme a_2) with delayed annual installments for the interest, for the given debt we have to pay €16,575.00 at the end of each of the first 5 years, adding at the end of the 5th year the repayment of the debt. Adopting, instead, advance annual installments for the interest, we have to pay €15,563.38 at the beginning of each of the first 5 years, with the repayment of the debt after 5 years. Case b) considers in general terms the *gradual amortization,* the form of which at fixed rate will be

1 An amortization in a continuous scheme, instead, would lead to consideration of a continuous annuity for the debt amortization. Such scheme is possible but has no practical relevance.

described in section 6.2. The periodicity of the installments is usually annual or semi-annual (but sometimes it is quarterly or monthly) and delayed[2]; less frequent are advance payments.

6.2. Gradual loan amortization at fixed rate

6.2.1. *Gradual amortization with varying installments*

The gradual amortization, once the initial debt S, the per period rate i and the length (or number of periods) n are given, the installments (also simply named payments) R_h in the delayed case or $\ddot{R}_h$ in the advance case (where h indicates the integer date of payments between 0 and n), must satisfy the constraint of financial closure expressed in the two cases by one of the equations in (5.23) where in the left side, S instead of v_0 or $\ddot{V}_0$ is used, obtaining

$$S = \sum_{h=1}^{n} R_h \ (1+i)^{-h} \tag{6.2}$$

$$S = \sum_{h=0}^{n-1} \ddot{R}_h \ (1+i)^{-h} \tag{6.2'}$$

The amortization annuities are, in fact, in the two cases: $\bigcup_{h=1}^{n}(h, R_h)$, $\bigcup_{h=0}^{n-1}(h, \ddot{R}_h)$.

The aforesaid installments, altogether equivalent to the initial debt, are divided into two amounts:

– the *"principal repaid"*, c_h in the delayed case or $\ddot{C}_h$ in the advance case, that decreases the debt;

– the *"interest paid"* i_h in the delayed case or $\ddot{I}_h$ in the advance case, which is a gain for the creditor and is proportional to the level of remaining debt (i.e. the outstanding loan balance, defined below)[3].

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2 In the practice of bank loans there can be a "pre-amortization" phase (see also the following footnote 3), from the day the loan is granted to the end of the first period, in which the debtor pays only the accrued interest and the amortization begins, the times of which are parts of a calendar year.

3 To avoid the remaining debt overcoming the initial loan during the amortization, something that the lender cannot allow (due to the consequent lack of guarantees), the principal repayments must never be negative, i.e. the installment are at least at the level of the amount of the interest paid. It is in particular verified the equality "installments = interest paid" ie the absence of principal repayments, during an interval of "pre-amortization" in the initial phase. The creditor can allow such facility in special cases, for instance when the investment financed by the loan implies a delay in the return and then an initial lack of liquidity for the debtor. In such cases, if the pre-amortization lasts for *m* periods, then R_1 =...= R_m = Si and the real amortization is included in the following *n-m* periods. Observe that the *amortization with*

By definition, the principal repayments must satisfy the constraints of *elementary closure* expressed by

$$\sum_{h=1}^{n} C_h = S \;\; ; \;\; \sum_{h=0}^{n-1} \ddot{C}_h = S \tag{6.3}$$

As time increases during the gradual amortization, it is necessary to account at the internal due dates (= end of periods) the *outstanding loan balance* (or *outstanding balance*, or simply *balance*) D_h and the *discharged debt* $E_h = S - D_h$.

Let us consider the *delayed gradual amortization*. This results in

$$D_0 = S \;\; ; \;\; D_h = S - \sum_{k=1}^{h} C_k \, ; \;\; (h = 1, ..., n) \tag{6.4}$$

and then $D_n = 0$ owing to the 1st part of (6.3).

We have already seen that in section 5.4 amortization as the inverse problem of calculating the initial value (*IV*) of an annuity with varying installments, that the solution is not unique, having n-1 degrees of freedom: we have only one constraint on the n unknowns R_h. In order that the number of such degrees be zero, so as to have a unique solution, we need to introduce other n-1 constraints that are linearly independent. This can be carried out in an infinite number of ways: one of which is the imposition of installments in arithmetic or geometric progression, as was shown in section 5.5. However, in general, we can fix the installments under the constraint in (6.2) or the principal repayments under the constraint in the 1st part of (6.3), taking into account the needs of both parties to the contract.

The solution can be found recursively from the system of $3n$ equations

$$(h = 1, ..., n) \begin{cases} D_h = D_{h-1} - C_h \\ I_h = i \, D_{h-1} \\ R_h = C_h + I_h \end{cases} \tag{6.4'}$$

in the $3n$ unknowns D_h, I_h, R_h, $(h = 1, ..., n)$ with the initial condition $D_0 = S$[4].

---

only one final repayment of type a_2 can be considered as a total pre-amortization until the end of the loan.

4 This means that, taking into account the initial condition $D_0 = S$, if the installments R_h are given, I_1 is found from the 2nd part of (6.5), C_1 from the 3rd part, D_1 from the 1st part, and we repeat such a procedure by increasing h. Instead, if the principal repayments C_h, $(h = 1, ..., n)$, are given, it is found I_1 from the 2nd part of (6.5), D_1 from the 1st part, R_1 from the 3rd part, and we repeat such a procedure by increasing h.

From (6.4') the recursive relation is found

$$D_h = D_{h-1}(1+i) - R_h \tag{6.5}$$

that gives the following alternative for the calculation of R_h:

$$R_h = D_{h-1}(1+i) - D_h = (D_{h-1} - D_h) + i\,D_{h-1} \tag{6.5'}$$

Both (6.5) and (6.5') have an expressive financial meaning referring to the dynamic of amortization.[5]

It is convenient, for calculation in case of assignments or advance discharge, to give the reserve defined in Chapter 4. At the rate i (which can be the one in the contract or a different one for the evaluation in k) and at the integer due date h the retro-reserve is

$$M_h = S(1+i)^h - \sum_{k=1}^{h} R_k (1+i)^{h-k} \tag{6.6}$$

while the pro-reserve is

$$W_h = \sum_{k=h+1}^{n} R_k (1+i)^{-(k-h)} \tag{6.6'}$$

We can easily verify that, due to the fairness of the amortization operation expressed by (6.2),[6] using for the valuation the loan rate i, $M_h = W_h = D_h$ follows.[7]

---

5 Let us verify that (6.6) is equivalent to (6.2), i.e. implies the financial closure, and let us give the closed expression for the balances. From (6.6) we find: $D_{h-1} = (R_h+D_h)v$ and when using it for decreasing values of h the following is obtained: $D_{n-1} = R_n v$; $D_{n-2} = (R_{n-1}+D_{n-1})v = R_{n-1}\,v + R_n\,v^2$; ... ; $D_{n-h} = (R_{n-h+1}+D_{n-h+1})\,v = \sum_{k=1}^{h} R_{n-h+k} v^k$;....; $S = D_0 = (R_1+D_1)\,v = \sum_{k=1}^{n} R_k v^k$ and vice versa. We can also write: $D_h = \sum_{k=1}^{n-h} R_{h+k} v^k$, which gives the remaining debt at the h^{th} due date as a function of the installments following R_h.

6 The fairness can be controlled in an alternative way, but which is equivalent, from the final debt. In fact it can soon be seen that if $D_n = 0$ is satisfied, *the operation consisting of the loan of S amortized with the sequence* $\{R_h\}$ *with delayed due dates is fair at rate i.* Otherwise, while the pro-reserve is zero due to the absence of remaining obligation, the retro-reserve would not become zero and the final spread D_n, positive or negative, would make the operation favourable respectively for the borrower or for the lender.

7 The retrospective and prospective reserves, evaluated in whichever integer time $h \in [0,n]$, maintain their meaning of evaluation of the net obligation before and after h, even if in h it is

Furthermore, bare ownerships and usufructs, as defined in Chapter 4, at the integer times h and with discontinuous formation of interests, being $I_k = iD_{k-1} = i\sum_{s=k}^{n} C_s$, are given by

$$(h=1,\dots,n)\begin{cases} P_h = \sum_{k=h+1}^{n} C_k (1+i)^{-(k-h)} \\ U_h = \sum_{k=h+1}^{n} I_k (1+i)^{-(k-h)} = i\sum_{k=h+1}^{n} (1+i)^{-(k-h)} \sum_{s=k}^{n} C_s \end{cases} \tag{6.7}$$

If we use the CCI regime, it would be possible to define the reserves at each intermediate time between two due dates in succession, to calculate in the exact way the assignment or discharge value for whichever time $t = k+s$ (where k = integer part of t; s = decimal part of t). We obtain

$$M(t) = M_k (1+i)^s \quad ; \quad W(t) = W_k (1+i)^s$$

Let us consider briefly the variation in an *advance gradual amortization*. Analogously to the delayed case, the solution can be obtained considering recursively the system of $3n$ equations

$$(h=0,\dots,n-1)\begin{cases} D_{h+1} = D_h - \ddot{C}_h \\ \ddot{I}_h = d\ D_{h+1} \\ \ddot{R}_h = \ddot{C}_h + \ddot{I}_h \end{cases} \tag{6.4''}$$

in the $3n$ unknowns d_{h+1} (= outstanding balance after h and until h+1), $\ddot{I}_h$, $\ddot{R}_h$, (h = 0, ..., n-1) with the usual initial condition $D_0 = S$.

---

adopted for the evaluation of a rate i different from that established at the inception of the loan. However, in such a case we lose the equality between prospective reserve and outstanding balance and also that between prospective reserve and retrospective reserve because, if the sequence of the installments R_h is unchanged, (6.2) does not hold any more and the fairness of the whole operation is lost. The same considerations hold for the case of advance amortization, considered later.

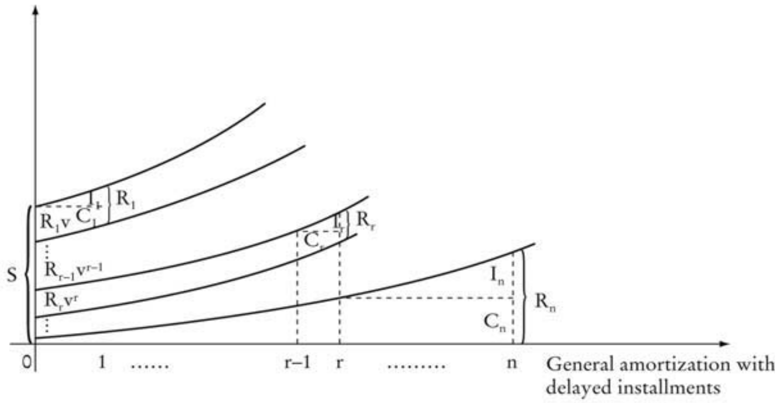

Figure 6.1. *Plot of delayed amortization*

From (6.4") we obtain

$$\ddot{R}_h = D_h - D_{h+1}\, v \tag{6.5"}$$

from which the recursive relation results:

$$D_h = D_{h+1}\,(1\text{-}d) + \ddot{R}_h \tag{6.5'''}$$

being $D_n = 0$ for the 2[nd] of (6.3).

Furthermore, at the delayed loan interest $i=d/(1\text{-}d)$ and at the due integer date h the retro-reserve is

$$M_h = S(1+i)^h - \sum_{k=0}^{h-1} \ddot{R}_k (1+i)^{h-k} \tag{6.6"}$$

while the pro-reserve is

$$W_h = \sum_{k=h}^{n-1} \ddot{R}_k (1+i)^{-(k-h)} \tag{6.6'''}$$

and due to (6.2') we obtain the fairness.

Furthermore, bare ownerships and usufructs at the integer times h and with discontinuous formation of interests, as $\ddot{I}_k = dD_{k+1} = d\sum_{s=k+1}^{n-1} \ddot{C}_s$, are given by

$$(h=0,\dots,n-1)\begin{cases} P_h = \sum_{k=h}^{n-1} \ddot{C}_k (1+i)^{-(k-h)} \\ U_h = d \sum_{k=h}^{n-1} (1+i)^{-(k-h)} \sum_{s=k+1}^{n-1} \ddot{C}_s \end{cases} \tag{6.7'}$$

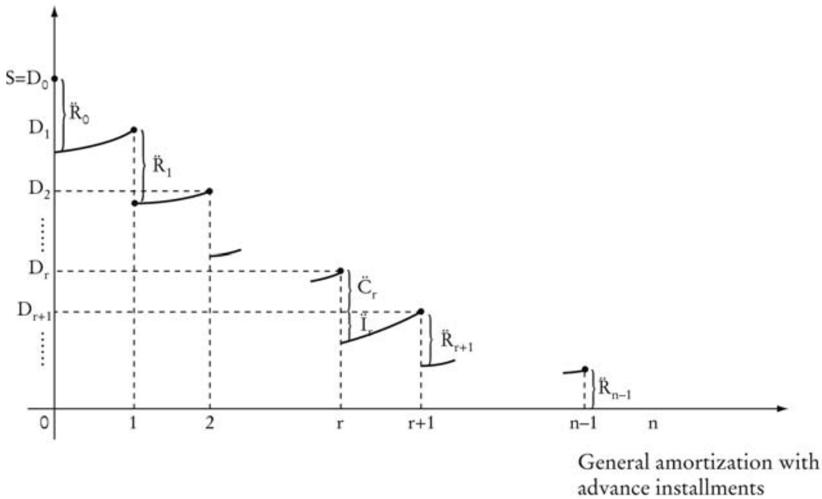

Figure 6.2. *Plot of advance amortization*

Debt amortization schedules

In the operative practice an amortization schedule is summarized in a table in which for each payment is reported on the same row: 1) period, 2) outstanding balance at the beginning of the period, 3) principal repaid, 4) interest paid, 5) installment, 6) outstanding balance at the end of the period. In the case of *delayed installments* the following table is obtained, for h=1,...,n.

(1)	(2)	(3)	(4)	(5)	(6)
…	…	…	…	…	…
h	D_{h-1}	C_h	$I_h = i\, D_{h-1}$	$R_h = C_h + I_h$	D_h
…	…	…	…	…	…

Table 6.1. *Amortization schedule*

Here, it is enough to consider only one of the columns (2) or (6), which coincide except for the displacement of one period. This is easy to change in the case of *advance installments,* in which case $\ddot{I}_h = d\ D_h$.

Example 6.2

What we discussed regarding a gradual amortization with delayed or advance payments is numerically explained here. Consider the delayed or advance amortization of a debt of €90,000 in 5 years at the annual delayed rate i = 5.50% with principal repayment given by:

$$C_1 = \ddot{C}_0 = C_5 = \ddot{C}_4 = 16,000 \quad ; \quad C_2 = \ddot{C}_1 = C_4 = \ddot{C}_3 = 19,000 ; C_3 = \ddot{C}_2 = 20,000 .$$

The elementary closure is verified, as it can be easily observed.

By applying (6.4') for the delayed case and using D_0 = €90,000, the following schedule is recursively obtained, by using a calculator or an *Excel* spreadsheet, as explained below

Debt = 90000		Rate i = 0.055		Length = 5	
h	D_{h-1}	C_h	I_h	R_h	D_h
1	90000.00	16000.00	4950.00	20950.00	74000.00
2	74000.00	19000.00	4070.00	23070.00	55000.00
3	55000.00	20000.00	3025.00	23025.00	35000.00
4	35000.00	19000.00	1925.00	20925.00	16000.00
5	16000.00	16000.00	880.00	16880.00	0.00

Table 6.2. *Example of gradual amortization with delayed payments*

The *Excel* instructions are as follows. B1: 90000; D1: 0.055; F1: 5; using the first two rows for data and column titles, from the 3rd row we have:

column A (year): A3: 1; A4=A3+1; copy A4, then paste on A5 to A7
column B (outstanding balance ante): B3: = B1; B4: = F3; copy B4, then paste on B5 to B7;
column C (principal repaid): from C3 to C7: (insert given data);
column D (interest paid): D3: = D$1*B3; copy D3, then paste on D4 to D7;
column E (installments): E3: = C3+D3; copy E3, then paste on E4 to E7;
column F (outstanding balance post): F3 = B3-C3; copy F3, then paste on F4 to F7;

For the advance case, being d = 5.21327%, by applying (6.4"), and using D_0 = €90000, the following schedule is obtained by using a calculator or an *Excel* spreadsheet, as explained below.

Debt = 90000		Rate d = 0,052133		Length = 5	
h	D_h	$\ddot{C}_h$	$\ddot{I}_h$	$\ddot{R}_h$	D_{h+1}
0	90000.00	16000.00	3857.82	19857.82	74000.00
1	74000.00	19000.00	2867.30	21867.30	55000.00
2	55000.00	20000.00	1824.64	21824.64	35000.00
3	35000.00	19000.00	834.12	19834.12	16000.00
4	16000.00	16000.00	0.00	16000.00	0.00

Table 6.3. *Example of gradual amortization with advance payments*

The *Excel* instructions are as follows. B1: 90000; D1: 0.055; F1: 5. Using the first two rows for data and column titles, from the 3rd row we have:

column A (year): A3: 0; A4=A3+1; copy A4, then paste on A5-A7;
column B (balance ante): B3: = B1; B4: = F3; copy B4, then paste on B5 to B7;
column C (principal repaid): from C3 to C7: (insert given data);
column D (interest paid): D3: = $D$1*F3; copy D3, then paste on D4 to D7;
column E (installment): E3: = C3+D3; copy E3, then paste on E4 to E7;
column F (balance post): F3: = B3-C3; copy F3, then paste on F4 to F7.

For the manual calculation of the polynomials in v in (6.2) and (6.2') it is enough to alternate multiplications by $v = 0.9478763$ and installment additions:

calculating: $\{([(R_5 v + R_4)v + R_3]v + R_2)v + R_1\}v$ in the delayed case, and $\{[(\ddot{R}_4 v + \ddot{R}_3)v + \ddot{R}_2]v + \ddot{R}_1\}v + \ddot{R}_0$ in the advance case, 90,000 is obtained.

When stopping the calculation after k installments from below, the backwards outstanding balances $D_{5\text{-}k}$ are obtained, i.e. 16,000; 35,000; 55,000; 74,000.

Exercise 6.1

We have to discharge a loan of 45 million monetary units (MU) for the financing of the building of an industrial plan which, owing to long assembly time, will give net profits only 2 years and 6 months from the inception date of the loan. Furthermore, having the possibility of increasing in time the accumulation of capital for the repayment, the parts agree that, after a pre-amortization with 5 semi-annual installments, the loan is amortized in 7 years with delayed semi-annual installments, increasing in arithmetic progression at the rate of 5% per half-year. Annual rates of 8% for the pre-amortization and 7% for the amortization, are agreed. Calculate the pre-amortization and amortization installments in the two alternatives:

a) the rates are effective annually,

b) the rates are nominal annual 2-convertible,

using for the evaluation the effective amortization rate.

A. Assuming the half-year as unit measure and using S = 45,000,000; R = base of installment ; D = ratio = 0.05 R ; i_1 = semiannual effective pre-amortization rate; i_2 = semiannual effective amortization rate; the equivalence relation must hold (at the amortization rate):

$$S = S\, i_1\, a_{\overline{5}|i_2} + (1+i_2)^{-5}[R\ a_{\overline{14}|i_2} + D\,(Ia_{\overline{14}|i_2}\,]$$

For *case a)*:

$i_1 = 0.0392305$; $i_2 = 0.0344080$; $(1+i_2)^{-5} = 0.8443853$; $a_{\overline{5}|i_2} = 4.5226323$; $a_{\overline{14}|i_2} = 10.9640169$; $(Ia_{\overline{14}|i_2} = 76.2254208$; then the relation becomes

$$45{\cdot}10^6 = 1765372{\cdot}4.5226323 + 0.8443853\, R\, [10.9640169 + 0.05{\cdot}76.2254208]$$

then: R = 2,966,957.60; D = 148,347.90. Therefore:

– pre-amortization installments: $S\, i_1$ = 1,765,372.50;

– amortization installments: R_1 = 3,115,305.50 ; R_2 = 3,263,653.40 ; R_3 = 3,412,001.30 ; ...; R_{14} = 5,043,828.20.

In *case b)*:

$i_1 = 0.04$; $i_2 = 0.035$; $(1+i_2)^{-5} = 0.8419732$; $a_{\overline{5}|i_2} = 4.5150524$; $a_{\overline{14}|i_2} =$
$= 10.5691229$;
$(Ia)_{\overline{14}|i_2} = 75.8226691$; then the relation becomes

$$45.10^6 = 1800000.4.5150524 + 0.8419732\, R\, [10.5691229 + 0.05.75.8226991]$$

then: R = 2,966,983.60; D = 148,349.20. Therefore:

pre-amortization installments: $S\, i_1$ = 1,800,000.00;

amortization installments: R_1 = 3,115,332.80; R_2 = 3,263,682.00; R_3 = 3,412,031.20 ; ...; R_{14} = 5,043,872.40.

6.2.2. *Particular case: delayed constant installment amortization*

Having developed the general case, it is enough to consider briefly the more diffused cases of the amortization of unshared loans at fixed rates. Let us start from the classical case, in which a loan of amount S is paid back in n periods (annual or shorter) with *constant delayed installments* R calculated on the basis of DCI law at the rate per period i. The equivalence constraint is the particular case of (6.2), as it is given on the basis of the symbols defined in Chapter 5, by:

$$S = R\, a_{\overline{n}|i} \quad \text{from which} \quad R = S\, \alpha_{\overline{n}|i} \tag{6.8}$$

The 2nd part of (6.8) gives univocally the amortization installment as a function of S, n, i. We obtain here a particular case of system (6.4') by using $R_h = R$.

An important property of such amortization, also called *French amortization*, that justifies the name of *progressive amortization*, consists of the fact that *the principal repayments increase in geometric progression (GP) with ratio* $(1+i)$.

Proof. Particularizing (6.6) for consecutive values h and h+1, it is found that:

$$R = D_{h-1}\,(1+i) - D_h \;;\; R = D_h\,(1+i) - D_{h+1}\,.$$

Subtracting term by term, we find

$$0 = (D_{h-1} - D_h)(1+i) - (D_h - D_{h+1})$$

from which

$$C_{h+1} = C_h \ (1+i) \ , \quad h = 1, \ldots, n-1 \tag{6.9}$$

As C_{h+1}/C_h is independent from h, this proves the evolution of C_h in GP.

Starting from the value of the installment given in (6.8) we easily obtain the following French amortization schedule[8].

Debt (S)	Rate (i)	Length (n)	Installment ($R = S\ \alpha_{\overline{n}\vert i}$)
Period (h)	**Principal (C_h)**	**Interest (I_h)**	**Balance (D_h)**
1	$R\ v^n$	$R(1 - v^n)$	$R\ a_{\overline{n-1}\vert i}$
2	$R\ v^{n-1}$	$R(1 - v^{n-1})$	$R\ a_{\overline{n-2}\vert i}$
..	....	....	....
h	$R\ v^{n-h+1}$	$R(1 - v^{n-h+1})$	$R\ a_{\overline{n-h}\vert i}$
..	....	....	....
n-1	$R\ v^2$	$R(1 - v^2)$	$R\ a_{\overline{1}\vert i}$
n	$R\ v$	$R(1 - v)$	0

Table 6.4. *French amortization*

In fact, applying (6.4') recursively with $R_h = R$, it results in:

$I_1 = R\ i\ a_{\overline{n}|i} = R(1 - v^n)$; $C_1 = R - I_1 = Rv^n$; $D_1 = R\ a_{\overline{n}|i} - Rv^n = R\ a_{\overline{n-1}|i}$,

$I_2 = R\ i\ a_{\overline{n-1}|i} = R(1 - v^{n-1})$;$C_2 = R - I_2 = Rv^{n-1}$; $D_2 = R\ a_{\overline{n-1}|i} - Rv^{n-1} = R\ a_{\overline{n-2}|i}$; etc. The GP behavior of C_h is confirmed.

---

8 It is easy to calculate the principal repaid, the interest paid and the outstanding balance as functions of the debt S. Due to (6.9) and the 1st part of (6.3), we find:

$$S = \sum_{h=1}^{n} C_h = C_1 \sum_{h=1}^{n} (1+i)^{h-1} = C_1\ s_{\overline{n}|i}$$

i.e.: $C_1 = S\ \sigma_{\overline{n}|i}$, $C_h = S\ \sigma_{\overline{n}|i}(1+i)^{h-1} = S\ \alpha_{\overline{n}|i} v^{n-h+1}$; $D_h = \sum_{k=1}^{n-h} C_{h+k} = S\ a_{\overline{n-h}|i} / a_{\overline{n}|i}$; $I_h = i\ D_{h-1} = = S\,(1 - v^{n-h+1}) / a_{\overline{n}|i}$.

Exercise 6.2

Make the schedule of a French amortization with annual installments for a debt of €255,000 to pay back in 5 years at the rate $i = 0.065$ (same data as Example 6.1).

A. The constant annual installment of amortization for (6.8) is R = €61,361.81 and the schedule can be obtained using $D_0 = 255{,}000$ and using the recursive system: $I_h = iD_{h-1}$, $C_h = R\text{-}I_h$, $D_h = D_{h-1}\text{-}C_h$ (h=1,...,5). The following amortization schedule with annual due date is obtained, using an *Excel* spreadsheet:

Debt = 255000.00 Rate i = 0.065

Length = 5 Installment = 61361.81

Year h	Interest I_h	Principal C_h	Installment I_h+C_h	Balance D_h
1	16575.00	44786.81	61361.81	210213.19
2	13663.86	47697.95	61361.81	162515.24
3	10563.49	50798.32	61361.81	111716.93
4	7261.60	54100.21	61361.81	57616.72
5	3745.09	57616.72	61361.81	0.00

Table 6.5. *Example of French amortization*

The *Excel* instructions are as follows: the first 3 rows are used for data and column titles; B1: 255000; E1: 0.065; B2: 5; E2: = B1*E1/(1-(1+E1)^-B2$B$2.

From the 4th row:

column A (year): A4: 1; A5: = A4+1; copy A5, then paste on A6 to A8.
column B (interest paid): B4: = E1*B1; B5: = $E$1*E4; copy B5, then paste on B6 to B8.
column C (principal repaid): C4: = $E$2-B4; copy C4, then paste on C5 to C8.
column D (installment): D4: = B4+C4; copy D4, then paste on D5 to D8.
column E (balance): E4: = B1-C4; E5: = E4-C5; copy E5, then paste on E6 to E8.

Calculation of usufructs and bare ownerships in French amortization

Sometimes it is necessary to distinguish in the attribution to the entitled parties the values due to interest and due to principal transaction, i.e. usufructs and bare ownerships (see Chapter 4). As known, we can make two different hypotheses on the formation of interest that brings about the evaluation of usufruct:

a) *continuous formation* of interest, with intensity δ;

b) *periodic formation* of interest at the end of the period at the per period rate $i = e^{\delta} - 1$.

Case a) gave rise to the general formule for $\tilde{U}(t)$ and $\tilde{P}(t) = W(t) - \tilde{U}(t)$ developed in section 4.3 and valid in the exponential financial regime.

In case b), given that interest is formed with impulsive flow only at the time of payments, the usufruct *U(t)* is simply the sum of the discounted interest payments, and the bare ownership *P(t)* is the sum of the discounted principal repayments.

The distinction between cases a) and b) can be applied to a generic financial operation with discrete distribution of payments and the differences in results have little relevance. To illustrate, let us develop the comparison between $\tilde{U}(t)$ and *U(t)*, evaluated according to the same δ, in the particular case of remaining payments of an annuity with constant periodic installments *R*, in the case of fair operation: it is enough to consider the French amortization of an amount *S*, with the constraint (6.8) between *S* and *R*. Then:

a) with *continuous* formation of interest, at a generic time $t=h+s \in \Re$, with $0<s<1$, the following is obtained:

$$\tilde{U}(t) = \sum_{k=1}^{n-h} \delta R(k-s) e^{-\delta(k-s)} = \delta R e^{\delta s} [(\mathrm{I}a)_{\overline{n-h}|i} - s\, a_{\overline{n-h}|i}] =$$

$$= \frac{\delta R}{id}\left[(1-ds)e^{\delta s} - \left\{1 + d(n-t)\right\}e^{-\delta(n-t)}\right] \tag{6.10}$$

From (6.10), with $s \to 0$, we find $\tilde{U}(t)$ at integer time *h*, obtaining

$$\tilde{U}_h = \frac{\delta R}{id}\left[1 - \{1 + d(n-h)\}e^{-\delta(n-h)}\right] \tag{6.10'}$$

The bare ownership $\tilde{P}(t)$ is easily obtained as the difference between $W(t) = W_h e^{\delta s}$ and (6.10);

b) with *periodic* formation of interest, it is meaningful to calculate usufruct and bare ownership only at the integer time *h*. Using the loan rate, we obtain

$$P_h = \sum_{k=h+1}^{n} C_k v^{k-h} = (n-h)C_h = (n-h)Rv^{n+1-h} \tag{6.11}$$

As $W(k) = D_k$, the usufruct is obtained as difference:

$$U_h = D_h - P_h = \frac{R}{i}\left[1 - \{1 + d(n-h)\}v^{n-h}\right] \tag{6.12}$$

and from the comparison with (6.10') it results in:

$$\tilde{U}_h = U_h \, \delta/d \tag{6.13}$$

Then, for the French amortization case, the spread of the usufructs is small, giving a value $\tilde{U}(h)$ slightly bigger than U_h and proportional to the coefficient δ/d.

6.2.3. *Particular case: amortization with constant principal repayments*

In such a form of amortization with delayed installments[9], also called *uniform* or *Italian*, given the debt S, the number of periods n and the per period rate i, as a main feature the *principal repayments* C_h, $(h=1,...,n)$, are constant in time, and then the outstanding balances D_h linearly decrease. The following recursive relations hold, with the initial condition $D_0 = S$:

$$(h=1,...,n)\begin{cases} C_h = D_{h-1} - D_h = \dfrac{S}{n} \\ I_h = iD_{h-1} \\ R_h = C_h + I_h \end{cases} \tag{6.14}$$

from which we obtain the following closed forms according to S:

$$(h=1,...,n)\begin{cases} C_h = \dfrac{S}{n} \quad ; \quad D_h = \dfrac{n-h}{n} S \\ I_h = \dfrac{n-h+1}{n} S\, i \; ; \; R_h = \dfrac{1+(n-h+1)i}{n} S \end{cases} \tag{6.14'}$$

Equation (6.14) enables us to perform the Italian amortization schedule. Furthermore, with the *periodic* formation of interests, the bare ownership P_h and the usufruct U_h at the loan rate i are:

$$P_h = \frac{S}{n} a_{\overline{n-h}|i} \; ; \; U_h = \frac{S}{n} (n - h - a_{\overline{n-h}|i}) \; ; \; (h = 1,...,n) \tag{6.15}$$

9 The amortization with constant principal repaid and advance installments, as well as advance interest paid, is seldom used.

Exercise 6.3

Prepare the schedule for the Italian amortization with annual installments for a debt of €255,000 to be paid back in 5 years at the rate $i = 0.065$ (the same data as in Exercise 6.2).

A. By applying (6.14) and using *Excel*, the following amortization schedule with annual due dates is obtained:

Debt = 255000.00 Rate = 0.065

Length = 5

Year	*Principal*	*Interest*	*Installment*	*Balance*
1	51000.00	16575.00	67575.00	204000.00
2	51000.00	13260.00	64260.00	153000.00
3	51000.00	9945.00	60945.00	102000.00
4	51000.00	6630.00	57630.00	51000.00
5	51000.00	3315.00	54315.00	0.00

Table 6.6. *Example of Italian amortization*

The *Excel* instructions are as follows: B1: 255000; E1: 0,065; B2: 5; using the first 3 rows for data and column titles, from the 4th row we have:

column A (year): A4: 1; A5: = A4+1; copy A5, then paste on A6 to A8.
column B (principal repaid): B4: = B$1/B$2; copy B4, then paste on B5 to B8.
column C (interest paid): C4: = E1*B1; C5: = $E$1*E4; copy C5, then paste on C6 to C8.
column D (installment): D4: = B4+C4; copy D4, then paste on D5 to D8.
column E (outstanding balance): E4: = B1-B4; E5: = E4-B5; copy E5, then paste on E6 to E8.

6.2.4. *Particular case: amortization with advance interests*[10]

For the general case of advance interest, let J_h be the advance interest paid for the period $(h,h+1)$, C_h be the delayed principal repaid for the period $(h\text{-}1,h)$ and R_h^* be the total amount paid in h, $(h=0,1,...,n)$. Comparing with (6.4') and (6.4'') we have:

$$J_h = v\,I_{h+1} = d\,D_h \quad (h=0,1,...,n\text{-}1)\ ;\ J_n = 0 \tag{6.16}$$

$$R_h^* = \begin{cases} J_0 = S\ d\ , & if\ \ h=0 \\ J_h + C_h = D_{h-1} - vD_h = \ddot{R}_h, & if\ \ h=1,...,n \end{cases} \tag{6.16'}$$

10 This is a classic case, even if seldom used, of amortization.

The operation is fair at the rate i[11].

In the particular hypothesis in which the advance installments following the first, made up only by interest paid in 0, are equal to the constant R^*, the amortization is called *German*; in such a case the delayed principal repaid increases in geometric progression with ratio (1+i), as in the French amortization. Therefore, the first is $C_1 = S\sigma_{\overline{n}|i}$ and $R^* = S\ \ddot{\alpha}_{\overline{n}|i}$.

Proof: If $R_h^* = R^*$, (h=1,...,n), by writing the relation $K_h = K_{h-1}(1+i) - R^*$ for consecutive values of h and subtracting, this results for h=1,...,n-1:

$C_{h+1} = D_h - D_{h+1} = (1+i)(K_h - K_{h+1}) = (1+i)^2 (K_{h-1} - K_h) = (1+i)(D_{h-1} - D_h) = (1+i)C_h$.

Exercise 6.4

With the same data as in Exercise 6.2, apply the German amortization with constant annual installment $R_h^* = R^*$ ($h \geq 1$) to obtain the amortization schedule.

A. By applying (6.16), (6.16'), and using *Excel*, we can obtain the amortization schedule at the annual due date. We find: C_1 = 44,786.81; R^* = 57,616.72. The following schedule is the result:

Debt = 255,000			Delayed rate = 0.065	
Length = 5		Advance rate = 0.061033		57,616.72
h	*C_h*	*D_h*	*j_h*	*R*_h*
0	0.00	255000.00	15563.38	15563.38
1	4478.81	210213.19	12829.91	57616.72
2	47697.95	162515.24	9918.77	57616.72
3	50798.32	11171.93	6818.40	57616.72
4	54100.21	57616.72	3516.51	57616.72
5	57616.72	0.00	0.00	57616.72

Table 6.7. *Example of German amortization*

The *Excel* instructions are as the follows. We use the first three rows for titles, data and basic calculations; B1: 255000; E1: 0.065; B2: 5; D2: = E1/(1+E1) (= advance rate) ; E2: = B1*C2/(1-(1+E1)^-B2) (= installment at h=1,...,5); from the 4th row, we have:

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11 *Proof:* using $K_h = D_h - J_h = vD_h$,(h=0,1,...,n), this results in: $K_h = K_{h-1}(1+i) - R_h^*$, ($h$ = 1,...,n). From here, given that $K_n = 0$, we obtain

$$\sum_{h=1}^{n} R_h^*(1+i)^{-h} = \sum_{h=1}^{n}\left[K_{h-1}(1+i)^{-(h-1)} - K_h(1+i)^{-h}\right] = K_0 = S - R_0^*, \quad \text{i.e.}$$

$S = \sum_{h=0}^{n} R_h^*(1+i)^{-h}$. Then the installments R_h^* amortize fairly at the rate i the debt S. For $h \geq 1$ they coincide with the advance installments R_{h-1}^* paid the previous period.

column A (time): A4: 0; A5: = A4+1; copy A5, then paste on A6 to A8.
column B (principal repaid): B4: 0; B5: = B1*E1/((1+E1)^B2-1); B6: =B5*(1+$E$1); copy A5, then paste on A6 to A8.
column C (outstanding balance): C4: = B1; C5: = C4-B5; copy C5, then paste on C6 to C8.
column D (interest paid): D4: = D$2*C4; copy D4, then paste on D5 to D8.
column E (installment): E4: = B4+D4; copy E4, then paste on E5 to E8.

6.2.5. *Particular case: "American" amortization*

To introduce *American* amortization let us consider a variation of the form a_2) of amortization as seen in section 6.1. In such a form the debtor could have difficulties in preparing a large amount as a lump-sum final payment; as guarantee for the creditor, it could be appropriate to agree that the debtor makes constant periodic payments into a bank account so that at the end of the loan the debtor has the amount to be paid back.

In the resulting scheme, the accumulation fund to pay back the debt is called a *sinking fund* (see section 6.4) and such a structure gives rise to American amortization that provides for three economic agents: 1) the *creditor* or *lender*; 2) the *debtor* or *borrower;* 3) the *bank* (or other financial institution) managing the *funding*.

For a debt of amount S to be paid back in n periods, we have to fix a *reward rate i,* i.e. the rate of the loan, which rules the periodic interest paid by the borrower, different from (and usually higher than) the *accumulation rate* $i*$, which rules the interest earned by the borrower on the funding[12]. On the basis of such elements:

– the debtor at the end of each period pays to the creditor the accrued interest Si and pays into the sinking fund *the periodic funding installment* $S\,\sigma_{\overline{n}|i*}$ in order to reach at maturity the amount S that the bank, instead of the debtor, will pay to the creditor; then the debtor against the initial cash inflow $(0,+S)$ pays $\bigcup_{h=1}^{n}(h,-R(i,i*))$, where

$$R(i,i*) = S\,(i+\sigma_{\overline{n}|i*}) \tag{6.17}$$

– the creditor, due to (6.1), has the cash-flow

$$(0,-S)\cup(1,Si)\cup\ldots\cup(n-1,Si)\cup(n,S(1+i))$$

– the bank manages in the interval of n periods the sinking fund at the rate $i*$ with periodic inflow $+S\,\sigma_{\overline{n}|i*}$ and the final outflow $-S$.

12 In fact, it is well known that, for obvious market reasons, for a private operator against a bank the allowed rates are lower than charged rates.

The role of the bank as a broker allows the different structure of the cash-flow following the inception, for both lender and borrower. For the lender, case a_2), of one lump-sum repayment, holds, while for the borrower we have periodic constant payments, as in progressive amortization. It is then useful to find the *cost rate z* for the debtor of the American amortization in the usual hypothesis: $i > i*$. z is the solution of the equation

$$\alpha_{\overline{n}|z} = i + \sigma_{\overline{n}|i*} \tag{6.18}$$

obtained making the constant installment of the French amortization at rate z equal to that of the American amortization and then dividing by S. The problem leads back to the search of the internal rate implied by the cost of a constant annuity (see section 5.2).

Observe that the right side of (6.18) can be written: $(i\text{-}i*) + \alpha_{\overline{n}|i*}$ (alternative formula for the American installment of the unitary debt) and then (6.18) becomes:

$$\alpha_{\overline{n}|z} = (i\text{-}i*) + \alpha_{\overline{n}|i*} \tag{6.18'}$$

If $i>i*$, $\alpha_{\overline{n}|z} > \alpha_{\overline{n}|i*}$ results, and then, $\alpha_{\overline{n}|z}$ being an increasing function of z, we obtain: $z>i*$; if instead $i<i*$, we obtain: $z<i*$. Furthermore, $\sigma_{\overline{n}|z}$ being a decreasing function of z, if $i>i*$, recalling (5.9) we obtain $\alpha_{\overline{n}|z} = i + \sigma_{\overline{n}|i*} > i + \sigma_{\overline{n}|i} = \alpha_{\overline{n}|i}$ and then, due to the behavior of $\alpha_{\overline{n}|z}$, we have: $z>i$; if instead $i<i*$, using analogous developments we obtain: $z<i$.

In conclusion, *z is external (and not internal mean) to the interval between i and i**, being the only alternative between $i*<i<z$ (usual case) and $i*>i>z$ (exceptional case). In the usual case the American amortization is more expensive for the borrower than the French at rate i, because the borrower must accumulate the amount for the repayment at the earned interest rate $i*<i$.

American amortization with equality of rates

It is appropriate to consider the case $i=i*$ in the American amortization[13]. With regard to the cost rate z for the debtor, $i*=i=z$ results. In addition, for (5.9), the

---

13 We have has to mention that the "geographical" terms for the different amortization that are usually used to differentiate are not always unique; it can be preferred to use technical adjectives (progressive and uniform instead of French and Italian). de Finetti (1969) (cited for deeper investigation, together with Volpe di Prignano (1985)) uses "English" amortization for the scheme, which we here call "American", with two different rates and "American" when the rates are the same.

periodic payment $S(i+\sigma_{\overline{n}|i})$ of the debtor coincides with $S\alpha_{\overline{n}|i}$, payment that he would have on the basis of the French amortization at the rate *i*. However, this is the same as the total payment that he would have if he pays the interests *Si* to the lender and accumulates the repayment capital *S at the same rate i agreed for the payment*. This situation can be realized if the bank[14] that manages the sinking fund at the rate $i^*<i$ is not present. It could be the same lender, if a financial institution gives loans at the rate *i*, to operate at the reciprocal rate *i* with the borrower for a deposit operation as guarantee for the loan. In such a case the American amortization with sinking fund is managed by the lender, because (5.9) is substantially reduced to the progressive amortization at the rate *i*.

However, this is not the case for the formal aspects. If the sinking fund at rate *i* is not managed by the lender, or it is but with separate accounting until maturity, then the periodic payment $S\sigma_{\overline{n}|i}$ for the sinking fund is not "principal repaid", because it does not reduce the debt that always remains at the level *S,* but "accumulation amount"; in the same way *Si* is not French "interest paid" but is constant "reward amount". However, if the accumulation payments are accounted periodically at the rate *i* to the lender to reduce the debt, so that at the due date *h* it becomes $Sa_{\overline{n-h}|i}/a_{\overline{n}|i}$, a sinking fund does not arise and we lead back to installment $S\alpha_{\overline{n}|i}$ decomposition in principal repaid and interest paid of a progressive amortization,[15] varying with *h* and given by

$$C_h = S\sigma_{\overline{n}|i}(1+i)^{h-1},\ I_h = S\alpha_{\overline{n}|i}(1-v^{n-h+1}).$$

In such cases, the American amortization does not hold.

Exercise 6.5

We have to amortize the amount $S = 35000$ in $n = 10$ years with the sinking fund method with two different rates; $i = 7.2\%$ for debt repayment, $i^*_a = 3.6\%$ in case a) and $i^*_b = 4.7\%$ in case b) for accumulation. Calculate the delayed annual payment *R(i,i*)* for the borrower and the constant rate *z* solution of (6.18) in the two cases.

A. On the basis of (6.17), the annual payment is given:
- in case a) by $R(0.072; 0.036) = 2520.00 + 2969.69 = 5489.69$
- in case b) by $R(0.072; 0.047) = 2520.00 + 2821.86 = 5341.86$

then being the amount in the sinking fund equal to 2969.69 and 2821.86 in the two cases. For the calculation of the cost rate *z*, we can use the numerical methods

14 See de Finetti (1969).
15 Recall that in such amortization the principal repaid is $S\,\sigma_{\overline{n}|i}$ only for the 1st period, after that it increases in GP while the interest paid decreases proportionally to the outstanding balance.

described in section 5.2 to obtain exact results through iterative methods until convergence, or with lower approximation if we use the linear interpolation;

– in *case a)*, using equation: $\alpha_{\overline{10}|z} = 5489.69/35000 = 0.1568483$ and a financial calculator with: [n] = 10; [pv] = -35000; [pmt] = 5489.69; [fv]=0; comp[i]; we obtain $\hat{z}$ = 9.14969%. With the linear interpolation on (9%; 9.25%) the results are: $\alpha_{\overline{10}|9\%} = 0.1558201$; $\alpha_{\overline{10}|9.25\%} = 0.1575389$; then

$$\bar{z} = 0,09 + \frac{10282}{17188}\ 0.0025 = 0.0914955$$

We can also apply the classic iteration method, using *Excel* starting from $z_0 = \bar{z}$ =0.0914955. Since the equation $\alpha_{\overline{10}|z}$= 0.1568483 has the form $g(z)=g_0$, with $g(z)= \alpha_{\overline{10}|z}$, g_0 = 0.1568483, we can go to the canonical form $f(z)= z$ using $f(z) = z\alpha_{\overline{10}|z}/g_0$. However, the iteration process on f diverges, resulting in: $|f'(\bar{z})|$=1.4400328>1. We then have to apply the transformation, analogous to that seen in case B of Example 4.3: $h(z)=[f(z) - mz]/[1-m]$, where $h(z)= z$ is equivalent to $f(z)= z$, using $m = |f'(\bar{z})|$. Starting from $z_0 = \bar{z}$, the following expansion, rapidly converging to $\hat{z}$, is obtained.

	$[g_0, z_0, f'(z_0)]$ =	0.15684830	0.09149550	1.44003280
k	*zk*	*G(zk)*	*f(zk)*	*h(zk)*
0	0.09149550	0.15684730	0.09149492	0.09149683
1	0.09149683	0.15684821	0.09149678	0.09149694
2	0.09149694	0.15684829	0.09149694	0.09149695
3	0.09149695	0.15684830	0.09149695	0.09149695
4	0.09149695	0.15684830	0.09149695	0.09149695

Table 6.8. *Calculation of cost rate by iteration*

The *Excel* instructions are as follows. The first two rows are used for data and titles; A1: 10 (= length); C1: 0.1568483 (= $\alpha_{\overline{10}|z}$); D1: 0.0914955 (= $\bar{z}$); E1: 1.44003280 (=$|f'(\bar{z})|$); from the 3rd row:

column A (step k); A3: 0; A4: = A3+1; copy A4, then paste on A5 to A7;
column B (approximate rate z_h); B3: = D1; B4: = E3; copy B4, then paste on B5 to B7;
column C ($g(z_k)$); C3: = B3/(1-(1+B3^-A$1); copy C3, then paste on C4 to C7;
column D ($f(z_k)$); D3: = B3*C3/C$1; copy D3, then paste on D4 to D7;
column E ($h(z_k)$); E3: = (D3-E$1*B3)/(1-E$1); copy E3, then paste on E4 to E7;

– in *case b)*, the solving equation is: $\alpha_{\overline{10}|z}$ = 5341.86/35000 = 0.1526246; using the financial calculator with: [n]=10; [pv]=-35000; [pmt]=5341.86; [fv]=0; comp [i], we obtain z = 0.0853193. With the linear interpolation on (8.5%; 8.75%) the results are: $\alpha_{\overline{10}|8.5\%}$ = 0.1524077; $\alpha_{\overline{10}|8.75\%}$= 0.1541097; then

$$z = 0.085 + \frac{2169}{17020} 0.0025 = 0.0853186$$

By using the *Excel* spreadsheet, it is sufficient to change the rate $i*$.

6.2.6. *Amortization in the continuous scheme*

A gradual amortization scheme that is widely used for theoretical aims is produced using a continuous annuity.

Let us consider briefly such a case, assuming a continuous flow $\alpha(t)$ of payments covering interest, used to amortize in a temporal interval $I(t_1)$ from time 0 to t_1 the amount S. It is not restrictive to assume for simplicity that S=1 (otherwise it is enough to multiply the results by S). In addition, let us assume a financial law strongly decomposable with intensity $\delta(t)$, that, as known, is a function only of the varying time t (in particular $\delta(t)$=δ if the exponential law is assumed). Using:

$$\varphi(t) = \int_0^t \delta(z)dz \quad , \quad t \in I(t_1) \tag{6.19}$$

$\varphi(t)$ is the natural logarithm of the accumulation factor from 0 to t. With such positions, the flow $\alpha(t)$ can be fixed varying in the interval $I(t_1)$, but must satisfy the constraint of financial closure:

$$\int_0^{t_1} \alpha(t) e^{-\varphi(t)} dt = 1 \tag{6.20}$$

If $\alpha(t) = \alpha$ constant and $\delta(t) = \delta$ constant, due to (5.16) and (6.20),

$$\alpha = 1/a_{t_1|i}^{(\infty)} \tag{6.20'}$$

holds, with extension of the meaning of the symbol $a_{t_1|i}^{(\infty)}$ if t_1 is not integer.

In addition, let us define:

- $c(t)$ = amortization flow (for the principal repayment) at time t;
- $j(t)$ = interest flow (allowed for the borrower) at time t;
- $B(t,t_1)$ = discharged debt at time t;
- $D(t,t_1) = 1 - B(t,t_1)$ = outstanding balance at time t;
- $A(t,t_1)$ = initial value of the payments of the borrower made from 0 to t.

Such quantities are linked by the following relations, of trivial interpretation, that determine them completely:

$$\alpha(t) = c(t) + j(t) \quad , \quad t \in I(t_1) \tag{6.21}$$

$$\int_0^{t_1} c(t)dt = 1 \tag{6.22}$$

$$B(t,t_1) = \int_0^t c(z)dz \quad , \quad t \in I(t_1) \tag{6.23}$$

$$D(t,t_1) = \int_t^{t_1} c(z)dz \quad , \quad t \in I(t_1) \tag{6.24}$$

$$j(t) = [1 - B(t,t_1)]\delta(t) \quad , \quad t \in I(t_1) \tag{6.25}$$

$$A(t,t_1) = \int_0^t \alpha(z)e^{-\varphi(z)}dz \quad , \quad t \in I(t_1) \tag{6.26}$$

The value $M(t,t_1) = [1 - A(t,t_1)]e^{\varphi(t)}$ represents the *retrospective reserve* (or *retro-reserve*, at credit for the lender) at time t with the meaning defined in Chapter 4. In addition, $W(t,t_1) = \int_t^{t_1} \alpha(z)e^{-\int_t^z \delta(\tau)d\tau} dz$ expresses the *prospective reserve* (or *pro-reserve*). Maintaining in $t \in I(t_1)$ the decomposable financial $\delta(t)$ initially adopted that assures the validity of (6.20), i.e. the fairness of the amortization operation, the following equalities hold:

$$M(t,t_1) = W(t,t_1) = D(t,t_1) \quad ; \quad t \in I(t_1) \tag{6.27}$$

then from (6.20) follows

$$[1 - A(t,t_1)]e^{\varphi(t)} = \int_t^{t_1} \alpha(z)\, e^{[\varphi(z)-\varphi(t)]}dz \quad , \quad t \in I(t_1)$$

and $D(t,t_1)$ is also the amount which can be fairly cashed in t instead of the payments with flow $\{\alpha(t)\}$ in the interval (t,t_1)[16].

16 In the continuous scheme the observations in footnote 7 on the lack of inequality between the prospective reserve, the outstanding balance and the retrospective reserve calculated in

6.3. Life amortization

6.3.1. *Periodic advance payments*

So far we have analyzed the debt amortization methods in case of certainty, then through annuities certain, not considering randomness in the repayment of the loan or on the interest payment. This is assuming that, in the event of the borrower dying, his heirs or other people must enter into and fulfill his obligations.

We can take into account the risk of death of the borrower and the difficulty for his heirs to pay back the loan, excluding, due to the contract, the continuation of the repayment in case of death of the borrower and then establishing for the borrower the debt amortization through a temporary life annuity of n years (= life of the loan). In such a way the debt is discharged by means of a *life amortization*[17] and the contract becomes stochastic, as with an insurance contract: the financial equivalence is obtained only as average, i.e. it has an actuarial nature.

We have to take into account the uncertainty on the borrower's survival, the probability of which is considered to depend only on his age x at the inception date of the amortization. This is obtained by replacing the financial discount factors $(1+i)^{-h}$ by the demographic-financial ones ${}_hE_x$[18].

$t \in I(t_1)$ remain valid, when at such a time an interest intensity $\delta^*(\tau)$ in $\tau \in I(t_1)$ different from the intensity $\delta(\tau)$ initially fixed is used, maintaining unchanged in $I(t_1)$ the flow $\alpha(\tau)$. In fact, in such case, (6.20) does not hold.

17 For a better understanding of life amortization see Boggio, Giaccardi (1969) and also Volpe di Prignano (1985).

18 We recall here that – with the symbols used in actuarial mathematics and assuming the discrete time scheme, starting from a demographic table of *survival* $\{l_x\}$ as a function of age (integer) x of a generic member of the community – the *survival probability* for h years of a person aged x is introduced and it is indicated with ${}_hp_x$ resulting in ${}_hp_x = l_{x+h}/l_x$; in particular for surviving one year we put $p_x = {}_1p_x$. In addition, considering the *financial discount factor* $(1+i)^{-h}$ for h years at the annual rate i, we introduce the value ${}_hE_x = {}_hp_x(1+i)^{-h}$ which is called *demographic-financial (or actuarial) discount factor* and is the mean present value of the unitary amount payable within h years only in case of the survival of a person aged x, i.e. the amount that it is fair to pay with certainty today, at age x, to receive the unitary amount within h years only in case of survival. It is obvious that ${}_0E_x = 1$ and we use ${}_1E_x = E_x$. Also, we introduce, for a person aged x, the mean present value of a unitary life perpetuity-due or -immediate, denoted respectively by $\ddot{a}_x$ or a_x, and also of a unitary life annuity-due or -immediate for n years, denoted respectively by ${}_{/n}\ddot{a}_x$ or ${}_{/n}a_x$. Such perpetuities or annuities give the unitary annual amount until death or at most for n years. This is, obviously:

Let us describe the operation with integer length n at fixed rate for the initial debt $D_0 = S$, incepting at time 0, with the borrower aged x (integer).

To discharge the loan the borrower pays a periodic life annuity-immediate, in particular annual, n-temporary with varying installments ${}_z\ddot{\alpha}_{x,n,S}$, payable at times $z = 0, 1, n\text{-}1$, referring to the periods $(z, z+1)$. For the congruity of the amortization, the constraint of actuarial equivalence

$$\sum_{z=0}^{n-1} {}_z\ddot{\alpha}_{x,n,S} \ {}_zE_x = S \tag{6.28}$$

has to be satisfied. Equation (6.28) generalizes (5.23) of Chapter 5. Therefore the sequence $\left\{ {}_z\ddot{\alpha}_{x,n,S} \right\}$ can be chosen with n-1 degrees of freedom[19].

We can immediately verify that by the installment ${}_z\ddot{\alpha}_{x,n,S}$ (or briefly: $\ddot{\alpha}_z$, omitting x,n,S) the borrower pays:

1) the advance *principal repaid* ${}_z\ddot{c}_{x,n,S}$ (or briefly: $\ddot{c}_z$);

2) the advance *financial interest paid* dD_{z+1} on the outstanding balance D_{z+1} in $z+1$;

3) and also – and here is the difference of the life amortization compared to the certain amortization – the *insurance natural premium* for the year $(z,z+1)$. Recalling that $v=(1+i)^{-1}=1-d$ and using: $q_y = 1-p_y = 1-l(y+1)/l(y)$ (= death probability between ages $y,y+1$), such a premium is given by $vq_{x+z}D_{z+1}$, proportional to the outstanding balance D_{z+1} that the borrower will not discharge in case of his death at the year $(z,z+1)$, leaving such duty to the lender, which in this aspect acts as insurer.[20]

The three installment's components make it possible to understand how the life amortization can be interpreted as a normal gradual amortization together with an insurance policy in case of the death of the borrower, which lasts for the length of the loan, and with varying capital given by the current outstanding balance, the premium of which is an addition of the financial installment.

$${}_nE_x = \prod_{k=0}^{n-1} E_{x+k} \ (n>0); \ {}_{/n}\ddot{a}_x = \sum_{h=0}^{n-1} {}_hE_x \ ; {}_{/n}a_x = \sum_{h=1}^{n} {}_hE_x \ ; \ddot{a}_x = \sum_{h\geq 0} {}_hE_x \ ; a_x = \sum_{h\geq 1} {}_hE_x$$

and for $k<n$ results in: ${}_{/n}\ddot{a}_x = {}_{/k}\ddot{a}_x + {}_kE_x \ {}_{/n-k}\ddot{a}_{x+k}$.

19 The inequality constraints that can be introduced for the non-negativity of the principal repayments do not reduce the number of degrees of freedom, because such a decrease holds only by the equality constraints.

20 If the lender does not manage the insurance himself, he can transfer the premium to an insurance company that accept the same technical bases to cover the risk.

We can define as *actuarial interest paid* for the year (z,z+1), indicating it with ${}_z\ddot{j}_{x,n,S}$ (or briefly: $\ddot{j}_z$), the amount $[dD_{z+1} + vq_{x+z}D_{z+1}] = (1\text{-}\,E_{x+z})D_{z+1}$, the sum of the amounts defined in points 2) and 3) above. Therefore, as in the certain case, the installment is divided into principal repaid and interest paid, but the interest paid is actuarial.

A more precise argument leads to the conclusion that the two components in the expression for $\ddot{j}_z^{i',i''} = \ddot{j}_z$, i.e. dD_{z+1} and $vq_{x+z}D_{z+1}$, are antithetic with respect to the rate: in the first, the rate is at debt for the borrower; in the second, it is at credit. If due to market law we keep them separate, indicating them with i' and i'', ($i' > i''$), then the actuarial interest amount is

$$\ddot{j}_z^{i',i''} = \left\{\left[1-(1+i')^{-1}\right]+(1+i'')^{-1}q_{x+z}\right\}D_{z+1}$$

We obtain $\ddot{j}_z^{i',i''} = \ddot{j}_z$ if $i' = i''$. Therefore, analogously to what happens for the American amortization, indicating with x the cost rate for the borrower, the result is: $x > i' > i''$. This scheme, which leads to further complications, is not discussed further.

To better clarify, let us consider the dynamic aspect of the life operation, assuming as already fixed the principal repayments $\ddot{c}_z$, which are under the *elementary closure* constraint

$$\sum_{z=0}^{n-1}\ddot{c}_z = S \tag{6.29}$$

– in the first year the actuarial interest paid is $\ddot{j}_0 = (1\text{-}\,E_x)D_1$, where $D_1 = D_0 - \ddot{c}_0$, and $\ddot{\alpha}_0 = \ddot{j}_0 + \ddot{c}_0 = ... = D_0 - E_x D_1$;

– in the second year the development, starting from the debt D_1, is repeated; we obtain: $\ddot{j}_1 = (1-E_{x+1})D_2$, where

$$D_2 = D_1 - \ddot{c}_1\text{, and } \ddot{\alpha}_1 = \ddot{j}_1 + \ddot{c}_1 = ... = D_1 - E_{x+1}D_2\,;$$

– and in general, due to: $D_{z+1} = D_z - \ddot{c}_z$ and $\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$, we obtain for the year (z,z+1), where $z+1 \le n$,

$$\ddot{j}_z = (1-E_{x+z})D_{z+1}\;\; \ddot{\alpha}_z = D_z - E_{x+z}D_{z+1}\,.^{21} \tag{6.30}$$

If instead the installments $\ddot{\alpha}_z$ under constraint (6.28) are fixed in advance, then for the actuarial equivalence $\forall z = 0,1,...,n\text{-}1$, (6.28) is generalized in

21 These formulae generalize (6.5) and (6.6'''), which hold in the absence of death. In fact, if $l_z =$ constant, $p_z = 1,\ \forall z$, results.

$$D_z = \sum_{k=z}^{n-1} \ddot{\alpha}_k \;\; {}_{k-z}E_{x+z} \tag{6. 31}$$

By obtaining D_z and D_{z+1} from (6.31) it is immediately verified that (6.30) is satisfied for $\ddot{\alpha}_z$, and also for $\ddot{j}_z$, given by definition $\ddot{c}_z = D_z - D_{z+1}$. Therefore the components of $\ddot{\alpha}_z$ are obtained from

$$\ddot{c}_z = D_z - D_{z+1} \quad ; \quad \ddot{j}_z = (1 - E_{x+z})D_{z+1} \tag{6.32}$$ [22]

The *life amortization with advance installments* schedule has in the row relative to the period (z,z+1), (z=0,...,n-1), the following elements

- payment time: z
- principal repaid: $\ddot{c}_z$
- discharged debt (after payment in z): $B_{z+1} = \sum_{k=0}^{z} \ddot{c}_k$
- outstanding balance (after payment in z): $D_{z+1} = \sum_{k=z+1}^{n-1} \ddot{c}_k$
- actuarial interest paid $\ddot{j}_z = (1 - E_{x+z})D_{z+1}$
- installment $\ddot{\alpha}_z = \ddot{j}_z + \ddot{c}_z = D_z - E_{x+z}D_{z+1}$

Making successive substitutions on D_{z+1} in the expression for $\ddot{\alpha}_z$ for z=0,...,n-1 and taking into account ${}_zE_x = \prod_{k=0}^{z-1} E_{x+k}$ we obtain:

$$D_0 = S = \sum_0^{k-1} \ddot{\alpha}_z \;\; {}_zE_x + {}_kE_xD_k ,$$

from which, due to k=n, (6.28) follows.

The expression

$$M_k = \frac{S - \sum_{z=0}^{k-1} \ddot{\alpha}_z \;\; {}_zE_x}{{}_kE_x} \tag{6.33}$$

can be interpreted as *retro-reserve* at time k of the life amortization operation from 0 to n, extending what is seen in Chapter 4 to the mean values obtaining, in the

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22 Note that: $\ddot{\alpha}_z = D_z - E_{x+z}(D_z - \ddot{c}_z) = E_{x+z}\ddot{c}_z + (1 - E_{x+z})D_z$. Therefore, $\ddot{\alpha}_z$ is the weighted mean of $\ddot{c}_z$ and D_z. In addition, in the particular case where z=n--1, as D_n = 0, $\ddot{\alpha}_{n-1} = D_{n-1} = \ddot{c}_{n-1}$ results, and then $\ddot{j}_{n-1} = 0$, in accordance with the fact that for the period (n-1,n) both the financial interest and the insurance premium are zero.

actuarial sense an *insurance retro-reserve*. In fact, such a formula gives the difference between the expected supplies of the lender and the borrower between the dates 0 and k, evaluated actuarially at time k. Instead, the *insurance pro-reserve*, considered as the difference between the expected obligations of the borrower and the lender (the latter are absent, because the lender's supply occurs only at the inception of the loan) between the dates k and n, evaluated actuarially at time k, is given by

$$W_k = \sum_{z=k}^{n-1} \ddot{\alpha}_z \; {}_{z-k}E_{x+k} \tag{6.33'}$$

Maintaining in $k \in (0,n)$ the actuarial base $\{i, l_x\}$ initially adopted that ensures the validity of (6.28), i.e. the actuarial fairness of the operation of life amortization, $M_k = W_k = D_k \; ,\forall k$, hold, taking into account (6.28) and the fact that D_k is the amount in k that finds a fair counterpart in the payments of $\ddot{\alpha}_z$ ($z = k,...,n$-1).[23]

If the life amortization is carried out with constant installment $\ddot{\alpha}_z$, for (6.28) it must be $\ddot{\alpha}_z = S \;/\; {}_{/n}\ddot{a}_x$ and the outstanding balances are given by

$$D_z = \frac{S}{{}_{/n}\ddot{a}_x} \; \frac{{}_{/n}\ddot{a}_x - {}_{/z}\ddot{a}_x}{{}_zE_x} = \mathrm{S} \; {}_{/n-z}\ddot{a}_{x+z} \,/\, {}_{/n}\ddot{a}_x \tag{6.31'}$$

Equation (6.32) is still applied for the calculation of the principal repayments and interest payments.

Exercise 6.6

We have to make a life amortization of €95,000 with advance annual installments for 10 years at rate i = 4.50% on a borrower aged 42 years. Calculate the amortization schedule on the basis of principal repayments assigned.

A. The survival probability is found on suitable tables for an age x = 42. We can apply the formulations on footnote 18 and in (6.30), using a calculator or an *Excel* spreadsheet. The values E_{42+z} are calculated and the principal repayments for z = 0, 1, ..., 9, the sum of which is 95000, are assigned. Then we find the discharged debts and the outstanding balances for 10 years, and also the actuarial interest payments and the advance installments. With both procedures the following schedule is found, with obvious meaning:

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23 Due to its decomposability the simplification effects of the actuarial law that leads to the discount ${}_kE_x$ and accumulation $1/\,{}_kE_x$ factors are obvious. Extended for the retro-reserve and pro-reserve in actuarial sense, the considerations of footnote 16 if at time k are adopted the technical bases $\{i,l(x)\}$ different from the ones initially used to prepare the life amortization .

	Debt = 95000			Rate = 0.045			Length = 10	
z	l_{42+z}	E_{42+z}	c_z	B_z	D_z	j_z	*Installment* c_z+j_z	*Installment* (6.30)
0	96400	0.955230	8700	0	95000	3863.62	12563.62	12563.62
1	96228	0.955128	8650	8700	86300	3484.32	12134.32	12134.32
2	96046	0.954975	9800	17350	77650	3054.94	12854.94	12854.94
3	95849	0.954761	9600	27150	67850	2635.15	12235.15	12235.15
4	95631	0.954486	9300	36750	58250	2227.90	11527.90	11527.90
5	95386	0.954189	10100	46050	48950	1779.76	11879.76	11879.76
6	95112	0.953879	9700	56150	38850	1344.42	11044.42	11044.42
7	94808	0.953647	9750	65850	29150	899.24	10649.24	10649.24
8	94482	0.953302	9200	75600	19400	476.32	9676.32	9676.32
9	94123	0.953105	10200	84800	10200	0.00	10200.00	10200.00
10	93746			95000	0			
total			95000					

Table 6.9. *Example of general life amortization*

The *Excel* instructions are as follows. The 1st row contains data: C1: 95000; F1: 0.045; I1: 10. The 2nd row is for column titles. The values for the year z are in the row z+3 and are as follows:

column A (time). A3: 0 ; A4: = A3+1; copy A4, then paste on A5 to A13;
column B (l_{42+z}). from B3 to B13: demographic data $l_{42},...,l_{52}$
column C (E_{42+z}). C3: = B4*(1/(1+F$1))/B3; copy C3, then paste on C4 to C12;
column D ($\ddot{c}_z$). from D3 to D12; principal repayments; D14:= SUM(D3:D12) (= C1 to control);
column E (B_z). E3: 0; E4: = E3+D3; copy E4, then paste on E5 to E13;
column F (D_z). F3: = C1; F4: = F3-D3; copy F4, then paste on F5 to F13;
column G ($\ddot{j}_z$). G3: = (1-C3)*F4; copy G3, then paste on G4 to G12;
column H ($\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$). H3: = D3+G3; copy H3, then paste on H4 to H12;
column I ($\ddot{\alpha}_z$ from (6.30)). I3: = F3 -F4*C3; copy I3, then paste on I4 to I12.

Exercise 6.7

Calculate an advance life amortization with the same data as in Exercise 6.6 for the debt amount, length, rate and the borrower data, but with constant installments. Calculate the installment amount and make the amortization schedule.

A. The survival probability is found on suitable tables for an age of x=42. Applying the formulae in footnote 17 and in (6.31) and (6.32), and using an *Excel* spreadsheet, where the debt is in C1 and the rate is in E1, the following schedule, divided into two parts, is set up.

	Debt	= 95000	Rate	= 0.045	Length =10
z	l_{42+z}	E_{42+z}	${}_zE_{42}$	${}_za_{42}$	$_{10-z}a_{42\|z}$
0	96400	0.955230	1.000000	0.000000	8.191301
1	96228	0.955128	0.955230	1.000000	7.528342
2	96046	0.954975	0.912367	1.955230	6.835045
3	95849	0.954761	0.871288	2.867598	6.110155
4	95631	0.954486	0.831872	3.738886	5.352285
5	95386	0.954189	0.794010	4.570757	4.559820
6	95112	0.953879	0.757636	5.364768	3.730728
7	94808	0.953647	0.722693	6.122404	2.862761
8	94482	0.953302	0.689194	6.845097	1.953302
9	94123	0.953105	0.657010	7.534291	1.000000
10	93746		0.626200	8.191301	0.000000
	Installment =	11597.67			

z	D_z	c_z	j_z	*Installment control* = $c_z + j_z$
0	95000.00	7688.78	3908.89	11597.67
1	87311.22	8040.63	3557.04	11597.67
2	79270.59	8407.04	3190.63	11597.67
3	70863.55	8789.52	2808.15	11597.67
4	62074.03	9190.75	2406.92	11597.67
5	52883.28	9615.53	1982.14	11597.67
6	43267.75	10066.40	1531.27	11597.67
7	33201.35	10547.61	1050.06	11597.67
8	22653.75	11056.08	541.59	11597.67
9	11597.67	11597.67	0.00	11597.67
10	0.00			

Table 6.10. *Example of life amortization with constant installments*

The *Excel* spreadsheet is set up in two parts.

In the top part, taking into account that the first two rows are for data and column titles, the values for the year z are in row z+3. The instructions are as follows:

column A (year). A3: 0 ; A4: = A3+1; copy A4, then paste on A5 to A13;
column B (l_{42+z}). demographic data $l_{42},\ldots,l_{52}$;

column C (E_{42+z}). C3: = B4*(1/(1+$E$1))/B3; copy C3, then paste on C4 to C13;
column D ($_zE_{42}$). D3: 1 ; D4: = C3*D3 ; copy D4, then paste on D5 to D13;
column E ($_{/z}\ddot{a}_{42}$). E3: 0 ; E4: = E3+D3 ; copy E4, then paste on E5 to E13;
column F ($_{/10-z}\ddot{a}_{42+z}$). F3: = ($E$13-E3)/D3 ; copy F3, then paste on F4 to F13.

In C14 the installment is calculated according to: $\ddot{\alpha}_z = S / {}_{/n}\ddot{a}_x$, then C14: = C1/E13.

In the bottom part, row 16 is for column titles and the values for year z are in row z+17 with the following instructions:

column A(year). A17: 0 ; A18: = A17+1; copy A18, then paste on A19 to A27;
column B (D_z). B17: = $C$1*F3/$E$13 ; copy B17, then paste on B18 to B27;
column C ($\ddot{c}_z$). C17: = B17-B18 ; copy C17, then paste on C18 to C26;
C28: = SUM(C17:C26) (= C1 to control);
column D ($\ddot{j}_z$). D17: = (1-C3)*B18 ; copy D17, then paste on D18 to D26;
column F ($\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$). F17: = C17+D17 ; copy F17, then paste on F18 to F26.

6.3.2. *Periodic payments with delayed principal amounts*

The life amortization, still with advance actuarial interest payments, can also be made with *delayed principal repayments* c_z. We then have an actuarial generalization of the scheme seen in section 6.2.4, in particular of the German scheme if the installment invariance is imposed.

Easy calculations lead to the conclusion that, when we have chosen the principal repayments c_z so that their sum is equal to the initial debt $D_0 = S$, the installments that realize the equivalence have the following values:

- $\hat{\alpha}_0 = \ddot{j}_0 = (1 - E_x)D_0$
- $\hat{\alpha}_1 = c_1 + \ddot{j}_1 = D_0 - D_1 + (d + vq_{x+1})D_1 = D_0 - E_{x+1}D_1$
- $\hat{\alpha}_z = c_z + \ddot{j}_z = D_{z-1} - D_z + (d + vq_{x+z})D_z = D_{z-1} - E_{x+z}D_z ; (z = 2,...,n\text{-}1)$ [24]

24 It is soon seen that the values $\hat{\alpha}_z$ paid in z , if introduced in (6.28), satisfy it, therefore realizing the actuarial congruity of this life amortization form. In fact, considering that $\sum_{z=1}^{n} c_z = S$ and $D_n = 0$, and that ${}_hE_x\, E_{x+h} = {}_{h+1}E_x$, $\forall h \geq 0$, the following formulae:
$\sum_{z=0}^{n} \hat{\alpha}_z\, {}_zE_x = (1\text{-}E_x)D_0 + (D_0 - D_1E_{x+1})E_x + \sum_{z=2}^{n-1}(D_{z-1} - D_zE_{x+z})\,{}_zE_x + D_{n-1}\,{}_zE_x =$
$= D_0 - D_0\, E_x + D_0\, E_x - D_1\, {}_2E_x + D_1\, {}_2E_x - ... - D_{n-1}\, {}_nE_x + D_{n-1}\, {}_nE_x = D_0 = S$
are obtained .

– $\hat{\alpha}_n = c_n = D_{n-1}$

The calculation of the retro-reserves and pro-reserves in z can be undertaken immediately, analogously to what was seen in section 6.3.1.

6.3.3. *Continuous payment flow*

In sections 6.3.1 and 6.3.2 we considered life amortization in the discrete scheme of periodic payments, in particular annual, for the loan. However, theoretically, for a limit case or as an approximation of a scheme with fractional payments with high frequency, for such a operation we can adopt a continuous payment flow, generalizing to the stochastic case the scheme considered in section 6.2.6.

Using the time origin in the inception date of the loan and assuming for the life amortization a length t^*, thus the time interval of the corresponding annuity is $I(t^*) = [0,t^*]$, we indicate with $\alpha(t;x,t^*)$, or more easily $\alpha(t)$, the payment flow[25] from the borrower for the loan and assuming a demographic technical base in the continuum[26]. It is then obvious that the actuarial equivalence constraint, i.e. the congruity of $\alpha(t)$ in order to realize the life amortization of the debt, that we assume unitary, in the interval $I(t^*)$, is expressed by

$$\int_0^{t^*} \alpha(t)\,{}_t\overline{E}_x\,dt = 1 \qquad (6.28')$$

25 The symbol α used for the payment flow is chosen in analogy with $\ddot{\alpha}$ and $\hat{\alpha}$ for the discrete case, stressing therefore the dimensional difference.

26 With such an aim we consider a survival law $\{l(x)\}$ as a function of the age $x \in \Re$ where $l(x) = l(a)e^{-\int_a^x \mu(y)dy}$ and $\mu(y) = -\frac{l'(y)}{l(y)}$ is the mortality intensity in y. Thus, the continuous actuarial discount factor, which is also dependent on the intensity $\delta(t)$ of the financial exchange law, that is assumed as strongly decomposable (in particular, $\delta(t)=\delta$ constant in the exponential case) it is written:

$$ {}_h\overline{E}_x = e^{-\delta h}\, l(x+h)/l(x) = e^{-\int_0^h [\delta+\mu(x+t)]dt}$$

while its reciprocal is the continuous actuarial accumulation factor. Furthermore the IV of a unitary life annuity paid in the interval $I(t^*)$ is expressed by ${}_{/t^*}\overline{a}_x = \int_0^{t^*} e^{-\int_0^h [\delta+\mu(x+t)]dt} dh$.

which can be written

$$\int_0^{t^*} \alpha(t) e^{-\psi(t)} dt = 1 \qquad (6.28'')$$

where $\psi(t) = \int_0^t [\delta + \mu(x+\tau)] d\tau$ is the natural logarithm of the actuarial accumulation factor, δ being the intensity of the exponential financial law and resulting, obviously, in $\psi(0) = 0$.

Proceeding as in section 6.2.6, let us define the following quantities:

- $c(t)$ = amortization flow (for principal repayment) at time t;
- $j(t)$ = actuarial interest flow at time t;
- $B(t,t^*)$ = mean discharged debt at time t;
- $D(t,t^*) = 1 - B(t;\, t^*)$ = mean outstanding balance at time t;
- $A(t,t^*)$ = mean initial value of the borrower payments made from 0 to t.

The following constraints are valid:

$$\alpha(t) = c(t) + j(t) \;, \quad t \in I(t^*) \qquad (6.21')$$

$$\int_0^{t^*} c(t) dt = 1 \qquad (6.22')$$

$$B(t,t^*) = \int_0^t c(z) dz \;, \quad t \in I(t^*) \qquad (6.23')$$

$$D(t,t^*) = \int_t^{t^*} c(z) dz \;, \quad t \in I(t^*) \qquad (6.24')$$

$$j(t) = D(t,t^*) \left[\delta + \mu(x+t)\right] \;, \quad t \in I(t^*) \qquad (6.25')$$

$$A(t,t^*) = \int_0^t \alpha(z) e^{-\psi(z)} dz \;, \quad t \in I(t^*) \qquad (6.26')$$

Evaluating at time t in the actuarial sense (i.e. acting on the mean values), the value

$$M(t,t^*) = \left[1 - A(t,t^*)\right] e^{\psi(t)}$$

expresses the *retro-reserve*, while

$$W(t,t^*) = \int_t^{t^*} \alpha(z) e^{-\int_t^z [\delta + \mu(\tau)] d\tau} dz$$

expresses the *pro-reserve*. Maintaining in $t \in I(t^*)$ the bases $\{\delta, \mu(x)\}$ fixed at the inception date, we obtain

$$M(t,t^*) = W(t,t^*) = D(t,t^*) \;, \quad t \in I(t^*)$$

given that, as $\psi(z)-\psi(t)=\int_t^z[\delta+\mu(x+\tau)]d\tau$ from (6.28"), it follows that

$$[1-A(t,t^*)]e^{\psi(t)} = \int_t^{t^*}\alpha(z)e^{-\int_t^z[\delta+\mu(\tau)]d\tau}dz$$

and, furthermore, that $D(t,t^*)$ is a fair actuarial counterpart for payments in the interval (t,t^*) with flow $\alpha(z)$.[27]

The previous formulations show that with a continuous payment flow we move from the certain amortization to the life one, substituting the purely financial intensity δ with the actuarial one $\delta+\mu(x+t)$ and then the function $\varphi(t)$, defined in (6.19), with $\psi(t)$.

We obtain easy generalizations by assuming, instead of the intensity δ of the exponential financial law, the intensity $\delta(t)$ of any decomposable financial law.

6.4. Periodic funding at fixed rate

6.4.1. *Delayed payments*

We saw in section 5.1 that the final value of an annuity on the basis of a given law can be considered as the final result of a funding operation on a saving account with such a law. Let us develop here in detail such an operation considering how it is done in the most important cases, starting from that of *delayed* payments.

Let us consider a generic operation of *funding* in n periods (years) of a capital S by means of accumulation on a saving account at the per period rate i of the set of payments of amount R_h *at the end of the* h^{th} *period* (h = 1,...,n). Then in such an account a *sinking fund* is increasing.

The following constraint

$$S=\sum_{h=1}^{n}R_h(1+i)^{n-h} \tag{6.34}$$

must then be satisfied. It implies the financial equivalence between the set of supplies (h,R_h) of the investor and the dated amount (n,S) that is the result of the investment operation.

Different from the discrete amortization schemes described previously, the *principal amount* C_h is the increase of the fund at time h and then is obtained

27 In the continuous case the considerations in footnote 23 also hold if at time z the technical bases $\{\delta, l(x)\}$ different from those initially assumed for the continuous life amortization are adopted.

adding, and not subtracting, to the installment R_h the *interest amounts* I_h earned by the investor on the sinking fund in the period (h-1,h) and proportional to the amount accumulated in h-1. Indicating with G_h the level of the *sinking fund* at the integer time h on the saving account due to the operation (then $G_0 = 0$) and having fixed in advance the principal amount C_h non-negative and satisfactory, as for the amortization, the constraint of the 1st of (6.3), the following recursive relations hold:

$$\left.\begin{aligned} G_h &= G_{h-1} + C_h \\ I_h &= i\, G_{h-1} \\ R_h &= C_h - I_h \end{aligned}\right\} (h = 1,\dots,n) \tag{6.35}$$

Starting from the initial condition $G_0 = 0$, all the values $\{I_h\},\{R_h\},\{G_h\}$ are obtained and in particular, due to the 1st part of (6.3): $G_n = S$, i.e. the requested funding. In the dynamics of the operation, the fundamental recursive relation holds

$$G_h = G_{h-1}(1+i) + R_h \;\;,\;\; (h = 1,\dots,n) \tag{6.36}$$

and can be written as

$$R_h = (G_h - G_{h-1}) - i\, G_{h-1} \;\;,\;\; (h = 1,\dots,n) \tag{6.36'}$$

The *retro-reserve* $M(h;i)$ and the *pro-reserve* $W(h;i)$ (at credit for the investor) of the operation, at time h and at rate i (the rate chosen at the beginning or adjusted in h) are given by the expression

$$\begin{aligned} M(h;i) &= \sum_{s=1}^{h} R_s (1+i)^{h-s} \\ W(h;i) &= S(1+i)^{-(n-h)} - \sum_{s=h+1}^{n} R_s (1+i)^{-(s-h)} \end{aligned} \tag{6.37}$$

and, if i is the rate of the law initially adopted for the funding, the result is

$$M(h;i) = W(h;i) = G_h \;\; ^{28} \tag{6.38}$$

As in gradual amortization, if the CCI regime is adopted, the reserves in each intermediate time between consecutive payments can be defined (for example to calculate exactly the assignment value of the credit) at each time $t=k+s$ (where k = integer part of t; s = decimal part of t). We find

$$M(t;i) = M(k;i)\,(1+i)^s \;\;;\;\; W(t;i) = W(k;i)\,(1+i)^s \tag{6.39}$$

28 Also for the funding, the considerations for retro-reserve and pro-reserve found in footnote 16 are extended if at time k different rates are adopted from the one initially chosen.

By varying t in the real numbers between 0 and n, in $[0,n]$ we obtain two functions, M and W, coincident if the funding operation between 0 and n is fair, discontinuous (right-continuous) at integer time k.

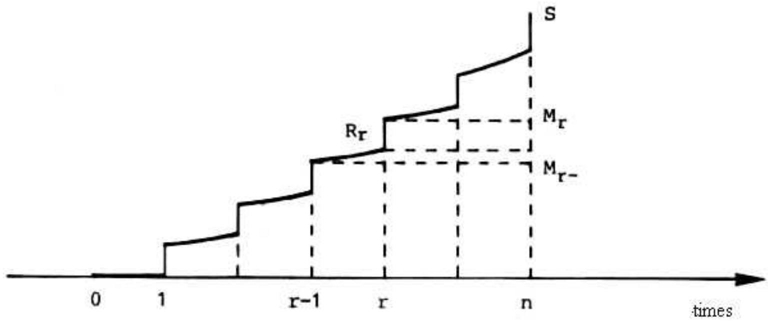

Figure 6.3. *Plot of delayed funding*

If the funding is made with constant delayed payments $R_h = R$[29], all the relations are adopted with this position. In particular, the equivalence constraint between S and R is given by

$$S = R\, s_{\overline{n}|i} \quad \text{or} \quad R = S\, \sigma_{\overline{n}|i} \tag{6.40}$$

By adopting (6.36), and using it for consecutive values of h and subtracting, it is verified that, as in French amortization, the principal amount changes in geometric progression with ratio (1+i), resulting in

$$C_h = R(1+i)^{h-1} \; ; \; G_h = R s_{\overline{h}|i} = S \frac{s_{\overline{h}|i}}{s_{\overline{n}|i}} \tag{6.41}$$

The retro-reserve and the pro-reserve in h at rate i are expressed by

$$M(h;i) = R\, s_{\overline{h}|i} \; ; \; V(h;i) = S(1+i)^{-(n-h)} - R\, s_{\overline{n-h}|i} \tag{6.37'}$$

29 An example of term funding by means of delayed constant periodic payments has been encountered in the American amortization considered in section 6.2.5.

Exercise 6.8

We have to form a capital at maturity of €25,500 in 5 years on a saving account at the annual delayed rate of 5.25%, with annual delayed payments corresponding to the following sequence of principal repayments, the sum of which is 25,500:

$$C_1 = 4{,}500\ ;\ C_2 = 5{,}300\ ;\ C_3 = 5{,}600\ ;\ C_4 = 6{,}000\ ;\ C_5 = 4{,}100.$$

Calculate the funding schedule.

A. Applying (6.35) on an Excel spreadsheet, from the given value $\{C_h\}$ in the 2nd column are found the end of year balances $\{G_h\}$; from here we find the earned interest $\{I_h\}$ and the installments $\{R_h\}$ to be paid by the investor. The following schedule is obtained.

DELAYED FUNDING WITH GIVEN PRINCIPAL AMOUNTS

Capital = 25,500 Rate = 0.0525

h	*Ch*	*Gh*	*Ih*	*Rh*
1	4,500.00	4,500.00	0.00	4,500.00
2	5,300.00	9,800.00	236.25	5,063.75
3	5,600.00	15,400.00	514.50	5,085.50
4	6,000.00	21,400.00	808,50	5,191.50
5	4,100.00	25,500.00	1123,50	2,976.50

Table 6.11. *Example of delayed funding*

The *Excel* instructions are as follows: the 1st, 2nd and 4th rows are for data and titles: B2: 25500; D2: 0.0525; the 3rd and 5th rows are empty. Starting from the 6th row:

column A (year): A6: 1; A7:= A6+1; copy A7, then paste on A8 to A10;
column B (principal amount): insert data on B6 to B10; sum in: B2);
column C (accumulated amount): C6:= C5+B6; copy C6, then paste on C7 to C10;
column D (interest amount): D6:= C5*D$2; copy D6, then paste on D7 to D10;
column E (installment): E6:= B6-D6; copy E6, then paste on E7 to E10.

Exercise 6.9

With the same data as exercise 6.8 for the amount at maturity, for the length and the rate, calculate the funding schedule imposing the installments invariance.

A. By applying (6.35) and the 2nd part of (6.40), and by using an *Excel* spreadsheet the following schedule is found.

DELAYED FUNDING WITH CONSTANT INSTALLMENT

Capital = 25,500 Rate = 0.0525

Years = 5 Installment = 4,591.87

h	*Ch*	*Ih*	*Gh*
1	4,591.87	0.00	4,591.87
2	4,832.94	241.07	9,424.81
3	5,086.67	494.80	14,511.48
4	5,353.72	761.85	19,865.21
5	5,634.79	1,042.92	25,500.00

Table 6.12. *Example of delayed funding*

The *Excel* instructions are as follows. Rows 1, 2, 3 and 5 are for data, titles and one calculation: B2: 25,500; D2: 0.0525; B3: 5; D3:= B2*D2/((1+D2)^B3-1); rows 4 and 6 are empty. From row 7:

column A (year):	A7:= A6+1; copy A7, then paste on A8-A11;
column B (principal amounts):	B7:= D$3*(1+D$2)^A6; copy B7, then paste on B8 to B11;
column C (interest amounts):	C7:= B7-D$3; copy C7, then paste on C8 to C11;
column D (sinking fund):	D7:= D6+B7; copy D7, then paste on D8 to D11.

6.4.2. *Advance payments*

Let us consider briefly the variations in relation to section 6.4.1 when the payments, indicated using $\ddot{R}_h$, are made at integer time h referring to the period $(h,h+1)$, $(h = 0, 1,..., n\text{-}1)$, and therefore are called *advance payments*. The closure constraint with the amount S to be formed at time n becomes

$$S = \sum_{h=0}^{n-1} \ddot{R}_h (1+i)^{n-h} \tag{6.42}$$

The recursive relations regarding the accumulated capitals G_h at time h, the principal amounts $\ddot{C}_h$ subject, as for the amortization, to the 2nd of (6.3), the interest amounts $\ddot{I}_h$ and the installments $\ddot{R}_h$, starting from the initial condition $G_0 = 0$, are:

$$\left.\begin{aligned} G_{h+1} &= G_h + \ddot{C}_h \\ \ddot{I}_h &= d\ G_{h+1} \\ \ddot{R}_h &= \ddot{C}_h - \ddot{I}_h \end{aligned}\right\} (h = 0,...,n-1) \tag{6.43}$$

where $d=i/(1+i)$ (obtaining, in particular, $G_n = S$) and the recursive relation on the accumulated amount is found to be

$$G_{h+1} = (G_h + \ddot{R}_h)(1+i) \quad , \quad (h = 0,...,n\text{-}1) \tag{6.44}$$

and the decomposition is found to be

$$\ddot{R}_h = (G_{h+1} - G_h) - d\ G_{h+1} \quad , \quad (h\text{=}0,...,n\text{-}1) \tag{6.44'}$$

For the retro-reserve $M(h;i)$ and the pro-reserve $W(h;i)$ at time h the following expressions hold:

$$M(h;i) = \sum\nolimits_{s=0}^{h-1} \ddot{R}_s (1+i)^{h-s}$$

$$W(h;i) = S(1+i)^{-(n-h)} - \sum\nolimits_{s=h}^{n-1} \ddot{R}_s (1+i)^{-(s-h)} \tag{6.45}$$

which are equal to each other and to G_h if i is the rate initially adopted for the funding.

If the CCI regime is adopted, in the advance case we can also define the reserves in whichever non-integer time $t=k+s$ (where k = integer part of t; s = decimal part of t), resulting in

$$M(t;i) = M(k+1;i)\,(1+i)^{-(1-s)} \ ; \ W(t;i) = W(k+1;i)\,(1+i)^{-(1-s)} \tag{6.39'}$$

By varying t in the real numbers between 0 and n we obtain in $(0,n)$ two functions, M and W, coincident if the funding operation between 0 and n is fair, discontinuous (continuous to left) at the integer time k.

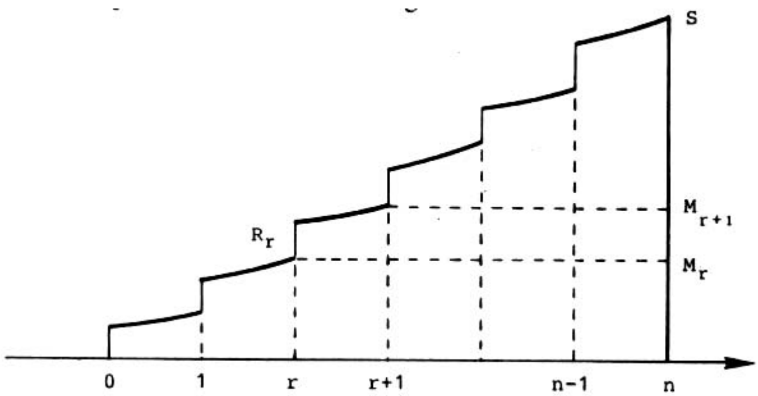

Figure 6.4. *Plot of advance funding*

In the case of constant advance payments it is enough to put $\ddot{R}_h = \ddot{R}$ constant in the previous formulation. The following is then obtained:

$$\ddot{S} = \ddot{R}\ \ddot{s}_{\overline{n}|i} \qquad \text{or} \qquad \ddot{R} = S\ \ddot{\sigma}_{\overline{n}|i} \tag{6.40'}$$

and, with $G_0 = 0$:

$$G_{h+1} = (G_h + \ddot{R})(1+i)\ \ ,\ \ (h=0,...,n-1) \tag{6.46}$$

from which

$$\ddot{R} = (G_{h+1} - G_h) - d\ G_{h+1},\ \ (h=0,...,n-1) \tag{6.46'}$$

Also in this case the principal amount varies in geometric progression with ratio $(1+i)$, resulting in:

$$\ddot{C}_h = \ddot{R}(1+i)^{h+1}\ \ ;\ \ G_h = \ddot{R}\ \ddot{s}_{\overline{h}|i} = S\frac{\ddot{s}_{\overline{h}|i}}{\ddot{s}_{\overline{n}|i}} \tag{6.47}$$

The retro-reserve and pro-reserve in h are

$$M(h;i) = \ddot{R}\ \ddot{s}_{\overline{n}|i}\ \ ;\ \ W(h;i) = S(1+i)^{-(n-h)} - \ddot{R}\ s_{\overline{n-h}|i} \tag{6.45'}$$

which are equal to each other and to G_h if i is the initially adopted rate for the funding.

Exercise 6.10

With the same data as in Exercise 6.8 for the capital at maturity, for the length and the rate, calculate the advance funding schedule imposing the invariance of installments.

A. Applying (6.43) and the 2nd of (6.40') we obtain on an *Excel* spreadsheet the following schedule.

ADVANCE FUNDING WITH CONSTANT INSTALLMENT

Capital = 25,500 Delayed rate = 0.0525

Years = 5 Installment = 4,362.82

h	*Ch*	*Ih*	*Gh*
0	4,591.87	229.05	0.00
1	4,832.94	470.12	4,591.87
2	5,086.67	723.85	9,424.81
3	5,353.72	990.90	14,511.48
4	5,634.79	1,271.97	19,865.21
5	0.00	0.00	25,500.00

Table 6.13. *Example of advance funding*

The *Excel* instructions are as follows: the first 3 row and the 5th row are for data, titles and one calculation:
B2: 25500; D2: 0.0525; B3: 5; D3:= B2*D2/(1+D2)/((1+D2)^B3-1);
the 4th row is empty; from the 6th row:
column A (year): A6: 0; A7:= A6+1; copy A7, then paste on A8 to A11;
column B (principal amount): B6:= D$3*(1+D$2)^A7; copy B6, then paste on B to B10; B11: 0;
column C (interest amount): C6:= B6-D$3; copy C6, then paste on C7 to C10 C11: 0;
column D (sinking fund): D6: 0; D7:= D6+B6; copy D7, then paste on D8 to D11.

6.4.3. *Continuous payments*

Analogous to the continuous amortization scheme (see section 6.2.6) is that of the certain funding[30] of a capital *S* by means of an continuous annuity with flow $\sigma(t)$ in the time interval $I(t_1)$ from 0 to t_1 .

30 Together with the classification of amortizations and for reasons of completeness we should briefly mention the funding by means of payments that are conditioned to an investor's survival, i.e. by a life annuity. However, it is evident such a scheme coincides with that of life

This issue has been already discussed in general terms in Chapter 4, where the value of the accumulated amount in (4.14') has been found as the solution of the differential equation (4.13) in which the flow that leads to the variation of the accumulated amount is the sum of the interest flow and of the increasing flow for the net payment -φ(t) (negative from the viewpoint of the cash) to the fund to be formed. It will be enough to mention it briefly in order to highlight conditions by which a payment flow $\sigma(t) = -\varphi(t)$ is used to form in t_1 a capital S. Let us assume for simplicity S=1 and a financial low strongly decomposable with intensity $\delta(t)$ ($\delta(t) = \delta$ constant if the low is exponential). Using:

$$\chi(t) = \int_t^{t_1} \delta(z)dz \ , \ t \in I(t_1) \tag{6.48}$$

to form the unitary capital at time t_1 the flow σ(t) varying in $I(t_1)$ must satisfy the constraint of financial closure:

$$\int_0^{t_1} \sigma(z)e^{\chi(z)}dz = 1 \tag{6.49}$$

If $\sigma(t) = \sigma$ constant and $\delta(t) = \delta$ constant, due to (5.16) and (6.49),

$$\sigma = 1/s_{\bar{t}_1|i}^{(\infty)} \tag{6.49'}$$

must hold, extending the meaning of the symbol $s_{\bar{t}_1|i}^{(\infty)}$ if t_1 is not an integer.

Using:

$G(t)$ = sinking fund formed in t;
$c(t)$ = flow in t of variation of the sinking fund;
$j(t)$ = flow in t of interest (received for the investor);

the following recursive relations hold, starting from $G(0)$=0

$$\left.\begin{aligned} j(t) &= \delta(t)G(t) \\ c(t) &= j(t) + \sigma(t) \\ \int_0^t c(z)dz &= G(t) \end{aligned}\right\} \forall t \in I(t_1) \tag{6.50}$$

In the further hypothesis of constant payment flows, it is possible to extend (5.9) to the continuous scheme. Considering (6.20') and (6.49') and also the relation $1/s_{\bar{t}_1|i}^{(\infty)} + \delta = 1/a_{\bar{t}_1|i}^{(\infty)}$, that can be immediately verified, we find:

$$\sigma + \delta = \alpha \tag{6.51}$$

---

insurances with endowment (temporary or perpetual) policies. Then for life funding it is enough to refer to a treatise on life insurances.

This relationship links the constant flows of unitary amortization and funding (then intensities from the dimensional point of view, having divided the flows by the amount *S*) in operations of the same length in an exponential regime.

6.5. Amortizations with adjustment of rates and values

6.5.1. *Amortizations with adjustable rate*

For the reasons explained in Chapter 1, the quantifications discussed so far consider monetary amounts. This is not only for homogenization of values, but it can be used to settle obligations because money is the legal measure of wealth.

The phenomenon of monetary inflation or other causes that lead to variations (more often a decrement) of the purchasing power of money, which is now no longer linked to gold or any other assets with stable and intrinsic value, is more and more widespread in the presence of macroeconomic imbalances.

Due to this phenomenon, loan operations and the following amortization, fair in monetary terms at a given rate, are not fair in real terms, i.e. considering the purchasing power of the traded sums. Then the receiver of the sums with future maturity is substantially damaged if the variation of the purchasing power is a decrement. Therefore, in recent times, which are characterized by permanent inflation, financial schemes for amortization have been developed which are used to correct its distorting effects by means of opportune variations in the aforementioned methods. Such schemes are not only useful to neutralize these negative effects for the investor, of monetary depreciation, but more generally are used to reduce the risk of oscillation of the financial market in both directions.

The *first variation* consists of making the rate fluctuate up and down, adjusting it to the current rate for new operations in the financial market, without changing the outstanding loan balance. With this procedure the interest amount of one period is calculated by multiplying per period the updated rate by the outstanding balance at the beginning of the period.

Limiting ourselves to the delayed installment case, let us consider two forms of amortization with adjustable rate, highlighting that the rate variations are not known at the beginning but are fixed in the h^{th} period in relation with the aforementioned phenomena, regarding the inflation and the following depreciation of money. Therefore, it is not possible to fix at the inception date of the loan the effective amortization plan that will be adopted.

a) French amortization with adjustable rate

In this form, we proceed initially with the progressive method described in section 6.2.2, calculating the installments by means of (6.8). The installments remain unchanged for the following periods if the rate is not adjusted, but, in the case of variations, new installments are calculated, using the adjusted rate, the outstanding loan balance and the remaining time, on the basis of (6.8).

In formulae, indicating with $i^{(1)},\ldots,\ i^{(n)}$ the rates (not necessarily different) that will be applied in the subsequent periods 1,..., *n,* the installments and the outstanding balances of each period are obtained recursively from the following equation system (where D_0=S)

$$(h=1,\ldots,n)\begin{cases} R_h = D_{h-1}\ \alpha_{\overline{n-h+1}|i^{(h)}} \\ D_h = R_h\ a_{\overline{n-h}|i^{(h)}} \end{cases} \tag{6.52}$$

Obviously the interest payments and the principal repayments are calculated using

$$I_h = D_{h\text{-}1}\, i^{(h)}\ ;\ C_h = R_h - I_h = D_{h\text{-}1} - D_h \tag{6.52'}$$

From (6.52) it follows that the installments remain unchanged between two subsequent rate variations; furthermore the installment variations are concordant to the rate variation, if it changes. To prove this statement, we can observe that the recursive relation

$$R_{h+1} = R_h \frac{a_{\overline{n-h}|i^{(h)}}}{a_{\overline{n-h}|i^{(h+1)}}}\ ,\ h = 1,\ldots.,n\text{-}1 \tag{6.52''}$$

on the installments follows from (6.52), and that $a_{\overline{m}|i}$ is a decreasing function o rate i. In addition, the principal repaid in h+1 is

$$C_{h+1} = D_h \sigma_{\overline{n-h}|i^{(h+1)}} = R_{h+1}(1+i^{(h+1)})^{-(n-h)} \tag{6.53}$$

and $\sigma_{\overline{n}|i}$ decreases with the rate. Therefore, the variation of the principal repayment due to the rate variation is discordant to it; the result is that a rate increment slows down the amortization, giving rise to higher outstanding balances and higher installments than those in the absence of adjustments, even if the “closure” remains unchanged, i.e. the debt becomes zero at the end of the loan.

b) Amortizations with adjustable rate and prefixed principal amount

In the previous form of amortization, a) in the case of rate adjustments there is a novation of the contract on the outstanding loan balance and remaining length, such that with respect to the progressive scheme at fixed rate not only are the sequences of interest payments subject to variations, but also those of principal repayments and then of the outstanding balances.

Or it can be agreed that the principal repaid remains unchanged in case of adjustment of the contractual rate, so as to eliminate the uncertainty of the principal repayments and to reduce that of the interest payments, obtained multiplying the rate, varying with h in a way not previously foreseen, for the prefixed outstanding loan balances. Thus we lead back to the recursive system (6.4'), modified to take into account the rate variability in the period h, i.e.

$$(h=1,\dots,n)\begin{cases} D_h = D_{h-1} - C_h \\ I_h = i^{(h)} D_{h-1} \\ R_h = C_h + I_h \end{cases} \qquad (6.4''')$$

which, using D_0=S, enables the calculation of the interest payments, the installments and the outstanding loan balances in the following periods.

Example 6.3

This example clarifies the comparison, set out in the following table, between the amortizations in 5 years of the amount S = €100,000 in the three different forms:

1) "French" at rate i = 0.05 that gives the delayed constant installment R = 23,097.48;

2) form a) with rates $i^{(h)}$ specified in the table;

3) form b) with the same $i^{(h)}$ and constant principal amount C_h = 20,000.00.

	French		form a)			form b)	
h	D_h	$i^{(h)}$	R_h	D_h	I_h	R_h	D_h
1	81,902.52	0.05	23,097.48	81,902.52	5,000.00	25,000.00	80,000.00
2	62,900.16	0.07	24,179.93	63,455.77	5,600.00	25,600.00	60,000.00
3	42,947.69	0.07	24,179.93	43,717.75	4,200.00	24,200.00	40,000.00
4	21,997.60	0.05	23,511.62	22,392.02	2,000.00	22,000.00	20,000.00
5	0.00	0.05	23,511.62	0.00	1000.00	21,000.00	0.00

Table 6.14. *Comparison of different amortization rules*

It can be seen that in form a) the installment of the 1st year coincides with the installment R of the French amortization at rate 5% but in the 4th year, after two years of increasing rates, even if the rate returned back to the initial level, due to the higher outstanding balance, $R_4 > R$ results. Thus, with $i^{(5)} = i^{(4)}$ we have $R_5 = R_4$.

6.5.2. *Amortizations with adjustment of the outstanding loan balance*

The rate adjustment considered in section 6.5.1 solves, in an approximated manner, the problem of money depreciation (or, more generally, of the variation of purchasing power of money) because it acts only by an additive variation of rate which does not exactly reflects Fisher's equation. A procedure to fully solve this problem is that of *indexation* of the prefixed outstanding loan balances, obtained by multiplying such balances by coefficients derived from a series of statistical indices measuring the mean prices varying with the same periodicity as the redemption payments.

In such a way, the installments and their components for interest and amortization, that are proportional to the outstanding balances, will be modified multiplicatively according to the same coefficients, where the constraint of elementary closure, which assumes the non-modifiability of the principal payments and thus of the outstanding balances, is not satisfied.

Let us formalize the procedure, limiting ourselves to the adjustment of the French amortization[31] of the landed amount $S = D_b$ at time b in n periods at the per period rate i by means of installments that, if the index remains constant, would all assume the value $R = S\, i/(1-(1+i)^{-n})$.

Let $\{Z_h\}$, $(h = b, b+1,...,b+n-1)$ be, the series of statistical indices needed for the adjustment in n periods, with the same periodicity of payments. The updating coefficient between time h and h+1 is $K_{h+1} = Z_{h+1}/Z_h = 1+p_{h+1}$, where p_{h+1} is the corresponding per period updating rate; therefore

$$\pi_b = 1 \;\; ; \;\; \pi_h = \prod_{j=b+1}^{h} K_j = Z_h / Z_b \; ; \;\; (h = b+1,...,b+n-1) \qquad (6.54)$$

are the global updating factors for h-b periods to be used in the calculations. In the absence of adjustments the outstanding loan balances at time r would be

$$D_h = R\, a_{\overline{b+n-h}|i} \qquad (6.55)$$

while, due to what has been said, the updating modifies the sequence $\{D_h\}$ in $\{D'_h\}$ defined by

$$D'_h = D_h\, \pi_h \qquad (6.56)$$

31 The same conclusions hold with different amortization schemes that give rise to any development of the outstanding balances before the updating.

It is clear that $b+n-h$ delayed payments of constant amount R'_{h+1} (= updated installment of the period h+1) would amortize D'_h in the absence of further updating. Thus $D'_h = R'_{h+1}\, a_{\overline{b+n-h}|i}$ and then due to (6.22): $D'_h/D_h = R'_{h+1}/R = \pi_h$, which can be written

$$R'_{h+1} = R\,\pi_h \qquad (6.56')$$

Proceeding analogously, the updated interest paid is

$$I'_{h+1} = i\,D'_h = i\,D_h\,\pi_h = I_{h+1}\,\pi_h \qquad (6.57)$$

and subtracting (6.24) from (6.23') it is obtained for the updated principal repaid

$$C'_{h+1} = (R - I_h)\,\pi_h = C_{h+1}\,\pi_h \qquad (6.57')$$

Briefly, the outstanding loan balance after h-b periods from the inception and also the installment paid at the end of the period, i.e. at time h+1, and its principal and interest components are updated by means of the factor π_h given by (6.21).

Exercise 6.11

Amortize in 5 years the amount €80,000 loaned at time 6 at the annual rate of 4.5% with value adjustments according to the index $\{Z_h\}$, (h = 6, 7, 8, 9, 10), of the "cost of life" on the basis of the observed values, specified in Table 6.15.

A. On the basis of the data and using: $D_h = D_{h-1} - C_h$, the following amortization schedule is obtained, that compares the non-updated values of the French amortization and the updated values in the outstanding loan balances on the basis of $\{Z_h\}$. By using S=€80,000; n=5; i=0.045; R=€18,223.33, the following data is obtained (rounding off €amounts to no decimal-digit).

S = 80000		Start = 6		Length = 5		Rate = 0.045		Payment = 18223	
h	Z_h	π_h	I_h	I'_h	C_h	C'_h	R'_h	D_h	D'_h
6	120.0	1.0000						80,000	80,000
7	122.5	1.0208	3,600	3,600	14,623	14,623	18,223	65,377	66,739
8	125.7	1.0475	2,942	3,003	15,281	15,600	18,603	50,095	52,475
9	129.6	1.0800	2,254	2,361	15,969	16,728	19,089	34,126	36,856
10	133.2	1.1100	1,536	1,659	16,688	18,023	19,681	17,439	19,357
11			785	871	17,439	19,357	20,228	0	0

Table 6.15. *Amortizations with adjustment of the outstanding loan balance*

The *Excel* instructions are as follows. The first 2 rows are for data, titles and one calculation: B1: 80000; D1: 6; F1: 5; H1: 0.045; J1:= (B1*H1)/(1-(1+H1)^-F1). From the 3rd row:

column A (time):	A3:= D1; A4:= A3+1; copy A4, then paste on A5 to A(3+F1);
column B (Z_h):	from B3 to B7 insert periodic index numbers for F1 periods;
column C (π_h):	C3: 1; C4:= B4/B$3; copy C4, then paste on C5 to C7;
column D (I_h):	D4:= I3*H$1; copy D4, then paste on D5 to D8;
column E (I'_h):	E4:= D4*C3; copy E4, then paste on E5 to E8;
column F (C_h):	F4:= J$1-D4; copy F4, then paste on F5 to F8;
column G (C'_h):	G4:= F4*C3; copy G4, then paste on G5 to G8;
column H (R'_h):	H4:= E4+G4 (or := J$1*C3); copy H4, then paste on H5 to H8;
column I (D_h):	I3:= B1; I4:= I3-F4; copy I4, then paste on I5 to I8;
column J (D'_h):	J3:= I3*C3; copy J3-paste on J4 to J8.

6.6. Valuation of reserves in unshared loans

6.6.1. *General aspects*

The valuation of the pro-reserve $W(t,i*)$ at a given time t of a financial operation, obtained by discounting the supplies after t on the basis of a prefixed law, in particular the exponential one at a valuation rate $i*$ generally different from the contractual rate i originally agreed for the calculation of interest (because can be different the valuation time, the evaluating subject, the aims and the market conditions), is often interesting. We have such valuations when a company balance is prepared for internal or external/official use, or for the assignment of credits or for the carrying of debts regarding the operation.

We will consider the calculation of the pro-reserve and its components in relation to the gradual amortization of a debt during its development (or sometimes at the inception date). Using periodic then discrete payments, we can assume the conjugate of a DCI. law. We will complete this consideration with the development of the so-called *Makeham's formula* and the calculation of the usufruct in the discrete scheme, using any valuation rate $i*$, for the most important amortization methods.

In a gradual amortization with n periodic installments R_k delayed and varying, of the type seen in section 6.2.1 (with simple variations for the advance case) assuming a unitary period, the pro-reserve $W(t,i*)$ is the current value in t of the installments R_k with due dates $k \geq t$; it is equal to the outstanding loan balance $D_h = \sum_{k=h+1}^{n} R_k (1+i)^{-(k-h)}$ if $i* = i$ and $t = h$. If there is a need to distinguish between the value of the principal repayments C_k and that of the interest payments I_k (e.g. because the creditor of interest is different from that of the principal), we will have to evaluate separately at the rate $i*$, the usufruct $U(t,i*)$ and the bare ownership $P(t,i*)$, the sum of which is $W(t,i*)$[32].

Let us consider the position $t = h$ evaluating at integer time h the pro-reserve and its components *usufruct and bare ownership in the discrete*, then the present value of the interest payments and the principal repayments, at an evaluation rate $i*$[33].

In formulae with already defined symbols,

$$(h=1,...,n)\begin{cases} W_h^* = W(h,i*) = \sum_{k=h+1}^{n} R_k (1+i*)^{-(k-h)} \\ U_h^* = U(h,i*) = \sum_{k=h+1}^{n} I_k (1+i*)^{-(k-h)} \\ P_h^* = P(h,i*) = \sum_{k=h+1}^{n} C_k (1+i*)^{-(k-h)} \end{cases} \quad [34] \tag{6.58}$$

obtaining $W_h = D_h$, U_h , P_h as particular values when $i*=i$.

6.6.2. *Makeham's formula*

The *additivity* expressed by

$$W_h^* = U_h^* + P_h^* \ ; \ W_h = U_h + P_h \ ; \ (h = 1,...,n) \tag{6.59}$$

is obvious (and it has already been found).

The following *Makeham's formula*, which links values at rate $i*$ to those at the contractual rate i, also holds:

$$W_h^* = P_h^* + \frac{i}{i*}(D_h - P_h^*) \tag{6.60}$$

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32 The examined valuation is apparently an operation with two rates, i and $i*$, but looking at it more closely, the only rate $i*$ is used as a variable with the meaning of discount rate of the amounts – principal repayments, interest payments, installments, etc. – that at the valuation time are already fixed as a function of the original data, between which there is the repayment rate i.

33 They will be initial values if $h= 0$, residual values if $h = 1,…,n$.

34 The values for non-integer time t in exponential regime, using $t=h+s$ $(0<s<1)$, are obtained from those in (6.25) multiplying by $(1+i)^s$.

and from which, due to (6.59), the following expression to evaluate U_h^* as a function of P_h^* is found:[35]

$$U_h^* = \frac{i}{i*}(D_h - P_h^*) \tag{6.60'}$$

Proofs of Makeham's formula

1) A brief proof of Makeham's formula based on the equivalence at the rate $i*$ can be given. It is enough to observe that at the rate i the debt D_h is amortized with installments $R_s = C_s + I_s$ ($s = h+1, ...,n$), i.e. it is fair to exchange D_h with the installments R_s, while at rate $i*$, if the principal amounts C_s and then the outstanding loan balances D_s remain unchanged, to preserve the equivalence the interest payments must be: $I_s^* = i * D_{s-1} = I_s\, i*/i$, i.e. at rate $i*$ it is fair to exchange D_h with the installments $R_s^* = C_s + (i*/i)I_s$. It then follows that:

$$D_h = \sum_{s=h+1}^{n} R_s^*(1+i*)^{-(s-h)} = \sum_{s=h+1}^{n} C_s(1+i^*)^{-(s-h)} + \frac{i^*}{i}\sum_{s=h+1}^{n} I_s(1+i*)^{-(s-h)}$$

or, due to (6.58),

$$D_h = P_h^* + \frac{i*}{i}U_h^* \tag{6.61}$$

from which we obtain (6.60') and (6.60). ☼

2) Due to the closure equation, it follows that $D_h = \sum_{k=1}^{n-h} C_{h+k}$, i.e. the outstanding loan balance D_h at time h is decomposed in subsequent principal repayments C_{h+k}, (k =1,..,n-h), each of which leads to its refund after k years and the payment of interest iC_{h+k} for k years. The overall valuation in h of these obligations at rate $i*$ is W_h^*. Therefore, using $v_*^k = (1+i*)^{-k}$ the following is obtained

35 By adding and subtracting D_h in the 2nd part of (6.27), Makeham's formula becomes: $W_h^* = P_h^* + \frac{i}{i*}(D_h - P_h^*)$, which highlights the decreasing of W_h^* with respect to $i*$ (then the convenience for the debtor, that assigns the debt during the amortization, to evaluate it at the highest possible rate) and gives a measure of the spread between the valuation at rate $i*$ and that at rate i of the future obligation of the debtor, as $(W_h^* - D_h)$ has the sign of (i-$i*$). If , in particular, h=0, it is sufficient to put in the formulae D_h= S = landed capital, to evaluate the obligations at any rate since from inception. Given the biunivocity of the relations, we can exchange the role between $i*$ and W_h^*, assuming the value W_h^* fixed by the market and obtaining $i*$ that takes the meaning of internal rate of return (IRR).

$$W_h^* = \sum_{k=1}^{n-h} C_{h+k}(v_*^k + i\, a_{\overline{k}|i^*}) = \sum_{k=1}^{n-h} C_{h+k}\left[v_*^k + \frac{i}{i^*}(1-v_*^k)\right] = P_h^* + \frac{i}{i^*}(D_h - P_h^*)$$

i.e. (6.61), from which we obtain (6.60') and (6.60).

3) A purely analytical proof of Makeham's formula is obtained by applying Dirichlet's formula, i.e. summing by columns instead of by rows the elements $m_{s,k} = C_k(1+i^*)^{-(s-h)}$ of a triangular matrix. It follows that

$$U_h^* = \sum_{s=h+1}^{n} I_s(1+i^*)^{-(s-h)} = i\sum_{s=h+1}^{n} D_{s-1}(1+i^*)^{-(s-h)} =$$

$$= i\sum_{s=h+1}^{n}\sum_{k=s}^{n} C_k(1+i^*)^{-(s-h)} = i\sum_{k=h+1}^{n} C_k \sum_{s=h+1}^{n}(1+i^*)^{-(s-h)} =$$

$$= \frac{i}{i^*}\sum_{k=h+1}^{n} C_k\left[1-(1+i^*)^{-(k-h)}\right] = \frac{i}{i^*}(D_h - P_h^*)$$

i.e. we obtain (6.60') and (6.60). ☼

Observations

1) By adding and subtracting D_h on the right side of (6.60) Makeham's formula becomes:

$$W_h^* = D_h - \frac{i^* - i}{i^*}(D_h - P_h^*) \tag{6.60''}$$

which, as $P_h^* < D_h$, highlights the increasing of W_h^* with respect to i^*. Thus, W_h^* is the assignment value of the residual credit of the lender at the integer time h (then the debtor that assigns the debt during the amortization has the convenience of evaluating at the highest possible rate) and gives a measure of the spread between the valuation W_h^* of the outstanding loan balance and its *nominal value* D_h if $i^* \neq i$, because we obtain $W_h^* < D_h$ or $W_h^* > D_h$ if $i^* > i$ or $i^* < i$ respectively. If h=0, it is sufficient to use $D_h = S$ in (6.60'').

2) Given the biunivocity of the relations, we can exchange in (6.60) or in its transforms the roles of i^* and W_h^*, assuming the latter as the value given exogenously by the market laws and obtaining i^* that assumes the meaning of return rate for the investor lender or cost rate for the financed borrower (see section 4.4.1).

3) A recurrent relation analogous to (6.6) also holds for W_h^*. In fact, as it is easily verifiable, it results in:

$$W_h^* = W_{h-1}^*(1+i^*) - R_h \tag{6.62}$$

4) New expressions of U_h^* and P_h^* are obtained by considering the variation of W_h^* due to that of the rate $i*$ and finding U_h^* and P_h^* from the system of equations (6.59) and (6.60') with $W_h = D_h$. This results in:

$$U_h^* = -i\,\frac{W_h^* - W_h}{i* - i} \quad , \quad P_h^* = \frac{i*W_h^* - i\,W_h}{i* - i} \tag{6.63}$$

and therefore U_h^* is the partial difference quotient of W_h^* in the variation from i to $i*$ multiplied by $-i$, while P_h^* is the partial difference quotient of $i^*W_h^*$ in the same variation. Taking the limit for $i* \to i$ on the differentiable functions W_h^* and $i^*W_h^*$, we obtain the following result

$$U_h = \lim_{i*\to i} U_h^* = -i\left(\frac{\partial W_h^*}{\partial i^*}\right)_{i*=i} \quad , \quad P_h = \lim_{i*\to i} P_h^* = \left(\frac{\partial (i^* W_h^*)}{\partial i^*}\right)_{i*=i} \tag{6.64}$$

6.6.3. *Usufructs and bare ownership valuation for some amortization forms*

In the concrete case of amortization, we are also interested in the valuation of the residual installments and their components for interest and for amortization at any rate $i*$ and at any time $t=h+s \in \Re$, with $0<s<1$, to which the additivity, espressed by (6.59), is extended. Let us note that, given the delayed or advance payments at integer times h, we obtain (see footnote 34):

$$\begin{cases} W(t,i*) = W(h,i*)(1+i*)^s, & \text{with delayed payments} \\ W(t,i*) = W(h+1,i*)(1+i*)^{-(1-s)}, & \text{with advance payments} \end{cases} \tag{6.65}$$

using analogous formulae for $U(t,i*)$ and $P(t,i*)$.

We can then limit ourselves to the calculation for integer time h, making explicit the valuations of usufruct and bare ownership (from which summing we find the pro-reserves) in the following usual forms of amortization. As a function of parameters $S, n. i,$ and evaluating at the rate $i*$ we easily obtain, using (6.63):

a) Amortization with one final lump-sum refund and periodic delayed interest

$$U(h,i*) = S\,i\,a_{\overline{n-h}|i*}\ ; \quad P(h,i*) = S(1+i*)^{-(n-h)} \tag{6.66}$$

b) Delayed amortization with constant principal repayments

$$\begin{cases} U(h,i*) = \dfrac{S\,i}{\mathrm{n}}\left[(n-h+1)a_{\overline{n-h}|i*} - (\mathrm{I}a)_{\overline{n-h}|\,i*}\right] \\ P(h,i*) = \dfrac{S}{n} a_{\overline{n-h}|i*} \end{cases} \tag{6.67}$$

c) Amortization with constant delayed installments

$$\begin{cases} U(h,i*) = \dfrac{R\,i}{\mathrm{i\text{-}i*}}\left[a_{\overline{n-h}|i*} - a_{\overline{n-h}|i}\right] \\ P(h,i*) = \dfrac{R}{\mathrm{i\text{-}i*}}\left[(1+i*)^{-(n-h)} - (1+i)^{-(n-h)}\right] \end{cases} \tag{6.68}$$

where $R = S\alpha_{\overline{\mathrm{n}}|\mathrm{i}}$.

Example 6.4: *application of Makeham's formula and comparisons*

Let us apply in this example Makeham's formula for the calculation of usufruct, starting from that of bare ownership and using any valuation rate, in the customary amortization forms for unshared loans, comparing the results with those obtainable using the closed formulae (6.29), (6.30) and (6.31):

a) Amortization with one final lump-sum refund and annual delayed interest.

Let us use:
S = €2,000 (debt); n = 10 year; i = 5.5% (annual contractual rate);
$i*$ = 6.2% (annual valuation rate).

With formula (6.29), the initial valuation (h=0) is obtained:

$$U_0^* = 110\ a_{\overline{10}|0.062}\ ;\ P_0^* = 2{,}000\ (1.062)^{-10} = 1{,}095.94\,;W_0^* = 1{,}897.93$$

at time h=5 the result is:

$$U_5^* = 110\ a_{\overline{5}|0.062}\ ;P_5^* = 2{,}000(1.062)^{-5} = 1{,}095.94\,;W_5^* = 1{,}941.35$$

By applying Makeham's formula in h=0 and h=5, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \tfrac{0.055}{0.062}(2000.00 - 1095.94) = 801.99;$$
$$U_5^* = \tfrac{0.055}{0.062}(2000.00 - 1480.50) = 460.85\cdot$$

b) Annual amortization with constant principal repayments

Let us use:

$S =$ €1,500 (debt); $n = 4$ years; $i = 6\%$ (annual contractual rate);
$i* = 5.2\%$ (annual valuation rate).

We then obtain the following amortization schedule.

Year	*Principal repaid*	*Interest paid*	*Installment*	*Balance*
1	375.00	90.00	465.00	1,125.00
2	375.00	67.50	442.50	750.00
3	375.00	45.00	420.00	375.00
4	375.00	22.50	397.50	0.00

Table 6.16. *Example of amortization with constant principal repayments*

For the initial valuation (h=0) with a direct calculus for U_0^* and using formula (6.30) for P_0^* we obtain:

$$U_0^* = 90.00 \bullet 1.052^{-1} + 67.50 \bullet 1.052^{-2} + 45.00 \bullet 1.052^{-3} + 22.50 \bullet 1.052^{-4} = 203.57$$

$$P_0^* = 375\ a_{\overline{4}|0.052} = 1,323.58\,;\ W_0^* = 1,527.15$$

For h=2 we find

$$U_2^* = 45.00 \bullet 1.052^{-1} + 22.50 \bullet 1.052^{-2} = 63.11\,;\ P_2^* = 375\ a_{\overline{2}|5.2\%} = 695.31\,;$$

$$W_2^* = 758.42$$

Applying Makeham's formula for h=0 and h=2, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \frac{0.06}{0.052}(1500.00 - 1323.58) = 203.57\ ;\ U_5^* = \frac{0.06}{0.052}(750.00 - 695.31) = 63.11$$

c) Annual amortization with constant installments

Let us use, as in b):

$S =$ €1,500 (debt); $n = 4$ years; $i = 6\%$ (annual contractual rate);
$i* = 5.2\%$ (annual valuation rate); then $R = 432.89$.

We obtain the following amortization schedule.

Year	*Principal repaid*	*Interest paid*	*Outstanding balance*
1	342.89	90.00	1,157.11
2	363.46	69.43	793.65
3	385.27	47.62	408.38
4	408.38	24.51	0.00

Table 6.17. *Example of amortization with constant installments*

Using (6.31), for the initial valuation (h=0) we obtain

$$U_0^* = \tfrac{432.89 \cdot 0.06}{0.008}\left[3.529538 - 3.465106\right] = 209.19\,;$$

$$P_0^* = \tfrac{43289}{0.08}\left[1.052^{-4} - 1.06^{-4}\right] = 1,318.71\,;\; W_0^* = 1,530.80$$

For h=2 we find

$$U_2^* = \tfrac{25.9734}{0.008}\left[1.854154 - 1.833393\right] = 67.41\,;$$

$$P_2^* = \tfrac{432.89}{0.08}\left(1.052^{-2} - 1.06^{-2}\right) = 735.23\,;\; W_2^* = 802.64.$$

By applying Makeham's formula for h=0 and h=2, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \frac{0.06}{0.052}(1500.00 - 1318.71) = 209.19;\;\; U_2^* = \frac{0.06}{0.052}(793.65 - 735.24) = 67.41$$

6.7. Leasing operation

6.7.1. *Ordinary leasing*

It is appropriate, for completeness, to mention briefly an operation which can be a convenient investment for a financial company and at the same time a form of financing, often preferred by firms to other forms considered in this chapter.

Let us summarize this operation as follows. A company working in leasing is a broker between the owner of an asset or real estate and the lessee firm, in the sense that it gives the financial means for the purchase and, maintaining the property of the asset, grants its use against payment. For this company the costs are those related to the purchase of the asset, while the returns are the payments for the leasing, which are called rent and form a periodic annuity.

On the opposite side, the lessee company, against the use of the asset, pays a periodic rent for the whole length of the contract and also pays an *earnest payment*, usually a multiple of the rent; furthermore the possibility of *redemption,* i.e. the purchase by the lessee of the leased asset, usually at a price strongly reduced and prefixed at the beginning of the lease, it is often provided at expiry[36].

There is then the issue of comparing it with the loan operation for purchasing the asset. From this comparison follows a problem of choosing between alternative loans. In fact, we have to compare, on one side, the purchase of the property of the asset using his own means and loaned capital, with the resulting lost profit for the self-financing part that was invested at a return rate i_1 and the emerging cost for the loaned part, at a cost rate i_2; and on the other side, the leasing operation that implies the payment of advance, periodic rents and the possible final redemption. The maintenance expenses, in both cases, are paid by the company that uses the asset.

The leasing rent cannot be limited only to remuneration, at the contractual per period rate i, of the amount S used by the lessor for the purchase, at net for the advance and the discounted redemption, because being assets with a limited economic life (due to wear, obsolescence, etc), it must take into account an amount for the funding of the used capital for the renewal. There is then a situation analogous to the American amortization with two coincident rates, where on the basis of (5.9) the rent C is given by $S(i+\sigma_{\overline{n}|i}) = S\alpha_{\overline{n}|i}$, where S is the net amount already specified. Therefore, the rent, if constant and not indexed, *is calculated as the progressive amortization installment of a loaned principal equal to the aforementioned net amount.*

In formulae, if the operation, with a length of n periods, is not indexed and it is provided for a value F, an advance A and also a redemption at expiry R, the delayed per period rent C if constant[37] is obtained from the following relation, justified on the basis of the equivalence principle:

$$F = A + C\ a_{\overline{n}|i} + R\,(1+i)^{-n} \tag{6.69}$$

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36 It is suitable to mention briefly the “real estate leasing”. The length is usually long and the redemption value has to take into account that the real estate is not subject to the same depreciation that other assets or industrial equipment are subject to. In addition, there are the taxation problems particular to such leasing arrangements.

37 A financial calculator with the keys (n), (i), (pv), (pmt) and (fv) allows for the immediate automatic calculation of one of the quantities n, i, $(F\text{-}A)$, C, R, given the others, because (6.27) can be written: $-(F-A) + C\,a_{\overline{n}|i} + R\,(1+i)^{-n} = 0$. In addition, if α is known, we obtain: $F = (F-A)/(1-\alpha)$; $A = F-(F-A)$.

Using $\alpha = A/F$ (= advance quota) and $\rho = R/F$ (= redemption quota), from (6.69) we find the expression for the periodic rent C[38]

$$C = F\,[1 - \alpha - \rho\,(1+i)^{-n}]\,/\,a_{\overline{n}|i} \qquad (6.69')$$

If the first m rents are paid at the beginning, they form the advance, then C is found from (6.69) using $A = mC$ and $a_{\overline{n-m}|i}$ instead of $a_{\overline{n}|i}$. Therefore

$$C = \frac{F\left[1 - \rho\,(1+i)^{-n}\right]}{m + a_{\overline{n-m}|i}} \qquad (6.69'')$$

Exercise 6.12

1) The lessor gives a plant, the total cost of which is €24,000, for leasing with delayed monthly rents for 5 years and with a redemption equal to the 5% of the cost and

a) an advance of 8% of the cost; or
b) an advance equal to 3 rents.

Calculate the rent for the two cases in the hypothesis that an annual remuneration rate 12-convertible of 9.5% is applied.

A. In case a) used in (6.28'): F=24000, n=60, α=0.08, ρ=0.05 and using months as the unit measure for time, the monthly rate is $i_{1/12}$ = 0.007917 and the rent C (that can be found with a financial calculator as in footnote 37) is

$$C = 24{,}000\,(1 - 0.08 - 0.05 \cdot 1.007917^{-60})/\ a_{\overline{60}|0.007917} = 448.02$$

In case b), used in (6.27") the previous data and m=3, we obtain[39]

$$C = 24000\,(1 - 0.05 \cdot 1.007917^{-60})/(3 + a_{\overline{57}|0.007917}) = 477.16$$

2) The lessor gives a plant for 3 years, with advance monthly rent, without earnest, providing the redemption as 2% of the price and with a clause for a decrement of 40% of the rent after 20 month. Calculate the corresponding rents, considering that the price of the plant is €16,500 and the nominal rate 12-convertible is 11.20%.

A. The equivalent monthly rate is 0.009333, the equation to find the rent C for the first 20 months is given by

38 Footnote 37 also holds for (6.69').

39 It has been agreed that the redemption is paid in the month of the last rent; the length is then reduced to 57 months. In this case the rent is:

$$C = 24{,}000\,(1 - 0.05 \cdot 1.007917^{-57})/(3 + a_{\overline{57}|0.007917}) = 476.80$$

$$-16{,}500.00 + C\,(\ddot{a}_{\overline{36}|0.009333} - 0.40\;_{20/}\ddot{a}_{\overline{16}|0.009333}) + 330.00 \cdot 1.009333^{-36} = 0$$

from which: C = 610.95. Therefore, the first 20 rents are €610.95 and the following 16 are €366.57.

6.7.2. *The monetary adjustment in leasing*

In section 6.5, which was dedicated to the adjustment and indexation in the amortization of an unshared loan, we considered the remedies to cover the creditor from monetary depreciation in a long-term operation. As the leasing can also be considered as a pluriennial loan, for this problem the same remedies can be applied, then we refer to those, limiting ourselves here to a brief discussion.

For the phenomenon of the purchase power variation, and in particular of depreciation, two remedies are used:

1) *line interest compensation,* through a procedure of varying rates that are the sum of a fixed real remuneration share i_h and a varying share Δi_h of compensation nature if it is adjusted to the level of the monetary depreciation rate;

2) *line value compensation,* if the same plant value (which is under a real financial amortization given the criteria for the calculation of the rent) is indexed proportionally to a statistical series of prices representing the interested phenomenon.

6.8. Amortizations of loans shared in securities

6.8.1. *An introduction on the securities*

In the previous chapter we examined methods to manage the remuneration and repayments of loans with two contractual parts: lender and borrower. However, loans of a large amount to relevant companies frequently occur. Then it is practically impossible to realize such operations by only one lender, and therefore many lenders will share the debt.

Such operations are then realized in the following ways:

1) many private lenders, which give the money against an obligation of repayment and a credit security;

2) brokerage by third party, in the sense that a bank or a group of banks formalize the obligations and securities, collect the money in the "stock market" of the subscribers of the credit securities (using its own organization through a Stock Exchange and its own branches), and give the debt sum in one or more "slices";

3) public guarantees, in the case of loans for public enterprise.

The stock market offers many possibilities for financial investments, typical or not. We will consider here only credit securities for which the principal to be paid back is well determined (even if interest can be paid according to varying rates). We will then not consider:

– equity shares, that from the juridical viewpoint are joint ownership stocks;

– "investment funds" which are prevailingly formed by mixtures of shares and bonds, that have risky elements and are sometimes linked to an insurance component;

– values due to rights linked to share exchanges, that have their own specificity and autonomy and are traded in the "derivative market".

The description of most of these financial products can be found in the second part of this book. However, for further information the interested reader can refer to specific books.

A fundamental distinction between credit instruments placed against a shared loan between many creditors is that between:

a) *Treasury Bonds* (placed by the State) with one maturity;

b) *bonds*, which can have different type of redemption. For these, we must make a further distinction:

b_1) bonds with redemption at only one maturity for all creditors; and

b_2) bonds with redemption at different maturities amongst the creditors.

If the length of the operation is not longer than one year, the return for the investor is obtained through a purchase cost discounted with respect to the redemption amount. This cost can depend on the dynamics of the negotiation during the "auction" in which the bonds are placed. The financial regime that follows is that of the rational discount[40] (see Chapter 3).

If the length of the operation is pluriennial, and *n* is the number of years, the interest (through coupons) with delayed semiannual or annual due date on the basis of nominal rate – also termed *coupon rate*, constant or varying according to a prefixed rule – is usually paid. In this case the interest is a form of *"detached return"* of the security. We must distinguish for each security between the *issue value p* and *the redemption value c*, which we assume coincident with the *nominal*

40 If 100 is the redemption value of the bond, not considering taxes, the purchase price A is linked to the annual rate i and to the days of investment g by the relation: $A = 100/(1+ig/360)$.

value on the security (if the last two are different, for financial purposes, only the redemption value must be considered):

– if $p<c$, we talk about *issue at a discount* or *below par;*

– if $p=c$, we talk about *issue at par;*

– if $p>c$, we talk about *issue at a premium* or *above par*.

6.8.2. *Amortization from the viewpoint of the debtor*

The debtor (issuer) must plan an *amortization schedule* for the whole debt with one of the methods considered before for the unshared loan. The presence of many creditors is irrelevant from a financial point of view; there are only the practical complications of dividing amongst them the payments for redemptions and interest, called *coupons*. Let us assume 0 as the issue time of the loan and suppose the absence of adjustment. Furthermore, let N be the *number* of issued bonds, each with an *issue value* p and *redemption value* c, and j the annual coupon rate for the computation of delayed interest, that is *nominal 2-convertible* if the coupons are semiannual.

Given that, *in case* b_1 we can apply the scheme, seen in section 6.1, of one final lump-sum at maturity n and periodic payment of interest, dividing both of them amongst the issued bonds. Therefore, in this case we can immediately verify that for the issuer against the income supply $(0,+Np)$, the amortization consists of the outflow supplies:

– $(1,-Ncj)\cup(2,-Ncj)\cup\ldots\cup(n-1,-Ncj)\cup(n,-Nc(1+j))$, for annual coupons;

– $(1/2,-Ncj/2)\cup(1,-Ncj/2)\cup\ldots\cup(n-1/2,-Ncj/2)\cup(n,-Nc(1+j/2))$, for semiannual coupons.

In case b_2) we can apply, for the issuer, the general scheme of gradual delayed amortization seen in section 6.2, fixing the redemption plan, i.e. the number N_h of securities to redeem completely at the end of each year h, with the obvious constraint: $N=\sum_{h=1}^{n}N_h$. In fact, a gradual amortization for each bond is inconvenient. We can then calculate the numbers

$$L_h = N - \sum_{k=1}^{h} N_k \; ; \; h = 1,\ldots,n \tag{6.70}$$

which identify the numbers of bonds "alive" soon after the h^{th} gradual redemption, i.e. not redeemed at times $k \leq h$. For L_h the recursive relation holds

$$L_h = L_{h\text{-}1} - N_h \quad ; \; L_0 = N \, ; \; \text{then} \; L_n = 0 \qquad (6.70')$$

It is clear that the issuer must also pay the annual interest *cj* or semiannual *cj/*2 on each of the alive bond. Therefore, against the income supply (0,+*Np*) the amortization consists of the outflow supplies:

- $\bigcup_{h=1}^{n}(h, -N_h \; c - L_{h-1} \; c\, j)$, (annual coupons);
- $\left[\bigcup_{h=1}^{n}(h, -N_h \; c - L_{h-1} \; c\, j/2)\right] \bigcup \left[\bigcup_{h=1}^{n}(h-1/2, -L_{h-1} \; c\, j/2)\right]$, (semiannual coupons).

To summarize, with annual coupons the installment to be paid by the issuer is

$$R_h = N_h \; c + L_{h-1} \, c\, j \quad , \quad (h = 1....,n) \qquad (6.71)$$

while for semiannual coupons, the interest is divided into two equal amounts.

The one lump-sum redemption of all securities implies a large financial need for the issuer at time *n*, that – if not covered by a previous new bonds issue – can be very difficult to realize for a private company without adequate means and guarantees, which can also be used to become trusted by the creditor; therefore form b_1 is more adequate for *Treasury Bonds* or public securities. On the contrary, form b_2 allows for a gradual repayment, by choosing in a suitable way the sequence $\{N_h\}$ in relation to the incomes following the investments financed by such loan, and it is suitable for loans to companies with private structure.

6.8.3. *Amortization from the point of view of the bondholder*

Referring to the bondholders-creditors, we need to distinguish case b_1 from case b_2 and the following considerations hold.

In case b_1 the number of creditors does not change the amortization procedure, in the sense that for the bondholder of each of the *N* bonds the amortization is with one final lump-sum at maturity *n*, the same for all bondholders, with periodic payment of interest on the basis of the same parameters. The financial operation is obtained from the one described in section 6.6.2 for the issuer dividing it into *N* equal parts (with administrative complications due to the large number of counterparts[41]) and

41 Such complications disappear when the bond loan is entirely subscribed by a large company, public or private. In such cases, the operation is equivalent to an unshared loan,

changing the sign, i.e. dividing by $-N$. Then each bond at the issue date[42], against the payment of the amount p, must receive the supplies:

– $(1,cj)\cup(2,cj)\cup\ldots\cup(n\text{-}1,cj)\cup(n,c(1+j))$, for annual coupon;

– $(1/2,cj/2)\cup(1,cj/2)\cup\ldots\cup(n\text{-}1/2,cj/2)\cup(n,c(1+j/2))$, from semiannual coupon.

The case b_2, usual for pluriennial bonds of large amount, which are sold at the inception to a great number of private investors, implies for the bondholder of each of the N bonds an amortization with one final lump-sum redemption, but *with staggered redemption dates*. The rate evolution on the stock market is the cause of a continuous and varying spread between the current rate, for reinvestment after redemption, and the nominal one on the current loan. Therefore, at time h, according to the sign of the spread, all (if the spread is positive) or none (if the spread is negative) of the $L_{h\text{-}1}$ residual bondholder are interested to be included amongst the N_h redeemed. To avoid complications and to obtain the fairness among the creditors with a symmetric situation between the residual bondholders, the system of *amortization by drawing* is common, in the sense that the *repayment schedule* becomes a *drawing schedule* to concretely find at time h the N_h bonds (simple, i.e. not considering possible grouping in multiple bonds). The bonds subject to this type of management are termed *drawing bonds.*

In this form, while from the viewpoint of the issuer the financial operation is certain, from the point of view of the bondholder for each security we have a *stochastic maturity,* then *the amortization cash-flow is stochastic in length, with one lump-sum redemption and periodic (annual or semiannual) inflow of interest.*

6.8.4. *Drawing probability and mean life*

Proceeding with the consideration of hypothesis b_2 that implies for the bondholder the randomness due to the drawable bond system for redemptions, it is appropriate to find the drawing probability at a given integer time $h \leq n$. For reasons of symmetry the probability, valuated at issue date, of drawing a bond at time h (i.e. of a life of h years from the issue) can be assumed equal to N_h/N, ratio of bonds issued that are redeemed after h years, while the probability that a bond still not

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where bonds are only used for tax advantages and the possibility of placing the bonds in the exchange market. Another form that simplifies the loan amortization is that, which is widely applied in mature economies, of the purchase of their own bonds on the exchange market, which is convenient when the current cost rate is lower than the loan rate.

42 If the bondholder is *incoming*, buying the security at integer time r (simplifying hypothesis which ignores here the “day-by-day interest”) at price p_r and if the bondholder waits for the maturity without selling, the inflow operation is for him:

– $(r+1, cj)\cup(r+2, cj)\cup\ldots\cup(n\text{-}1, cj)\cup(n, c(1+j))$, with annual coupons;

– $(r+1/2, cj/2)\cup(r+1, cj/2)\cup\ldots\cup(n\text{-}1/2, cj/2)\cup(n, c(1+j/2))$, with semiannual coupons.

drawn at time r has a residual life of h years can be assumed equal to N_{r+h}/L_r, the ratio of residual bonds at time r that are redeemed after other h years.

It is also interesting to consider, in order to summarize with just one number the length of the investment for the bondholder as it occurs in the case of certain maturity, the *mean life* for the generic bond of a given loan[43].

We can calculate the *mean life at issue date* as a weighted arithmetic average of the lengths, expressed by the formula

$$e_0 = \sum\nolimits_{h=1}^{n} h \frac{N_h}{N} \tag{6.72}$$

It is also useful to evaluate, in the case of purchase or assignment r years after the issue date, the variation of *residual mean life* of a bond still not drawn at time r, expressed by

$$e_r = \sum\nolimits_{h=1}^{n-r} h \frac{N_{r+h}}{L_r} \tag{6.72'}$$

Example 6.5

Let us consider an amortization for a bond loan, gradual for the issuer and then with a drawing plan for the bondholder, issued *at a discount*. Let us take, with amounts in €:

- p = 1,760 = issue value;
- c = 2,000 = nominal and redemption value;
- j = 6.2% = annual coupon rate;
- N = 10,000 = number of issued bonds;
- n = 5 = length of the loan;
- $\{N_h\}$ = {1,500, 1,800, 2,500, 1,600, 2,600} = draws plan.

It follows that the number of residual bonds after each draw is: L_1 = 8,500, L_2 = 6,700, L_3 = 4,200, L_4 = 2,600 and L_5 = 0. The inflow for the issuer at 0 is 17,600,000 gross of inflow costs, while the whole debt is €20million, not considering the management costs.

With an annual coupon, their value is €124.00 and the annual installments for the payment to the creditor are:

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43 The bond mean life is a concept analogous to the mean life of a person, valuated at his birthday, for which mortality is measured by means of a demographic table. In probabilistic terms, the bond mean life is the expected value of its random length. In fact, (6.72) expresses it as the ratio between the whole life length of all bonds, according to the redemption schedule, and the numbers of issued bonds.

$R_1 = 2{,}000\ (1{,}500 + 0.062 \cdot 10000) = 4{,}240{,}000.00;$

$R_2 = 2{,}000\ (1{,}800 + 0.062 \cdot 8{,}500) = 4{,}654{,}000.00;$

$R_3 = 2{,}000\ (2{,}500 + 0.062 \cdot 6{,}700) = 5{,}830{,}000.00;$

$R_4 = 2{,}000\ (1{,}600 + 0.062 \cdot 4{,}200) = 3{,}720{,}000.00;$

$R_5 = 2{,}000\ (2{,}600 + 0.062 \cdot 2{,}600) = 4{,}240{,}000.00.$

With a semiannual coupon, their value is €62.00 and the semiannual installments for the payment to the creditor are:

$R_{1/2}$	$= 2{,}000\ (0.031 \cdot 10{,}000)$	=	620,000.00;
R_1	$= 2{,}000\ (0.031 \cdot 10{,}000 + 1{,}500)$	=	3,620,000.00;
$R_{3/2}$	$= 2{,}000\ (0.031 \cdot 8{,}500)$	=	527,000.00;
R_2	$= 2{,}000\ (0.031 \cdot 8{,}500 + 1{,}800)$	=	4,127,000.00;
$R_{5/2}$	$= 2{,}000\ (0.031 \cdot 6{,}700)$	=	415,400.00;
R_3	$= 2{,}000\ (0.031 \cdot 6{,}700 + 2{,}500)$	=	5,015,400.00;
$R_{7/2}$	$= 2{,}000\ (0.031 \cdot 4{,}200)$	=	260,400.00;
R_4	$= 2{,}000\ (0.031 \cdot 4{,}200 + 1{,}600)$	=	3,460,400.00;
$R_{9/2}$	$= 2{,}000\ (0.031 \cdot 2{,}600)$	=	161,200.00;
R_5	$= 2{,}000\ (0.031 \cdot 2{,}600 + 2{,}600)$	=	5,361,200.00

The mean life at the issue date, due to (6.72), is 3.2 years = 2y+2m+12d, while the residual mean life at time 3, due to (6.72'), is 1.619 years = 1y+7m+13d.

6.8.5. *Adjustable rate bonds, indexed bonds and convertible bonds*

Introduction

Modern capitalistic economies are characterized by a strong dynamism, by a wide variety of technical schemes for investments also by monetary systems, which are subject to variations of the purchasing power from which the investors must protect himself. Thus, even the management of shared loans, as that of unshared loans, considered in sections 6.2 and 6.3, is subject to adjustments and variations that make them more interesting for investors.

The listing and description of such investments would be too long if we wanted to consider all the modalities that sometimes have a very short life, because due to needs which are not valid any more.

It is then sufficient to briefly consider a few types, general and consolidated, which are widely applied.

Bonds with adjustable rate

As for the unshared loans, the issue of bonds can provide, as safeguard against inflation or better adjustment of the investment to the evolving market conditions, for an adjustable nominal rate, according to an appropriate linking rule to external parameters that allow not only the recovering of inflation and/or the adjustment due to the market measured at the issue date, but also during the time towards maturity.

We can then provide in the previous formulae for the substitution of the fixed rate i with a varying rate $i^{(h)}$ with current year h, and then adapt all the results.

Indexed bonds

Due to the requirement of protection against inflation, we can prefer, for a better recovery both on interest and principal, to leave the bond rate to a real return level and make the nominal value c varying and adjustable substituting it in the previous developments, both for redemption and for the calculation of semiannual or annual interest, with an amount $c^{(h)}$ varying with the current year h, indexed proportionally to an appropriate statistical series, for example to the consumer price index.

A more detailed formulation on the valuation of updated rates and indexed bonds will be given in section 6.9.4.

Convertible bonds

Convertible bonds are more complex and require a more in-depth discussion.

We can limit ourselves, here, by saying that a firm that wants to increase its capital, can initially collect money in the loan market as *credit capital* leaving the possibility to the subscribers – with appropriate limits and according to prefixed exchange ratios – to convert, in a given temporal interval, the *credit capital* into *risky capital*. In this way they become shareholders, then co-owners and partners in the enterprise. This is due to a number of reasons of convenience, also tax reasons, that allows the redemption of the debt by means of compensation with capital increasing.

6.8.6. *Rule variations in bond loans*

Bond loans often provide for variations that modify the cost and return parameters and that must then be taken into account. Leaving to the reader the easy calculation of the financial effect of such variations, we limit ourselves here to listing the most frequently used variations, warning that it is almost impossible to give a complete view of this topic.

1) *Redemption value higher than nominal value*

This is an additional premium and higher cost for the debtor. To calculate it, it is enough to take into account this redemption price, no longer the nominal value.

2) *"All inclusive" bonds with premium*

For such bonds, there is no payment of interest in the drawn year. Our formulae are adapted to this case decreasing the redemption value by the amount of the coupon.

3) *Bonds with premium*

A total premium amount $P^{(h)}$ can be provided for bonds drawn at year h. The debtor must take them into account adding C to $N_h c$ while for the bondholder the redemption value c is on average increased by $P^{(h)}/N_h$.

4) *Bonds with incorporated interest (= full accumulation)*

The loan can provide for the absence of coupons and a redemption value increasing with time, together with interest. It is obvious that the return for the different length h is found by considering the redemption value as an accumulated value after h years of the purchase price.

5) *Bonds with pre-amortization*

It can happen that there are no redemptions for the first h years, i.e. $N_1 = N_2 = ... = N_h = 0$. In this case, not having redemptions, the cost of interest for the debtor concerns all the issued bonds for the whole length of the pre-amortization.

6.9. Valuation in shared loans

6.9.1. *Introduction*

In section 6.8 we examined, from an *objective* point of view, the problem of management and amortization of loans shared in bonds. In this section 6.9 we will consider the problem of *subjective* valuation, made at inception or during the loan life, of the residual rights connected to owning the bonds, from the point of view of the creditor bondholder, the debtor (issuer) and a potential buyer. The logic is then that behind Makeham's formula and the more convenient choice between alternative investments on the basis of comparison rates fixed by the decision-maker.

We limit ourselves to the case of gradual amortization of a bond loan that, as we have seen, implies a pluriennial repayment plan by means of draws and randomness for the bondholder (but not for the issuer) of the values, and also of the usufructs and bare ownerships. Also for the valuation this is the more interesting case that gives rise to higher complexity.

Indeed, if the *issue is not at par*, only *ex-post*, after the draw, it is possible for the investor to calculate the effective return exactly[44]. In fact, the difference (positive or negative) *c-p* between return at redemption and purchasing cost is a "capital gain" or "capital loss", i.e. an "incorporated" return component that, from a previous point of view, is gained (or lost) in a random number of years *T*, where $(c-p)/(pT)$, i.e. the intensity, is also random. It follows that the IRR of the given bond investment is random. We will consider, for valuations and choices, an appropriate functional average, called the *ex-ante mean rate of return* for the bond, coinciding with *j* for at par issue.

In the not at par issues we can highlight the *immediate rate of return* or *current yield*, given by *cj/p*, that measures the return of the investment *p* given by the coupon, but not considering the capital gain or loss[45].

Everything will be clarified in what follows, starting from the case of a bond with a given maturity.

6.9.2. *Valuation of bonds with given maturity*

Let us consider first the model that follows from the hypothesis of certainty of the length, i.e. assuming that the bond has a given maturity. This can occur:

a) if all the bonds have a common maturity. This is case b_1 of amortization with one lump-sum redemption, where for both parties length and returns are certain;

b) only for the bonds that will be redeemed at a given maturity, in the drawing bond case.

Valuing from the bondholder point of view, let us consider the bonds that will be called after *s* years from the issue date, i.e. all in case a) with common maturity *s*, or only the N_s , $(1 \le s \le n)$, defined in case b).

With the usual symbols, in the case of annual coupons, with *i* being the effective delayed annual evaluation rate (subjectively chosen according to the market

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44 We highlight that if the issue is at par, the randomness of the length does not imply the randomness of the IRR, that coincides with the coupon rate, as it is obvious for the financial equivalence principle. Analytically we can deduce that the *issue at par is a necessary and sufficient condition such that IRR = j*. *Proof*: necessity: if IRR = *j*, let *T* be the random length, *b* the issue price and *c j* being the coupon, must be: $-b + cj\,[1-(1+j)^{-T}]/j + c\,(1+j)^{-T} = 0$ $\forall T$, then $b = c$. *Proof*: sufficiency. If the issue is at par, $-c + cj[1-(1+x)^{-T}]/x + c(1+x)^{-T} = 0$, where x = IRR, then: $cj[1-(1+x)^{-T}]/x = c[1-(1+x)^{-T}]$; $j/x = 1$, i.e. $x = j$.

45 In the not at par issues the immediate rate is obviously always between the coupon rate and the mean rate (or certain rate) of effective return. In at par issues, all the aforesaid rates coincide.

behavior and to the returns of alternative investments), indicating with $W_0(i)$ the valuation at issue date (in r=0) on the basis of the expected encashment of the bondholder, dependent on the return rate i, results in

$$W_0(i) \;=\; c\,j\; a_{\overline{s}|i} + c\,(1+i)^{-s} \tag{6.73}$$

(independent of the issue date because we adopted a uniform financial law). The symbol W means that this value coincides with the pro-reserve evaluated just after the purchase.

Assuming a market logic (a topic which will be more fully developed in Chapter 7), we indicate with $z^{(s)}$ the purchase price of the bond at issue date ($z^{(s)}<c$ if at a discount, $z^{(s)}=c$ if at par, $z^{(s)}>c$ if at a premium). Thus, solution x, existing and unique, of the equation in i

$$c\,j\,a_{\overline{s}|i} + c\,(1+i)^{-s} \;=\; z^{(s)} \tag{6.74}$$

(that, due to (6.73), expresses the equality between the value $V_0(i)$ and the price $z^{(s)}$ at time 0) is the IRR, the rate to which the mean return rate is taken back, given that in the bond investment with certain length the return rate is not random but certain, even with not at par issues.

Given that $W_0(i)$ is a decreasing function of i and that $i=j$ if the bond is issued at par, it is obvious that in the *at discount* case the solution for i in (6.74) is $x>j$, while in the *at premium* case the solution for i in (6.74) is $x<j$.

Constraint (6.74) between price $z^{(s)}$ and IRR in case of a given maturity acts biunivocally: given the wanted IRR, we obtain the corresponding issue price; and conversely, given the price $z^{(s)}$, we find the IRR as rate x that makes fair the operation to pay $z^{(s)}$ and to cash s annual coupon cj and the redemption c after s years. Clearly, at fixed c and j, the IRR is a decreasing function of $z^{(s)}$.

In (6.74), using the solution value x instead of i, then $z^{(s)}$ is obviously also the *value* $W_0(x)$ of the bond at rate x, while the two addenda at the left side form, respectively, the *usufruct* and *bare ownership* of the bond at rate x.

If, instead, at issue date the valuation is made at an intermediate time (integer) $r>0$, then $z^{(s)}$, to be written $z^{(s-r)}$, becomes the "forward" in r of the security on the exchange market and it is enough to substitute the residual life s-r instead of maturity s[46].

With a semiannual coupon, it is enough to consider in (6.74) the fractional annuity $a^{(2)}_{\overline{s}|x}$ and make the appropriate changes.

Example 6.6

For valuations connected to the return, it is enough to consider a single bond, even better a virtual share, putting the nominal value (that we suppose equal to the redemption value) equal to 100. Let us consider a security with certain maturity, which pays semiannual coupons at the nominal rate 7% (semiannual convertible) and is redeemed after 8 years. Let us assume the time unit is a half-year and let us put the time origin 0 at purchase (at issue date or a following one in the market of issued securities) of this share; then put in 16 the redemption time.

The purchase price P that assures an annual effective return of 6% (being $i_2 = \sqrt{1.06} - 1 = 0.029563$ the semiannual rate equivalent to 6% annually) is given by

$$P = 3.5\ (1 - 1.029563^{-16})/0.029563 + 100 \cdot 1.029563^{-16} = 106.85$$

then the purchase is "at premium", given that the annual effective return rate of 6% below the coupon annual effective rate, equal to 7.1225% corresponding to a nominal rate of 7%. The usufruct is the first addend of the right side, whose value is 44.11. The bare ownership is the second addend, whose value is 62.74.

Example 6.7

Let us consider a bond with certain maturity and the following data: nominal value and also redemption value at 9 years after the purchase = 100; annual coupons at rate of 6%; purchase price = 94.65, then the bond is "at discount". The current yield is by definition: 6/94.65 = 6.3391%. The IRR, that measures the effective return with the "capital gain", is solution x of the equation in i

$$-94.65 + 6\,[1 - (1+i)^{-9}]/i + 100\,(1+i)^{-9} = 0$$

46 Precisely the pro-reserve W_r in $r > 0$, dependent on i, is obtained from the right side of (6.73), using s-r instead of s. A simple calculation shows that the following recursive between subsequent values of W_r, dependent on the IRR of the security: $W_r = (1+i)^{-1}(cj+W_{r+1})$ with $W_s = C$ (thus putting the redemption soon after time s). In fact, in $r < s$ the bond with value W_r gives right after one year, accumulating at rate i, to the coupon cj and to further rights valuated W_{r+1} at time r+1. Such a simple formula is useful to calculate, using *Excel*, the sequence of residual values at integer times between 0 and s. From another point of view W_r is in $r = 0$ the *spot price* at issue date and in $r > 0$ the *forward price*, which are found from the right side of (6.73), in biunivocal correspondence with the value $I = x =$ IRR.

It is found with appropriate methods (it is sufficient a financial calculator) to be:

$$\text{IRR} = 6.8147\% > 6.3391\% \text{ (= current yield)} > 6\% \text{ (= coupon rate)}$$

6.9.3. *Valuation of drawing bonds*

inlet us consider case b_2 with repayments in n years randomly, according to a draw of $N_1,.., N_n$ drawing bonds in the years 1,…, n. Thus after the h^{th} draw the number of bonds L_h is given by (6.70). Therefore, extending the considerations of section 6.9.2, given the symmetry between the securities, the issue price z is found to equal the price $N z$ of the whole of the bond issue to the sum of the present values, calculated according to the prefixed IRR x, of the number of bonds which have to redeem at different maturities s, the number of which N_s is previously known.

Thus, the following relation holds

$$N z = \sum_{s=1}^{n} N_s \, c \, j \, a_{\overline{s}|x} + \sum_{s=1}^{n} N_s \, c \, (1+x)^{-s} \tag{6.75}$$

i.e.

$$z = \sum_{s=1}^{n} N_s \, z^{(s)} / N \tag{6.75'}$$

that expresses z as the weighted mean of $z^{(s)}$ with weights N_s/N, which express the probabilities, valued at issue date, of draw after s years.

In (6.75) the 1st addendum of the right side expresses the usufruct and the 2nd addendum the bare ownership, referred to the whole of the N bond issue. Therefore, we find, dividing by N, the *mean usufruct* u_0 and the *mean bare ownership* np_0 of a single bond at time 0.

Equation (6.75), with given z and unknown x, is also the equation that gives (univocally for the algebraic properties of (6.75)) the IRR as the *mean effective yield rate*[47] of the investment at price z. Instead, the *ex-post yield rate,* in the case of a draw after s years, is found by solving (6.75) with respect to the unknown rate x, with the value of s corresponding to the verified time of draw.

47 We must highlight that the *mean effective yield rate* is not the real profit rate for the investor in a bond, taking into account incorporated revenues and costs; this is the *ex-post* rate, valuable only after the bond call. In fact, the *mean effective yield rate* is a suitable functional mean of feasible *ex-post* rates owing to drawing.

If the valuation is performed at time $h \geq 0$, in (6.75') it is enough to add from 1 to n-h and substitute N_{h+s}/L_h to N_s/N. In this way the mean values z_h, the mean usufructs u_h and the bare ownerships np_h for each bond still alive at time h are obtained, resulting in

$$\begin{cases} \mathrm{u_h} = \displaystyle\sum_{s=1}^{n-h} \frac{N_{h+s}}{L_h} \, c\, j\, a_{\overline{s}|x} \;\; ; \;\; \mathrm{np_h} = \displaystyle\sum_{s=1}^{n-h} \frac{N_{h+s}}{L_h} c(1+x)^{-s} \\ z_h = \displaystyle\sum_{s=1}^{n-r} \frac{N_{h+s}}{L_r}\left(c\, j\, a_{\overline{s}|x} + c(1+x)^{-s}\right) = \mathrm{u_h} + \mathrm{np_h} \end{cases} \tag{6.76}$$

Example 6.8

Let us value the prices, the mean usufructs and bare ownerships, at issue and after 2 years, of the drawing bonds loan, the data of which are:

$n = 5$; $N = 1{,}000$; $c = €5{,}000$; $j = 5.60\%$; $x = 6.14\%$;
$N_1 = 150$; $N_2 = 170$; $N_3 = 200$; $N_4 = 230$; $N_5 = 250$.

To apply the resolving formulae we build the following table.

s	$a_{\overline{s}\|x}$	$(1+x)^{-s}$	$z^{(s)}$	N_s/N	L_s
	(1)	(2)	(3)	(4)	(5)
1	0.942152	0.942152	4,974.56	0.15	850
2	1.829802	0.887650	4,950.59	0.17	680
3	2.666103	0.836301	4,928.01	0,20	480
4	3.454026	0.787923	4,906.74	0.23	250
5	4.196369	0.742343	4,886.70	0.25	0

Table 6.18. *Elements for calculating values, usufructs and bare-ownerships*

– The price z_0 at issue, corresponding to IRR 6.14%, is the arithmetic weighted mean of values $z^{(s)}$, obtainable as a scalar product of vectors (= component product sum) given by columns 3 and 4: $z_0 = €4{,}923.61$;

– We obtain the mean usufruct at issue by scalar product of column vectors 1 and (4), then multiplying by $c\,j = 280$: $u_0 = €792.16$;

– We obtain the mean bare ownership at issue by the scalar product of column vectors 2 and 4, then multiplying by $c = 5{,}000$: $np_0 = €4{,}131.45$.

$u_0 + np_0 =$ €4,923.61 gives the value z_0 in another way.

Valuing after 2nd refund (r=2), with residual time length of the loan = 3, we have to repeat the procedures of calculation already shown, but limit ourselves to the averages of the first three elements of columns 1 and 2, taking as weights the redemption percentages $N_3/L_2 = 200/680 = 0.294118$; $N_4/L_2 = 230/680 = 0.338235$; $N_5/L_2 = 250/680 = 0.367647$. We obtain:

$$u_2 = 525.33 \; ; np_2 = 4{,}424.01 \; ; z_2 = u_2 + np_2 = 4{,}949.34.$$

Particular case: constant principal repayments

If N is a multiple of n, we can choose N_r = const. = N/n. By introducing this value in the 3rd equation into (6.76), we find

$$\frac{z_h}{c} = \frac{a_{\overline{n-h}|x}}{n-h} + \left(1 - \frac{a_{\overline{n-h}|x}}{n-h}\right)\frac{j}{x} \quad \text{[48]} \qquad (6.76')$$

Exercise 6.13

Let us consider a bond loan of €750,000 shared into 750 bonds redeemable at nominal value according to draw, with constant principal repayments and annual coupons, with the following parameters:

- length in years $n = 10$
- coupon rate $j = 5.5\%$
- mean effective yield rate $x = 6\%$

calculate for one bond the issue price and the forward price after the 3rd draw, which realize the assigned yield at 6%.

A. The unitary result does not depend on the number of issued bonds. Applying (6.76'), the following is obtained:

at issue date (h=0): $$\frac{z_0}{1000} = \frac{7.3600871}{10} + \left(1 - \frac{7.3600871}{10}\right)\frac{5.5}{6.0}$$

$$z_0 = 1000\,(0.7360087 + 0.2639913 \cdot 0.9166667) = 978.0007$$

after 3 years (h=3): $$\frac{z_3}{1000} = \frac{5.5823814}{7} + \left(1 - \frac{5.5823814}{7}\right)\frac{5.5}{6.0}$$

$$z_3 = 1000\,(0.7974831 + 0.2025169 \cdot 0.9166667) = 983.1236$$

48 Equation (6.76') shows that z_h/c is a weighted mean between 1 and j/x with weights varying with h. Therefore $z_h<c$ iff $j<x$ (at discount) while $z_h>c$ iff $j>x$ (at premium).

Particular case: debtor installments (almost) constant

The redemption of a bond loan can also be made with constant annual delayed payments for the issuer on the model of the French amortization. Therefore a constant installment for the loan of the type $R = Nc/a_{\overline{n}|j}$ is to be valued. Then, the value of redemption of each security being constant c, both the total redemption amount at time r and the numbers N_r of redeemed bonds have to increase in geometric progression with ratio $(1+j)$. Thus, it must be $N_r = k(1+j)^r$ and from $\sum_{r=1}^{n} N_r = N$ follows:

$$k = N\, \ddot{\sigma}_{\overline{n}|j} \quad ; \quad N_r = N\, \ddot{\sigma}_{\overline{n}|j}\, (1+j)^r$$

to substitute in (6.76) for h=0. An easy calculation leads to the formula

$$\frac{z_0}{c} = \frac{j}{x} + \left(1 - \frac{j}{x}\right) np_0 \qquad (6.76")$$

where in this case the total redemption shares discounted are constant and then: $np_0 = nc\, \ddot{\sigma}_{\overline{n}|j}$, highlighting that z_0/c is a weighted mean between 1 and np_0. The changes to make the calculation for z_h with h>0 are obvious.

In addition, we have to observe that the values N_r previously obtained are always integer. Therefore, this scheme must be corrected by approximating for each year the theoretical number N_r, by its floor and transferring to the following year in acc/repayments the accumulated value of the not amount used, then valuing the new number of bonds to redeem, always rounding off at integer, and carrying on this way till the term.[49]

Exercise 6.14

Let us consider the bond loan with data of Exercise 6.13 but ruled by constant installments. Not considering the rounding off to obtain integer numbers, calculate such theoretically drawn numbers and also the issue price of one bond.

A. Using the formulae discussed above, as $\ddot{\sigma}_{\overline{10}|0.055} = 0.073619$, we find:

$N_1 = 750\cdot 0.073619\cdot 1.055 = 58.250827$; $N_2 = 1.055\cdot N_1 = 61.454622$;
$N_3 = 1.055\ N_2 = 64.834626$; $N_4 = 1.055\ N_3 = 68.400531$;
$N_5 = 1.055\ N_4 = 72.162560$; $N_6 = 1.055\ N_5 = 76.131501$;
$N_7 = 1.055.N_6 = 80.318733$; $N_8 = 1.055.N_7 = 84.736263$;
$N_9 = 1.055.N_8 = 89.396758$; $N_{10} = 1.055.N_9 = 94.313580$.

49 It is obvious that this rounding operation changes the mean yield rate and the ex-post rates very little with respect to the calculated ones according to the theoretical redemption with exactly constant installments.

To check, adding the numbers above, we obtain 750.

As $np_0 = nc\ddot{a}_{\overline{10}|0.055}$ = 10·1000·0.073619 = 736.19, the issue value is

$$z_0 = 1000\,(0.916667 + 0.083333 \cdot 0.73619) = 978.02.$$

The approach for management years

Equation (6.76) is obtained using a total direct valuation but it is also possible to use the *management years approach*, that offers the advantage of analyzing the temporal development and easily enables a generalization for the hypothesis of varying rate and adjustment of the values.

Proceeding for management years, we find for the year $h+s$ the total amount for the paid coupon by the debtor as interest and for redemptions as principal. This amount originates from the L_h bonds circulating at time h (or from the N bonds issued, if h=0). If it is divided by L_h we obtain for symmetry reasons the mean amount s years after h for the generic purchased bond. Then, the value z_h assigned to each bond, on the basis of an appropriate valuation rate x, is given by

$$z_h = \frac{1}{L_h}\sum_{s=1}^{n-h}(L_{h+s-1}ci + N_{h+s}c)(1+x)^{-s} \tag{6.77}$$

It is easy to show algebraically the equivalence between the last equation in (6.76) and (6.77). Furthermore in (6.77) L_{h+s-1}/L_h and N_{h+s}/L_h are respectively the probabilities to be drawn for bonds not drawn till h, of no drawing for another $s-1$ years and to be drawn in the following year. Given that $L_{h+s-1} = L_{h+s} + N_{h+s}$, the total amount of year s can be written as $L_{h+s-1}ci + N_{h+s}c(1+i)$, distinguishing for the circulating bonds at the beginning of the year $h+s$ the amount for interest for the bonds not drawn in the year and the amount for interest and redemptions for the drawn bonds[50].

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50 Let us find here a relevant property for z_h. Indicating with $c^* = ci/x$ the capital that reproduces the annual coupon given at rate x and resulting for equivalence

$$\sum_{s=1}^{n-h}(L_{h+s-1}c^*x + N_{h+s}c^*)(1+x)^{-s} = c^*L_h,$$

(6.77) can be written as

$$z_h = c^* + (c - c^*)\sum_{s=1}^{n-h} N_{h+s}(1+x)^{-s}/L_h,$$

where, with an obvious financial interpretation, the result of Σ is: $0<\Sigma<1$. Therefore, z_h is always between c^* and c.

If we use the delayed semiannual interest coupon, the same correction factor $x/2(\sqrt{1+x}-1)$, that transforms the value of the delayed constant annual annuity in to that of the semiannual fractional annuity (see Chapter 5), must be introduced in the valuation.

By introducing a direct argument, if we are using a semiannual coupon we have to replace ci by the accumulated value at the year's end at rate x of the two semiannual coupon $ci/2$, i.e. the value $cix[\sqrt{1+x}-1]/2$.

In practice, with a semiannual coupon it is enough to substitute in (6.77) the annual coupon rate i for its transformed one $i' = ix/2(\sqrt{1+x}-1)$.

Recursive relation of a bond value at fixed coupon rate

In addition, for the valuation of bond loans with drawing redemption at any rate x^* we can consider the dynamic aspect on the basis of the management years approach. Using the symbols already defined, the relation[51] between subsequent values z_h valued at rate x is as follows:

$$L_h z_h = (1+x)^{-1}(cN_{h+1} + cjL_h + z_{h+1}L_{h+1}) \ , \quad (h = 0,...,n\text{ - }1) \qquad (6.78)$$

Equation (6.78) extends, to the loans shared in bonds, the recursive relation examined in section 6.2 for the unshared loan and is based on a principle of preserving the value in equilibrium conditions, expressing the equality between the valuation of residual securities at time h, and the sum of the differently used amount of such securities in h+1, soon after the $(h+1)^{\text{th}}$ draw, valued in h. In fact, considering that $N_{h+1}+L_{h+1} = L_h$, at the right side of (6.78) are added for the total loan: 1) the payment in principal for the redemption of drawing bonds in h+1; 2) the payments of interest for the living bonds between h and h+1; 3) the valuation of residual bonds in h+1.

Mathematical life and Achard's formula

Let us define *mathematical life* at time r and rate x the exponential mean of the possible residual life length of a bond still not drawn in r, on the basis of the repayment plan; this indicated by em_r, is implicitly defined by

$$(1+x)^{-em_r} = \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r}(1+x)^{-s} \qquad (6.79)$$

51 The considered recursive relation, concerning random values due to the call, shows an analogy with the known *Fouret's equation* about life insurance theory.

Defining $a_{\overline{s}|x} = (1 - (1+x)^{-s})/x$ as well for non-integer times, the mean value z_r taken from (6.76), given (6.77), can be transformed in

$$z_r = c\, j\, a_{\overline{em_r}|x} + c\,(1+x)^{-em_r} \tag{6.80}$$

The right side of (6.80) can be split into mean usufruct and mean bare ownership, i.e. $u_r = c\, j\, a_{\overline{em_r}|x}$; $np_r = c\,(1+x)^{-em_r}$. Therefore, the mean valuation of usufruct and bare ownership, in uncertainty conditions following the repayment plan, are equivalent to the certain ones with length em_r. In other words, the mean financial valuation for random maturity is equivalent to the one that would be obtained with a maturity certain at time $r+em_r$, i.e. after a time equal to the mathematical life.

For the expression of u_r and np_r taken from (6.76), the mean usufruct of a bond with nominal and redemption value c can be expressed according to the mean bare ownership in the form:

$$u_r = \frac{j}{x}\left[c - np_r\right] \tag{6.81}$$

that is *Achard's formula*[52]. It particularizes the Maheham's formula on a single bond, given that, as the amortization with one lump-sum redemption at maturity, the intermediate outstanding balances remains always equal to the redemption value c.

6.9.4. *Bond loan with varying rate or values adjusted in time*

It is known that, to face monetary variations or to adjust pluriennial operations to the changing of market conditions, it is possible to adopt in the management of loans, varying coupon interest rates or indexed outstanding loan balance.

Sometimes such schemes are also adopted in bond loans. In particular, for the valuation considered in this chapter, it is possible to formalize such a scheme if we use the approach for management years described in section 6.9.3.

Let us refer to formula (6.77) and observe that if, due to the varying rates and/or to indexing of values, we assume a sequence of coupon interest rates $i^{(s)}$ and/or a

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52 The proof follows from:

$$\sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} c\, j\, a_{\overline{s}|x} = c\, j \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r}\left[1-(1+x)^{-s}\right]/x =$$

$$= \frac{c\, j}{x} \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} - \frac{c\, j}{x} \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r}(1+x)^{-s} = \frac{j}{x}\left[c - c\sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r}(1+x)^{-s}\right]$$

sequence of redemption values $c^{(s)}$ corresponding to years s = 1,..., *n–h* starting from year $h \geq 0$, then it is enough to replace in (6.77) for each time *h*+*s* the coupon constant rate *i* by the varying rate $i^{(s)}$ and/or the constant unitary debt *c* by the indexed debt $c^{(s)}$. Considering that usually the indexing of debt is used as an alternative to the variation of coupon rate, the following formulae, that at an appropriate valuation rate *x* give the pro-reserve of the total outstanding balance at time $h \geq 0$, hold. In the case of *varying coupon rate* the pro-reserve is

$$W_h = \sum_{s=1}^{n-h} (L_{h+s-1} c i^{(s)} + N_{h+s} c)(1+x)^{-s} \tag{6.77'}$$

while in the case of *indexing of the outstanding balance* the pro-reserve is

$$W_h = \sum_{s=1}^{n-h} (L_{h+s-1} c^{(s)} i + N_{h+s} c^{(s)})(1+x)^{-s} \tag{6.77''}$$

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Chapter 7

Exchanges and Prices on the Financial Market

7.1. A reinterpretation of the financial quantities in a market and price logic: the perfect market

7.1.1. *The perfect market*

The relations examined in Chapter 2, which follow from indifference financial laws, give rise to models summarizing preferences in simple or complex exchange operations. Such models enable the measurement of the value given to the temporary availability of financial capital, by means of calculation of the interest on the landed principal or, more generally, of the return of a financial investment.

Many of these concepts can be reformulated more concretely, putting them in a *market logic*, in particular, of the exchange market, establishing the relations that link *prices* of assets, obtained by the meeting of global demand and supply in this market. Therefore, we consider now a different, but analogous, formulation of the theory of financial equivalence, which is helpful in understanding the exchanges taking place in the financial market. Exchange factors in the presence of effective transactions and the indifference relation (in particular, of equivalence if the right conditions, which realize the strong decomposability, exist) that links *financial values* of market referred to different times are considered in this formulation.

The point of view introduced here is therefore an inversion with respect to the settings of Chapter 2 and to the particular cases of Chapter 3, which gave rise to the results of Chapter 4, 5 and 6. In fact, in the previous chapters, on the basis of a *theoretical* approach, first a financial law is introduced; from there the *value* at a given time of the activities linked to an operation *O* net of the losses is obtained; finally the *price* is adapted to the found value. In this chapter we use an *empirical* approach in the sense that the initial input is the *price* of the activities in *O* net of the losses (price obtainable from "market surveys" in broad sense) and from it – on the basis of constraints and relations analogous to those that gave rise to the *value* – is found the coherent structure of return dynamics that links by equivalence the given price to *O*. From this last point of view, we can see, as a particular case of fixed rates, the calculation of the internal rate of return (IRR) of a *project O* (see section 4.4).

To fully understand the approach that we called *empiric,* it is convenient, for simplicity, to refer to the case of *bonds,* public or private, and to the specific market where they are traded. The management of loans shared in stocks and the connected financial valuation has already been discussed in Chapter 6, where, amongst other things, we considered drawing bonds. In addition, we analyze the properties of securities prices that come from a special hypothesis on the financial market, which enables us to speak about a *perfect capital market.*

We talk about a *perfect market* when it has the following features:

– *no friction,* i.e.:

- no transaction cost and taxes;

- the possibility of *short selling,* i.e. sale of securities not owned by the seller with delivery at sale date;

- no risk of default (thus certainty of results);

- homogenity of information;

– *continuity,* i.e.:

- securities are *infinitively divisible* and can be increased; there is no limitation in the trading quantities;

– *competitiveness,* i.e. each market operator:

- *maximizing his profit* – he prefers, all things being equal, to own higher quantities (see rule c of economic behavior in Chapter 1);

- is a *price taker*, i.e. he is a passive subject, not active, with respect to price, in the sense that his operation does not influence the stock price;

– *coherence,* i.e.:

– no-arbitrage opportunities[1].

We will call a market satisfying the coherence hypothesis *coherent*.

It is clear that the perfect market comes from ideal and theoretical conditions; such a market is a model for study. It will be interesting to analyze the properties that are valid for transactions in such markets, properties analogous to those considered for financial laws in a different content. Note that in a real market some of the hypotheses may not be true, as well as some of the properties.

7.1.2. *Bonds*

We will not consider in this chapter random operations, but only bonds with certain *maturity*, concentrating on the following basics types.

a) *Zero-coupon bond*

In such a security the investor returns are completely incorporated; from the financial viewpoint, the debtor, who is the issuer of N bonds with maturity t, issue value P and nominal (and redemption) value C, makes the pure exchange operation

$$(0, NP) \cup (t, -NC) \tag{7.1}$$

whereas each creditor, subscriber or purchaser of a bond, makes the operation

$$(0, -P) \cup (t, C) \tag{7.2}$$

Usually the zero-coupon bonds have maturities that are not too long. Referring to operation (7.2), the return rate for the length t is given by $i_t = (C-P)/P$. With reference to a regime of simple accumulation (being $t \leq 1$) and then to the intensity $j = i_t/t$, given on the basis of market considerations, we obtain

$$P = C/(1+jt) \tag{7.3}$$

1 To clarify this context, an operation O, defined in (4.1) or (4.1'), is called *arbitrage (non-risk)* if the amounts S_h, not all zero, have the same sign. Therefore, O is not fair with any financial law. There are two types of arbitrage:

a) purchase of non-negative amounts, with at least a positive one, at a non-positive price (for free or with an encashment);

b) purchase of non-negative amounts at a negative price (with an encashment). The market coherence is equivalent to the principle of “no arbitrage”.

i.e. the issue value is the discounted value of C in regime of rational discount (conjugated to the simple accumulation).

Example 7.1

In the issue of a semiannual zero-coupon bond for 181 days, let the purchase price for 100 nominal be 95.18. Not considering taxes, it follows that the per period return rate is (100-95.18)/95.18 = 5.0641% and has an intensity equal to 10.0722 years^{-1}.

Considering a taxation of 12.5%, the purchaser pays effectively 95.18+0.125 (100-95.18) = 95.7825, to which corresponds a net per period rate of 4.4032% and an intensity equal to 8.7578 years^{-1}.

b) *Coupon bond*

In such a security, described in Chapter 6 as a shared loan with certain maturity, the investor return has a component of *return* paid periodically, i.e. interest payments (*coupon* payment), to which can be added a component of *incorporated return,* positive or negative (the *capital gain* or *capital loss*, with issues or purchases respectively at discount or at premium). The investor lends to the issuer the amount *P (issue price)*, or buys from the previous investor on the exchange market paying the price *P (purchase price)*. In both cases he periodically receives, for the residual life, the payment $I = C'j$ of the coupons, with j being the coupon rate and C' the *nominal value* redeemed at maturity. If the redemption value C is different from the nominal value, one considers C in the financial valuation. Here, we consider fixed coupon bonds, deferring to the following section 8.5 for a brief introduction to bonds with varying coupons.

Therefore, indicating with n the *maturity* of the loan, the financial operation for the bondholder is given by

$$\mathbf{T\&S} = (t, t+1, \dots, n-1, n)\&(-P, I, \dots, I, C+I) \tag{7.4}$$

where we assume $t = 0$ in the case of subscription at the issue date, $t \in \mathcal{N}$ in case of later purchase. In all cases $n-t$ is the *length* of the investment, equal to the bond maturity, if $t = 0$.

Fixed coupon bonds are widely used for long-term investments.

Example 7.2

Let us consider a 5-year coupon bond with semiannual coupon and nominal annual rate 2-convertible of 6%, issue price 96.2 for 100 nominal. Not considering taxes, this results in **T&S** = (0; 0.5;1; 1.5; 2; 2.5; 3; 3.5; 4; 4.5; 5) & (-96.2; 3; 3; 3; 3; 3; 3; 3; 3; 3; 103).

The rates are: coupon rate = 3 %, current semiannual rate = 3/96.2 = 3.1185% i.e. annual rate of 6.3343%, effective semiannual rate (with the capital gain), i.e. the semiannual IRR of the operation, which is solution x of the following equation:

$$-96.2 + 3\ a_{\overline{10}|x} + 100\ (1+x)^{-10} = 0$$

This results in x = IRR semiannual = 3.4559%, annual IRR 7.0312%.

Let us also consider a 5-year coupon bond at the annual nominal rate of 6%. It is issued at September 1, 2001, then with a maturity date of September 1, 2006, at the price of 95.35. The semiannual coupons are paid on March 1 and September 1 of each year until maturity. In t = January 14, 2003 the *ex-interest price* (EIP), which assigns to the buyer a share of the current coupon after purchase is 95.75.

Calculation of residual life in t: 3 years+230 days.

Calculation of net coupon: with taxes at 12.5%, we have: 3 (1-0.125) = 2.625.

Calculation of net redemption value: with taxes at 12.5%, we have: 100-0.125·(100-95.35) = 99.419.

Calculation at time t of the price, called the *flat price* (FP), which assigns to the buyer the whole current coupon, so it is given by the *ex-interest price* (EIP) plus the "before day-by-day interests" (b.dbdi) from the last coupon payment (September 1, 2002) until t, then for 135 days; we have: FP = EIP + b.dbdi = 95.75 + 2.625·135/181 = 97.708.

Calculation of "ex-coupon price" at time t: paying the *ex-coupon price* ECP the buyer obtains the bond without current coupon; so ECP is given by EIP minus "after- day by day interests" (a.dbdi) from t until the next coupon payment (March 1, 2003), then for 46 days; we have: ECP = EIP - a.dbdi = 95.75 - 2.625· 46/181 = 95.08. Obviously it then also results: ECP = FP - (net) coupon = FP - 2.625.

Other types of bonds can depend on the variability or randomness of the coupon. In fact, we can have:

– a *coupon with varying rate* according to a previous agreed rule;

– a *coupon with indexed rate*, linked to the future evolution of market or macroeconomic indices.

7.2. Spot contracts, price and rates. Yield rate

Using the theory of financial contract, we will develop a parallel discussion to that in Chapter 2 that will consider the price formation, in a perfect market or at least under the coherence hypothesis, in conditions of certainty. To better clarify the analogy, we will use the same symbols, but with a different meaning.

Referring initially to a unitary zero-coupon bond (UZCB) as a fundamental element (given that more complex transactions can be obtained as linear combinations of UZCB with increasing lengths), we indicate with small letters the times, i.e. the distances from the chosen origin 0. If

$$v(y,z) \quad , \quad y \leq z \tag{7.5}$$

is the market price paid in y to purchase the unitary amount in z on the basis of a contract entered into at time y, then such a contract is called a *spot* contract and *$v(y,z)$ is the spot price (SP);* note that the supply $(y;v(y,z))$ can be exchanged with the supply (z;1). The interval (y,z) is called the *exchange horizon (e.h.).*

The analogy of $v(y,z)$ with the discount factor $a(z,y)$ defined in Chapter 2, going from *values*, following subjective valuations, to *prices*, following market laws, is obvious. The position of the variables, (y,z) instead of (z,y) for v, being $y<z$, is due to the prevalent use of operators that prefer a chronological order.

On the basis of the *money return principle* it follows that:

$$v(y,z) < 1 \quad , \quad \forall (y<z) \tag{7.6}$$

Although prices are formed in light of complex causes, the introduction of market hypothesis imposes conditions and constraints. Thus, from market coherence it follows that:

$$v(y,z) > 0 \quad \forall (y < z) \quad , \quad v(y,y) = 1 \tag{7.7}$$

In the same way, from coherence follows the *decreasing of prices with time to maturity* of the bond (that is the final time of the e.h), i.e.:

$$v(y,z') > v(y,z'') \; , \quad \forall (y \leq z' \leq z'' \tag{7.7'}$$ [2]

2 The proof follows *ab absurdo*, observing that if it were $v(y,z') \leq v(y,z'')$, the composition of the three operations:

1) purchase in y of UZCB with maturity z';

The return inherent in the exchange between [y,v(y,z)] and (z,1) can be measured by the rate, which is defined by:

$$i(y,z) \quad = \quad [v(y,z)]^{-1/(z-y)} - 1 \tag{7.8}$$

Equation (7.8) shows that in this context *the rate is not per period but is per unit of time,* i.e. *on unitary base*, i.e. *on a unitary basis,* in particular *on an annual basis* if the unit is a year.

By inversion of (7.8) the following is obtained:

$$v(y,z) \;=\; [1 + i(y,z)\,]^{-(z-y)} \tag{7.8'}$$

Observation

When the price v is a function of return variables, as in (7.8'), and such variables are expressed by the market, then v changes its nature, assuming that of value following a calculation.

We define intensity of return at maturity (intensity r.m.), referring to a spot contract, by the function:

$$\phi(y,z) \quad = \quad -\ln v(y,z)/(z-y) \tag{7.9}$$

By inverting (7.9) f(y,z) satisfies:

$$v(y,z) \;=\; e^{-\phi(y,\,z)(z-y)} \tag{7.10}$$

that is – recalling the definition of instantaneous intensity given in Chapter 2 for a discount law with two time-variables, to which those for the price formation are analogous – the intensity $\phi(y,z)$ coincides with the constant instantaneous intensity of the exponential law equivalent, in return terms, to the one obtainable from $v(y,z)$ on the e.h. (y,z). In addition, due to (7.9), being: $\ln v(y,z) = -\int_y^z \delta(y,u)du$, the formula $\phi(y,z) = (\int_y^z \delta(y,u)du)/(z-y)$ follows. Then the *intensity r.m.* $\phi(y,z)$ *is the mean of the instantaneous intensities* $\delta(y,u)$ *agreed in y and varying with u in the interval (y,z).*

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2) short sell in y of UZCB with maturity z'';
3) purchase in z' of UZCB with maturity z'';
is equivalent to the union of the supplies $[y,\ v(y,z'') - v(y,z')]$, $[z', 1-v(z',z'')]$. Taking into account (7.6), in the hypothesis to verify the amount of the former supply is non-negative and that of the second one is positive, in contrast with the *no arbitrage principle*.

It follows that in the continuous time approach the return structure of the spot market can be fully found from the assumption of the financial law $\delta(y,u)$ because from it we find for all intervals (y,z) the intensity r.m. and then the spot rates and prices according to the other constraints. In fact, taking into account (7.9) and (7.10) and the comparison between (7.8) and (7.9), we find the relation between rate and intensity r.m., expressed by:

$$\phi(y,z) = \ln[1+i(y,z)] \tag{7.11}$$

or, by inversion,

$$i(y,z) = e^{\phi(y,z)} - 1 \tag{7.11'}$$

From what has been said above, the analogy of (7.11) with that regarding the constant intensity of the exponential law as a function of the rate: $\delta = \ln(1+i)$ (i.e. *flat structure*) is obvious. Furthermore, due to (7.11), recalling the logarithmic series, it follows that:

$$\phi(y,z) = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{[i(y,z)]^k}{k}$$

where the quadratic approximation of $\phi(y,z)$ is: $i(y,z) - [i(y,z)]^2 < i(y,z)$. It follows that the exact value of $\phi(y,z)$ is between its quadratic approximation and $i(y,z)$.

Example 7.3

Assuming a (bank) year as the time unit, if 0.95 is the price paid on February 15, 2003 to purchase a UZCB with maturity June 30, 2003, the annual return rate of the operation is 14.6578% , while the intensity at maturity is 0.136782 per year.

In fact, we have:

$$v(y,z) = 0.95$$

$$\left.\begin{array}{r} y = (\text{February } 15,\ 2003) \\ z = (\text{June } 30,\ 2003) \end{array}\right\} \to z - y = \frac{135}{360} \text{ years}$$

from which, due to (7.8), it follows that:

$$i(y,z) = v(y,z)^{-\frac{1}{z-y}} - 1 = (0.95)^{-\frac{360}{135}} - 1 = 0.1465783$$

$$h(y,z) = -\ln\ v(y,z)/(z-y) = (-\ln\ 0.95)\frac{360}{135} = 0.1367821$$

Example 7.4

The SP of a quarterly UZCB with annual intensity of intensity r.m. at level of 0.03 is equal to 0.99252805. The annual return rate is 3.0454%.

In fact, we have:

$$\phi(y,z) = 0.03$$
$$z - y = \frac{1}{4} = 0.25 \text{ years}$$

from which, for (7.10), it follows that:

$$v(y,z) = e^{-\phi(y,z)(z-y)} = e^{-0.03 \cdot 0.25} = 0.99252805$$
$$i(y,z) = e^{\phi(y,z)} - 1 = (0.99252805)^{-4} - 1 = 0.03045453$$

If the zero-coupon bond is not of unitary type, having a redemption value S at maturity z, for the perfect market property (continuity and *price-taker*) it follows that the price of the security in y, indicated with $V(y,z;S)$, is equivalent to that of S UZCB, the price of which is $v(y,z)$, which then must be:

$$V(y,z;S) \quad = \quad S\ v(y,z) \tag{7.12}$$ [3]

Example 7.5

Using data of Example 7.3, on February 15, 2003 the SP of a zero-coupon bond with redemption value 100 and maturing on June 30, 2003 is 95.

In fact, we have:

$$V(y,z;100) = 100\ v(y,z) = 100 \cdot 0.95 = 95$$

Example 7.6

If the SP of a two-yearly zero-coupon bond with redemption value 200 is 150, the SP of a corresponding two-yearly UZCB is 0.75.

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3 We can prove (7.12) *ab absurdo*, on the basis of the no arbitrage principle, showing that the hypothesis of inequality of prices between the non-unitary zero-coupon bond, whose value is $V(y,z;S)$, and the S UZCB allows an arbitrage, obtained with the short selling of stocks with higher price. So that if, in (7.12), the results are $V(y,z;S) < S\ v(y,z)$, the arbitrage is obtained by buying in y the bond that gives (z,S) and short selling in y the S UZCB with maturity z. We obtain an analogous conclusion if $V(y,z;S) > S\ v(y,z)$.

In fact, we have:

$$v(y,z) = \frac{V(y,z;200)}{200} = \frac{150}{200} = 0.75$$

We have highlighted the analogy of *v(y,z)* with the discount factor *a(z,y)* of a financial law with two time variables. However, we can also work in a market logic with the analogy of the accumulation factor *m(y,z)* resulting from the conjugated law *a(z,y)*, then given by its reciprocal. We can precisely define: $m(y,z) = 1/v(y,z)$ as the ratio between the encashment *K* due to owning the bond at maturity *z* and the price *Kv(y,z)* paid for its purchase at a time $y<z$. It follows that $m(y,z)-1$ is the incorporated per period return rate, which refers to e.h. and is obtained by such investment.

Considering *complex securities* that regard inflow vectors $\{S_k\}$ of subsequent amounts according to the maturities $\{z_k\}$, i.e. operations that regard m the supplies:

$$\{(z_1, S_1), (z_2, S_2), \ldots.. , (z_n, S_n)\} \tag{7.13}$$

it is obvious that, given the infinite divisibility of securities in a perfect market, such amounts can also be obtained forming a *portfolio* ***S*** (= set of distinct securities) of $\sum_{k=1}^{n} S_k$ UZCB, divided amongst *n* maturities in order to have S_k UZCB with maturity z_k (k=1,2,...,n). If the operation is carried out at time *y*, the price of one UZCB that matures in z_k is given by $v(y,z_k)$, thus the price in *y* of the whole portfolio is:

$$\sum_{k=1}^{n} S_k \; v(y,z_k) \tag{7.14}$$

From the market coherence follows the *property of price linearity*: the price $V(y,\boldsymbol{S})$ of the portfolio *S*, i.e. of the complex security, must coincide with the value (7.14). In formula:

$$V(y,\boldsymbol{S}) = \sum_{k=1}^{n} V(y,z_k;S_k) = \sum_{k=1}^{n} S_k \; v(y,z_k)^{4} \tag{7.15}$$

4 For proof, it is enough to repeat for each maturity the argument in footnote 3: if (7.15) is not satisfied, there is arbitrage with the buying (selling) in *t* of the complex security and the selling (buying) in *t* of S_k UZCB with maturity z_k.

The supplies of the bonds with fixed coupon (*coupon bonds* or *bullet bonds*) or also bonds with varying coupon (for instance, if the coupon rate varies due to indexing or other reasons) are included in (7.13). Indicating with C the redemption value of the security at time z_n and with I_h the varying coupon at time z_h (I if constant), for such a security (7.13) becomes

$$\{(z_1, I_1), (z_2, I_2), \ldots.. , (z_{n-1}, I_{n-1}), (z_n, C_n + I_n)\} \tag{7.13'}$$

and the supplies can also be referred to the residual time after the buying on the market, not necessarily at the issue date. Therefore the value at time y, on the basis of (7.15), is given by

$$V(y,\mathbf{S}) = \sum_{k=1}^{n} I_k \, v(y,z_k) + C \, v(y,z_n) \tag{7.15'}$$

Until now we have considered prices and rates referred to a given maturity, to apply to securities already in the market. Let us now consider a change of the intensity r.m. ϕ regarding one security, or a set of homogenous securities, during its, or their, economic life. This is the *return rate* (or *yield rate*), defined as that rate, which, used to discount the cash flow produced by the security after its purchase and to its maturity, makes the result equal to its purchase *tel quel* price. Using, referring to the security purchased in 0:

- P = purchase *tel quel* price;
- n = residual length;
- Y = yield rate;
- S_k = net encashment at time $z_k > 0$ ($k = 1,\ldots,n$).

The rate Y is the solution of the equation:

$$P = \sum_{k=1}^{n} \frac{S_k}{(1+Y)^{z_k}} \tag{7.15''}$$

It is immediately verified that the *yield rate* is the IRR on a time interval to maturity and is reduced to the *spot rate* $i(0,n)$ if the security is a zero-coupon bond with life n.

Observing a given number of almost homogenous bonds and calculating for each length the yield rate corresponding to the market price according to (7.15''), on a Cartesian diagram we obtain a set of points with the same number of points as the observed bonds. By means of an appropriate interpolation we find the *yield curve*, putting on the abscissa the residual length and on the ordinate the interpolated yield. Such a curve is a model that gives information on the behavior of the observed bond market if a representative sample is used. Obviously the obtained yields for each length can be different from those effectively obtainable from each security on the market.

We can usually say that if for the security its measured point is above (below) the *yield curve*, it is overestimated (underestimated) by the market, with the following input to sell (buy) if there is assumed a tendency for equilibrium.

In conclusion, the indication obtainable from the *yield curve* dot does not have the same coherence as the spot rates. However, theoretically we can say that the yield rate Y of a single bond is a functional mean (according to Chisini 1929) of the spot rate applied for the valuation of such a bond. In a formula, indicating with S_k the expected inflow due to the bond at time z_k, by definition this constraint:

$$\sum_{k=1}^{n} \frac{S_k}{[1+i(0,z_k)]^{z_k}} = \sum_{k=1}^{n} \frac{S_k}{(1+Y)^{z_k}}$$

follows.

Example 7.7

A bond issued on January 1, 2003 gives the right to the encashment sequence: 4; 2; 101, and according to the time schedule July 1, 2003, October 1, 2003, and November 15, 2003. If, at the issue date and according to the same time schedule, the spot prices structure of the UZCB is (0.96; 0.94; 0.93), the price of the bond is 99.65.

In fact, indicating with:

$$S = (S_1, S_2, S_3) \qquad S_1 = 4 \,,\, S_2 = 2 \,,\, S_3 = 101$$

$$y = 1.1.2003 \,;\, z_1 = 1.7.2003 \,;\, z_2 = 1.10.2003 \,;\, z_3 = 15.11.2003$$

$$z_1 - y = \frac{6}{12}\,year \;;\; z_2 - z_1 = \frac{3}{12}\,year \;;\; z_3 - z_2 = \frac{1.5}{12}\,year$$

$$v(y,z_1) = 0.96 \quad ; \quad v(y,z_2) = 0.94 \quad ; \quad v(y,z_3) = 0.93$$

we have:

$$\begin{aligned} V(y;S) &= V(y,z_1;S_1) + V(y,z_2;S_2) + V(y,z_3;S_3) = \\ &= S_1 v(y,z_1) + S_2 v(y,z_2) + S_3 v(y,z_3) = \\ &= 4 \cdot 0.96 + 2 \cdot 0.94 + 101 \cdot 0.93 = 99.65 \end{aligned}$$

Example 7.8

If the price of the complex security in Example 7.7 is 100 on January 1, 2003, keeping all the other conditions the same, we would realize a secure profit of 0.35 using the following arbitrage strategy:

– short selling the complex bond, with a return of 100;
– buying 4 UZCB maturing on July 1, 2003, with a cost of 4.0.96 = 3.84;
– buying 2 UZCB maturing on October 1, 2003, with a cost of 2.0.94=1.88;
– buying 101 UZCB maturing 15.11.2003, with a cost of 101.0.93= 93.93.

As the result, we would have: 100 -3.84 -1.88 -93.93 = 0.35 > 0.

Example 7.9

Given the function $v(y,z)=\left[1+1.06^{z}-1.06^{y}\right]^{-1}$ that defines the SP of a UZCB, where time is measured in years, the intensity r.m. of the spot contract, expressed in years^{-1} is given, due to (7.9), by:

$$\phi(y,z)=\frac{\ln\left(1+1.06^{z}-1.06^{y}\right)}{z-y}$$

If z-y is small, $1.06^{z}-1.06^{y}<<1$ results, then a good approximation is MacLaurin's formula:

$$\phi(y,z) = \frac{1.06^{z}-1.06^{y}}{z-y} = \text{(incremental ratio of } 1.06^{x}\text{)}$$

Using y =3 , z = 5.5, $v(y,z)$ = 0.842622 results and we obtain:

$$\phi(y,z)=\frac{0.171237}{2.5}=0.068495$$

Instead, using y=3 ; z=3+1/12 = 3.083333, we obtain: $v(y,z)$ = 0.994236 , $\phi(y,z)$ = 0.069367 , approximated $\phi(y,z)=\frac{1.196813-1.191016}{0.083333}=0.069568$.

7.3. Forward contracts, prices and rates

We can now consider operations that include an exchange between two dates, both of which are after the time of the contract. Comparing with spot contracts, there is no more coincidence between the time of contract (in which the conditions are fixed) and the time of payment. In such a case, we talk about *forward contracts*, which give rise to *delayed sales,* agreed with time x and taking place in $y > x$. If, as we suppose, the sold asset, delivered and paid in y, is a security that gives right to an encashment at maturity $z \geq y$ (or many encashments at times $z_k \geq y$), then for each trade we consider three times, x, y, z. In addition, we can underline that in x (= contracting time) there is no money or asset transfer and that the price of the asset (in particular, of the security), agreed with x, is a *forward price* (f.p.).

The elementary contract that we consider here is the forward purchase, with conditions agreed at time x but with delivery and payment at time $y > x$, of a UZCB redeemed at time $z \geq y$. Let us indicate with:

$$s(x;y,z)\ , \quad x < y \leq z \tag{7.16}$$

[5]

the f.p., fixed in x, of the UZCB delivered in y and with maturity in z.

Also here is obvious the analogy of $s(x;y,z)$ with the continuing discount factor with the meaning specified in Chapter 2. Furthermore, for continuity reasons implied in the perfect market hypothesis for $x \to y$, it results in:

$$s(y;y,z) = v(y,z) \tag{7.17}$$

then the spot contract can be seen as a limit case of the forward one.

Example 7.10

It is agreed today to buy, after two months, at the price of 0.80, a UZCB with a residual life of four months at the time of purchase.

In symbols, expressing time in months, the agreed forward price is: $s(0;2,6) = 0.80$. The financial operation can be written as (0,2,6)&(0,-0.80,+1) or $(0,0) \cup (2,-0.80) \cup (6,+=1)$.

5 In general, we can put: $x \leq y \leq z$, meaning that if $x = y \leq z$, the contract is a spot contract.

Analogously to what was seen for the spot contracts, the return in the exchange between $[y,\ s(x;y,z)]$ and $(z,1)$ can be measured by the *rate* referred to a time unit, i.e. on unitary base, in particular annual, defined by:

$$i(x;y,z) \quad = \quad [s(x;y,z)]^{-1/(z-y)} - 1 \qquad (7.8'')$$

where, by inversion,

$$s(x;y,z) \ = \ [1 + \mathrm{i}(x;y,z)^{-(z-y)} \qquad (7.8''')$$

In addition we define *intensity r.m.*, referring to a forward contract, by the function:

$$\phi(x;y,z) \ = \ -\ln[s(x;y,z)]/(z-y) \qquad (7.9')$$

By inversion:

$$s(x;y,z) \ = \ e^{-\phi(x;y,z)(z-y)} \qquad (7.10')$$

therefore $\phi(x;y,z)$ coincides with the constant instantaneous intensity of the exponential equivalent law in terms of a return to the one obtainable from $s(x;y,z)$ on the e.h. (y,z). Furthermore, due to (7.9'), owing to

$$\ln s(x;y,z) = -\int_y^z \delta(x,u)du\ ,$$

we have $\phi(x;y,z) = (\int_y^z \delta(x,u)du)/(z-y)$. Thus, *the intensity r.m.* $\phi(x;y,z)$ *in forward contracts is the mean of the instantaneous interest intensity* $\delta(x,u)$ *fixed in x and varying with u in the interval (y,z).* Also in the forward market with a continuous time approach, the return structure is given starting from an instantaneous intensity function $\delta(x,u)$. In fact, from $\{\delta(x,u)\}$ we find $\phi(x;y,z)$ on the basis of the aforementioned formula. From the comparison of (7.8'') and (7.9') we find the relation between $\phi(x;y,z)$ and rate, expressed by:

$$\phi(x;y,z) \quad = \ \ln[1 + i(x;y,z)] \qquad (7.11'')$$

or, by inversion:

$$i(x;y,z) \;=\; e^{\,\phi(x;y,z)} - 1 \qquad (7.11''')$$

On the analogy of the conclusions obtained about the spot contracts and the (7.11), for $\phi(x;y,z)$, due to (7.11"), the quadratic approximation of the logarithmic series holds; therefore $\phi(x;y,z)$ is included from its quadratic approximation $i(x;y,z)-[i(x;y,z)]^2$ to $i(x;y,z)$.

Furthermore, for forward contracts, as well as for the spot contracts, we can work in a market logic in terms of accumulation and accumulation factors on the basis of conjugated law. Therefore, we can define, analogously to the continuing accumulation factor defined in Chapter 2, the factor $r(x;y,z) = 1/s(x;y,z)$ defined as the ratio between the encashment K, due to the ownership of the bond at maturity z, and the price $Ks(x;y,z)$ paid for its purchase at time $y<z$ in a forward contract with conditions agreed in x. It follows that $r(x;y,z)$-1 is the per period incorporated return rate, referred to the e.h., obtained from such an investment.

Example 7.11

Considering with the same spot price function defined in Example 7.10 the forward contract with $x = 1$; $y = 3$; $z = 5.5$, the intensity r.m. is given by

$$\phi(x;y,z) = -\ln \frac{v(x,z)}{v(x,y)} \Big/ (z-y) = \frac{-\ln\ 0.852869}{2.5} = 0.063660$$

The corresponding annual interest rate $i(x;y,z)$, given in the forward contract, satisfies relation (7.11"), i.e.: $0.063660 = \ln\ \left[1+i(x;y,z)\right]$, from which: $i(x;y,z) = 0.065730 = 6.5730\%$

7.4. The implicit structure of prices, rates and intensities

It is fundamental that the following property of the *implicit structure*, if the market coherence holds true, links the parameters of forward contracts to those of spot contracts, propriety that can be summarized regarding prices with the formula:

$$s(x;y,z) \;=\; v(x,z)/v(x,y) \quad , \quad \forall\ (x \le y \le z) \qquad (7.18)$$

Equation (7.18) expresses briefly the fact that forward prices are obtained *implicitly* from the spot ones on the basis of the constraint

$$v(x,z) = v(x,y)\, s(x;y,z) \quad , \quad \forall\ (x \le y \le z) \tag{7.18'}$$

equivalent to (7.18) and analogous to that valid for continuing factors.

Thus, we speak about the *theorem of implicit prices*, observing that (7.18) follows from the coherence hypothesis[6]. This hypothesis leads us to assert that the *applied forward rates are those implicit in the spot structure*.

Following from (7.18) and (7.7'), the relations that summarize the main properties of f.p. implicit in SP are as follows:

$\forall(x \le y \le z)$: $s(x;y,z) > 0$ (positive f.p.)

$\forall\ (x \le y)$: $\begin{cases} s(x;y,z) < 1, & if \quad y < z \\ s(x;y,y) = 1 \end{cases}$ (f.p. not greater than the profit at maturity)

$\forall(x \le y' < y'' \le z)$: $s(x;\ y',\ z)\ <\ s(x;\ y'',\ z)$ (increasing of f.p. with initial time of e.h.)

$\forall(x \le y \le z' < z'')$: $s(x;\ y,\ z')\ >\ s(x;\ y,\ z'')$ (decreasing of f.p. with final time of e.h.)

Furthermore, the perfect market *hypothesis in conditions of certainty* implies the property, analogous to decomposability (for which, as specified in Chapter 2, the initial discount factor is equal to the continuing one), thus called: *independency from contractual time,* on the basis of which

$$s(x;y,z) = s(y;y,z) = v(y,z)\ ,\ \forall(x \le y \le z) \tag{7.19}$$

follows. Due to this equation, the f.p. $s(x;y,z)$ in x to pay in y the UZCB redeemed in z, must coincide with the SP $v(y,z)$ of such UZCB; this is according to the principle of *price uniqueness* of exchange on the horizon (y,z), i.e. of its invariance with respect to x.

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6 Also in this case the proof holds *ab absurdo*, observing that the lack of (7.18) effectiveness leads to certain profit. Indeed, if $v(x,z) > v(x,y)\ s(x;y,z)$, we would obtain a certain profit from the composition of the following three operations at time x:

1) short selling of UZCB redeemed in z;
2) spot purchase of $s(x;y,z)$ unit of UZCB redeemed in y;
3) forward purchase, with delivery in y, of the UZCB redeemed in z.

The result of this composition is a certain profit of the amount $v(x,z) - v(x,y) \cdot s(x;y,z)$ in x, owing to the set-off among other supplies. We obtain a certain loss in the hypothesis $v(x,z) < v(x,y)\ s(x;y,z)$, because there is a certain profit inverting the sign of each price. (7.18) is also justified by the fact that it must be equivalent to pay in x the spot price $v(x,z)$ to purchase a unitary amount in z or investing it in x to purchase in $y \le z$ at forward price $s(x;y,z)$, but to obtain the required amount $s(x;y,z)$ in y we have to pay in x the spot price $v(x,y) \cdot s(x;y,z)$.

In other words, in light of the hypothesis of independence from contractual time, (7.18) becomes

$$v(x,z) = v(x,y)\ v(y,z) \tag{7.20}$$

Then the financial law induced by the spot structure is decomposable.

Note that in practice the following can happen:

– at time $x<y$ the "future" price $v(y,z)$ has to be considered random, then (7.19) does not hold;

– we can find, *a posteriori*, SP and f.p. not satisfying (7.18);

– the f.p. are not implicit by SP, then there are arbitrage possibilities.

In this case, the ideal situation of a perfect market does not hold.

Example 7.12

Referring to data from Example 7.7, the structure of forward prices, implicit in that of the given spot prices, is:

$$s(y;y,z_1) = \frac{v(y,z_1)}{v(y,y)} = v(y,z_1) = 0.96 \quad (\text{being } v(y,y) = 1)$$

$$s(y;z_1,z_2) = \frac{v(y,z_2)}{v(y,z_1)} = \frac{0.94}{0.96} = 0.97916666$$

$$s(y;z_2,z_3) = \frac{v(y,z_3)}{v(y,z_2)} = \frac{0.93}{0.94} = 0.98936170$$

Obviously the price on January 1, 2003 of the complex examined security is always 99.65. In fact, we have:

$$\begin{aligned} V(y;S) &= S_1\ s(y;y,z_1) + S_2\ s(y;z_1,z_2)s(y;y,z_1) + S_3\ s(y;z_2,z_3)s(y;z_1,z_2)s(y;y,z_1) \\ &= 3.84000000\ + 1.88000000\ + 93.92999992\ = 99.64999992\ \cong\ 99.65 \end{aligned}$$

In light of forward prices $s(x;y,z)$ given by the market, we can define the *implicit forward rates,* considered as (mean) rates on unitary basis (in particular, annual), that express the return given by the market, and that are obviously linked to $s(x.y,z)$ by (7.8") and (7.8"').

If market returns are defined by means of spot rates (7.8), the implicit forward structure can be expressed in terms of rates using the following fundamental relation that follows from (7.18), applying (7.8) and (7.8"):

$$[1+i(x;y,z)]^{z-y} = \frac{[1 + i(x,z)]^{z-x}}{[1 + i(x,y)]^{y-x}} \tag{7.21}$$

As already mentioned, we can also define the rate structure as a function of the intensities r.m. defined in forward contracts, adopting suitable symbols and changing the definitions (7.9) and (7.9'). Thus, due to (7.11) and using natural logarithms in (7.21) we find

$$\phi(x; y, z)\,(z - y) = \phi(x, z)(z - x) - \phi(x, y)\,(y - x) \tag{7.22}$$

from which

$$\phi(x,z) = \phi(x,y)\,\frac{(y - x)}{(z - x)} + \phi(x;y,z)\,\frac{(z - y)}{(z - x)} \tag{7.23}$$

where the spot intensity r.m. in the total interval *(x,z)* is the weighted mean of the intensities r.m. (where the former is a spot intensity and the latter is a forward intensity) for the partial intervals *(x,y)* and *(y,z)* by which the total interval can be decomposed.

Example 7.13

Referring to the data from Example 7.7, the spot rate structure is:

$$i(y,z_1) = v(y,z_1)^{-\frac{1}{z_1-y}} - 1 = (0.96)^{-\frac{12}{6}} - 1 = 0.085069444$$

$$i(y,z_2) = v(y,z_2)^{-\frac{1}{z_2-y}} - 1 = (0.94)^{-\frac{12}{9}} - 1 = 0.085999258$$

$$i(y,z_3) = v(y,z_3)^{-\frac{1}{z_3-y}} - 1 = (0.93)^{-\frac{12}{10,5}} - 1 = 0.086474374$$

Obviously, rates increase with decreasing prices. The corresponding structure of implicit forward rates is:

$$i(y;y,z_1)=\frac{[1+i(y,z_1)]^{\frac{z_1-y}{z_1-y}}}{[1+i(y,y)]^{\frac{y-y}{z_1-y}}}-1=i(y,z_1)=0.08506944$$

$$i(y;z_1,z_2)=\frac{[1+i(y,z_2)]^{\frac{z_2-y}{z_2-z_1}}}{[1+i(y,z_1)]^{\frac{z_1-y}{z_2-z_1}}}-1=0.08786128$$

$$i(y;z_2,z_3)=\frac{[1+i(y,z_3)]^{\frac{z_3-y}{z_3-z_2}}}{[1+i(y,z_2)]^{\frac{z_2-y}{z_3-z_2}}}-1=0.089329438$$

or, equivalently, using the results from Example 7.12:

$$i(y;y,z_1)=s(y,y,z_1)^{-\frac{1}{z_1-y}}-1=i(y,z_1)=0.085069444$$

$$i(y;z_1,z_2)=s(y,z_1,z_2)^{-\frac{1}{z_2-z_1}}-1=0.087861277$$

$$i(y;z_2,z_3)=s(y,z_2,z_3)^{-\frac{1}{z_3-z_2}}-1=0.089329438$$

Recalling that in the hypothesis of deterministic perfect market we have:

$$s(y,y,z_1)=v(y,z_1)$$
$$s(y,z_1,z_2)=v(z_1,z_2)$$
$$s(y,z_2,z_3)=v(z_2,z_3)$$

it follows that:

$$i(y,z_1)\ =i(y,y,z_1)\ =0.08506944$$
$$i(z_1,z_2)=i(y,z_1,z_2)=0.08786128$$
$$i(z_2,z_3)=i(y,z_2,z_3)=0.08932944$$

This means that borrowing, at market conditions, the amount 99.65, which is needed to purchase the security, and paying back at due dates the amounts to which it is entitled, at the security's maturity, the debt will all be paid back in full without adding any cost or profit.

In fact, we have:

Time	*Outstanding balance*
January 1, 2003	99.65
July 1, 2003	$99.65\,[1+i(y,z_1)]^{z_1-y}-4 = 99.80208327$
October 1, 2003	$99.80208327\,[1+i(z_1,z_2)]^{z_2-z_1}-2 = 99.92553188$
November 15, 2003	$99.92553188\,[1+i(z_2,z_3)]^{z_3-z_2}-101 = 0$

In practice, the spot rates, that are realized on the market on the subsequent due dates, can be different from the foreseen ones on the basis of the above-mentioned hypothesis.

If, for example, the observed spot prices are higher than the foreseen rates and are equal to:

$$i_{eff}(y,z_1) = i(y,z_1) = 0.085069444$$
$$i_{eff}(z_1,z_2) = 0.088865467$$
$$i_{eff}(z_2,z_3) = 0.089432222$$

then the described operation would imply, for the debtor, a loss of 0.02495764 which is equal to the outstanding balance at maturity. In fact we have:

Time	*Outstanding balance*
January 1, 2003	99.65
July 1, 2003	$99.65\,[1+i(y,z_1)]^{z_1-y}-4 = 99.80208327$
October 1, 2003	$99.80208327\,[1+i(z_1,z_2)]^{z_2-z_1}-2 = 99.92553188$
November 15, 2003	$99.94904525\,[1+i_{eff}(z_2,z_3)]^{z_3-z_2}-101 = 0.02495764$

If the observed spot prices were lower than the foreseen ones, then the operation described above would imply a profit for the debtor.

7.5. Term structures

7.5.1. *Structures with "discrete" payments*

The previous formulae gave prices, rates and intensities for spot or forward contracts, related to payment dates in $\mathcal{R}$.

According to market practice, without loss of generality, we now suppose for the payment dates a discrete "lattice" type distribution, i.e. such that the payments are made at the beginning (or the end) of periods of the same length, that we assume to be unitary[7]. Then, referring to a contract time $t \in \mathcal{R}$, let us consider a complex security with n equi-staggered maturities, that we assume as positive integers, starting from t; then we use a payment schedule (t, t+1,..., t+n). It follows that for financial valuations made in the previous section, we are interested in *spot prices* $v(t,t+k)$ and *forward prices* $s(t;t+h,t+k)$, *spot rates* $i(t,t+k)$ and *forward rates* $i(t;t+h,t+k)$ (referred to the year or, as a general case, to any unit of time), *spot intensities r.m.* $\phi(t,t+k)$ and *forward intensities r.m.* $\phi(t;t+h,t+k)$, where:

$$h \in \mathcal{N} \ , \quad k \in \mathcal{N} \ ; \ \ 1 \le h \le k \le n \, . \tag{7.24}$$

In this case the definitions and coherence relations seen for the general case are valid. In particular, if $h=k$ the forward prices have value 1 and the forward rates 0.

If the referring time t (i.e. of contract or valuation) is only one, in the sense that no other date is simultaneously considered, it is convenient to put t=0. Such a choice

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7 No necessarily annual. As an example, with semiannual, quarterly, etc., due dates in the market, it is enough to assume semester, quarter, etc., as the unit of measure adjusting times and equivalent rates and assuming a semiannual, quarterly etc., structure of prices and rates. We will develop this in section 7.5.2 in more details. The only restriction to such measure variation is that the due dates are rational numbers. In such a case, written in the form m_i/d_i , (i =1....n) and indicating by *lcm* the least common multiple of the denominators $d_1,...,d_n$ (obtained, as known, decomposing them into factors and taking the product of common and non-common factors, each with the highest exponent), it is enough to reduce the unit of measure according to the ratio *lcm*, where – using k_i = *lcm* /d_i – the new maturities are the integers $m_i k_i$. By filling the tickler with all other integers in the interval where we put payments equal to zero, we obtain the wanted tickler with a unitary period. For example, assuming the maturities December 7, August 13, May 22, the *lcm* of denominators is $3.5.2^3$ = 120. Therefore, reducing by 120 the unit of measure, we have: k_1 = 10, k_2 = 15 , k_3 = 24 and the new maturities are 70, 195, 528. By completing with natural numbers the interval (70, 528) (in which, except for the three given times, we put no payments), we obtain the wanted distribution. In addition, we have to observe that more often the market gives returns by means of annual rates (or intensities) where in such cases we have to find the equivalent rates between year and the period here used as unitary, if it is subdivision of a year.

is not restrictive, if we consider the arbitrary time origin,[8] and it allows the reduction of the symbols, where the first variable is implicit and the other time variables are written by bottom indexes. It is then enough, with the aforesaid meaning of the symbols, to use[9]:

$$\begin{aligned} &v(t,t+k) = v_k\,;\;\; s(t;t+h,t+k) = s_{h,k} \\ &i(t,t+k) = i_k\,;\;\; i(t;t+h,t+k) = i_{h,k} \\ &\phi(t,t+k) = \phi_k\,;\;\; \phi(t;t+h,t+k) = \phi_{h,k} \end{aligned} \tag{7.25}$$

Then we assume, unless stated otherwise, the symbols in (7.25) and the hypothesis that encashment on securities can occur only on the dates

$$T = (1,...,k,...,n) \tag{7.26}$$

It follows that we measure e.h. with natural numbers. Let us also assume a market *complete* and *perfect* (or, at least, *coherent*) in the sense that there is the possibility of having a zero-coupon bond at each time in (7.26) and the known properties hold true, amongst which is the property of *independence from the amount and coherence*.

We can then outline the term structures for prices and rate *in case of discrete dates*, deducing the formulas that, referring to prices, rates and intensities express each of them as function of the others. They are obtained from those seen in section 7.2 and 7.3 considered for the discrete case, i.e. using x=0 and $y,z \in \mathcal{N}$. For simplicity, from now on we will assume in the application the annual unit of measure, but it is easy to also consider multiples or submultiples periods, as we will see in section 7.5.2.

Spot structures

The symbols in (7.12) and (7.15) give the *spot prices* (SP) $V(0,k;S_k)$ in t=0 of the zero-coupon bonds that pay the amount S_k in k. It follows that

$$v_k = V(0,k;S_k) \,/\, S_k \tag{7.27}$$

8 The position t=0 does not imply uniformity in time of the financial law underlying the rates term structure. However, if we assume uniformity of time, the financial results do not depend on the choice of the time origin. In more general cases, for instance, if we have to compare or take into account in the same context different structures with different transaction times t_1, t_2, ..., we have to refer to the general case defined above.

9 The single time subscript of the spot rate are not to be confused with those used in Chapter 3 and 5, which have a different meaning. In the same way the double subscript in the forward parameters with integer time are not to be confused with the pairs of variables of the spot parameters with real times seen in section 7.2.

is the SP of the corresponding UZCB. For the linearity property of price it is not restrictive to refer only to UZCB with SP v_k. Thus, it is clear that for a portfolio of n zero-coupon bonds with amounts S_k payable in k, (k = 1,...,n) – where the security that is entitled to the supply *(k,S_k)* is equivalent to a given number of zero-coupon bonds with amount payable in k the sum of which is S_k – the SP is:

$$V(0,\mathbf{S}) = \sum_{k=1}^{n} S_k\, v_k \qquad (7.27')$$

From the sequence $\{v_k\}$ we find the *rate structure (on annual bases) of spot interest* $\{i_k\}$ in t=0 by means of the following formula that describes equivalently the scenario of the SP:

$$i_k = v_k^{-1/k} - 1 \ , \ (k = 1, \dots ,n) \qquad (7.28)$$

If the spot rates are i_k in t=0, we find the unitary price, inverting (7.28):

$$v_k = [1 + i_k]^{-k} , \ (k = 1, \dots , n) \qquad (7.28')$$

From the sequence $\{v_k\}$ or the sequence $\{i_k\}$ we find the *instantaneous intensities r.m. structure* ϕ_k in t=0 for spot contracts. It is enough to modify (7.28) or (7.28') and consider the natural logarithm, resulting in:

$$\phi_k = -\ln v_k/k = \ln [1 + i_k] \qquad (7.29)$$

Inverting (7.29) we find:

$$v_k = e^{-k\,\phi_k} \ ; \ i_k = e^{\phi_k} - 1 \qquad (7.29')$$

Example 7.14

In the market of zero-coupon bond with redemption value 100, are fixed today (t=0) the following SP dependent on annual payments dates, which, divided by 100, define v_k:

96.28 with maturity 1; 93.71 with maturity 2;
90.08 with maturity 3; 87.88 with maturity 4.

The corresponding spot rates structure in 0 is as follows:

$i_1 = v_1^{-1} - 1 \quad = 0.038637 = 3.8637\%;$
$i_2 = v_2^{-0.5} - 1 \quad = 0.033016 = 3.3016\%$
$i_3 = v_3^{-0.333} - 1 \quad = 0.035437 = 3.5437\%$
$i_4 = v_4^{-0.25} - 1 \quad = 0.032827 = 3.2827\%$

The intensities r.m. structure is as follows:

$$\begin{aligned}
\phi_1 &= -\ln v_1 &&= \ln(1+i_1) = 0.037910\\
\phi_2 &= -\ln v_2/2 &&= \ln(1+i_2) = 0.032483\\
\phi_3 &= -\ln v_3/3 &&= \ln(1+i_3) = 0.033424\\
\phi_4 &= -\ln v_4/4 &&= \ln(1+i_4) = 0.032299\ .
\end{aligned}$$

From the sequence $\{v_k\}$ we find *the spot discount (or advance interests) rate (on an annual basis) structure* $\{d_k\}$ in $t = 0$ on the interval $(0,k)$ by means of the following formula:

$$d_k = 1 - v_k^{1/k}, \qquad (k = 1, \dots, n) \tag{7.28''}$$

which is obtained by inverting: $v_k = (1 - d_k)^k$.

By comparing (7.28') and (7.28") we find

$$d_k = i_k / (1+i_k), \qquad (k = 1, \dots, n) \tag{7.28'''}$$

that generalize a well known formula valid for flat structure, obtainable form (3.53).

Example 7.15

With the same value v_k as in Example 7.14, the annual spot discount rates are, according to (7.28"):

$$d_1 = 1 - v_1^{1.000} = 0.037200\ ;\ d_2 = 1 - v_2^{0.500} = 0.031961;$$

$$d_3 = 1 - v_3^{0.333} = 0.034225\ ;\ d_4 = 1 - v_4^{0.250} = 0.031783.$$

The results for the spot structure obtained in Examples 7.14 and 7.15 can be easily found using an *Excel* spreadsheet as shown below.

Maturity	Spot price %	Delayed spot rate	Spot intensity r.m.	Advance spot rate
1	96.28	0.0386373	0.0379096	0.0372000
2	93.71	0.0330160	0.0324826	0.0319607
3	90.08	0.0354375	0.0348240	0.0342246
4	87.88	0.0328268	0.0322995	0.0317834

Table 7.1. *Spot structure*

The *Excel* instructions are as follows. 2nd row: titles; from the 3rd row:

column A	A3: 1; A4:= A3+1; copy A4, then paste on A5 to A6;
column B	insert data (spot prices %) on B3-B6;
column C	C3:= (B3/100)^-(1/A3)-1; copy C3, then paste on C4 to C6;
column D	D3:= ln (1+C3); copy D3, then paste on D4 to D6;
column E	E3:= 1-(B3/100)^(1/A3); copy E3, then paste on E4 to E6.

Forward structures

The market fixes the *structure of prices, rates and intensities for forward contracts*. In a coherent market the *implicit* forward structure is assumed, i.e. derived from the spot structures on the basis of formulae that adapt (7.18) to (7.24).

Always using the contract time in t=0, we obtain the following basic relation that expresses *the forward price* (f.p.) *structure* $s_{k-1,k}$ of UZCB according to the spot structure v_k for annual e.h. (or *uniperiod*):

$$s_{k-1,k} = \frac{v_k}{v_{k-1}}, \qquad (k = 1,...,n) \tag{7.30}$$

which, for k=1, simply expresses the known relation for the SP: $s_{0,1} = v_1$.

The corresponding *structure of forward (implicit) interest rates* for annual e.h. is given by

$$i_{k-1,k} = s_{k-1,k}^{-1} - 1 = \frac{v_{k-1}}{v_k} - 1, \quad (k = 1,...,n) \tag{7.31}$$

By inversion we find

$$s_{k-1,k} = (1 + i_{k-1,k})^{-1} \tag{7.31'}$$

From (7.31), and recalling (7.28), the *implicit rates theorem,* which is expressed by the following equation, is obtained:

$$1 + i_{k-1,k} = \frac{\left[1+i_k\right]^k}{\left[1+i_{k-1}\right]^{k-1}}, \qquad (k = 1,...,n) \tag{7.32}$$

which gives the implicit forward structure according to the spot structure.

The forward market structure can be expressed according to the spot structure also using the *intensity r.m.* $\phi_{k\text{-}1,k}$, obtainable from $i_{k\text{-}1,k}$ and $s_{k\text{-}1,k}$ using

$$\phi_{k\text{-}1,k} \;=\; \ln(1+i_{k\text{-}1,k}) \;=\; -\ln s_{k\text{-}1,k}\,,\;\; (k = 1,\dots, n) \qquad (7.29'')$$

In fact it is possible to show the validity of the formula:

$$\phi_{k\text{-}1,k} = k\,\phi_k - (k\text{-}1)\,\phi_{k\text{-}1}\,,\quad (k = 1,\dots,n) \qquad (7.22')$$

which particularizes (7.22). Applying this formula recursively with varying *k,* by adapting (7.23) to discrete times, the following is obtained:

$$\phi_k \;=\; \sum\nolimits_{r=1}^{k} \phi_{r-1,r}\,/\,k\,,\;\; (k = 1,\dots, n) \qquad (7.23')$$

i.e. the spot intensity for k periods is the arithmetic mean of the forward intensities in the unitary periods of such horizon (spot in the first of them).

By applying (7.32) recursively, we finally find that

$$(1 + i_k)^k \;=\; \prod\nolimits_{r=1}^{k} (1+i_{r-1,r})\,,\;\; (k = 1,\dots n) \qquad (7.33)$$

i.e. the spot accumulation factor $1+i_k$, with reference to the horizon of k unitary consecutive periods, is the geometric mean of k forward accumulation factors relative to the single periods. In this sense the rate i_k on the e.h. $(0,k)$ in a coherent market is a functional mean, according to Chisini, of the forward rates $i_{r\text{-}1,r}$.

If, instead, the rates varying for unitary horizons are given as $i_{k\text{--}1,k},$ we implicitly find the spot prices, expressed by

$$v_k = (1 + i_k)^{-k} \;=\; \prod\nolimits_{r=1}^{k} (1+i_{r\text{-}1,r})^{-1}\,;\;\; (k = 1,\dots,n) \qquad (7.30')$$

Sometimes it is convenient to highlight the corresponding *forward discount (or advance interest) rate (implicit) structure* for annual e.h., expressed by

$$d_{k\text{--}1,k} \;=\; 1 - s_{k\text{--}1,k} \;=\; 1 - \frac{v_k}{v_{k\text{-}1}}\,,\quad (k = 1,\dots,n) \qquad (7.31'')$$

from which, recalling (7.28"), we find

$$1 - d_{k-1,k} = \frac{[1 - d_k]^k}{[1 - d_{k-1}]^{k-1}} , \qquad (k = 1,...,n) \qquad (7.32')$$

which links forward discount rate structure as a function of the spot ones. Applying recursively (7.32'), we finally find

$$[1 - d_k]^k = \prod_{r=1}^{k} [1 - d_{r-1,r}] , \qquad (k = 1,...,n) \qquad (7.33')$$

analogous to (7.33), i.e. the spot discount factor ($1-d_k$) referred to the horizon of k unitary consecutive periods is the geometric mean of the k forward discount factors of each period.

Example 7.16

In a coherent market the discount factors v_k , (k = 1,...,4), obtained from the spot prices for annual horizons up to 4 years, given in Example 7.14, are fixed. The forward price structure $s_{k-1,,k}$ for unitary securities for annual horizons is as follows:

$$s_{0,1} = 0.9628/1.0000 = 0.962800$$
$$s_{1,2} = 0.9371/0.9628 = 0.973307$$
$$s_{2,3} = 0.9008/0.9371 = 0.961263$$
$$s_{3,4} = 0.8788/0.9008 = 0.975577$$

The corresponding implicit forward interest rate structure is

$$i_{k-1,k} = [s_{k-1,k}]^{-1} - 1 = \frac{[1+i_k]^k}{[1+i_{k-1}]^{k-1}} - 1$$

and recalling the results of Example 7.14, the structure assumes the values:

$$i_{0,1} = 0.962800^{-1} - 1 = 0.038637 = \frac{1.038637}{1} - 1$$
$$i_{1,2} = 0.973307^{-1} - 1 = 0.027425 = \frac{1.033016^2}{1.038637} - 1$$
$$i_{2,3} = 0.961263^{-1} - 1 = 0.040298 = \frac{1.035437^3}{1.033016^2} - 1$$
$$i_{3,4} = 0.975537^{-1} - 1 = 0.025434 = \frac{1.032827^4}{1.035437^3} - 1$$

Let us verify (7.33) for the values obtained here:

k=1:	1.038637	$= 1.038637$
k=2:	1.033016^2	$= 1.038637 \cdot 1.027425$
k=3:	1.035437^3	$= 1.038637 \cdot 1.027425 \cdot 1.040298$
k=4:	1.032827^4	$= 1.038637 \cdot 1.027425 \cdot 1.040298 \cdot 1.025034$

The corresponding implicit forward discount rate structure (which is seldom used)

$d_{k\text{-}1,k} = 1 - s_{k\text{-}1,k}$ assumes the following values:

$$d_{0,1} = 0.037200; \quad d_{1,2} = 0.026693;$$
$$d_{2,3} = 0.038737; \quad d_{3,4} = 0.024423.$$

It is left as an exercise for the reader to verify (7.32') and (7.33'), recalling the results of Example 7.15.

The developments of the results obtained in Example 7.16, can be easily obtained using an *Excel* spreadsheet as follows, as can a comparison of the spot rates given by (7.28) and reported in Example 7.16.

Maturity	Spot price%	Fwd price	Spot rate	Fwd delayed rate	Fwd intensity	Fwd advance rate
1	96.28	0.962800	0.038637	0.038637	0.037910	0.037200
2	93.71	0.973307	0.033016	0.027425	0.027056	0.026693
3	90.08	0.961263	0.035437	0.040298	0.039507	0.038737
4	87.88	0.975577	0.032827	0.025034	0.024726	0.024423

Table 7.2. *Spot and uniperiod forward structure*

The *Excel* instructions are as follows: 2nd row: titles; from the 3rd row:

column A:	A3: 1; A4:= A3+1; copy A4-paste on A5 to A6;
column B:	insert date (spot price %) on B3 to B6;
column C:	C3:= B3/100; C4:= B4/B3; copy C4, then paste on C5 to C6;
column D:	D3:= (B3/100)^-(1/A3)-1; copy D3, then paste on D4 to D6;
column E:	E3:= (1/C3)-1; copy E3, then paste on E4 to E6;
column F:	F3:= ln (1+E3); copy F3, then paste on F4 to F6;
column G:	G3:= 1-C3; copy G3, then paste on G4 to G6.

The description of a forward structure can be completed with the extension to prices and interest rates for e.h. not only unitary but *of integer positive length* (then *uni- and multi-period*). The market gives at contract time t=0 the forward prices $s_{h,k}$ of the UZCBs paid in h and entitle us to the unitary amount in k, with h,k specified in (7.24).

Such prices $s_{h,k}$ can be expressed as elements of an $\mathrm{n} \times \mathrm{n}$ upper triangular matrix (if h is the row index and k the column index), i.e.

$$\left\| \begin{matrix} s_{1,1} & s_{1,2} & s_{1,3} & \cdot\cdot & s_{1,n} \\ & s_{2,2} & s_{2,3} & \cdot\cdot & s_{2,n} \\ & & s_{3,3} & \cdot\cdot & s_{3,n} \\ & & & \cdot\cdot & \cdot\cdot \\ & & & & s_{n,n} \end{matrix} \right\| \tag{7.34}$$

where:

$$s_{h,k} \begin{cases} = 1 \ , if \ \ h = k \\ < 1 \ , if \ \ h < k \end{cases} \quad , \qquad 1 \le h \le k \le n \tag{7.34'}$$

The number of elements in (7.34) is $n/(n$+1)/2, but the meaningful prices ($\neq 1$) are those for which $h<k$, the number of which is $n(n$-1)/2.

In the coherent market hypothesis, due to (7.18), for the dates (7.24) the *general formula* holds, that is the basis of a forward structure for a transaction made in t=0:

$$s_{h,k} = \frac{v_k}{v_h} \quad , \quad (1 \le h < k \le n) \tag{7.35}$$

Owing to:

$$\frac{v_k}{v_h} = \frac{v_k}{v_{k\text{-}1}} \frac{v_{k\text{-}1}}{v_{k\text{-}2}} \cdots \frac{v_{h+1}}{v_h}$$

we find, adapting the indices in (7.30), that:

$$s_{h,k} = \prod_{r=h+1}^{k} s_{r\text{-}1,r} = \prod_{r=h+1}^{k} (1+i_{r\text{-}1,r})^{-1} \tag{7.36}$$

If we want to express the constraint of the forward structure working on the rates, then indicating with $i_{h,k}$ the agreed interest rate in 0 on the e.h. (*h,k*), using (7.31) we obtain:

$$[1+i_{h,k}]^{k-h} = \frac{v_h}{v_k} = \left(\frac{v_k}{v_{k-1}}\,\frac{v_{k-1}}{v_{k-2}}\ldots\frac{v_{h+1}}{v_h}\right)^{-1} = \prod\nolimits_{r=h+1}^{k}\,[1\;+\;i_{r-1,r}] \qquad (7.37)$$

Equation (7.37) expresses the annual forward accumulation factor, averaged on e.h., as a geometric mean of the forward accumulation factors for each period and links it to the f.p. of the UZCB. In the last term of (7.37), we can read the *varying per period rates* applicable in the market; the comparison with the left side shows that $i_{h,k}$ is equivalent to the mean forward rate on the horizon (*h,k*) in the flat structure that follows from the exponential regime.

From (7.37) we find the delayed forward interest rate (annual base) on the e.h. (*h,k*):

$$i_{h,k} = s_{h,k}{}^{-1/(k-h)} - 1 \qquad (7.38)$$

or advance

$$d_{h,k} = 1 - s_{h,k}{}^{1/(k-h)} \qquad (7.39)$$

(obtaining this last formula using a generalization of (7.28")).

For a comparison between (7.38) and (7.39), we find the relation between rates, analogous to (7.28"')

$$d_{h,k} = i_{h,k}/(1+i_{h,k}) \qquad (7.39')$$

Example 7.17

Still using the structure of the SP given in Example 7.14, let us find the corresponding structure of the f.p. in a coherent market, leaving out the restriction of annual horizons. In this case, the upper triangular matrix $s_{h,k}$ ($1 \le h \le k \le 4$), with h = row index and k = column index, is of order 4 and assumes the values given below, found through (7.35).

Using v_1 = 0.9628; v_2 = 0.9371; v_3 = 0.9008; v_4 = 0.8788, let us find the prices matrix $s_{h,k}$ and forward rates $i_{h,k}$ by means of an *Excel* spreadsheet that has the following form.

Prices structure $s_{h,k}$

H	Price sp %	*k*=1	*k*=2	*k*=3	*k*=4
1	96.28	1.000000	0.973307	0.935604	0.912754
2	93.71		1.000000	0.961263	0.937787
3	90.08			1.000000	0.975577
4	87.88				1.000000

Rates structure $i_{h,k}$

H	*k*=1	*k*=2	*k*=3	*k*=4
1		0.027425	0.033841	0.030897
2			0.040298	0.032638
3				0.025034
4				

Table 7.3. *Uni- and multi-period forward structure*

The *Excel* instructions are as follows: 1st and 2nd row: empty; then:

price structure: from 3rd to 8th row. 3rd and 4th row: titles; rows 5-8:

column A	A5: 1; A6:= A5+1; copy A6, then paste on A7 to A8;
column B	insert data (spot prices %) on B5 to B8;
diagonal (k=h)	C5:1; D6:1; E7:1; F8:1;
1° supradiagonal (k=h+1):	D5:= \$B6/\$B5; copy D5, then paste on E6, F7;
2° supradiagonal (k=h+2)	E5:= \$B7/\$B5; copy E5, then paste on F6;
3° supradiagonal (h+2=k)	F5:= B8/B5;

rate structure: from 10th to 15th row. 10th and 11th row: titles; rows 12-15:

column A	A12: 1; A13:= A12+1; copy A13, then paste on A14 to A15;
1° supradiagonal (k=h+1)	D12:= D5^-(1/(\$A6-\$A5))-1; copy D12, then paste on E13, F14;
2° supradiagonal (k=h+2)	E12:= E5^-(1/(\$A7-\$A5))-1; copy E12, then paste on F13;
3° supradiagonal (h+2=k)	F12:= F5^-(1/(A8-A5))-1;

oher cells: empty.

In the price matrix the values on the first supradiagonal are obviously the prices $s_{k-1,k}$ obtained in Example 7.16. The corresponding implicit rates (excluding those that are reduced to spot rates) are found by means of (7.37).

Let us verify the properties of (7.37). With the numbers obtained by the previous matrix $\|i_{h,k}\|$ we find:

e.h. 1-3:	1.033841^2	=	$1.027425 \cdot 1.040298$
e.h. 2-4:	1.032638^2	=	$1.040298 \cdot 1.025034$
e.h. 1-4:	1.030897^3	=	$1.027425 \cdot 1.040298 \cdot 1.025034$

If we complete the matrix $\|i_{h,k}\|$ using $i_{h,k} = 0$ if $h \geq k$, and add 1 to each element of the square matrix $\|i_{h,k}\|$ thus obtained, then in each square submatrix, extracted from the new matrix and such that its main diagonal has elements $i_{h,k}$ satisfying $h<k$, the number written in the NE vertex is the geometric mean of those which appear on the main diagonal of the submatrix.

The previous considerations show how the gathering of market prices implicitly leads us to formalize on the given time horizon (h,k) a financial exchange law, defined only on integer time variables, that can be expressed by means of discount factors $s_{h,k}$ (<1 if $h<k$) defined in (7.5) or analogously by means of accumulation factors $r_{h,k} = 1/s_{h,k}$ or interest rates $i_{h,k}$ or discount rates $d_{h,k}$.

On the contrary, we can think – as was already observed at the beginning of this chapter – that the term structure valid in a market follows the definition of an empirical financial law that in a given time interval holds on the market for simple operations. Such a formulation can be extended to complex operations, in particular to *annuities* and *amortizations*. This will be considered in Chapter 8.

Observation

From the previous formulations, in particular from (7.33), it is obvious that the term structure maintains the principle of compound accumulation, even if in a more general way that leads to varying rates.

Building up the term structure of spot and forward rates.

Referring to the *bond market*, the use of spot rates implies that the financial flows generated by different securities are assumed to be discounted at the same rate. It is then essential to deduce from the available data the so-called *term structure of spot rates* applicable to all securities of the market as a function of the different evaluation time interval.

It is possible to find this structure, expressed by *spot rates* for the same interval terms, starting from an observation taken from the market on the issues prices of bonds with maturities increasing in arithmetic progression (according to natural numbers, on the basis of an appropriate choice of the unit time). We can then find a sequence of spot rates applicable to the examined market.

The calculation procedure can be described as follows: decide the time unit and change the rates accordingly; carry out a statistical observation of the issue prices (i.e. at time 0) of the securities in a bond market with different financial profiles; obtain a price for each of the maturities $h = 1,...,n$. If we refer to a *coupon bond with maturity h*, knowing the price $V_{(h)}$ and the coupon amounts $\{I_k\}$, $k = 1,...,h$, and the redemption C_h for the bond with maturity h [10] we can write the solving system, where the price $V_{(h)}$ of the bonds maturing after h periods is made equal to the present value according to the unknown rate structure. This system is given by:

$$V_{(h)} - H_{h-1} = \frac{I_h + C_h}{(1+i_h)^h} \quad , \quad h = 1,...,n \tag{7.40}$$

where:

$$H_0 = 0 \;\; , \;\; H_h = \sum_{k=1}^{h} \frac{I_k}{(1+i_k)^k} \;\; , \; h = 1,...,n\text{ -}1 \tag{7.41}$$

In (7.40), the unknowns i_h appear in a triangular way, in the sense that in the 1st equation (h=1) we have only i_1 which is then found directly, in the 2nd equation (h=2) we have i_1 and i_2 which are again found directly knowing i_1 in H_1; then, in the h^{th} equation the first term is found using (7.41) and in the second term the only unknown is i_h which is found immediately.

This procedure assumes the existence of a sequence of securities with maturities distributed at regular intervals and quoted at equilibrium prices. Note that, different from the *yield rates*, there exists for each time interval a biunivocal correspondence between *spot rates* and prices.

Example 7.18

Let us apply the procedure to build up the spot rate structure, starting from the price sequence, referring to five bond types (which are not all zero-coupon bonds),

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10 For the ZCB it is enough to set all the values $\{I_k\}$ at zero.

with equally spaced maturities. Data are summarized in the first four columns of the following tables; each row is referred to one bond; the first three are zero-coupon bonds and the other two have fixed coupons. The last column gives the results, obtained as specified below, i.e. the *spot* rates referred to the length of the bond specified in the here 2nd column (but valid in the market of the bonds considered).

Nominal value	Residual length (in years)	Semiannual coupon	Market price	Spot rate (%) on given length
100	0.25	0%	98.90	4.524
100	0.50	0%	97.64	4.893
100	1.00	0%	95.11	5.141
100	1.50	3%	100.84	5.495
100	2.00	2.50%	98.80	5.741

Table 7.4. *Computation of spot rate structure*

For each of the zero-coupon bonds, the price is given by $100 \cdot v_k$ where k=0.25; 0.50; 1.00 and the *spot* rate is found applying (7.8), i.e.

$$0.9890^{\,-1/0.25} = 0.04524\ ;\ 0.9764^{\,-1/0.50} = 0.04893\ ;\ 0.9511^{\,-1} = 0.05141$$

The first three *spot rates* are then obtained in the last column. The 4th bond, with fixed coupon, is entitled supplies: (0.5,3), (1,3), (1.5,103), and the price is the sum of the present values of each amount using the *spot* rate referred to its time. The first two rates (in 2nd and 3rd row) are already known: their values are $i_{0.50}$ = 4.893% and $i_{1.00}$ = 5.141%. The third, i.e. $i_{1.50}$, is obviously the solution to the following equation:

$$\frac{3}{1.04893^{\,0.50}} + \frac{3}{1.05141} + \frac{103}{(1+i_{1.50})^{1.50}} = 100.84$$

from which $i_{1.50}$ = 0.05495. The 5th bond, with fixed coupon, is entitled to the following supplies: (0.5; 2.5), (1; 2.5), (1.5; 2.5), (2; 102.5), and here the price is the sum of the discounted values with four spot rates referred to the intervals which are multiples of a half-year. The first three, indicated with $i_{0.50}$, $i_{1.00}$ and $i_{1.50}$, are already known. The fourth, i.e. $i_{2.00}$, is obtained analogously as the solution of the following equation:

$$\frac{2.5}{1.04893^{\,0.5}} + \frac{2.5}{1.05141} + \frac{2.5}{1.05495^{\,1.5}} + \frac{102.5}{(1+i_{2.00})^{2}} = 98.80$$

from which $i_{2.00}$ = 0.05741. In this way, substituting the results found in the subsequent equations for fixed coupon bonds, we find the whole *term structure* of *spot rates* corresponding to the price gathered on the market for the examined securities.

Also for a forward contract, we can build up a *term structure of forward rates*. It is enough to refer to the building up of a sequence of *spot rates* seen before and obtaining from them the implicit *forward* rates, on the basis of market coherence.

7.5.2. *Structures with fractional periods*

As already shown at the beginning of section 7.5.1, we clarified that the time structure in "discrete" scheme is referred to unitary periods, but the unit of times is not necessarily a year. In financial practice, there are market structures with a period which is not annual, but fractional, in which spot and forward prices, rates, intensities r.m have as a basis a fraction of a year (semester, quarter, month, etc.), while pluriennial periods are not used. In such a case, the e.h. and the bond maturities will be multiples of such fractional periods. Let us give a brief insight into this argument.

We must observe that the definition and transformation formulae given in section 7.5.1 are still valid, without any modification, with fractional periods, except for the time measure that is no longer a year, but a fraction of a year. Then the prices concern assets with fractional maturities and the rates refer to periods that are fractions of a year.

It is unnecessary to repeat here the formulae to adapt them to this case: it is enough to declare the different time unit. The argument will then be clarified developing, using *Excel,* Examples 7.19, 7.20 and 7.21, which closely follow Examples 7.15, 7.16 and 7.17, which refer to annual bases.

Example 7.19

On the UZCB market there is fixed today (t=0) the following SP as a function of the quarterly maturities, which define v_k , assuming the quarter as the unit to measure time:

– 0.9866 with maturity after one quarter; 0.9788 with maturity after two quarters;

– 0.9654 with maturity after three quarters; 0.9521 with maturity after four quarters.

In the following *Excel* table, with formulations analogous to those see in Example 7.15, the corresponding structures of spot delayed and advance rate and also of the intensity r.m. are set out.

Maturity	Spot price	Delayed spot rate	Spot intensity r.m.	Advance spot rate
1	0.9866	0.0135820	0.0134906	0.0134000
2	0.9788	0.0107716	0.0107140	0.0106568
3	0.9654	0.0118067	0.0117376	0.0116690
4	0.9521	0.0123469	0.0122713	0.0121963

Table 7.5. *Quarterly basis spot structure*

Comparing with the *Excel* instruction of Example 7.15, to go from the 2nd column to 3rd and 5th column we do not have to divide by 100, because they are prices of UZCB.

Example 7.20

Using the data on prices given in Example 7.19, in the following *Excel* table are fixed starting form the spot structure, with formulations analogous to what was seen in Example 7.16, the corresponding one period structure of forward prices and rates, delayed and advance, and also the intensity r.m.

Maturity	Spot price	Fwd price	Spot rate	Fwd delayed rate	Fwd intensity	Fwd advance rate
1	0.9866	0.986600	0.013582	0.013582	0.013491	0.013400
2	0.9788	0.992094	0.010772	0.007969	0.007937	0.007906
3	0.9654	0.986310	0.011807	0.013880	0.013785	0.013690
4	0.9521	0.986223	0.012347	0.013969	0.013872	0.013777

Table 7.6. *Quarterly basis spot and uni-period forward structure*

Comparing with the *Excel* instruction of Example 7.16 to go from the 2nd column to 3rd and 4th column we do not have to divide by 100, given that one considers prices of UZCB.

Example 7.21

Using the data on prices given in Example 7.19, in the following *Excel* table with formulations analogous to what seen in Example 7.17, the corresponding multiperiod structure of forward prices and rates is set out.

Prices structure $s_{0;h,k}$

h	Spot price	k=1	k=2	k=3	k=4
1	0.9866	1.000000	0.992094	0.978512	0.965031
2	0.9788		1.000000	0.986310	0.972722
3	0.9654			1.000000	0.986223
4	0.9521				1.000000

Rates structure $i_{0;h,k}$

H	k=1	k=2	k=3	k=4
1		0.007969	0.010920	0.011936
2			0.013880	0.013925
3				0.013969
4				

Table 7.7. *Quarterly basis uni- and multi-period forward structure*

The *Excel* instructions are those in Example 7.17.

Observations

In banks and Stock Exchange markets it is used to consider nominal annual return rates even in case of fractional structures. In the considered case, with quarterly structure and data from Example 7.20 (with structures of any frequency, it is enough to use m instead of 4), given the *uniperiod forward rates* in the 5th column of the following *Excel* table, it is enough to multiply by four to have (in the 6th column) the nominal annual return in the current quarters.

However, these values show on an annual basis the return of each quarter, but do not give the effective return rate obtained on an annual e.h. To obtain this, starting from an investment in 0 with a given structure, we proceed as follows. The return rate r_k on an e.h. of k periods is found from

$$1+r_k = (1 + i_k)^k = \prod_{r=1}^{k}(1+i_{r-1,r}) \tag{7.33''}$$

which extends (7.33) referring to fractional structures. For k=1,2,3,4, the values of i_k are the quarterly spot rates shown in the 4th column of the table, while the values of r_k are shown in the 7th column and are the return rates on the e.h. of the first k quarters, which is also the basis. In particular, for k=4 we obtain (with the data of Example 7.20 to which the table is referred) the rate 0.050310, which is the effective return rate for one year and on an annual basis, better than the nominal rate.

k	Spot price	Fwd price	Spot rate	Fwd rate	Nominal annual rate	Return rate 0-*k*
1	0.9866	0.986600	0.013582	0.013582	0.054328	0.013582
2	0.9788	0.992094	0.010772	0.007969	0.031876	0.021659
3	0.9654	0.986310	0.011807	0.013880	0.055521	0.035840
4	0.9521	0.986223	0.012347	0.013969	0.055876	0.050310

Table 7.8. *Nominal annual rates in the current quarters*

The *Excel* instructions for this table are as follows. After 2nd row for titles, the first 5 columns are the same as those in Example 7.20; in addition:
6th column: (nominal annual rates) F3:= 4*E3; copy F3, then paste on F4 to F6;
7th column: (return rates e.h. 0-k) G3:= (1+D3)^A3-1; copy G3, then paste on G4 to G6.

7.5.3. *Structures with flows "in continuum"*

Let us consider the case in which the flows are continuous (for instance a continuous trading market). Let us first observe that, assuming continuous time, the formulae (7.8) and (7.8") are enough to define the spot rate *i(x,y)* and the spot intensity $\phi(x,y)$ according to the SP *v(x,y)*.

In addition, with continuous payment flows, the implicit structure, corresponding to the spot structure, we have to consider infinitesimal e.h. *(y, y+dy)* where the forward prices *s(x;y,y+dy)* go to 1 and the implicit forward rates *i(x;y,y+dy)* go to 0, losing any meaning. It is then appropriate to refer directly to the *instantaneous discount intensity time structure*. The spot structure is found from formulae of the type of (2.23) (or inversely (2.24)) reinterpreted in market terms. The term structure is found from the spot structure based on the known constraints. The functions $\delta(x,y)$ are then the starting point of the term structure in continuous time.

With discrete schedules we can build up a term structure starting from an instantaneous intensity $\delta(x,y)$ that gives, always in symmetric hypothesis, an exchange law in continuum, from which are found spot and forward prices, rates and intensities r.m.. We show here the following formulae that are immediately justifiable (referring to a transaction time that is not to restrictive too put in $t = 0$):

$$v_k = e^{-\int_0^k \delta(0,u)du} \qquad (7.42)$$

$$s_{h,k} = e^{-\int_h^k \delta(0,u)du} \qquad (7.43)$$

$$i_k = e^{(\int_0^k \delta(0,u)du)/k} - 1 \tag{7.44}$$

$$i_{h,k} = e^{(\int_h^k \delta(0,u)du)/(k-h)} - 1 \tag{7.45}$$

$$\phi_k = \frac{1}{k}\int_0^k \delta(0,u)du \tag{7.46}$$

$$\phi_{h,k} = \frac{1}{(k-h)}\int_h^k \delta(0,u)du \tag{7.47}$$

Example 7.22

For investment on the horizon (0;5), the return financial law is ruled by the instantaneous intensity $\delta(0,u)$, according to current time u, $(0 \le u \le 5)$, for operations agreed in 0, defined by $\delta(0,u) = 0.04 + 0.00564\ u - 0.00033\ u^2$, where, for example: $\delta(0;0) = 0.04$; $\delta(0;2) = 0.05$; $\delta(0;5) = 0.06$.

On the horizon (0;5), by means of (7.47), we obtain the intensities r.m.

$\phi_{k\text{-}1,k}$ for annual intervals, that we will indicate with H_k . We obtain:

$$H_k = \int_{k-1}^k \delta(0,u)du = \left| const. + 0.04\ u + 0.00282\ u^2 - 0.00011\ u^3 \right|_{k-1}^k =$$

$$= 0.04 + 0.000282\ (2k - 1) - 0.00011\ (3k^2 - 3k + 1)$$

Thus:

$H_1 = 0.042710$; $H_2 = 0.047690$; $H_3 = 0.052010$; $H_4 = 0.055670$; $H_5 =$ 0.058670

from which, due to (7.43), the values:

$$s_{0,1} = e^{-H_1} = 0.958189\ ;\ s_{1,2} = e^{-H_2} = 0.953429\ ;\ s_{2,3} = e^{-H_3} = 0.949319\ ;$$
$$s_{3,4} = e^{-H_4} = 0.945851\ ;\ s_{4,5} = e^{-H_5} = 0.943018$$

follow. From the intensities $\phi_{k\text{-}1,k} = H_k$ for annual intervals we find, due to (7.45), the corresponding implicit forward rates $i_{k\text{-}1,k} = e^{-H_k} - 1$, obtaining:

$i_{1,2} = 0.048845$; $i_{2,3} = 0.053386$; $i_{3,4} = 0.057249$; $i_{4,5} = 0.060425$.

The spot intensities r.m. for k years are written:

$$\phi_k = \frac{1}{k}\sum_{r=1}^{k} H_r$$

and it follows that:

$\phi_1 = 0.042710$; $\phi_2 = 0.045200$; $\phi_3 = 0.047470$; $\phi_4 = 0.049520$; $\phi_5 = 0.051350$.

In addition, the forward intensities r.m. are given by:

$$\phi_{h,k} = \frac{1}{k-h}\sum_{r=h+1}^{k} H_r$$

With the given instantaneous intensity we obtain, for example:

$$\phi_{2;4} = (0.05201+0.05567)/2 = 0.05384.$$

The spot rates i_k on an horizon of k years, expressed by (7.44), but which can also be written in the form $e^{\phi(k)} - 1$, are:

$i_1 = 0.043635$; $i_2 = 0.046237$; $i_3 = 0.048615$; $i_4 = 0.050767$;
$i_5 = 0.052691$.

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Chapter 8

Annuities, Amortizations and Funding in the Case of Term Structures

8.1. Capital value of annuities in the case of term structures

In Chapter 5 the annuity evaluation, defined as a financial operation for which the amounts do not show any sign inversion, has been made in the case of a flat structure. Only in Chapter 6, regarding annuities formed by loan amortization installments and the management of bond loans, did we consider briefly the case of varying rates.

It is appropriate here to extend the scenario, assuming that such supplies are made in a perfect market, featured by a given term rate structure and then spot prices for goods with delayed delivery, obtained applying discount factors to the forward values of such goods. In a wider context than that of the security market and with the symbols introduced in Chapter 7, if $v(y,z)$ defined in (7.5) is used as a discount factor to apply to the value S_z an asset with purchase in y and delivery at time $z>y$ to have the *spot price* $P_{y,z}$, then

$$P_{y,z} \;=\; v(y,z)\, S_z \tag{8.1}$$

while for a transaction at time $x<y<z$, fixing the value defined in (7.16), the *forward price* $P_{x;y,z}$ in y of an asset of value S_z at delivery z is given by

$$P_{x;y,z} \;=\; s(x:y,z)\, S_z \tag{8.1'}$$

If the market is perfect (then the property of independence from the transaction time holds true, given by (7.19)), we have $\forall x$: $s(x;y,z) = v(y,z)$, where $P_{x;y,z} = P_{y,z}$. However, the market *coherence* property, defined in Chapter 7, is enough for the developments of this chapter.

Let us consider a complex operation O whose amounts have the same sign and are payable according to a tickler with n dates in a given interval; we have seen already that it is not reductive to assume this tickler is equally spaced[1]. The rates are per period in the case of a given term structure. For simplicity we will refer mainly to annual rate structures and to annual periods, unless otherwise stated, for which what has been said in section 7.5.2 holds.

Then O has a tickler on a time horizon of n years, that can be written: $(T,T+1,....,T+n)$; let us indicate the corresponding amounts with R_0, R_1,...,R_n, assuming them to be all negative and at least one positive. It is known that this operation O is called *annuity,* temporary for n years[2] (R_h are the *installments* of the annuity) and it can never be fair. In an annuity with delayed payments it is with certainty $R_0 = 0$; if the payments are advance, it definitely is $R_n = 0$.

Generalizing the formulation seen in Chapter 5, where we assumed a *flat rate structure*, we can here evaluate the annuity O at any time on the basis of a *discrete term* according to what was specified in section 7.5.1. It is clear that the results that will be obtained in this chapter – where we generalize those obtained in Chapters 5 and 6 considering annuities, amortizations of shared and unshared loans and funding, evaluated on the basis of varying rates according to term structures – are meaningful only if we can assume that the rate structure, introduced at the starting time, remains valid for the whole time horizon of the considered operation. On the contrary a periodic adjustment of the structure is necessary to evaluate the pro-reserves.

It is convenient here to reinterpret the spot and forward prices defined in Chapter 7 also in terms of *discount factors* for the evaluation. In addition, recalling an observation introduced in section 7.5.3, it is convenient, also with discrete ticklers, to obtain the term structure following from a function (integrable) of *instantaneous intensity* $\delta(x,y)$. We suppose that this intensity on the considered time horizon (assuming the reflexivity and symmetry, with the meaning specified in Chapter 2, and in particular cases also the strong decomposability) holds.

1 This topic has been discussed in section 7.5.1.

2 We could consider the case of perpetuities as the limit case for $n \to \infty$, but it is hard to introduce a term structure on an infinite interval.

For known results, generalizing the formulae from (7.42) to (7.47) we find the following *spot* and *forward* elements of the structure in the discrete scheme, i.e. with the constraints (7.24):

- *the spot present value*

$$v(T,T+k) = a\,(T+k,T) = 1/m(T,T+k) = e^{-\int_T^{T+k}\delta(T,\tau)d\tau} \tag{8.2}$$

- *the delayed interest spot rate (on the unitary base)*

$$i(T,T+k) = m(T,T+k)^{1/k} - 1 = v(T,T+k)^{-1/k} - 1 = \left(e^{\int_T^{T+k}\delta(T,\tau)d\tau}\right)^{1/k} - 1 \tag{8.3}$$

- *the spot return at maturity*

$$\phi(T,T+k) = \left(\int_T^{T+k}\delta(T,\tau)d\tau\right)/k \tag{8.4}$$

- *the advance interest spot rate (on the unitary base)*

$$d(T,T+k) = 1 - v(T,T+k)^{1/k} = 1 - \left(e^{-\int_T^{T+k}\delta(T,\tau)d\tau}\right)^{1/k} \tag{8.5}$$ [3]

- *the forward present value*

$$s(T;T+h,T+k) = v(T,T+k)/\,v(T,T+h) = e^{-\int_{T+h}^{T+k}\delta(T,\tau)d\tau} \tag{8.2'}$$

- *the delayed interest forward rate (on the unitary base)*

$$i(T;T+h,T+k) = s(T,T+h,T+k)^{-1/(k-h)} - 1 = \left(e^{\int_{T+h}^{T+k}\delta(T,\tau)d\tau}\right)^{1/(k-h)} - 1 \tag{8.3'}$$

- *the forward return at maturity*

$$\phi(T;T+h,T+k) = \left(\int_{T+h}^{T+k}\delta(T,\tau)d\tau\right)/\,(k-h) \tag{8.4'}$$

- *the advance interest forward rate (on unitary base)*

$$d(T;T+h,T+k) = 1 - s(T;T+h,T+k)^{1/(k-h)} = 1 - \left(e^{-\int_{T+h}^{T+k}\delta(T,\tau)d\tau}\right)^{1/(k-h)} \tag{8.5'}$$

Generalizing what was seen in Chapter 4, where a flat rate structure is considered, the value in T^* of O is called *capital value* of the annuity; or, more

3 From (8.3) and (8.5) it follows that *i(T,T+h)* and *d(T,T+h)* are (mean) annual rates in the interval from *T* to *T+h*.

precisely: *initial value* or *present value of prompt annuity* if $T^* = T$; *present value of deferred annuity* if $T^* < T$; *final value* if $T^* = T+n$. The *present value* (then with $T^* \leq T$) is expressed by

$$V_a(T^*) = \sum_{h=0}^{n} R_h\, v(T^*, T+h) = \sum_{h=0}^{n} R_h\, [1 - d(T^*, T+h)]^{T+h-T^*} \quad (8.6)$$

The *final value* is expressed by

$$V_f(T^*) = \sum_{h=0}^{n} R_h\, m(T+h, T^*) = \sum_{h=0}^{n} R_h\, [1 + i(T+h, T+n)]^{n-h} \quad (8.7)$$

Relations (8.6) and (8.7) take into account the spot rates that are valid on the respective payment time.

On the basis of considerations discussed for financial operations (see Chapter 4), the operations $O'' \cup (-W_a(T^*), T^*)$ with $T^* \leq T$ and $O'' \cup (-W_f(T^*), T^*)$ with $T^* = T+n$ are fair according to the adopted financial laws.

Clearly if T^* and T are integers we can give an *integer term structure* of rates and values for n unitary periods. Let us assume such a position, adopting the formulations (7.25) and the positions from (7.26) to (7.39) and indicating the times (that are also the distances with sign from the origin 0, that we choose as the reference time for the rates structure) with lower case letters.

Given the above, the initial value of the prompt annuity on the horizon $[0,n]$, obtainable from (8.6) with $T^* = 0$, is given by

$$V_a(0) = \sum_{k=0}^{n} R_k v_k = \sum_{k=0}^{n} R_k\, (1+i_k)^{-k} \quad (8.6')$$

or, according of the forward rates in the term structure[4],

$$V_a(0) = \sum_{k=0}^{n} R_k \prod_{r=1}^{k} 1/(1+i_{r-1,r}) \quad (8.6'')$$

Example 8.1

Using n=5, let us assign on the market at time 0 the structured system of interest rates on annual periods:

$i_{0,1} = 0.0418$; $i_{1,2} = 0.0461$; $i_{2,3} = 0.0524$; $i_{3,4} = 0.0485$; $i_{4,5} = 0.0432$

---

4 In (8.6'') when k=0, the product is put equal to 1. The same observation is valid for some other following formula.

Let us consider an annual annuity-immediate on the horizon (0,5), formed by the monetary amounts

R_1 = 1,250 ; R_2 = 1,389 ; R_3 =1,450 ; R_4 = 1,310 ; R_5 = 1,100

By using such data applied directly by (8.6") we obtain the present value of the prompt annuity

$V_a(0) = R_1(1.0418)^{-1} + R_2(1.0418 \cdot 1.0461)^{-1} + R_3(1.0418 \cdot 1.0461 \cdot 1.0524)^{-1} +$
$+ R_4(1.0418 \cdot 1.0461 \cdot 1.0524 \cdot 1.0485)^{-1} + R_5(1.0418 \cdot 1.0461 \cdot 1.0524 \cdot 1.0485 \cdot 1.0432)^{-1} =$
= 5,704.78.

If the spot rates i_k , that we find by coherence with the previous ones, were given directly, resulting in

$i_1 = 0.0418$; $i_2 = 0.0439$; $i_3 = 0.0468$; $i_4 = 0.0472$; $i_5 = 0.0464$

(from which the prices would be

$v_1 = 0.59877$; $v_2 = 0.917577$; $v_3 = 0.871890$; $v_4 = 0.831559$; $v_5 = 0.797123$)

we could apply (8.6') still obtaining 5704.78.

To obtain the initial value of annuities-deferred of m years, assuming a rate structure for $m+n$ years, it is enough to multiply $V_a(0)$ by a discount factor relative to the deferment. Such a discount factor is given by v_m and therefore the present value in 0 of the annuity-deferred by m years with installments R_k at times $m+k$, ($k = 0,\dots,n$), is expressed, according to the rate structure, by

$$_{m/}V_a(0) = \prod_{r=1}^{m} (1 + i_{r-1,r})^{-1} \sum_{k=0}^{n} R_k \prod_{r=1}^{k} (1 + i_{m+r-1,m+r})^{-1} \qquad (8.8)$$

Example 8.2

Let us consider at 0 the annual annuity-due, deferred by 3 years, consisting of 3 payments:

R_3 = 10,500 ; R_4 =11,600 ; R_5 = 40,300

The varying rates structure

$i_{0,1} = 0.030$; $i_{1,2} = 0.035$; $i_{2,3} = 0.040$; $i_{3,4} = 0.037$; $i_{4,5} = 0.034$

The present value of the deferred annuity is then

$_{m/}V_a(0) = (1.030 \cdot 1.035 \cdot 1.040)^{-1}$ [10500 + 11600 $(1.037)^{-1}$ +
+ 40300 $(1.037 \cdot 1.034)^{-1}$] = 53459.71

Using the same hypothesis, the final value of the annuity at time n is expressed as a function of the accumulation forward factor $r_{k,n} = r(0;k,n)$ by

$$V_f(n) = \sum_{k=0}^{n} R_k \, r_{k,n} = \sum_{k=0}^{n} R_k \,(1+i_{k,n})^{n-k} \tag{8.7'}$$

or, as a function of the forward rates in the term structure

$$V_f(n) = \sum_{k=0}^{n} R_k \prod_{r=k+1}^{n} (1+i_{r-1,r}) \tag{8.7''}$$

Relations (8.6"), (8.8) and (8.7") directly use the varying rates that come from the market conditions.

Example 8.3

On a three year interval, assuming the semester as a time unit, let us assign the (spot and forward) interest rates structure on a semiannual base as well as the semiannual annuity-immediate, whose payments are

$R_1 = 8{,}500$; $R_2 = 9{,}250$; $R_3 = 8{,}620$; $R_4 = 12{,}628$; $R_5 = 4{,}644$; $R_6 = 6{,}240$

Let us find the final value, extracting from the structure the following uniperiod forward rates:

$i_{1,6} = 0.0490$; $i_{2,6} = 0.0475$; $i_{3,6} = 0.0465$; $i_{4,6} = 0.0450$; $i_{5,6} = 0.0445$

The final value of the annuity is then

$$V_f(6) = 8500 \cdot 1.0490^5 + 9250 \cdot 1.0475^4 + 8620 \cdot 1.0465^3 + 12628 \cdot 1.0450^2 +$$
$$+ 4644 \cdot 1.0445 + 6240 = 56693.59$$

8.2. Amortizations in the case of term structures

Extending what has been said in Chapter 6, with the positions and symbols defined above, we can develop the theory of amortizations assuming a financial law obtained according to a term structure. To remain closer to the financial market behavior, *we will not assume the independence of the structure from the referring time.*

The amortization with varying rates has been considered in section 6.5 only for the case of uniperiod *spot* rates. It is appropriate to refer to this scheme when it is not realistic to assume the validity of the structure for the whole length of the amortization. For the opposite assumption, we assume then the variability of the rates according to a more general rate structure scheme fixed at time 0 of the loan inception, where the amortization flow is, technically, an “annuity” for which the initial value calculated on the basis of such structure is equal to the debt to be amortized. The amortization installments are mostly periodic, thus annual, semiannual, etc.

We will refer to cases of the annual period; for a period of a different length it is sufficient to change the unit of measure. In the presence of pre-amortization, it is sufficient to refer to the true amortization interval, in which the principal repayments are paid, following the one in which only interest is paid.

However, in the case of varying installments (as far as the outstanding loan balance will not increase with time) it is obvious that, if the initial debt, the length of the amortization and the rate structure valid in the same interval are given, infinite solutions exist for the installments vector used to amortize the debt. This means that, from the lender point of view[5], the payment of the lent amount and the encashment of such installments form altogether a fair operation in relation to the given rate structure. Instead, if the installment invariance is postulated, then the financial equilibrium equation gives the constant installment as the only unknown.

We will limit our analysis to the following cases of amortization:

− the general case of varying installments;
− the particular case of constant installments;
− the particular case of constant principal repayments;
− the case of life amortization.

8.2.1. *Amortization with varying installments*

Let there be the initial debt S to be amortized in n unitary periods (in particular annual), according to a term structure given at initial time 0, for which formulations (7.25) and relations (7.26) and (7.39) hold true. The equivalence between debt and vector $\{R_k\}$, (k=0,1,....,n), of the installments paid at the assigned dates gives the constraint that defines a solution $\{R_k\}$ for the amortization. This is found from (8.6') or (8.6") putting $V_a(0) = S$. We then obtain the following relation, that is the constraint of *financial closure* between debt and amortization installments:

5 We highlight the lender point of view that is usually the “stronger party” in the contract. It is clear that the borrower adapts himself to the conditions fixed by the lender and accepts the contract if, on the basis of his utility and his alternative possibilities on the market, he considers convenient the conditions offered by the lender.

$$S = \sum_{k=0}^{n} R_k \, v_k = \sum_{k=0}^{n} R_k \, (1+i_k)^{-k} = \sum_{k=0}^{n} R_k \prod_{r=1}^{k} (1+i_{r-1,r})^{-1} \quad (8.9)$$

where v_k and i_k correspond to the given $i_{h,k}$ on the basis of the known relations. Therefore, given S, n and the term structure, a vector of components R_k, (k=0,1,....,n), that satisfies (8.9) gives a solution to the amortization[6]. As already mentioned, some restrictions on the arbitrariness of $\{R_k\}$ follow from the eventual constraint of the outstanding loan balance not increasing in time. In addition, we talk about amortizations

- with *delayed* installments, if $R_0 = 0$;
- with *advance* installments, if $R_n = 0$.

Due to (8.9) it is obvious that $\{i_{r-1,r}\}$ gives a rate structure of cost for the borrower, i.e. a generalized internal rate of return (GIRR) in the sense set out in section 4.4.2.

As happens for a constant rate, each installment is divided into *principal repaid* and *interest paid*, and can, by convention, be paid by the debtor in a delayed or advance way: if both are paid delayed or advance, one has amortization with *delayed* or, respectively, *advance* installments.[7]

Amortization with delayed installments

The development of the delayed amortization schedule includes the interest amounts I_k , the principal repayments C_k and the outstanding balances D_k at time k, that follows from the following equations system

$$(k = 1, \dots, n) \begin{cases} I_k = D_{k-1} \, i_{k-1,k} \\ D_k = D_{k-1} - C_k \\ R_k = I_k + C_k \end{cases} \quad (8.10)$$

6 It is obvious that independence from the transaction time, an assumption that in fact is not very realistic, could lead to the equality between forward and spot rates, i.e.: $\{i(0;h;k)\} = \{i(h,k)\}$, $\forall h,k$, from which it would follow that the outstanding amounts and outstanding loan balances expressed by (8.11) or (8.11') would coincide. However, if the amortization is agreed with indexed interests on the basis of the resulting market rates (where the agreed schedule in 0 on the basis of the term structure at this time is only an estimated calculatation), if the market does not behave as a "perfect market", the inequality $i(r\text{-}1,r) \neq i(0;\, r\text{-}1.r)$ can follow at the r^{th} year with possible differences between estimated and final balance. In such a case some adjustments are needed.

7 We do not consider here the case of advance interest payments and delayed principal repayments – sometimes used in the past in the particular case of *German amortization* (see Chapter 6) – because this scheme is not used often.

with the initial condition $D_0 = S$. From here follows:

Theorem. For a delayed amortization in the case of a term structure, from the recursive relations (8.10) *for the outstanding loan balances we obtain*

$$D_h = \sum_{k=h+1}^{n} R_k\, s_{h,k} = \sum_{k=h+1}^{n} R_k \prod_{r=h+1}^{k} ([1+i_{r-1,r})^{-1} \qquad (8.11)$$

*that extends to the case h>*0 *the relation of financial closure* (8.9) *with* $R_0 = 0$ (case h=0, S=D_0). *Therefore,* $\forall k$ *the exchange of the outstanding balance in k with the outstanding installment flow at their respective due dates is fair, i.e. on the basis of the given structure the outstanding balance coincides with the pro-reserve.*

Proof. Proceeding by induction, let us verify (8.11) for h=1. Since D_0 is given by (8.9) with $R_0 = 0$ and taking into account (7.28'''), we obtain:

$I_1 = i_1\, D_0 = d_1(R_1 + \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r})$;

$C_1 = R_1 - I_1 = R_1\, v_1 - d_1 \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}$

$D_1 = D_0 - C_1 = (R_1\, v_1 + \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}) - (R_1\, v_1 - d_1 \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}] =$

$= \sum_{k=2}^{n} R_k\, (\prod_{r=1}^{k} s_{r-1,r} + d_1 \prod_{r=2}^{k} s_{r-1,r}) = \sum_{k=2}^{n} R_k \prod_{r-2}^{k} s_{r-1,r}$,

because v_1+d_1 = 1.

Let us then verify that if (8.11) is true for $h \geq 1$, it is also true for h+1. Recalling (7.39'), if $D_h = \sum_{k=h+1}^{n} R_k \prod_{r=h+1}^{k} s_{r-1,r}$ then $I_{h+1} = i_{h,h+1} D_h = d_{h,h+1}(R_{h+1}$ $+ \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r})$;

$C_{h+1} = R_{h+1} - I_{h+1} = R_{h+1}\, s_{h,h+1} - d_{h,h+1} \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r}$; then $D_{h+1} =$ $D_h - C_{h+1} = (R_{h+1} s_{h,h+1} + \sum_{k=h+2}^{n} R_k \prod_{r=h+1}^{k} s_{r-1,r}) - (R_{h+1}\, s_{h,h+1} -$ $d_{h,h+1} \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r}) = \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}$,

because $s_{h,h+1} + d_{h,h+1} = 1$.

Example 8.4

Let there be underwritten at time 0 a loan contract for €86000 to be amortized with delayed varying annual installments over 10 years, on the basis of a term structure expressed by spot factors v_k (k=1,...,10) fixed at 0, from which by means of (7.31) we find the forward rates to apply annually. These rates are indicated in the 2nd column of the following *Excel* amortization schedule, the quantities of which follow from recursive relations (8.10) starting from a given principal repaid C_k indicated in the 3rd column, according to a choice that gives higher payments in the central years. On the contrary, if we assigns the installments, satisfying (8.9) as in column 5, are used, obtaining the outstanding balance by (8.11), we find the interest paid (4th column) and then the principal repaid (3rd column).

	Debt = 86000		*Length* = 10		
Year	Forward rate	Principal repaid	Interest amount	Installment	Outstanding balance
K	$I_{k-1,k}$	C_k	I_k	R_k	D_k
0					86000.00
1	0.050	5000.00	4300.00	9300.00	81000.00
2	0.048	6000.00	3888.00	9888.00	75000.00
3	0.046	7000.00	3450.00	10450.00	68000.00
4	0.044	8000.00	2992.00	10992.00	60000.00
5	0.042	12000.00	2520.00	14520.00	48000.00
6	0.040	15000.00	1920.00	16920.00	33000.00
7	0.043	12000.00	1419.00	13419.00	21000.00
8	0.046	8000.00	966.00	8966.00	13000.00
9	0.049	7000.00	637.00	7637.00	6000.00
10	0.052	6000.00	312.00	6312.00	0.00
		86000.00			

Table 8.1. *Example of delayed amortization*

The *Excel* instructions are as follows. The first three rows are for data and titles; C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty. 5th to 14th rows:

column A (years):	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	from B5 to B14: insert data;
column C (principal repayments):	from C5 to C14: insert data with constraint: "SUM(C5:C14)"= C1 in C15 (to control);
column D (interest payments):	D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (installments):	E5:= C5+D5; copy E5, then paste on E6 to E14;
column F (outstanding balances):	F5:= F4-C5; copy F5, then paste on F6 to F14.

Amortization with advance installments

The development of the advance amortization schedule includes the installments $\ddot{R}_k$ made by the interest payments $\ddot{I}_k$ and by the principal repayments $\ddot{C}_k$, payable for the $(k+1)^{th}$ period soon after the integer time k, and also the outstanding balances D_k at time k, that results from the following equation system

$$(k=0,\dots,n-1)\begin{cases} \ddot{I}_k = D_{k+1}\ d(0;k,k+1) \\ D_{k+1} = D_k - \ddot{C}_k \\ \ddot{R}_k = \ddot{I}_k + \ddot{C}_k \end{cases} \tag{8.10'}$$

using the initial constraint $D_0 = S$. Owing to (8.10') we can deduce the following theorem.

Theorem. For an advance amortization in the case of a rates term structure, from the recursive relations (8.10') *we obtain for the outstanding balances the expression*

$$D_h = \sum_{k=h}^{n-1} \ddot{R}_k\ s_{h,k} = \sum_{k=h}^{n-1} \ddot{R}_k \prod_{r=h+1}^{k} (1-d_{r-1,r}) \tag{8.11'}$$

This formula extends to the case h>0 *the relation of financial closure* (8.9) *with* R_n=0 (case h=0, S=D_0). *Therefore,* $\forall h$ *the exchange of the outstanding balance in h with the flow of outstanding installments at their due dates is fair, i.e. in the case of this structure the outstanding balances coincide with the pro-reserve.*

The proof of this theorem, that gives rise to (8.11'), proceeds by induction analogously to the one that leads to (8.11), taking into account the identities

$$1 - d_{r-1,r} = s_{r-1,r} = (1 + i_{r-1,r})^{-1}$$

which give a value equal to v_r/v_{r-1} in the perfect market hypothesis. Although for sake of brevity it is omitted, we would say that in the induction it is convenient to proceed backwards, i.e. verifying (8.11') for h=n-1 and proving that if it holds true for an index h (with 1≤h≤n-1), it is also true for h-1.

Example 8.5

Let us consider again Example. 8.4 assuming a loan for the same amounts, length and distribution of principal repayments, but advance installments and then advance forward rates $d_{k-1,k}$, choosing those equivalent to the delayed rates in Example 8.4. By working in *Excel* we easily obtain the following table.

Debt = 86000.00 *Length* = 10

Year	Delayed forward rate	Advance forward rate	Outstanding balance	Principal Repaid	Interest paid	Installment
k	$i_{k-1,k}$	$d_{k-1,k}$	D_k	ant. C_k	ant. I_k	ant. R_k
0			86000.00	5000.00	3857.14	8857.14
1	0.050	0.047619	81000.00	6000.00	3435.11	9435.11
2	0.048	0.045802	75000.00	7000.00	2990.44	9990.44
3	0.046	0.043977	68000.00	8000.00	2528.74	10528.74
4	0.044	0.042146	60000.00	12000.00	1934.74	13934.74
5	0.042	0.040307	48000.00	15000.00	1269.23	16269.23
6	0.040	0.038462	33000.00	12000.00	865.77	12865.77
7	0.043	0.041227	21000.00	8000.00	571.70	8571.70
8	0.046	0.043977	13000.00	7000.00	280.27	7280.27
9	0.049	0.046711	6000.00	6000.00	0.00	6000.00
10	0.052	0.049430	0.00			
				86000.00		

Table 8.2. *Example of advance amortization*

The *Excel* instructions are as follows. The first three rows are for data and titles; C1: 86000; E1: 10. 4th to 14th rows:

column A (year) A5:= A4+1; copy A5, then paste on A6 to A14;
column B (delayed forward rate) B4 empty; from B5 to B14 insert date;
column C (advance forward rate) C4:= 1-(1+B5)^-1; copy C4, then paste on C5 to C13;
column D (outstanding loan balance) D4:= C1; D5:= D4-E4; copy D5, then paste on D6 to D14;
column E (advance principal E4 to E13 insert data with the constraint "SUM(E4:E14)"=C1 in E15 (check);
column F (advance interest paid) F4:= D5*C5; copy F4, then paste on F5 to F13;
column G (installment) G4:= E4+F4; copy G4, then paste on G5 to G13.

Observation

In the delayed amortizations, from system (8.10) the following corollary holds.

Corollary. If we have fairness, each vector $\{R_k\}$ *of delayed amortization installments satisfies, for* k=1,...,n,

$$R_k = D_{k-1}\, i_{k-1,k} + (D_{k-1} - D_k) \tag{8.12}$$

i.e. the following recursive relation holds

$$D_k = D_{k-1}(1 + i_{k-1,k}) - R_k \tag{8.13}$$

Analogously in advance amortizations from system (8.10') we can deduce the following corollary:

Corollary. If we have fairness, each vector $\{\ddot{R}_k\}$ of advance amortization installments, satisfies, for k=0,...,n-1

$$\ddot{R}_k = D_{k+1}\, d_{k,k+1} + (D_k - D_{k+1}) \tag{8.12'}$$

i.e. the following recursive relation

$$D_k = D_{k+1}(1 - d_{k,k+1}) + \ddot{R}_k \tag{8.13'}$$

holds.

Proof. Since, owing to the fairness of this operation, D_n = 0 holds true, if in the delayed case we write (8.13) for k=1,...,n, with subsequent substitutions we obtain the relation of financial closure and, writing such relation for k=h+1,....,n, we easily obtain (8.11). Analogously if in the advance case we write (8.13') for k=0,...,n-1, with subsequent substitutions we obtain the relation of financial closure and, writing it for k=h,....,n-1 we easily obtain (8.11').

8.2.2. *Amortization with constant installments*

The conclusions for this major case of refund techniques are obtained from the results in section 8.2.1 using R_k or $\ddot{R}_k$ = constant = R. Therefore, given the initial debt S to be amortized in n periods, according to a given (or assumed) term structure at initial time 0, for which formulations (7.25) and the relations from (7.26) to (7.39) hold true, the installment solution is deduced introducing constraint (8.9). Therefore we obtain the following relation:

Delayed case

Using R_0 = 0, R_k = R, (k=1,...,n), in the financial closure relation (8.9), the installment R is given by

$$R = S / \sum_{k=1}^{n} \prod_{r=1}^{k} (1 + i_{r-1,r})^{-1} \tag{8.14}$$

Recursive relations (8.10) hold, where the outstanding balances D_h are expressed by

$$D = R \sum_{k=h+1}^{n} \prod_{r=h+1}^{k} (1 + i_{r-1,r})^{-1} \tag{8.15}$$

Exercise 8.1

Considering again the loan in Example 8.4 with the data given there, find the amortization schedule under the constraint that the delayed installment is constant.

A. Finding the installment by means of (8.9) we obtain R=10916.95; from here, applying recursively (8.10) with $R_k = R$, we obtain, using *Excel* for the following schedule.

debt = 86000		*length* = 10		*installment*= 10916.95	
Year	Forward rate	Discount factor	Interest paid	Principal repaid	Outstanding balance
K	$i_{k-1,k}$	Π by (8.14)	I_k	C_k	D_k
0		1			86000.00
1	0.050	0.952381	4300.00	6616.95	79383.05
2	0.048	0.908760	3810.39	7106.56	72276.48
3	0.046	0.868796	3324.72	7592.23	64684.25
4	0.044	0.832180	2846.11	8070.84	56613.41
5	0.042	0.798637	2377.76	8539.19	48074.22
6	0.040	0.767920	1922.97	8993.98	39080.24
7	0.043	0.736261	1680.45	9236.50	29843.74
8	0.046	0.703883	1372.81	9544.14	20299.60
9	0.049	0.671003	994.68	9922.27	10377.33
10	0.052	0.637836	539.62	10377.33	0.00
		7.877658			

Table 8.3. *Example of amortization with constant delayed installments*

The *Excel* instructions are as follows. The first three rows are for data, columns titles and one calculation: B1: 86000; D1: 10; F1:= B1/C15. 4th row: A4: 0; C4: 1; G4:= B1; 5th to 15th rows:

column A (years):	A5: = A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	insert forward rates (see Example 8.4);
column C (discount factors):	C5:= C4*(1+B5)^-1; copy C5, then paste on C6 to C14; C15:= SUM(C5:C14).
column D (interest payments):	D5: = F4*B5; copy D5, then paste on D6 to B14.
column E (principal repayments):	E5: = $F$1-D5; copy E5, then paste on E6 to E14.
column F (outstanding balances):	F5: = F4-E5; copy F5, then paste on F6 to F14.
other cells	empty.

Observation

An amortization that keeps constant installments in a varying rate regime is possible only in the case that the rate structure is agreed in 0 (or that the perfect market assumption hold true), as assumed in the previous examples. If these assumptions fail and the amortization proceeds in time in a flexible form on the basis of annual varying spot rates $i_{k-1,k}$ (with complete notation) not predictable in 0 and that will be different from $i_{0;k-1,k}$, then the schedule cannot be fixed in advance and we have to proceed as discussed in section 6.5, point a. We adopted in this section the complete formulation of the rate structure because many contracting times are here considered here.

In particular, we can proceed for subsequent renovations of the contract, calculating the installment and its elements each year that the rate changes (using (6.52) and (6.52')) on the basis of the new rate, the outstanding balance, and the remaining length. This procedure is consistent with the constant installment scheme, because if after the renovation the rate no longer changes, the new installment will remain constant, as can be seen from equation $D_h = R\, a_{\overline{n-h}|i}$.

Example 8.6

Let us give an example of the second procedure, that uses the spot rates $i(k\text{-}1;k\text{-}1,k)$. For an easy comparison, let us use the input data of Example 8.4, obtaining the following *Excel* table.

	Debt = 86000		*Length* = 10		
Year	spot rate	installment	interest paid	principal repaid	outstanding balance
K	*I(k-1; k-1, k)*	*Rk*	*Ik*	*Ck*	*Dk*
0					86000.00
1	0.050	11137.39	4300.00	6837.39	79162.61
2	0.048	11038.42	3799.81	7238.61	71924.00
3	0.046	10948.97	3308.50	7640.46	64283.53
4	0.044	10869.12	2828.48	8040.65	56242.89
5	0.042	10798.96	2362.20	8436.76	47806.13
6	0.040	10738.55	1912.25	8826.31	38979.82
7	0.043	10814.58	1676.13	9138.45	29841.37
8	0.046	10875.97	1372.70	9503.27	20338.10
9	0.049	10922.44	996.57	9925.87	10412.24
10	0.052	10953.67	541.44	10412.24	0.00

Table 8.4. *Example of delayed amortization with spot rates*

The *Excel* instructions are as follows. The first three rows are for data and columns titles; C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty; 5th to 15th rows:

column A (years):	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	insert data from B5 to B14;
column C (installments):	C5:= F4*B5/(1-(1+B5)^-($E$1+1-A5)); copy C5, then paste on C6 to C14;
column D (interest payments):	D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (principal repayments):	E5:= C5-D5; copy E5, then paste on E6 to E14;
column F (outstanding balances):	F5:= F4-E5; copy F5, then paste on F6 to F14.

Advance case

Using $R_n = 0$, $R_k = \ddot{R}$, (k=0,...,n-1), in the relation of financial closure (8.9), the installment $\ddot{R}$ is given by

$$\ddot{R} = S \Big/ \left[1 + \sum_{k=1}^{n-1} \prod_{r=1}^{k} (1\text{-}d_{r-1,r})\right] \tag{8.14'}$$

Recursive relations (8.10') hold true, where the outstanding balances D_h are expressed by

$$D_h = \ddot{R} = \left[1 + \sum_{k=h+1}^{n-1} \prod_{r=h+1}^{k} (1 - d_{r-1,r})\right] \tag{8.15'}$$

Exercise 8.2

Let us consider a loan of €45,000 with varying rates, a length of 5 years and forward rates, fixed when the contract is signed. Calculate the amortization schedule, where the rates are specified and where the advance installments are constant.

A. To calculate the installment, apply relation (8.14') and for the principal and interest payments (that cannot be calculated starting form the initial debt) we first have to calculate the outstanding balances at the intermediate integer times by means of (8.15') using the identity:

$$\prod_{r=h+1}^{k} (1 - d_{r-1,r}) = \prod_{r=1}^{k} (1 - d_{r-1,r}) / \prod_{r=1}^{h} (1 - d_{r-1,r})$$

We then take into account the 2nd of (8.10') for the principal repayments and the 1st and 3rd of (8.10') for the interest paid. Proceeding with *Excel*, we obtain the following schedule with two sections, where the second is an instrument to calculate on single columns the outstanding balances.

	Debt = 45000.00		*Length* = 6		*Installment*= 9865.56	
Year	Delayed forward rate	Advance forward rate	Spot discount factor	Outstanding balance	Principal repaid	Interest paid
k	$i_{k-1.k}$	$d_{k-1.k}$	Π by(8.14')	D_k	*Ant* C_k	*Ant* I_k
0			1	45000.00	8108.84	1756.72
1	0.050	0.047619	0.952381	36891.16	8568.33	1297.23
2	0.048	0.045802	0.908760	28322.82	9016.53	849.03
3	0.046	0.043977	0.868796	19306.29	9440.73	424.83
4	0.045	0.043062	0.831384	9865.56	9865.56	0.00
5	0.047	0.044890		0.00		
			4.561321		45000.00	

Year	Outstanding balance 1	Outstanding balance. 2	Outstanding balance 3	Outstanding balance.4	Outstanding balance 5
k	D_1	D_2	D_3	D_4	D_5
1	1.000000				
2	0.954198	1.000000			
3	0.912236	0.956023	1.000000		
4	0.872953	0.914854	0.956938	1.000000	
	36891.16	28322.82	19306.29	9865.56	0.00

Table 8.5. *Example of amortization with constant advance installments*

The *Excel* instructions are as follows.

1st sector. C1: 45000; E1: 6; G1:= C1/D10; other cells: empty; 2nd and 3rd rows for titles; 4th to 10th rows:

column A (year):	A4: 0; A5:= A4+1; copy A5, then paste on A6 to A9;
column B (delayed forward rate):	B4 empty; insert data from B5 to B9;
column C (advance. forward rate):	C4 empty; C5:= 1-(1+B5)^-1; copy C5, then paste on C6 to C9;
column D (discount factor 0-k):	D4: 1; D5:= D4(*(1-C5); copy D5, then paste on D6 to D8; D9 empty; D10:= SUM(D4:D9);
column E (outstanding debt):	E4:= C1; E5:= B19; E6:= C19; E7:= D19; E8:= E19; E9:= F19
column F (principal repaid):	F4: E4-E5; copy F4, then paste on F5 to F8;
column G (interest paid):	G4:= $G$1-F4; copy G4, then paste on G5 to G8.

2nd sector. 13th and 14th rows for titles; 15th to 19th rows:

column A (year):	A15: 1; A16:= A15+1; copy A16, then paste on A17 to A18; A19 empty;
column B (outstanding balance 1):	B15:= D5/D$5; copy B15, then paste on B16 to B18; B19:= $G$1*SUM($B$15:$B$18);
column C (outstanding balance 2):	C15 empty; C16:= D6/D$6; copy C16, then paste on C17 to C18; C19:= $G$1*SUM($C$16:$C$18);
column D (outstanding balance 3):	D15,D16 empty; D17:= D7/D$7; copy D17, then paste on D18; D19:= $G$1*SUM($D$17:$D$18);
column E (outstanding balance 4)	E15,E16,E17 empty; E18:= E8/E$8; copy E18, then paste on E18; E19:= $G$1*$E$18;
column F (outstanding balance 5)	F15,F16,F17,F18 empty; F19:= $G$1-F8 .

8.2.3. *Amortization with constant principal repayments*

In this case, if the structure of the per period forward rates $\{i_{h,k}\}$ is given, according to the installment due dates on the time interval from 0 to n for the debt S to be amortized, the calculation of such installments gives a unique solution, in the following way.

First of all, the constant principal repaid of the n installments is calculated, which is simply S/n. This implies that the outstanding balances decrease in arithmetic progression with ratio S/n; then after h payments we have an outstanding balance of $S(n\text{-}h)/n$.

For each period the interest rate is found from the vector $\{i_{k\text{-}1,k}\}$, (k=1....,n) and then the installments R_k are

– *in the delayed case:*

$$R_0 = 0\ ;\ R_k = \frac{S}{n}\ [1 + (n - k + 1)\ i_{k\text{-}1,k}]\ \ ,(k = 1,...,n) \tag{8.16}$$

– *in the advance case:*

$$\ddot{R}_k = \frac{S}{n}\,[1 + (n - k - 1)\, d_{k,k+1}]\ \ ,(k = 0,...,n - 1)\ ;\ \ddot{R}_n = 0 \tag{8.16'}$$

Exercise 8.3

Let us consider again the problem of amortization and the data used in Example 8.4, but now applying the method with constant principal repayments. Using (8.16) we obtain the following *Excel* table.

	Debt = 86000		*Length* = 10		
Year	Forward rate	Principal repaid	Interest paid	Installment	Outstanding balance
k	$i_{k-1.k}$	C_k	I_k	R_k	D_k
0					86000.00
1	0.050	8600.00	4300.00	12900.00	77400.00
2	0.048	8600.00	3715.20	12315.20	68800.00
3	0.046	8600.00	3164.80	11764.80	60200.00
4	0.044	8600.00	2648.80	11248.80	51600.00
5	0.042	8600.00	2167.20	10767.20	43000.00
6	0.040	8600.00	1720.00	10320.00	34400.00
7	0.043	8600.00	1479.20	10079.20	25800.00
8	0.046	8600.00	1186.80	9786.80	17200.00
9	0.049	8600.00	842.80	9442.80	8600.00
10	0.052	8600.00	447.20	9047.20	0.00

Table 8.6. *Example of amortization with constant principal repayments*

The *Excel* instructions are as follows. The first three rows are for data and titles. C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty. 5th to 14th rows:

column A (years): A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rate): B5 to B14: insert data;
column C (principal repaid): C5:= C$1/E$1; copy C, then paste on C6 to C14;
column D (interest paid): D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (installment): E5:= C5+D5; copy E5, then paste on E6 to E14;
column F (outstanding balance): F5:= F4-C5; copy E5, then paste on F6 to F14.

8.2.4. *Life amortization*

Having fully described this actuarial operation in section 6.3, we limit ourselves here to briefly considering the variations linked to the introduction into a scheme of *advance life amortization* of a discrete term structure that can be identified by a uniperiod forward rates $\{i_{r\text{-}1,r}\}$ agreed at time 0, indicating with $*$ the quantities that depend on it.

Let S be the debt of the annual loan; n the length in years; $\{i_{r\text{-}1,r}\}$ the structure of the adopted rates, that gives rise to a law that generalizes the IRR of the lender-insurer; x the integer age of the borrower at the drawing up of the contract. In addition, let us indicate the actuarial discount factor on the interval $(z,z+1)$ for the borrower with

$$_1E_x^* = \frac{l_{x+z+1}}{l_{x+z}}(1+i_{z,z+1})^{-1} \qquad (8.17)$$

The actuarial discount factor on the interval (0,z) is given by

$$_zE_x^* = \prod_{r=0}^{z-1} \frac{l_{x+r+1}}{l_{x+r}}(1+i_{r,r+1})^{-1} \qquad (8.18)$$

Then $_zE_x^* = \prod_{r=0}^{z-1} {}_1E_{x+r}^*$ holds true. Let us now take into account now that the uniperiod discount forward rates $d_{r\text{-}1,r}$ are linked to the interest forward rates by the relation

$$1-d_{r-1,r} = (1+i_{r-1,r})^{-1} = s_{r-1,r} \qquad (8.19)$$

Thus, the constraint of financial closure on the advance installments $\ddot{\alpha}_z^*$ that generalizes (6.28) is written as (considering (8.18)):

$$\sum_{z=0}^{n-1} \ddot{\alpha}_z^* \, {}_zE_x^* = S \qquad (8.20)$$

Proceeding analogously to section 6.3.1, if the *principal repayments* $\ddot{c}_z$ are given under the constraint (6.29) we find the outstanding balances D_z on the basis of the 1st part of (6.32), from which the advance *actuarial interest payments* $\ddot{j}_z^*$ comes, is given by

$$\ddot{j}_z^* = \left[d_{z,z+1} + (1-d_{z,z+1})q_{x+z}\right]D_{z+1} = (1- {}_1E_{x+z}^*)D_{z+1}\,,\ z = 0,...,n\text{-}1 \qquad (8.21)$$

and using (8.21) we obtain the advance *installments*

$$\alpha_z^* = \ddot{c}_z + \ddot{j}_z^* = D_z - {}_1E_{x+z}^* D_{z+1}\,, \quad z = 0,...,n\text{-}1 \qquad (8.22)$$

If, instead, the installments $\ddot{\alpha}_z^*$ are given subject to (8.20), as far as the outstanding balances the formula

$$D_z = \sum_{k=z}^{n-1} \ddot{\alpha}_k^* \, {}_kE_x^*\,, \quad z = 0,...,n\text{-}1 \qquad (8.23)$$

that generalizes (6.31) holds true. The values (8.23) thus allow us to calculate $\ddot{c}_z$ using the 1st of (6.32) and $\ddot{j}_z^*$ using (8.21).

If in z the technical bases, fixed in 0, are not changed, (8.23) also gives the *prospective reserves* W_Z while the *retrospective reserves* are expressed by

$$M_z = \frac{S - \sum_{k=0}^{z-1} \ddot{\alpha}_k^* \, {}_k E_x^*}{{}_z E_x^*} \quad , \quad z = 1,\ldots,n\text{-}1 \tag{8.24}$$

Exercise 8.4

Using the financial data in Example 8.4, calculate the advance life amortization schedule with the demographic data in Exercise 6.6.

A. On the basis of the advance uniperiod forward rates deducible from the delayed ones, assigned in the following 3rd column, we obtain the required schedule.

	Debt =	86000			*Length =*	10	
Year	Survival table	Forward rate	Actuarial discount factor	Principal repaid	Outstanding balance	Interest paid	Installment
Z	l_{42+z}	$i_{z-1.z}$	E^*_{42+z}	C_z	D_z	J_z	α_z
0	96400		0.950682	5000	86000	3994.78	8994.78
1	96228	0.050	0.952394	6000	81000	3570.47	9570.47
2	96046	0.048	0.954062	7000	75000	3123.78	10123.78
3	95849	0.046	0.955676	8000	68000	2659.45	10659.45
4	95631	0.044	0.957234	12000	60000	2052.76	14052.76
5	95386	0.042	0.958776	15000	48000	1360.38	16360.38
6	95112	0.040	0.955708	12000	33000	930.13	12930.13
7	94808	0.043	0.952736	8000	21000	614.44	8614.44
8	94482	0.046	0.949667	7000	13000	302.00	7302.00
9	94123	0.049	0.946763	6000	6000	0.00	6000.00
10	93746	0.052		86000	0		
total					86000		

Table 8.7. *Example of life amortization*

The *Excel* instructions are as follows. The first three rows are for titles and data. C1: 86000; G1: 10. 4th to 14th rows:

column A (year): A4: 0; A5:= A4+1; copy A5, then paste on A6 to A14;
column B (survival table): insert data from B4 to B14;
column C (forward rate): insert data from C5 to C14;
column D (actuarial discount factors): D4:= B5*(1/(1+C5))/B4; copy D4, then paste on D5 to D13; D14 empty;
column E (principal repaid): insert data from E4 to E13 with the constraint: "SUM (E4:E13)" = C1, in E15;

column F (outstanding balance): F4:= C1; F5:= F4-E4; copy F5, then paste on F6 to F14;

column G (interest paid): G4:= (1-D4)*F5; copy G4, then paste on G5 to G13; G14 empty;

column H (installment): H4:= E4+G4; copy H4,then paste on H5 to H13; H14 empty.

8.3. Updating of valuations during amortization

We can generalize to the case of varying rates, according to a term structure, the considerations developed in section 6.6 about residual valuations (pro-reserves) of financial operations with rates changed to the initial rates. Such observations were useful about calculations regarding assignments of a credit, firm valuations, etc. with the application of rates used on the market at the time of calculation. If we are talking about residual valuations regarding gradual amortizations, we use *Makeham's formula* (see section 6.6.2).

With reference to the general amortization of a loan drawing up in 0, shown in section 8.2.1, we can calculate at time $t \in \mathcal{N}$ the loan *pro-reserve* W_t , *usufruct* U_t and *bare ownership* P_t. However it is important that such valuations often have to be made according to the term structure given by the market at time t, summarized − using the complete notation, because of plurality of reference times − by $\{i(t;h,k)\}$,$(t \leq h < k)$ that, under the hypothesis of *dependence on valuation time*, differs from that valid at the loan issue, summarized by $\{i(0;h,k)\}$, according to which the installments, interest and principal payments have been calculated.

Let us refer to a delayed amortization (but the changes for the case of advance amortization are easy) and assigning the payments R_k satisfying (8.9) as well as the interest paid I_k and the principal repaid C_k , satisfying the recurrent system (8.10) and then coherent with the structure $\{i(0;h,k)\}$. Then the pro-reserve W_t at time $t \in \mathcal{N}$, valued according to forward rate structure $\{i(t;h,k)\}$ equivalent to that of spot prices $v(t,k)$, is given by

$$W_t = \sum_{k=t+1}^{n} R_k \, v(t,k) = \sum_{k=t+1}^{n} R_k \prod_{r=t+1}^{k} [1+i(t;r\text{-}1,r)]^{-1} \tag{8.25}$$

having considered the constraints between prices and rates, effective in a coherent market. The pro-reserve W_t is the sum of usufruct U_t, the present value of residual interest payments I_k, and bare ownership P_t, present value of residual principal payments C_k , valued according to the updated structure $\{i(t;h,k)\}$. Then we have:

$$U_t = \sum_{k=t+1}^{n} I_k \, v(t,k) \quad ; \quad P_t = \sum_{k=t+1}^{n} C_k \, v(t,k) \tag{8.26}$$

where I_k and C_k are obtained using (8.10).

If the payments subject to constraint (8.9) and the forward rates' structure are agreed in advance, owing to (7.36)

$$U_t = \sum\nolimits_{k=t+1}^{n} v(t,k)\, i(0;k-1,k) \sum\nolimits_{u=k}^{n} R_u\ s(0;k-1,u) \tag{8.27}$$

holds for the usufruct;

$$P_t = W_t - U_t = \sum\nolimits_{k=t+1}^{n} v(t,k)\ \left[R_k - i(0;k-1,k)\sum\nolimits_{u=k}^{n} R_u\ \ s(0;k-1,u)\right] \tag{8.27'}$$

holds for the bare ownership.

Example 8.7

Let us apply the previous formulae on a delayed amortization with the given principal repaid on a debt of €100.000 and time length 5 years, for valuing pro-reserves, split into usufruct and bare ownership components, in the rate structure hypothesis changing at each end of year.

Using an *Excel* table, in the first part we calculate the delayed amortization schedule plan of € 100.000 in 5 years, having assigned the principals repaid and rate structure. In the second part, recalling relation (8.25) between unit spot prices and forward rates, we obtain pro-reserves as well as usufructs and bare ownerships according to modified rates, using (8.25) and (8.26) under the hypothesis that in each year all the varying rates after the first increase of 0.2%. The obtained pro-reserves can be compared with outstanding loan balances, reminding us that if the rate change does not occur, in each period we should have equality. Carrying out the calculations we obtain the following table.

PART 1

Debt = 100000 *Length* = 5

Year	Forward rate	Principal repaid	Interest paid	Installment	Outstanding balance
K	$i_{k-1,k}$	C_k	I_k	R_k	D_k
0					100000.00
1	0.040	10000.00	4000.00	14000.00	90000.00
2	0.043	20000.00	3870.00	23870.00	70000.00
3	0.046	30000.00	3220.00	33220.00	40000.00
4	0.044	30000.00	1760.00	31760.00	10000.00
5	0.042	10000.00	420.00	10420.00	0.00

PART 2

Calculus of spot rates

Year	Modified forward rate	Spot price	Spot price	Spot price	Spot price
K		$V_{1,k}$	$V_{2,k}$	$V_{3,k}$	$V_{4,k}$
1		1.000000			
2	0.045	0.956938	1.000000		
3	0.048	0.913109	0.954198	1.000000	
4	0.046	0.872953	0.912236	0.956023	1.000000
5	0.044	0.836162	0.873789	0.915731	0.957854

Calculation of pro-reserves, usufructs and bare ownerships

Year	Pro-reserve	Usufruct	Bare ownership
k	W_k	U_k	P_k
1	89613.36	8531.14	81082.21
2	69775.96	5045.05	64730.91
3	39905.20	2067.21	37838.00
4	9980.84	402.30	9578.54

Table 8.8. *Calculation of pro-reserves, usufructs and bare-ownerships*

The *Excel* instructions for the first part are analogous to that specified in Example 8.4 which works out this type of amortization kind. The instructions for the second part are as follows.

12th to 15th rows:	titles
16th to 20th rows:	calculation of unit prices (as discount factors):
column A (year):	A16: 1; A17:= A16+1; copy A17, then paste on A18 to A20;
column B (updated fwd rate):	B16 empty; input of data from B17 to B20;

column C (v(1,k)): C16: 1; C17:= C16*(1+$B17)^-1; copy C17, then paste on C18 to C20;
column D (v(2,k)): D16 empty; D17: 1; copy C17, then paste on D18 to D20;
column E (v(3,k)): E16, E17 empty; E18: 1; copy C17, then paste on E19 to E20;
column F (v(4,k)): F16, F17, F18 empty; F19: 1; copy C17, then paste on F20.
21th row: empty; 22th to 24th rows: titles.
25th to 28th rows: calculation of pro-reserves, usufructs and bare ownerships:
column B (year): B25: 1; B26:= B25+1; copy B26, then paste on B27 to B28;
in the following right-side columns we calculate "scalar products between vectors" using *Excel* function "MATR-SUM-PRODUCT" here abreviated as MSP:
column C (pro-reserve = scalar product between installments and prices)
C25 := MSP(E7:E10;C17:C20); C26 := MSP(E8:E10;D18:D20);
C27 := MSP(E9:E10;E19:E20); C28 := MSP(E10;F20);
column D (usufruct = scalar product between interest paid and prices)
D25 := MSP(D7:D10;C17:C20); D26 := MSP(D8:D10;D18:D20);
D27 := MSP(D9:D10;E19:E20); D28 := MSP(D10;F20);
column E (bare ownership = scalar product between principal repaid and prices)
E25 := MSP(C7:C10;C17:C20); E26 := MSP(C8:C10;D18:D20);
E27 := MSP(C9:C10;E19:E20); E28 := MSP(C10;F20).

8.4. Funding in term structure environments

We can generalize the problem already considered in section 6.4, by assigning the equivalence relation between:

- a monetary amount that has to be set up at a given maturity t;
- a concordant payments set, then an annuity, with tickler before t and embedded into a financial structure giving accrued interest, fit to give such an amount at t.

For the sake of simplicity we assume periodic payments as in section 8.1 and for the annuity a horizon of *n* periods (in particular, *n* years). Moreover, let us settle the term structure giving the uniperiod forward immediate rates $\{i_{r-1,r)}\}$. Then the funding problem is solved if, having fixed the capital G_n at maturity *n*, in (8.7") we put $V_f(n) = G_n$.

If this funding is made by payments at the end of the period (*delayed payments*), it is enough to put $R_0 = 0$. Then the constraint between G_n that is to be set up in *n* and a vector $\{R_k\}$ of payments suitable for the funding is

$$G_n = \sum_{k=1}^{n-1} R_k \prod_{r=k+1}^{n} (1 + i_{r-1,r}) + R_n \tag{8.28}$$

Similarly if the sinking fund is accumulated with payments at the beginning of the period (*advance payments*), it is enough to put $R_n = 0$. Then the constraint by G_n in n and a vector $\{\ddot{R}_k\}$ of suitable payments is

$$G_n = \sum_{k=0}^{n-1} \ddot{R}_k \prod_{r=k}^{n-1} (1 + i_{r,r+1}) \tag{8.28'}$$

The accumulated capital sum at time $h<n$ with *delayed* payments is

$$M_h = G_h = \sum_{k=1}^{h-1} R_k \prod_{r=k+1}^{h} (1 + i_{r-1,r}) + R_h \tag{8.28''}$$

and, by advance payments, it is

$$M_h = G_h = \sum_{k=0}^{n-1} \ddot{R}_k \prod_{r=k}^{h-1} (1 + i_{r,r+1}) \tag{8.28'''}$$

For distinguishing the *principal shares* from *interest shares*, as $G_0 = 0$, in the *delayed* case such shares, denoted by C_h and I_h , are constrained by the system

$$(h = 1,\dots,n)\begin{cases} C_h = G_h - G_{h-1} \\ I_h = G_{h-1}\ i_{h-1,h} \\ C_h = R_h + I_h \end{cases} \tag{8.29}$$

which implies the recursive equation

$$G_{h-1}(1+i_{h-1,h}) + R_h = G_h \tag{8.30}$$

that allows us to find (8.28) and (8.28") again. In the *advance case*, denoting the principal repaid and interest paid with $\ddot{C}_h$ and $\ddot{I}_h$ and recalling (8.19), they are constrained by the system

$$(h = 0,\dots,n-1)\begin{cases} \ddot{C}_h = G_{h+1} - G_h \\ \ddot{I}_h = G_{h+1}\ d_{h,h+1} \\ \ddot{C}_h = \ddot{R}_h + \ddot{I}_h \end{cases} \tag{8.29'}$$

which implies the recursive equation

$$G_h + R_h = G_{h+1}\,(1\text{-}d_{h,,h+1}) \tag{8.30'}$$

for which it is possible to find (8.28') and (8.28''') again.

If we consider *constant delayed payments* R, given as G_n and according to (8.28), they are obtained by

$$R = G_n \, / \, \{1 + \sum_{k=1}^{n-1} \prod_{r=k+1}^{n} (1+i_{r-1,r}) \,\} \qquad (8.31)$$

If we consider *constant advance payments*, denoting them by $\ddot{R}$ and according to (8.28'), the result is

$$\ddot{R} = G_n \, / \sum_{k=0}^{n-1} \prod_{r=k}^{n-1} (1+i_{r,r+1}) \qquad (8.31')$$

Exercise 8.5

Mr. John wishes to obtain €100.000 by annual constant payments in advance during 5 years, on a savings account yielding according to given forward rates. Let us calculate the constant payment and the sequence of balances.

A. The given rates are written in the 2nd column of the following *Excel* table to carry out the calculations. According to (8.31') the 3rd column allows us to calculate the constant payment $\ddot{R}$, which results in €17595.14. The 4th column gives the balances (i.e. the retro-reserves) at the end of each year.

	FUNDING IN ADVANCE DURING		5	YEARS	
Capital =	100000		Installment =	17595,14	
Year	Delayed forward rate	Accumulation factor	Retro-reserve	Interest paid	Principal repaid
k	$i_{k,k+1}$	$\Pi\ k,4$	G_k	I_k	C_k
0	0.040	1.233106	0.00	703.81	18298.95
1	0.043	1.185679	18298.95	1543.45	19138.59
2	0.046	1.136797	37437.54	2531.50	20126.65
3	0.044	1.086804	57564.18	3307.01	20902.15
4	0.041	1.041000	78466.34	3938.52	21533.66
5		1.000000	100000.00		
Σ		5.683387		12024.29	100000.00

Table 8.9. *Example of funding in advance*

The *Excel* instructions are as follows. The first 5 rows devoted to data, titles and calculus of constant installments. D1: 5; B2: 100000; E2:= B2/C12; 6th to 11th rows:

column A (year): A6: 0; A7:= A6+1; copy A7, then paste on A8-A11;
column B (delayed forward rate): data input from B6 to B10;

column C (accum. fact. (*k*,4)):	C11: 1; C10:= C11*(1+B10); copy C10, then paste on C9 to C6;
column D (retro-reserve):	D6: 0; D7:= (D6+E$2)*(1+B6); copy D7, then paste on D8 to D10;
column E (interest paid):	E6:= (1-1/(1+B6))*D7; copy E6, then paste on E7 tE10;
column F (principal repaid):	F6:= E$2+E6; copy F6, then paste on F7 to F10.
row 12 (totals):	C12:= SUM(C6:C10); copy C12, then paste on E12 to F12.

Other cells: empty.

8.5. Valuations referred to shared loans in the term structure environment

In sections 6.8 and 6.9, all questions concerning the issue and management of bonds have been considered, from the point of view of the organization of the operation and of the valuation of reserves, usufructs and bare ownerships, with special reference to relations between bond prices and rates of return.

The previous investigation has been carried out assuming constant rates, both coupon rate and return rate. In this section we ought to complete this investigation in the term structure environment, supposing the structure to be assigned at an evaluation time put in 0, i.e. at bonds issue. For such structures, and assuming a coherent market, we shall use (7.25) and relations (7.26) to (7.39). As occurs for unshared loans amortization, when the change of rates can be performed over current time, the current uniperiods spot rates must be used by rules shown in section 6.9.4.

The treatment of the previous topics can be restricted in few words if we observe that a lot of schemes regarding bond management, shown in sections 6.8 and 6.9, is still valid in the new context. Indeed it is enough *to replace the constant coupon rate j with uniperiod forward coupon rates*, varying over the time interval, the structure of which we will denote by $\{j_{r-1,r}\}$. In addition, we will introduce, instead of only a valuation (or return) rate, an uniperiod forward return rate's structure, that we apply to give the value in 0 by discounting the cash-inflow subsequent to 0[8], or otherwise the price in 0 that assures the yield given by the given structure, that we denote $\{i_{r-1,r}\}$.

Unless stated otherwise the bonds have coupons, the period and payment times are annual and so are the rates. In case of semiannual coupons, it is sufficient to halve the coupon each year.

8 The inverse problem, of calculating a balanced return structure according to purchase price, gives infinite solutions. Then it has a theoretic importance, linkable with Generalized Discounted Cash Flow (GDCF) questions seen in section 4.4.2.

8.5.1. *Financial flows by the issuer's and investor's point of view.*

Generalizing what was shown in section 6.8, we must distinguish the case of *only one maturity* for all bonds from that of *different maturities* with refunds according to a drawing plan:

a) Assumption of bonds with only one maturity

Let us recall some symbols specified in section 6.8, using:

- s = maturity (or life) of bonds, all issued in 0;
- c = redemption value of each bond (usually equal to par value);
- N = number of issued bonds;
- p_0 = purchase price of a bond at issue;
- p_r = purchase price of a bond at time r>0.

In addition, we use coupon $\{j_{r-1,r}\}$ and yield $\{i_{r-1,r}\}$ rate structure.

On the basis of such assumptions the parties make the following operations:

i) *issuer:* $(0,Np_0)\cup(1,-Ncj_{0,1})\cup...\cup(n-1,-Ncj_{n-2,n-1})\cup(n,-Nc(1+j_{n-1,n}))$

ii) *buyer in* 0: $(0,-p_0)\cup(1,cj_{0,1})\cup...\cup(n-1,cj_{n-2,n-1})\cup(n,c(1+j_{n-1,n}))$

iii) *buyer in r*: $(r,-p_r)\cup(r+1,cj_{r,r+1})\cup...\cup(n-1,cj_{n-2,n-1})\cup(n,c(1+j_{n-1,n}))$

Such results hold under *annual coupons*. In the case of *semiannual coupons*, at k^{th} year for each bond we obtain two equal coupons whose amount is $cj_{k-1,k}/2$;

b) Assumption of different bonds maturities with refunds according to draw

Let n be the given loan time length with gradual refunds according to the following *drawing plan*

$$\{N_s\},\ \text{sub} \sum\nolimits_{s=1}^{n} N_s = N \tag{8.32}$$

In such a assumption the issuer is the debtor on a gradual amortization whereas the investors are creditors on an amortization with random time length and only one final refund after the payment of periodical interest. In detail, using (6.70) the operations are the following:

i) *issuer:* $(0,Np_0)\cup\left[\bigcup\nolimits_{s=1}^{n}(s,-N_s c-L_{s-1}cj_{s-1,s})\right]$

ii) *buyer in* 0 *with drawing and refund in s*>0:

$$(0,-p_0)\cup(1,cj_{0,1})\cup...\cup(s-1,cj_{s-2,s-1})\cup(s,c(1+j_{s-1,s}))$$

iii) *buyer in r with drawing and refund in s>r*:

$$(r,-p_r)\cup(r+1,cj_{r,r+1})\cup...\cup(s-1,cj_{s-2,s-1})\cup(s,c(1+j_{s-1,s}))$$

8.5.2. *Valuations of price and yield*

In section 6.9 valuations of bonds as a function of a given rate were performed; furthermore we have seen the correspondence between prices and discount rates that, given the prices, signify yield rates of the consequent investment operation.

We have to recall that, in a constant rate context, the correspondence between present values (or initial prices) and rates is biunique. On the other hand, in a varying rate context according to term structures the correspondence is only univocal, in the way that "term structure $\Rightarrow$ price", as soon as the bond loan parameters are assigned (see footnote 8). Then let us restrict ourselves, in this section devoted to valuations, to the calculation of the formula giving the balanced purchase price in the two schemes of loan management.

a) Assumption of bonds with only one maturity

Generalizing the results in section 6.9.2 and in (6.74) under coupon $\{j_{k-1,k}\}$ and return $\{i_{k-1,k}\}$ rates structures, with the symbols used in section 8.5.1 under a), the *purchase price* in 0 of bonds with life s is given by[9]

$$z_0^{(s)} = c\left[\sum_{h=1}^{s} j_{h-1,h}\prod_{k=1}^{h}(1+i_{k-1,k})^{-1} + \prod_{k=1}^{s}(1+i_{k-1,k})^{-1}\right] \qquad (8.33)$$

Furthermore, the *bond purchase price in r* (0<r<s), with unchanged term structures in (0,s) interval, is given by

$$z_r^{(s)} = c\left[\sum_{h=r+1}^{s} j_{h-1,h}\prod_{k=r+1}^{h}(1+i_{k-1,k})^{-1} + \prod_{k=r+1}^{s}(1+i_{k-1,k})^{-1}\right] \qquad (8.33')$$

b) Assumption of drawing bonds

Generalizing the results of section 6.9.3 and (6.75') under coupon $\{j_{k-1,k}\}$ and yield $\{i_{k-1,k}\}$ rates structures, with the symbols used in section 8.5.1 under b) the *bond purchase price in* 0 is now the arithmetic mean, weighed by N_s , of bonds' prices having life s. Therefore it is worth

$$z_0 = \sum_{s=1}^{n}\frac{N_s z^{(s)}}{N} = \sum_{s=1}^{n}\frac{N_s\, c}{N}\left[\sum_{h=1}^{s} j_{h-1,h}\prod_{k=1}^{h}(1+i_{k-1,k})^{-1} + \prod_{k=1}^{s}(1+i_{k-1,k})^{-1}\right]$$

(8.34)

9 Equation (8.33) shows that the inverse problem "price $\rightarrow$ structure $\{i_{k-1,k}\}$" gives infinite solutions of a difficult calculation in the generalized IRR environment.

Furthermore, the *bond purchase price in r* (0<*r*<*s*), with unchanged term structures in (0,*s*) interval, is given by

$$z_r = \sum_{s=r+1}^{n} \frac{N_s\ c}{L_r} \left[\sum_{h=r+1}^{s} j_{h-1,h} \prod_{k=r+1}^{h} (1+i_{k-1,k})^{-1} + \prod_{k=r+1}^{s} (1+i_{k-1,k})^{-1} \right] \quad (8.34')$$

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Chapter 9

Time and Variability Indicators, Classical Immunization

9.1. Main time indicators

Knowledge about the indicators of the time structure in the operation

$$O = \{t_h\}\&\{S_h\} \tag{9.1}$$

consisting of receipt (or payment) of amounts $S_1,...,S_n$ to times $t_1,...,t_n$ is important in the management of securities. Thus we preserve the assumption of the same sign into $\{S_h\}$ which are not all zero.[1] Therefore, O results are not fair (see Chapter 4).

Concerning the particular case of a bond, the amounts $\{S_h\}$ are the receipts owed to its owner, both as interest by coupon and as principal by refunds. The payment for the bond purchase is not considered; thus O is a generalized annuity, because the payment schedules can be not periodic.

We will now give a description of *time indicators* useful in financial management. They are in the time dimension, so are measured in the unit chosen in the tickler (usually a year). In addition, they are invariant under proportional

1 As we will see immediately, the time indicators represent "mean times" because they are means of the interval length between the reference instant (in particular, the purchase or evaluation instant) and the maturity of each receipt. Therefore, these time indicators have the feature of "internal means", i.e. are intermediate numbers between the lowest and the highest length of such intervals.

variations of S_h. Then if O is an annuity with constant payments, the indicators for O can be estimated on the corresponding unitary annuity.

9.1.1. *Maturity and time to maturity*

Maturity and time to maturity are the simplest time indicators of O. Using the previous symbols and denoting by t the reference instant (e.g. the purchase or valuation date) the *maturity* of O is t_n, and its *time to maturity* is t_n-t. It is evident that this is an indicator on complete information about the structure of time only on *zero-coupon bonds*, because it neglects the coupon distribution.

With regard to the following indicators, using (7.25), for the sake of simplicity we put at t=0 the reference instant, assuming $t_h \geq 0$, $\forall h$, and at least one $t_h > 0$; thus the time horizon of O is subsequent to 0. Therefore, if t=0 is the purchase or valuation instant of a bond m time units after the issue, this instant is $-m$ and only the payments subsequent to the reference instant are considered. With such an input, the maturity and the time to maturity coincide. It is evident with any t that it is sufficient to use (t_h-t) instead t_h in what follows.

9.1.2. *Arithmetic mean maturity*

This is defined as the arithmetic mean of the maturities t_h, weighted by the amounts S_h of O defined in (9.1), then calculable by the formula

$$\bar{t} = \frac{\sum_{h=1}^{n} t_h S_h}{\sum_{h=1}^{n} S_h} \tag{9.2}$$

The meaning of $\bar{t}$ in terms of mechanics is evident, as the center of mass about the system of S_h put in the points t_h of time axis. Obviously in (9.2) we can assume, instead of S_h, the standardized weights $S_h / \sum_{k=1}^{n} S_k$, that represent the cash-inflow shares at t_h. Then $\bar{t}$ is a synthetic indicator of the cash-flow timing.

9.1.3. *Average maturity*

We define *average maturity* z as the solution of the following equation, referred to (9.1):

$$(1+i)^{-z} \sum_{h=1}^{n} S_h = \sum_{h=1}^{n} S_h (1+i)^{-t_h} \tag{9.3}$$

depending on a given flat-rate *i.* From (9.3) we deduce the explicit form:

$$z = \frac{-\ln\left[\sum_h S_h (1+x)^{-t_h} / \sum_h S_h\right]}{\ln(1+x)} \tag{9.3'}$$

z is an *exponential mean* of t_h, obtained with the transformation of an arithmetic mean by the monotonic function $f(x)=(1+i)^{-x}$. Therefore, it is associative and (9.3) gives $\forall i$ its only solution $z=z(i)$. As $f(x)$ is a discount factor, the average maturity is the time, that, concentrating all payments in this time, we obtain the same present value obtainable according to the given tickler $\{t_h)$.

More generally, if we consider a financial discount law related to the structure of spot prices $v(0,t)$, the average maturity z, depending on $v(0,t)$, is the solution of

$$v(0,z) \sum_{h=1}^{n} S_h = \sum_{h=1}^{n} S_h \; v(0,t_h) \tag{9.3''}$$

The average maturity enables a thorough analysis of the feature and the return of a financial plan made by an operation O^* with amounts of any sign. Sharing the n supplies of O^* according to the amount sign, we obtain the outlays (usually called the *costs* of the plan) and the receipts (also called the *revenues*). Then, for every fixed h,

– if $S_h < 0$, we use $C_h = |S_h| = -S_h > 0$ (cost) and $t_h = t'_r$;

– if $S_h > 0$, we use $R_h = S_h > 0$ (revenue) and $t_h = t''_s$.

Then we obtain the sub-operations $O^{*\prime}$ of the n' costs and $O^{*\prime\prime}$ of the n'' revenues of O^* (being $n'+n''=n$) in their respective maturities. The value of O^* is the sum of the $O^{*\prime}$ and $O^{*\prime\prime}$ values. Then, using $C = \Sigma_r C_r$, $R = \Sigma_s R_s$, and denoting with z_C and z_R the mean maturities of $O^{*\prime}$ and $O^{*\prime\prime}$, and selecting a uniform discount law $v(t)$ (depending only on time t), the O^* value, using the new symbols, is

$$V_0 = -\sum_{r=1}^{n'} C_r \, v(t'_r) + \sum_{s=1}^{n''} R_s \, v(t''_s) = -C\,v(z_C) + R\,v(z_R)$$

Therefore, with the purpose of the valuation, the O^* plan is equivalent to the point input, point output (PIPO) plan $\{z_C, z_R\}\&\{-C, R\}$ obtained by concentrating all costs in z_C and all revenues in z_R . Using $\tau = z_R - z_C$, if $z_C < z_R$ so $\tau>0$, the plan O^* has the *investment feature*, since the costs on average occur before the revenues; but if $z_C > z_R$ i.e. $\tau<0$, instead, the costs on average occur after the revenues, then the plan O^* has the *loan feature.*

In the case of $z_C < z_R$ if we select $v(t)$ subject to strong decomposability, which implies symmetry, then the accumulation factor from z_C to z_R is $v(z_C)/v(z_R)$.

However, in this case, as known, the exchange law is exponential: $v(t) = (1+i)^{-t}$, where i is the interest rate. Then we obtain:

$$V_0 = -C\,(1+i)^{-z_C} + R\,(1+i)^{-z_R}$$

where z_C , z_R and τ depend on i. If $i=i^*=$ IRR of O^*, we obtain:

$$V_0(i^*) = (1+i^*)^{-z_R}\,[-C(1+i^*)^{\tau} + R] = 0, \quad \text{so: } C(1+i^*)^{\tau} = R$$

This formula clarifies, with reference to the PIPO plan equivalent to O^*, the meaning of the internal rate of return IRR and of the average time length τ.

9.1.4. *Mean financial time length or "duration"*

Given a term structure, defined by spot prices $v(0,t_h)$ in the valuation time 0 and an operation O set as (9.1), we define *duration*, denoted by D (see Macaulay, 1938) in a reference time put in 0, the *arithmetic mean of times t_h weighted by the present values $S_h \cdot v(0,t_h)$* of amounts S_h, that is *by the prices at* 0 of the *zero coupon bonds (*ZCB) that enable the buyer of the bonds to receive S_h at the times t_h, (h=1,...,n). Then the duration is univocally obtained by

$$D = \frac{\sum_{h=1}^{n} t_h S_h v(0,t_h)}{\sum_{h=1}^{n} S_h v(0,t_h)} \tag{9.4}$$

If the tickler has integer times $t_h = h$, then in (9.4) the unit price $v(0,h)$ can be expressed according to the implicit forward annual rates by (7.30').

Definition (9.4) shows that the duration is a mean of the times on the basis of the economic scenario valued in the reference instant. The h^{th} weight $S_h \cdot v(0,t_h)$ of the mean is the share of present value, or price, at 0 due to supply (t_h,S_h). It is also evident that D as the meaning of the *first moment*. Thus, it is the abscissa of the center of mass regarding the system $\{S_h v(0,t_h)\}$ of mass put on the time axis in the abscissas t_h.

If we assume, in order to obtain valuations, that the flat-yield structure will always be at level i, the duration, in this case named *flat yield curve duration* (*FYC duration*), depending on i or $\delta = \ln(1+i)$, becomes:

$$D = \frac{\sum_{h=1}^{n} t_h S_h (1+i)^{-t_h}}{\sum_{h=1}^{n} S_h (1+i)^{-t_h}} = \frac{\sum_{h=1}^{n} t_h S_h\, e^{-\delta\, t_h}}{\sum_{h=1}^{n} S_h\, e^{-\delta\, t_h}} \tag{9.5}$$

It is easy to prove the following theorem:

Theorem: *For any operation O having an annuity feature,* $\forall i > 0$

$$D \le z \le \bar{t} \tag{9.6}$$

results, holding the equalities only if O has only one amount at maturity t_n. *The inequalities are reversed if* $i<0$[2] .

Example 9.1

Let us consider the operation O given by the cash-inflows S_h: {10450, 12500, 8820, 56600} in the times: {1, 2.5, 3.75, 5 }, which are valued using the annual flat-rate $i = 4.75\%$.

Recalling formulae (9.2), (9.3), (9.5), O has the time parameters $\bar{t}$, z , D, defined above. We obtain

1) $\bar{t} = \dfrac{10,450 + 12,500 \cdot 2.5 + 8,820 \cdot 3.75 + 56,600 \cdot 5}{10,450 + 12,500 + 8,820 + 56,600} = \dfrac{357,775}{88,370} = 4,049$;

---

2 This theorem, formulated by E. Levi (1964), is proved here in the case of flat-yield structure taking into account known inequalities among means. *Proof*: with only one cash-inflow in t_n , (9.6) is trivial when it gives equalities. With many cash-inflows we firstly prove the strong inequality between $\bar{t}$ and z. Put: $v = 1/(1+i)$, we obtain

$$v^{\bar{t}} = v^{\sum_h t_h S_h / \sum_h S_h} = \left\{ \prod_h \left(v^{t_h} \right)^{S_h} \right\}^{1/\sum_h S_h}$$

Therefore, $v^{\bar{t}}$ is the geometric mean of the discount factors v^{t_h} with weights S_h, then it is smaller than their arithmetic mean with the same weights, which by (9.3) equals v^z. Owing to $v^{\bar{t}} < v^z$, we obtain $z < \bar{t}$ if $i>0$ (that is $v<1$); on the contrary we obtain $z > \bar{t}$ if $i<0$ (that is $v>1$). Moreover we prove the strong inequality between z and D: using $u=1+i$, we obtain

$$u^D = u^{\sum_h t_h S_h v^{t_h} / \sum_h S_h v^{t_h}} = \left\{ \prod_h (u^{t_h})^{S_h v^{t_h}} \right\}^{1/S_h v^{t_h}}$$

Therefore, u^D is the geometric mean of the accumulation factors u^{t_h} with weights S_h , then less than their arithmetic mean with the same weights, which equals u^z, considering the reciprocal in (9.3). Owing to $u^D < u^z$, we obtain $D<z$ if $i>0$ (that is $u>1$); on the contrary we obtain $D>z$ if $i<0$ (that is $u<1$). Finally, by the transitivity of "<" and ">", $D > \bar{t}$ follows if $i>0$, $D < \bar{t}$ follows if $i<0$ ←.

We can deduce these relations between D and $\bar{t}$ observing that if $i>0$ the discounting of S_h, made on D and not on $\bar{t}$, cause a reduction which is greater for the amounts S_h payable at times nearer to the last maturity, so the weighted arithmetic mean decreases. The opposite conclusion results if $i<0$; in this case we obtain a greater reduction for the payments closer to 0.

2) z is given by:

$88{,}370 \cdot 1.0475^{-z} = 10{,}450 \cdot 1.0475^{-1} + 12{,}500 \cdot 1.0475^{-2.5} +$
$+ 8{,}820 \cdot 1.0475^{-3.75} + 56{,}600 \cdot 1.0475^{-5}$

that is: $1.0475^{-z} = 73397.46 / 88370 = 0.830570$

$$z = \frac{-\log\ 0.830570}{\log\ 1.0475} = 4{,}000$$

3) the FYC duration D is given by

$$D = \frac{10{,}450 \cdot 1.0475^{-1} + 2.5 \cdot 125{,}00 \cdot 1.0475^{-2.5} + 3.75 \cdot 8{,}820 \cdot 1.0475^{-3.75} + 5 \cdot 56{,}600 \cdot 1.0475^{-5}}{10{,}450 \cdot 1.0475^{-1} + 12{,}500 \cdot 1.0475^{-2.5} + 8{,}820 \cdot 1.0475^{-3.75} + 56{,}600 \cdot 1.0475^{-5}} =$$

$$= \frac{289{,}991.80}{73{,}397.46} = 3.951$$

We can verify: $\bar{t} \geq Z \geq D$, according to $i > 0$.

Exercise 9.1

With the same cash-inflows virtue as in Example 9.1, let us consider a spot-prices structure $v(0,z) = \dfrac{30}{z+30}$ and calculate z and D.

A. We obtain: $v(0,1) = 0.967742$; $v(0,2.5) = 0.923077$; $v(0,3.75) = 0.888889$; $v(0,5) = 0.857143$

By virtue of (9.3), z is solution to

$$\frac{30}{z+30} = \frac{10{,}450 \cdot 0.967742 + 12{,}500 \cdot 0.923077 + 8{,}820 \cdot 0.888889 + 56{,}600 \cdot 0.857143}{10{,}450 + 12{,}500 + 8{,}820 + 56{,}600} =$$

$$= \frac{78{,}005.66}{88{,}370} = 0.882717 \text{ ; then } \quad z = 3.986$$

By virtue of (9.4), D is given by

$$D = \frac{10{,}450 \cdot 0.967742 + 2.5 \cdot 12{,}500 \cdot 0.923077 + 3.75 \cdot 8{,}820 \cdot 0.888889 + 5 \cdot 56{,}600 \cdot 0.857143}{10{,}450 \cdot 0.967742 + 12{,}500 \cdot 0.923077 + 8{,}820 \cdot 0.888889 + 56{,}600 \cdot 0.857143} =$$

$$= \frac{310{,}930.53}{78{,}005.66} = 3.986$$

The denominator is the value in 0 of this inflow operation.

For the *duration D* the following property is valid, and is very useful in the subsequent applications:

Let us consider two investments at 0 *in order to obtain the operations* O_1 *and* O_2 *made up respectively of cash-inflows* $\{a_h\}$ *at the maturities* $\{t'_h\}$ *and* $\{b_k\}$ *at* $\{t''_k\}$. *Let us also denote by* $A = \sum_h a_h v(0,t'_h)$ *and* $B = \sum_k b_k v(0,t''_k)$ *the values at* 0 *of* O_1 *and* O_2, *according to the spot prices structure* $v(0,t)$, *or the corresponding rates* $i(0,t)$. *Then the duration* D_{a+b} *of the operation* $O_1 \cup O_2$, *which includes together the cash-inflows of* O_1 *and* O_2 *in the respective maturities, is the arithmetic mean of the duration* D_a *of* O_1 *and* D_b *of* O_2, *weighted by the values A and B*[3] .

Then the following *mixing property* holds:

Suppose that it is possible to vary continuously and in a proportional way the amounts $\{a_h\}$ *and* $\{b_k\}$ *of two investments which give rise to the operations* O_1 *and* O_2, *so that the values A and B change, but not the durations of* O_1 *and* O_2. *Under this assumption we can continuously vary the shares A/(A+B) and B/(A+B) of two investments so as to obtain a duration of* $O_1 \cup O_2$ *however chosen in the interval between the durations of* O_1 *and* O_2.

The classical case concerns the assignment of the total amount $A+B$ to buy two kinds of securities. A and B are changed as written with $A+B$ = const., so as to obtain the desired duration D_{a+b}. This property can be extended to more than two operations.

In the applications the calculation of the FYC duration is useful for basic operations which are components of a complex portfolio management, when we assume a flat-yield structure and therefore a *FYC duration*. We use this calculation for the following operations.

O = temporary annuity-immediate with constant payments

In order to calculate the *FYC duration*, because of its invariance with respect to proportional variations of amounts, it is not restrictive to consider O as unit annuity. Moreover we assume unit periods and annually delayed payments. By virtue of (9.5) and the symbols in Chapter 5, we obtain

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3 The proof follows the associative feature of the arithmetic mean. Analytically, concerning the duration of $O_1 \cup O_2$ we can be written:

$$D_{a+b} = \frac{\sum_h t'_h\, a_h v(0,t'_h) + \sum_k t''_k\, b_k v(0,t''_k)}{A + B} = D_a \frac{A}{A+B} + D_b \frac{B}{A+B} .$$

$$D = \frac{\sum_{h=1}^{n} h(1+i)^{-h}}{\sum_{h=1}^{n}(1+i)^{-h}} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}} \tag{9.7}$$

where the denominator is the present value $a_{\overline{n}|i} = v\frac{1-v^n}{1-v}$ of the annuity and the numerator is the present value $(Ia)_{\overline{n}|i} = \frac{v}{1-v}[\frac{1-v^n}{1-v} - nv^n]$ of the increasing annuity.[4] We easily obtain the expression of D by i:

$$D = \frac{1+i}{i} - \frac{n}{(1+i)^n - 1} \tag{9.7'}$$

It is easy to verify that the duration given by (9.7') is a decreasing function of the annuity valuation's rate. Moreover, the value $n/[(1+i)^n-1]$ vanishes with diverging n and then the curve $D(n)$ is strictly increasing[5] and bounded by the asymptote $i/(1+i) = 1/d$. This level then gives the FYC duration of a perpetuity.

Example 9.2

Let us consider a semiannual annuity-immediate over 6 years, using the rate of 6.20%. With regard to the duration's calculation, it is equivalent to assume unit payments. Taking the half-year as the unit, we use (9.7') and n=12 half-years and i = 0.030534 (= six-month equivalent rate). The result is

$$D = \frac{1.030534}{0.030534} - \frac{12}{(1.030534)^{12} - 1} = 6.142$$

i.e., FYC duration = 3.071 years = 3y+0m+26d.

O = cash-inflows by zero-coupon bonds (ZCB)

Since the duration is a mean of the cash-inflows times and the ZCB gives only one encashment at maturity n, $D=n$ results. This number is the greatest value obtainable with respect to the durations of bonds with cash-inflows of any amount and period before maturity.

4 See. formulae (5.2) and (5.26) of Chapter 5.

5 To prove the increase of D with n, it is enough to verify that the subtrahend in (9.7') decreases. Indeed, since $(1+i)^{-x} > 1 - x\ln(1+i)$ (= its linear approximation), $\forall x > 0$, results, the derivative of $y = x/[(1+i)^{-x} - 1]$ here is negative.

The bonds have a redemption value C and coupon I for the unit period. Then

$$t_h = h, (h=1,...,n) \; ; S_h = I \;\; (\text{if } h=1,...,n\text{-}1), \; S_h = C+I \;\; (\text{if } h=n) \tag{9.8}$$

results, and the FYC duration is obtained taking into account the effect of (9.8) on (9.5). Then we obtain

$$D = \frac{I \; (\mathrm{I}a)_{\overline{n}|i} + n \; C(1+i)^{-n}}{I \; a_{\overline{n}|i} + C(1+i)^{-n}} \tag{9.9}$$

Equation (9.9) can be meaningfully obtained by the mixing property, pointing out that the operation here considered is the union of O' (= cash-inflows of coupons) and O'' (= cash-inflow of redemption principal). The value in 0 of O' is $A=I\cdot a_{\overline{n}|i}$; that of O'' is $B=C(1+i)^{-n}$; the FYC durations are respectively $(\mathrm{I}a)_{\overline{n}|i}/\,a_{\overline{n}|i}$ and n. Calculating their arithmetic mean with weights A and B we obtains (9.9), which is a function decreasing with respect to both the coupon rate I/C and the yield rate i.

In Figure 9.1 the curve of D, as a function of the time, tends to the asymptote $(1+i)/i$. It is strictly increasing only if $I/C \geq i$ (purchase at par or above par); otherwise (purchase below par) it increases up to local maximum $\hat{D} > (1+i)/i$ and then decreases towards the asymptote. However, it is to say that with the customary rates we obtain the local maximum point after a long time, then in the numerical interval of the usual maturities the duration D, as a function of the time t, increases.

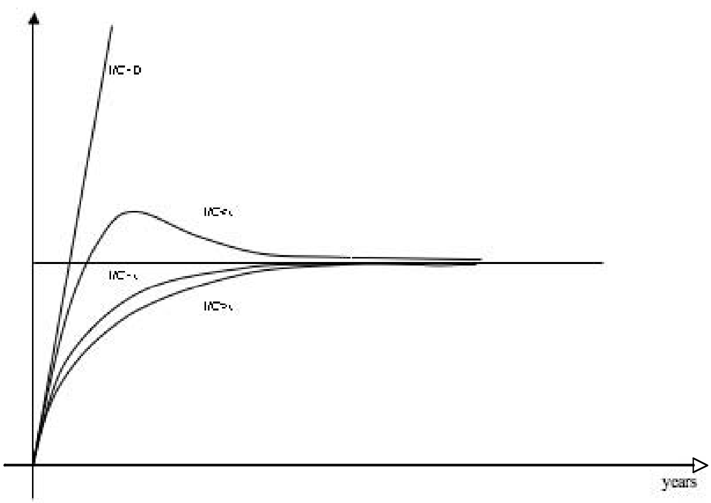

Figure 9.1. *Plot of D, function of the time t*

Example 9.3

Let us consider at t=0 a bond with redemption value 100 in t=5 and annual coupons whose amount is 6.50 payable in 1, 2, 3, 4 and 5. Let us assume the valuation rate = 7%.

The duration's calculation proceeds as follows:

$v = 1.07^{-1} = 0.934579$; n=5 ; C = 100 ; I = 6.50

$$(\mathrm{I}a)_{\overline{n}|i} = \frac{0.934579}{0.065421}\left[\frac{0.287014}{0.065421} - 3.564931\right] = 11.746862$$

$$a_{\overline{n}|i} = \frac{0.287014}{0.07} = 4.100197$$

hence by virtue of (9.9)

$$D = \frac{6.50 \cdot 11.746862 + 5 \cdot 100 \cdot 1.07^{-5}}{6.50 \cdot 4.100197 + 100 \cdot 1.07^{-5}} = 4.419 = 4\mathrm{y} + 5\mathrm{m} + 8\mathrm{d}$$

O = *cash-inflows by bond portfolio*

The previous calculation for the duration can be extended to the vectorial case, i.e. to a portfolio of m types of bonds whose purchase transfers the rights on m encashment operations, that we assume on the same tickler, e.g. on n years. These cash-inflows in such a tickler can be collected in a matrix $S = \{S_{kh}\}$. Therefore, $O = O_1 \cup ... \cup O_m$ where at any operation

$$O_k = \{S_{k1},...,S_{kn}\} \& \{t_1,...,t_n\},\ k=1,...,m,$$

which concerns a unit of the k^{th} bond, we join the initial value (or purchase price at 0)

$$P_k = \sum\nolimits_{h=1}^{n} S_{kh}(1+i)^{-t_h}; \quad (k = 1,...,m) \tag{9.10}$$

Let us now consider a portfolio obtained by λ_k units of the k^{th} bond. It is evident the cash-inflows due to the given portfolio set up the operation $O = \lambda_1 O_1 \cup ... \cup \lambda_m O_m$. Then the value (or price) P of O at 0 is the linear combination of the values (or prices) P_k of O_k with weights λ_k . In addition, at 0 the FYC duration D of O is the arithmetic mean of D_k, FYC durations of O_k , weighted by the values $\lambda_k P_k$ at 0 of the k^{th} bond's shares in the portfolio[6]. Such conclusions remain valid if,

6 Then it is possible to extend the mixing property for m>2 bonds. For the proof it is sufficient to use the linear algebra. Indeed, the cash-inflows of O in t_h are $P_h = \sum_k \lambda_k S_{kh}$. Then

1) using (9.10) it follows $A = \sum_{h=1}^{n} P_h (1+i)^{-t_h} = \sum_{h=1}^{n}\sum_{k=1}^{m} \lambda_k S_{kh}(1+i)^{-t_h} = \sum_{k=1}^{m} \lambda_k A_k$

instead of a flat-yield curve, we use any discount law (or unit prices structure) $v(0, t)$.

Example 9.4

Let us consider three kinds of bonds, and use 100 as the unit redemption value and 5.50% as valuation flat-rate:

– 1st bond: with constant coupon; maturity 4 years; annual coupon 5;

– 2nd bond: with zero-coupon; maturity 2 years;

– 3rd bond: with variable coupon; maturity 3 years; annual coupons with amounts: 5.40; 5.80; 5.60.

Denoting by λ_k the quantities of the bonds in the portfolio, let us consider two portfolio mix assumptions:

assumption α) $\lambda_1 = 25$; $\lambda_2 = 3$; $\lambda_3 = 10$;

assumption β) $\lambda_1 = 2$; $\lambda_2 = 28$; $\lambda_3 = 8$.

Then, assuming a unit times tickler, the cash-inflows tickler per bond unit and the possible mixing are the following:

$t_h =$	*1*	*2*	*3*	*4*	α	β
1st bond	5	5	5	105	25	2
2nd bond	0	100	0	0	3	28
3rd bond	5.4	5.8	105.6	0	10	8
Total					38	38

We could calculate the FYC duration portfolio by working on the total cash-flows, that in the two given hypotheses are written here below.

$t_h =$	*1*	*2*	*3*	*4*
α	179.0	483.0	1,181.0	2,625.0
β	53.2	2,856.4	854.8	210.0

We obtain

$$D_\alpha = \frac{12530.61630}{3728.320452} = 3.36093 \qquad D_\beta = \frac{8045.04565}{3514.24090} = 2.28927$$

---

2) using (9.4') and (9.10), $D_k = \dfrac{\sum_h t_h S_{kh} (1+i)^{-t_h}}{A_k}$; $D = \dfrac{\sum_h t_h P_h (1+i)^{-t_h}}{A} =$

$$= \frac{\sum_h t_h \sum_k \lambda_k S_{kh} (1+i)^{-t_h}}{A} = \sum_k \lambda_k \frac{A_k}{A} \frac{\sum_h t_h S_{kh} (1+i)^{-t_h}}{A_k} = \frac{\sum_k \lambda_k A_k D_k}{\sum_k \lambda_k A_k}$$ □

However, it is important to calculate the bond unit duration and make the linear combination for each mixing assumption. Denoting by D_s the FYC duration of the s^{th} bond, we easily obtain:

$$D_1 = \frac{365.52882}{98.24742} = 3.72049 \quad ; \quad D_2 = \frac{179.6904}{89.8452} = 2$$
$$D_3 = \frac{285.33174}{100.25991} = 2.84592$$

where in the denominators the values P_k of unit bonds appear. Since the portfolio duration is the arithmetic mean of unit bond durations weighted by the total values of each bond in the portfolio, we obtain

$$D_\alpha = \frac{25 \cdot 98.247424 \cdot 3.720493 + 3 \cdot 89.8452 \cdot 2 + 10 \cdot 100.259910 \cdot 2.845921}{25 \cdot 98.247424 + 3 \cdot 89.8452 + 10 \cdot 100.259910} =$$

$$= \frac{12530.61036}{3728.32030} = 3.36093$$

$$D_\beta = \frac{2 \cdot 98.247424 \cdot 3.720493 + 28 \cdot 89.8452 \cdot 2 + 8 \cdot 100.259910 \cdot 2.845921}{2 \cdot 98.247424 + 28 \cdot 89.8452 + 8 \cdot 100.259910} =$$

$$= \frac{8045.04317}{3514.23973} = 2.28927$$

i.e. we obtain the previous results. At the denominator of D_α and D_β we have the values of the two portfolios α and β, i.e.

$$P^\alpha = \sum_k \lambda_k^\alpha P_k = 3728.32 \quad ; \quad P^\beta = \sum_k \lambda_k^\beta A_k = 3514.24$$

9.2. Variability and dispersion indicators

9.2.1. *2nd order duration*

In the portfolio management it is useful to take into account the dispersion. To satisfy this need, we define the 2nd *order duration* at 0

$$D^{(2)} = \frac{\sum_{h=1}^{n} t_h^2 S_h v(0,t_h)}{\sum_{h=1}^{n} S_h v(0,t_h)} \tag{9.11}$$

which has the dimension of (time^2) and depends on the term structure of spot prices $v(0,t_h)$. Equation (9.11) shows that $D^{(2)}$ is the *second moment* of the mass system whose D is the first moment. $D^{(2)} \leq (t_n)^2$ always results.

In particular, in the case of a flat-yield structure the *2nd order FYC duration* takes the form of

$$D^{(2)} = \frac{\sum_{h=1}^{n} t_h^2 S_h e^{-\delta t_h}}{\sum_{h=1}^{n} S_h e^{-\delta t_h}} = \frac{\sum_{h=1}^{n} t_h^2 S_h (1+i)^{-t_h}}{\sum_{h=1}^{n} S_h (1+i)^{-t_h}} \qquad (9.11')^7$$

In addition, it is suitable to look over the consequences of interest rate variability, particularly in the case of investment rate of return (see the immunization theory in section 9.3). By working under a flat-rate, it is known that initial value $V(i)$ of a cash-inflow set due to an investment (or the price which allows a rate of return i) is a function that decreases and is a downward concave of i.

The reference to initial value (or price) $V(\delta)$ and to its derivatives depending on intensity $\delta=\ln(1+i)$ simplifies the following formulae. We obtain

$$V(\delta) = \sum_{h=1}^{n} S_h e^{-\delta t_h}\ ;\ V'(\delta) = -\sum_{h=1}^{n} t_h S_h e^{-\delta t_h}\ ;\ V''(\delta) = \sum_{h=1}^{n} t_h^2 S_h e^{-\delta t_h} \qquad (9.12)$$

resulting in: $V(\delta) > 0$; $V'(\delta) < 0$; $V''(\delta) > 0$. [8]

Example 9.5

Let us again use the cash-flow given in Exercise 9.1, i.e. the cash-inflows {10,450; 12,500; 8,820; 56,600} over the tickler {1; 2.5; 3; 3.75; 5}, valued by the law $v(0,z)=30/(z+30)$. We have seen that the value at 0 of the given cash-flow is 78,005.66 and its duration is 3.986 years.

Using some results of that exercise, we verify that the 2nd order FYC duration by virtue of (9.11') is given by

$$D^{(2)} = \frac{10{,}450\cdot 0.967742 + 2.5^2\cdot 12{,}500\cdot 0.923077 + 3.75^2\cdot 8820\cdot 0.888889 + 5^2\cdot 56{,}600\cdot 0.857143}{10{,}450\cdot 0.967742 + 12{,}500\cdot 0.923077 + 8{,}820\cdot 0.888889 + 56{,}600\cdot 0.857143} =$$

---

7 From a physical point of view, also with a flat-yield structure the duration D, given in this case by (9.5), is the first moment, thus the *center of mass*, of the distribution of the mass $S_h e^{-\delta t_h}$ put in t_h, whereas $D^{(2)}$ given by (9.11') is the second moment, that is the *moment of inertia* in a rotation around the origin. Moreover $\sigma^2 = D^{(2)} - D^2$ is the *variance,* i.e. the central second moment (or central moment of inertia), which is a *dispersion indicator*. In a more general approach with any term structure, the mass $S_h v(0,t)$ are taken, D is given by (9.4) and $D^{(2)}$ is given by (9.11) being valid analogous conclusions.

8 It is well known that the sign of second derivative measures, if this sign is positive, the punctual degree of upward concavity (or downward convexity) of a *f(x)* and, if this sign is negative, that of downward concavity (or upward convexity). The concavity and the convexity imply “downward”.

$$= \frac{1,405,335.65}{78,005.66} = 18.0158 \;\; \text{years}^2$$

9.2.2. *Relative variation*

Let us carry out a survey of variability indicators under the flat-yield structure. With reference to the function $V(\delta)$ and its first derivative (see (9.12)), we can define an index of *relative variation* by

$$\frac{V'(\delta)}{V(\delta)} = \frac{d}{d\delta} \ln V(\delta) < 0 . \tag{9.13}$$

Recalling (9.5), which gives the FYC duration D, the basic formula

$$V'(\delta)/V(\delta) \;=\; -D \tag{9.13'}$$

that identifies in absolute value the quickness of relative variation of V with respect to δ, with the FYC duration, holds.[9]

Note

Among the consequences of rate fluctuations there is also that of the same duration change, which in previous approximations is neglected. Under a flat-yield structure the quickness and the direction of such a variation are measured by the derivative of D. Using (9.12) this results in:

$$\frac{\partial D}{\partial \delta} = \frac{\partial}{\partial \delta} \frac{\sum_{h=1}^{n} t_h S_h e^{-\delta t_h}}{\sum_{h=1}^{n} S_h e^{-\delta t_h}} = \frac{-\sum_{h=1}^{n} t_h^2 S_h e^{-\delta t_h} . V(\delta) + (\sum_{h=1}^{n} t_h S_h e^{-\delta t_h})^2}{V^2(\delta)} =$$

$$-[D^{(2)} - D^2] = -\sigma^2 < 0 \tag{9.14}$$

Therefore $\partial D / \partial \delta$ is a meaningful *volatility indicator* of times with respect to mean time D. By virtue of (9.14) it follows that D decreases when intensity or rate increases. We obtain the following equation

$$\frac{\partial D}{\partial i} = \frac{\partial D}{\partial \delta} \frac{d\delta}{di} = -v\,\sigma^2 < 0 \tag{9.14'}$$

9 A type of duality holds between duration and interest instantaneous intensity: intensity (=time^{-1}) is the derivative of value's logarithm (pure number because it is an exponent) with respect to time; duration (=time) is the derivative of value's logarithm (pure number because it is an exponent) with respect to intensity (time^{-1}).

To conclude: distribution variance $\Rightarrow$ quickness of $D(\delta)$ variation $\Rightarrow$ quickness of $D(i)$ variation.

9.2.3. *Elasticity*

In the flat-yield structure assumption, we define *elasticity* η_δ of a bond value (or price) at 0 with respect to δ[10] the limit ratio on vanishing $\Delta\delta$ between the relative variations $\Delta V/V$ and $\Delta\delta/\delta$. The result is

$$\eta_\delta = \lim_{\Delta\delta \to 0} \frac{\Delta V/V}{\Delta\delta/\delta} = \delta\frac{V'(\delta)}{V(\delta)} = -\delta D \tag{9.15}$$

Denoting by η_i the elasticity with respect to $i = e^\delta - 1$, the result is:

$$\eta_i = \lim_{\Delta i \to 0} \frac{\Delta V/V}{\Delta i/i} = i\frac{V'(i)}{V(i)} = -\frac{i}{1+i}D \tag{9.15'}$$

9.2.4. *Convexity and volatility convexity*

Under the flat-yield structure assumption, let us introduce two further indicators linked to second derivative (>0) of value (or price) V. The former indicator, called *convexity*, is the level of convexity per unit of value. The convexity can be expressed as a function of the intensity δ, called δ-*convexity* and denoted by γ_δ, as well as by a function of rate i, called *i-convexity* and denoted by γ_i. Due to (9.11'), the δ-*convexity* coincides with the *2nd order FYC duration*. Using symbols, the two indicators valued at 0 are:

$$\gamma_\delta = \frac{\sum_{h=1}^n t_h^2\, S_h\, e^{-\delta t_h}}{\sum_{h=1}^n S_h\, e^{-\delta t_h}} = D^{(2)} = \frac{V''(\delta)}{V(\delta)} \tag{9.16}$$

$$\gamma_i = \frac{\sum_{h=1}^n t_h\,(t_h+1)\,S_h\,(1+i)^{-t_h}}{\sum_{h=1}^n S_h\,(1+i)^{-t_h}} = \frac{V''(i)}{V(i)}(1+i)^2 \tag{9.16'}$$

10 In general, given two variables x, y functionally linked by $y=f(x)$ (continuous and derivable), we define *elasticity* of y with respect to x, here denoted by η, the punctual relative increment of y with respect to x, that is the limit ratio between their relative variations. Using symbols

$$\eta = \lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) - f(x)\}/f(x)}{\Delta x/x} = \frac{x}{f(x)} \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = x\frac{f'(x)}{f(x)} = \frac{d[\ln f(x)]}{d(\ln x)}$$

The latter indicator, called *volatility-convexity*, means the convexity per unit of value variation. The *volatility-convexity* can be expressed according to the intensity δ, called δ-*volatility-convexity* and denoted by γ_δ^*, or dependent on rate *i*, called *i-volatility-convexity* and denoted by γ_i^*. Using symbols:

$$\gamma_\delta^* = -\frac{\sum_{h=1}^n t_h^2 S_h e^{-\delta t_h}}{\sum_{h=1}^n t_h S_h e^{-\delta t_h}} = -\frac{D^{(2)}}{D} = \frac{V''(\delta)}{V'(\delta)} \tag{9.17}$$

$$\gamma_i^* = -\frac{\sum_{h=1}^n t_h (t_h+1) S_h (1+i)^{-t_h}}{\sum_{h=1}^n t_h S_h (1+i)^{-t_h}} = \gamma_\delta^* - 1 = \frac{V''(i)}{V'(i)}(1+i) \tag{9.17}$$

Comparing (9.5) with (9.16) and (9.16') we obtain the important simple formula: $\gamma_i = \gamma_\delta + D$, which enables us to easily calculate one of the quantities having been given the others. In addition, such indicators are applied in the theory of classical immunization, which we address in section 9.3.

Exercise 9.2

Given the inflows operation *J* with amounts [8,520; 11,400; 6,450; 61,800] and tickler [0.5; 2; 3.5; 5.25], due to a previous investment with amount calculable by (5.23), let us calculate the duration, the convexity and the volatility-convexity at 0, with respect to δ and *i*, valuing by *i* = 4.75% or by the corresponding δ.

A. Using an *Excel* spreadsheet, we draw up the following table which gives the asked solutions by working on the data of *J*. The constraints among *i*-convexity, δ-convexity and FYC duration are verified.

CALCULUS OF DURATION, CONVEXITY AND VOLATILITY-CONVEXITY

Depending on δ			$\delta =$	0.046406	
t_h	S_h	$v_h=\exp(-\delta t_h)$	$S_h v_h$	$t_h S_h v_h$	$t_h^2 S_h v_h$
0.50	8,520	0.977064	8,324.58	4,162.29	2,081.15
2.00	11,400	0.911364	10,389.55	20,779.10	41,558.20
3.50	6,450	0.850082	5,483.03	19,190.60	67,167.11
5.25	61,800	0.783775	48,437.29	254,295.76	1,335,052.72
Σ			72,634.45	298,427.76	1,445,859.19

$V =$	72634.45	$D =$	4.1086
$\gamma_\delta =$	19.9060	$\gamma^*_\delta =$	-4.8449

Depending on i			$i = e^{\delta}-1 =$	0.047500	
t_h	S_h	$v_h = (1+i)^{-t_h}$	$S_h v_h$	$t_h S_h v_h$	$t_h(t_h+1) S_h v_h$
0.50	8,520	0.977064	8,324.58	4,162.29	6243.44
2.00	11,400	0.911364	10,389.55	20,779.10	62337.31
3.50	6,450	0.850082	5,483.03	19,190.60	86357.72
5.25	61,800	0.783775	48,437.29	298,427.76	1589348.48
Σ			72,634.45	298,427.76	1,744,286.94

$V =$	72634.45	$D =$	4.1086
$\gamma_i =$	24.0146	$\gamma^*_I =$	-5.8449

The constraint $\gamma_i = \gamma_\delta + D$ is verified

Table 9.1. *Example of calculus of duration, convexity and volatility-convexity*

The *Excel* instructions are the following. With regard to non-empty cells, we have:

E14: input of annual rate; E3:= ln(1+E4).

Depending on δ :

from row 5 to 8:

column A: maturity: input from A5 to A8;
column B: flow: input from B5 to B8;
column C: unit spot price: C5:= EXP(-$E$3*A5); copy C5, then paste on C6 to C8
column D: present value: D5:= B5*C5); copy D5, then paste on D6 to D8;
column E: present value · maturity: E5:= A5*D5); copy E5, then paste on E6 to E8;
column F: present value · maturity^2: F5:= A5*E5); copy F5, then paste on F6 to F8;
row 9: sums: D9:= SUM(D5:D8); copy D9, then paste on E9 to F9;
row 11: value, duration: C11:= D9 ; F11:= E9/D9;

row 12: δ-convexity, δ-volatility-convexity: C12:= F9/D9 ; F12:= -F9/E9;
Depending on i:
from row 16 to 19:
column A: maturity: copy from A5 to A8, then paste on A16 to A19;
column B: flow: copy from B5 to B8, then paste on B16 to B19;
column C: unit spot price: C16:= (1+E$14)^-A16; copy C16, then paste on C17 to C19;
column D: present value: D16:= B16*C16); copy D16, then paste on D17 to D19;
column E: present value · maturity: E16:= A16*D16); copy E16, then paste on E17 to E19;
column F: present values · maturity · (maturity+1): F16:= E16*(A16+1); copyF16, then paste on F17 to F19;
row 20: sums: D20:= SUM(D16:D19); copy D20, then paste on E20 to F20;
row 22: value, duration: C22:= D20 ; F22:= E20/D20;
row 23: *i*-convexity, *i*-volatility-convexity: C23:= F20/D20 ; F23:= -F20/E20.

9.2.5. *Approximated estimations of price fluctuation*

Let us explain, using the assumption of a flat-yield structure, an alternative interpretation of FYC duration and convexity. Multiplying by a small enough spread $d\delta$ we obtain the approximate formula:

$$\frac{\Delta V(\delta)}{V(\delta)} \cong \frac{V'(\delta)}{V(\delta)} d\delta = -D\, d\delta \tag{9.18}$$

which gives a significant sense of FYC duration. Indeed, since $\Delta V(\delta)/V(\delta)$ gives the rate of $V(\delta)$ variation, by multiplying D by a small increase (or small decrease) of δ, we obtain in an approximate way the corresponding relative decrease (or relative increase) of $V(\delta)$.[11] For this reason D is a *1st order sensitivity indicator* of price with respect to rate changes. By virtue of (9.18), we deduce the simple formula

$$V(\delta_0 + d\delta) \cong V(\delta_0)(1 - D\, d\delta) \tag{9.18'}$$

obtained by the Taylor expansion, restricted to the 1st order, over $V(\delta)$. It allows an approximate estimate of new price consequent to a market rate change in regard to bond, whose price and duration are given according to a previous rate.

In addition, let us observe that the convexity is a *2nd order sensitiveness indicator* of price with respect to rate changes. Along with duration, it enables us to improve the rough valuation of variation of values (or prices) depending on the variation of

11 Therefore, with the same change of δ, in a bond having high (or low) duration, we obtain a high (or low) relative change of price, having an opposite sign with respect to that of $d\delta$. Thus, this rule follows: it is better to invest in bonds with low duration in case of expectation of increasing rates; on the contrary, to invest in bonds with high duration in case of expectation of decreasing rates.

market intensity, expanding the Taylor formula $V(\delta)$ up to 2nd order. Then we obtain the following improved estimate:

$$V(\delta_0 + d\delta) \cong V(\delta_0) + V'(\delta_0)d\delta + V''(\delta_0)(d\delta)^2/2 =$$

$$= V(\delta_0)(1 - D\,d\delta + \gamma_\delta(d\delta)^2/2) \tag{9.19}$$

and then the consequent relative variation

$$\frac{\Delta V(\delta)}{V(\delta)} \cong -D\,d\delta + \gamma_\delta(d\delta)^2/2 \tag{9.19'}$$

Example 9.6

Let us consider at 0 a bond that gives rise to the distribution of J specified in Exercise 9.2. Under the annual rate i_0 = 4.75% or the corresponding intensity δ_0 = 0.049406, the values D = 4.1086; γ_δ = 19.9060 have been obtained. Let us calculate by an *Excel* spreadsheet, given below, the value (or price) at 0 corresponding to δ_0 and the values (or prices) at 0 corresponding to spreads $d\delta$ = +0.003 and $d\delta$ = -0.004.

CALCULUS OF BOND PRICES BY DURATION (given δ)

Duration =	4.1086		δ-convexity =	19.9060
	Intensity δ =	0.046406	0.049406	0.042406
Amounts	*Maturities*	*Values at* 0	*Values at* 0	*Values at* 0
8.520.00	0.50	8,324.59	8,312.11	8,341.25
11.400.00	2.00	10,389.56	10,327.41	10,473.01
6.450.00	3.50	5,483.04	5,425.77	5,560.34
61.800.00	5.25	48,437.38	47,680.47	49,465.32
True initial price =		72,634.56	71,745.75	73,839.92
Initial price using (9.18') =		72,634.56	71,739.28	73,828.27
Initial price using (9.19) =		72,634.56	71,745.79	73,839.84
True ΔV/V =		0.000000	-0.012237	0.016595
Approximate ΔV/V using (9.18) =		0.000000	-0.012326	0.016434
Approximate ΔV/V using (9.19') =		0.000000	-0.012236	0.016594

Table 9.2. *Example of calculus of bond prices*

In this table, after data inputs (duration and three intensities) the subsequent four rows give by column the amounts, the maturities and the inflow present value depending on the three intensities. Then in the following rows the prices at 0 are calculated by adding up, by column, and, for each intensity, are compared with their estimates according to (9.18') and (9.19). The subsequent three rows give comparisons among the relative variations of true prices and those deduced by (9.18) and (9.19').

The *Excel* instructions for non-empty cells are as follows: duration in B3 and three intensities in C4, C5, C6. Rows from 7 to 10:

column A:	inflow data;
column B:	time data;
columns C,D,E	(cash-inflows present values): C7:= $A7*EXP(-C$4*$B7); copy C7, then paste on C8-C10, on D7-D10, on E7-E10;
row 12:	C12:= SUM(C7:C11); copy C12, then paste on D12-E12;
row 13:	C13:= $C12*(1-$B3*(C4-$C4)); copy C13, then paste on D13-E13;
row 14:	C14:= $C12*(1-$B3*(C4-$C4)+$E3*(C4-$C4)^2/2); copy C14, then paste on D14-E14;
row 16:	C16:= C12/$C12-1; copy C16, then paste on D16-E16;
row 17:	C17:= -$B3*(C4-$C4); copy C17, then paste on D17-E17;
row 18:	C18:= -$B3*(C4-$C4)+$E3*(C4-$C4)^2/2; copyC18, then paste on D18-E18.

Let us now reconsider the previous expansions, assuming the rate i to be a variable of yield (let us recall (9.16') and (9.17')). In this case, taking into account the formulae

$$V(i) = \sum_{h=1}^{n} S_h (1+i)^{-t_h}, \quad V'(i) = -\sum_{h=1}^{n} t_h S_h (1+i)^{-t_h - 1}$$

we immediately obtain:

$$\frac{V'(i)}{V(i)} = \frac{-D}{1+i} = -D\,v \tag{9.20}$$

that also follows from (9.13') by observing that $\frac{d\delta}{di} = \frac{d\ln(1+i)}{di} = v$ and then $\frac{1}{V}\frac{dV}{di} = \frac{1}{V}\frac{dV}{d\delta}\frac{d\delta}{di} = \frac{-D}{1+i}$. Therefore, to make the previous approximations with use of the annual rate, the same expansions can be repeated using $D^*=D/(1+i)$ (called *modified duration* or *volatility*) instead of D. In particular, (9.18) becomes

$$\frac{\Delta V(i)}{V(i)} \cong \frac{V'(i)}{V(i)} di = d\ln V(i) = \frac{-D}{1+i} di \tag{9.20'}$$

and (9.18') becomes

$$V(i_0 + di) = V(i_0)\left[1 - \frac{D}{1+i_0}\, di\right] \tag{9.20''}$$

Also, for *V(i)* we can find a better approximation of its change estimate by also considering (9.16') and Taylor expansion up to 2nd order. Thus, we obtain a better estimate by

$$V(i_0 + di) \cong V(i_0) + V'(i_0)di + V''(i_0)(di)^2/2 =$$

$$= V(i_0)\left[1 - \frac{D}{1+i_0}\, di + \frac{\gamma_\delta}{2(1+i_0)^2}(di)^2\right] \tag{9.21}$$

and by (9.21) the consequent relative variation depending on *i*:

$$\frac{\Delta V(i)}{V(i)} \cong -\frac{D}{1+i}\, di + \frac{\gamma_i}{2(1+i)^2}\gamma_\delta (di)^2 \tag{9.21'}$$

Example 9.7

Let us again take Example 9.6 with the same cash-inflow distribution, but considering rate variations. Under the annual rate i_0 = 4.75% we obtained in Exercise 9.2 the following values: D = 4.1086; γ_i = 24,0146. Let us now calculate, using *Excel* table below, the value (or price) at 0 corresponding to i_0 and the values (or prices) at 0 corresponding to rate variations di = 0.004 and di = -0,004 as well as the relative variations.

CALCULUS OF BOND PRICES BY DURATION (given *i*)

Duration =	4.1086		*i*-convexity=	24.0146
	Rate *i* =	0.0475	0.0515	0.0435
Amounts	*Maturities*	*Values at* 0	*Values at* 0	*Values at* 0
8,520.00	0.50	8,324.58	8,308.74	8,340.52
11,400.00	2.00	10,389.55	10,310.66	10,469.36
6,450.00	3.50	5,483.03	5,410.37	5,556.95
61,800.00	5.25	48,437.29	47,477.71	49,420.04
True initial price =		72,634.45	71,507.48	73,786.87
Initial price using (9.20") =		72,634.45	71,494.88	73,774.03
Initial price using (9.21) =		72,634.45	71,507.60	73,786.74
True ΔV/V =		0.000000	-0.015516	0.015866
Approximate ΔV/V using (9.20') =		0.000000	-0.015689	0.015689
Approximate ΔV/V using (9.21') =		0.000000	-0.015514	0.015864

Table 9.3. *Example of calculus of bond prices*

The *Excel* instructions are as follows. Duration in B3 and the three rates in C4, C5, C6. Rows 7 to 10:

column A: inflow data;
column B: time data;
columns C,D,E (inflows present values): C7:= $A7*(1+C$4)^-$B7; copy C7, then paste on C8-C10, on D7-D10, on E7-E10;
row 12: C12:= SUM(C7:C11); copy C12, then paste on D12-E12;
row 13: C13:= $C12*(1-$B3*(C4-$C4)/(1+$C4)); copy C15, then paste on D15-E15;
row 14: C14:=$C12*(1-$B3*(C4--$C4)/(1+$C4)++$E3*(C4$C4)^2/(2*(1+$C4)^2)); copy C14, then paste on D14-E14;
row 16: C16:= C12/$C12-1; copy C16, then paste on D16-E16;
row 17: C17:= -$B3*(C4-$C4)/(1+$C4); copy C17, then paste on D17-E17;
row 18: C18:=-$B3*(C4-$C4)/(1+$C4)+$E3*(C4--$C4)^2/(2*(1+$C4)^2); copy C18, then paste on D18-E18.

A generalization

We can analyze the change of value (or price) in more general assumptions, using the symbols in (7.28) and assuming a spot-price structure

$$\{v(0,h)\} = \{[1+i_h]^{-h}\}.$$

For the sake of simplicity we consider a bond implying cash-inflow due to varying coupons I_h and redemption in C. The price (or value) at 0 of such a bond is given by

$$V = \sum_{h=1}^{n} I_h (1+i_h)^{-h} + C(1+i_n)^{-n} \tag{9.22}$$

The duration D at 0 on the basis of this structure by virtue of (9.4) is

$$D = \left\{\sum_{h=1}^{n} h\, I_h (1+i_h)^{-h} + n\, C(1+i_n)^{-n}\right\} / V\,. \tag{9.23}$$

V can be considered a function of spot-rates $i_1, i_2, ..., i_n$. Its total differential, corresponding to increments of spot-rates all equal to Δ, is

$$dV = -\left\{\sum_{h=1}^{n} h\, I_h (1+i_h)^{-h-1} + n\, C(1+i_n)^{-n-1}\right\} \Delta = -D^* V\, \Delta \tag{9.24}$$

depending on a *modified duration* D^*, that here is equal to

$$D^* = \left\{\sum_{h=1}^{n} h\, I_h (1+i_h)^{-h-1} + n\, C(1+i_n)^{-n-1}\right\} / V\,. \tag{9.23'}$$

By dividing the sides of (9.24) by V, we obtain the relative variation

$$\frac{dV}{V} = -D^* \Delta \tag{9.25}$$

that generalizes (9.13') and highlights that D^* is a sensitivity index. From (9.25) we find that

$$V(i_1 + \Delta, ..., i_n + \Delta) = V(i_1, ..., i_n)(1 - D^* \Delta) \tag{9.25'}$$

which generalizes (9.20'') and easily gives the new price corresponding to a uniform variation of rate structure.

9.3. Rate risk and classical immunization

9.3.1. *An introduction to financial risk*

Among the more frequently discussed problems concerning risk theory in finance are those of *interest rate risk*. Such a risk also appears in operations agreed under certainty and considered safe from risks, such as the investments in bonds. To clarify the problems of the risk theory we refer only to investments in bonds, bearing in mind that the application's field is much wider.

As shown in sections 6.9 and 6.10, in a bond loan where all the securities have the same maturity (and we talk about only one maturity) the *rate of return* (IRR) is defined as that rate at which is zero the present value, calculated at the issue, of the algebraic sum of the cash-flow owing to the buyer of the bonds. In case of differentiated maturities, e.g. by a draw rule, the *ex-ante yield* is a mean value in relation to the redemption maturities of the bonds. We define the bond *ex-post rate* the as the real rate achieved according to the date of refund and then to the realized length of life. We saw that the ex-post rates always coincide with the coupon rate for the bonds whose purchase value coincides with the par and redemption value (i.e., *par bonds*).

Examining this more closely, because in a financial operation's valuation it is necessary to take into account all the payments made in the time horizon of such an operation, then referring to only the coupon bond (or more than one coupon bond, but where all the bonds have the same maturity) it is necessary to distinguish three types of yield:

a) the *initial yield*, i.e. the IRR, also called the *ex-ante rate* and denoted by r_i , which is the rate that makes the present value (at the moment issue or purchase) of both receipts and payments equal. Then r_i is obtained not considering the reinvestment of coupons cashed during the bond's lifetime, or else considering them, but – as it will soon be proved – supposing that the reinvestments are profitable according to a rate equal to IRR (then supposing that the curve of the market rates is *flat-yield curve* throughout the bond's lifetime). Moreover, this rate coincides with the *yield rate* defined in section 7.2 in the case of bonds with a certain return and constant coupon or ZCB;

b) the *yield at maturity*, here denoted by r_m , i.e. an ex-post rate realized on a bond at its maturity, taking into account the reinvestment rates obtained on the cashed coupons;

c) the *yield in advance*[12], here denoted by r_a , which is analogous to r_m but referred to a sale and realization before the *maturity*.

12 Obviously the yield in advance has not to mistake for the discount rate (or advance interest rate) defined in Chapter 3.

Let us prove the equivalence stated in a) and summarized as the following:

Theorem A. Let us suppose that issue (or purchase) price at 0, nominal value and redemption value of a bond are equal to C, so that $r_i = i$ (= coupon rate). If in bond management we also consider the reinvestments of coupons as cashed up to maturity and their yield is r_i , then $r_m = r_i$ holds true. On the contrary, without reinvestments, $r_m < r_i$ holds true.

Proof. The latter point is evident after proving the former one. For this purpose we observe that each of n coupons is equal to $R = Ci$. Let $F(n)$ be the accumulated value of cash-inflows. Using the given assumptions and with C as the redemption value, we obtain

$$F(n) = C + R + R(1+r_i) + ... + R(1+r_i)^{n-1} = C + R\frac{(1+r_i)^n - 1}{r_i}$$

In addition, with C as the purchase price and r_i as the coupon rate, then $R = Cr_i$,

$$(1+r_m)^n = \frac{C + R\left[(1+r_i)^n - 1\right]/r_i}{C} = 1 + r_i\frac{(1+r_i)^n - 1}{r_i} = (1+r_i)^n$$

results. Thus $r_m = r_i$. ←

In light of the previous reasoning, it is evident that the bondholder must have to consider as random the return of reinvestment revenue due to future cashed coupons as well as the bond price in the case of future sale before the fixed maturity, which is calculated by discounting, at the time of sale, the future flows due to the buyer as coupons and redemption. Hence the *financial rate risk*, which is of two types:

1) *reinvestment risk*, which is the due to the future random fluctuation of market rate on the reinvestment of cashed coupons;

2) *realization risk*, which is the due to the future random fluctuation of the same market rate on the bond price in case of sale in advance.

The effects of two risks are not in accordance with each other; then we obtain a partial compensation, whose degree depends on sale time $t'' \in [0,n]$, where $[0,n]$ is the time interval of investment.

Let us explain the problem with reference to an investment operation O in $[0,n]$ with the only outcome being $-P$ at 0 and receipts being $R_h > 0$ at time $t_h \in [0,n]$ where $t_n = n$. Such quantities enable the valuation, at 0, of the rate of return r_i. Let $r(t)$ be the rate of return, generally varying with respect to the time. It is evident that $r(0) = r_i$.

In the ideal assumption that the market rate be invariant in the whole interval $[0,n]$, the yield of O retains the level of the rate $r(0)$, since at such a rate we can reinvest the intermediate revenues R_h[13]. In case of selling in advance, the transferor's and transferee's returns depend on the transfer price. However, if at this price the seller retains the rate of return x, such a rate is also valid for the buyer[14].

However, if $\exists\, t > 0$ such that $r(t) \neq r_i$, then owing to market rate variation regarding reinvestments and price of realization in advance, the performances change and a decrease is possible, and the expectations, which were valued at purchase time, fail. Then the problem of immunizing arises, i.e. of neutralizing the effects of risk due to rate $r(t)$ fluctuations.

Limiting ourselves to operation $J = \{t_h\} \,\&\, \{R_h\}$ of inflows, regarding its value $V(t,r)$ at t, subject to $(t_k \le t \le t_{k+1})$ under rate r, the result is: $V(t,r) = F(t,r) + P(t,r)$, where

$$F(t,r) = \sum\nolimits_{h=1}^{k} R_h (1+r)^{t-t_h} \tag{9.26}$$

is the *accumulated amount* at t, on reinvesting under rate r the cash-inflows before t, and

$$P(t,r) = \sum\nolimits_{h=k+1}^{n} R_h (1+r)^{-(t_h-t)} \tag{9.26'}$$

is the *present value* at t under rate r of cash-inflows after t, then the *price of realization in advance* at t. Obviously this results in

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13 Let us use as an example a bond as specified in section 6.10, bought in 0 at the price z (so generalizing the previous theorem) with c as the redemption at time n and annual coupons according to the rate i. By the defining equation, whose solution is the (initial) yield rate x, then written as: $-z+ci\ a_{\overline{n}|x} + c(1+x)^{-n} = 0$, we obtain, multiplying by $(1+x)^n$: $ci\ s_{\overline{n}|x} + c = z(1+x)^n$. The left side is the economic outcome in n of z invested in 0, with reinvestments according to the rate x of coupons as cashed. Since it equals the right side $z(1+x)^n$, the *ex-post* yield is x. The opposite is also true.

14 Referring to the bond in footnote 13, in case of a sale after only m years with price p, and of coupon reinvestment at rate x both by the seller and by the buyer, the fairness equation of O on x, quoted in footnote 13, can be written (multiplying by $(1+x)^m$ and considering that, if $n>m$, $a_{\overline{n}|x} = a_{\overline{m}|x} + (1+x)^{-m} a_{\overline{n-m}|x}$), as:

$$[-z(1+x)^m + ci\ s_{\overline{m}|x} + p] + \{-p + ci\ a_{\overline{n-m}|x} + c(1+x)^{-(n-m)}\}$$

The F quantity in square brackets is the value in m of the transferor's O' operation, whereas the P quantity in curly parentheses is the value in m of the transferee's O'' operation. If p is such that F=0, i.e. it is the retrospective reserve in m, O' is fair under rate x; then x is the transferor's rate of return. However, because of the O fairness the price p is also the prospective reserve in m, then P=0 and then O'' is fair under rate x; then x is also the transferee's rate of return.

$$V(t,r) = F(t,r) + P(t,r) = \sum_{h=1}^{n} R_h (1+r)^{t-t_h} \qquad (9.27)$$

Given t, it is evident (and immediately verified, using the derivative with respect to r) that $F(t,r)$, obtained by accumulating, is an increasing function of r, whereas $P(t,r)$, obtained by discounting, is a decreasing function of r.

Let us assume, for the sake of simplicity, that in $[0,n]$ the function $r(t)$ is subject to only one variation in $t' \leq t_1$, changing from $r(0)$ to $r^* = r(0) + \Delta r$ (where $\Delta r > 0$ or $\Delta r < 0$). Under such a change, assuming $t_1 \leq t \leq t_n$, if t is close to t_1 , the variation of F is small whereas that, opposite in sign, of P is large. Then by virtue of (9.27) the V variation has the sign of the P variation. On the contrary, if t is close to t_n , the variation of F is large whereas that, opposite in sign, of P is small. Then due to (9.27) the V variation has the sign of the F variation. Owing to the continuity of such functions, this result implies the existence of a *critical time* $\hat{t}$ regarding the sale in advance, which produces opposite values of F and P variations. Then V remains unchanged. Using symbols we have: $V(\hat{t}, r^*) = V(\hat{t}, r(0))$. Thus, we obtain a thorough neutralization of $r(t)$ variation's effects on such values, then on r_a rate, which would agree with $r_i = r_m$ without following the variations of the initial market rate $r(0)$. The calculation of such a critical time is based on classic immunization theory, which will be addressed in section 9.3.2.

The following examples, which recall an exercise given in Devolder (1993), refer to different settings of realization time t'' from that of market rate change (assumed to be only one) t' and the maturity n of a bond with annual coupons; for simplicity they all refer to the purchase of a security at issue (at 0) with purchase price = par value = redemption value = 100, then $r_i = r(0)$ = coupon rate.

Example 9.8. Sale in advance at time $t''=t'=2$ *of a bond with maturity* $n=10$.

Let us put $r(0) = r_i = 0.05 = 5\%$ and assume that the set Ω of "states", concerning the dynamics of the market rate $r(t)$ into the interval [0,10], is given only by the following events:

ω_0= (no change of $r(t)$ at $t \in [0,10]$);

ω_1= (only one change of $r(t)$ at $t_0 = 2$, given by $\Delta = +0.01 = +1\%$);

ω_2= (only one change of $r(t)$ at $t_0 = 2$, given by $\Delta = -0.01 = -1\%$);

Clearly, if ω_0 is true, it results in $r_a = r_m = r_i = 0.05$. Let us consider two other events ω_1 and ω_2, denoting by (ω) the dependence on the Ω state.

The sum $F_\omega(2)$, accrued by an investor owing to cashed coupons at periodic maturities and reinvested up to sale at $t''=2$, do not depend on the Ω state, because changes of $r(t)$ into [0,2) do not occur. The sum is given by

$$F_\omega(2) = 5\,(1.05) + 5 = 10.25$$

The sale price $P_\omega(2)$ follows by rates *r(t)* in [2,10], thus depends on the Ω state:

$$P_\omega(2) = 5a_{\overline{8}|r(\omega)} + 100[1 + r(\omega)]^{-8}$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $P_{\omega_1}(2) = 31.05 + 62.74 = 93.79$

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $P_{\omega_2}(2) = 33.66 + 73.07 = 106.73$

The seller's total revenue at $t'' = 2$ is $S_\omega(2) = F_\omega(2) + P_\omega(2)$. Then

– if $\omega = \omega_1$: $r(\omega_1) = 0{,}06$, $S_{\omega_1}(2) = 104.04$

– if $\omega = \omega_2$: $r(\omega_2) = 0{,}04$, $S_{\omega_2}(2) = 116.98$

The yield in advance $r_a(\omega)$ depends on Ω state, as it is solution of

$$100\,[1 + r_a(\omega)]^2 = S_\omega(2)$$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.020000$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.081573$, then $r_a(\omega_1) < r_a(\omega_2)$ with a large difference among them and r_i which is in the middle. As $t'' = t'$, a reinvestment risk does not exist, because the coupons are reinvested in [0,2] under certain rate $r(0) = 0.05$ whereas the risk of realization exists with a large decrease (increase) of the sale price and of the yield in advance when the market rate increases (decreases).

Example 9.9. Sale in advance of a bond with maturity n=10 at time t''=6 in the middle from t' and n.

On the basis of the data and events set out in Example 9.8, except for $t''=6$, we obtain the following results.

The sum $F_\omega(6)$, accrued by the investor due to cashed coupons at periodic maturities and reinvested up to sale at $t'' = 6$, depends on the Ω state and is given by

$$F_\omega(6) = 5\ \{1.05\,[1 + r(\omega)]^4 + s_{\overline{5}|r(\omega)}\}$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $F_{\omega_1}(6) = 5\,(1.325601{+}5.637093) = 34.81$;

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $F_{\omega_2}(6) = 5\,(1.228351{+}5.416323) = 33.22$.

The sale price $P_\omega(6)$ depends on the Ω state and is given by

$$P_\omega(6) = 5\ a_{\overline{4}|r(\omega)} + 100\,[1 + r(\omega)]^4$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $P_{\omega_1}(6) = 17.32 + 79.21 = 96.53$;

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $P_{\omega_2}(6) = 18.15 + 85.48 = 103.63$.

The seller's total revenue at $t'' = 6$ is $S_\omega(6) = F_\omega(6) + P_\omega(6)$. Then

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $S_{\omega_1}(6) = 131.34$;
– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $S_{\omega_2}(6) = 136.85$.

The yield in advance $r_a(\omega)$ depends on the Ω state, as it is the solution of

$$100\,[1 + r_a(\omega)]^6 = S_\omega(6)$$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.046485$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.053677$.

Compared to the results of Example 9.8, the difference between $r_a(\omega_1)$ and $r_a(\omega_2)$ is much reduced, since these rates are approaching the value of the initial market rate, 0.05. As $t'<t''<n$, both the reinvestment risk on cashed coupons from time 2 to 6, and the realization risk exist, owing to the advance of the sale in respect to the maturity, which implies a discount from time 10 to 6 under a random market rate.

*Example 9.10. Realization of a bond at maturity n=*10

On the basis of the data and events set out in Example 9.8, except for $t'' = 10$, we obtain the following results.

The sum $F_\omega(10)$, accrued by the investor due to cashed coupons at periodic maturities and reinvested up to realization at time 10, depends on the Ω state and is given by

$$F_\omega(10) = 5\ \{(1.05)[1 + r(\omega)]^8 + s_{\overline{9}|r(\omega)}\}$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $F_{\omega_1}(10) = 5\,(1.673540+11.491316) = 65.82$;
– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $F_{\omega_2}(10) = 5\,(1.368569+10.582795) = 59.76$.

The realization value is certainly $P_\omega(10)=100$; it does not depend on the Ω state, as it lacks a discount under a random rate.

The seller's total revenue at $t'' = 10$ is $S_\omega(10) = F_\omega(10) + P_\omega(10)$. Then
– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $S_{\omega_1}(10) = 165.82$;
– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $S_{\omega_2}(10) = 159.76$.

The yield in advance $r_a(\omega)$ becomes yield to maturity $r_m(\omega)$ because the realization occurs at fixed maturity; it depends on Ω state, as it is the solution of

$$100\,[1 + r_m(\omega)]^{10} = S_\omega(10)$$

If $\omega = \omega_1$, we obtain: $r_m(\omega_1) = 0.051874$; if $\omega = \omega_2$: $r_m(\omega_2) = 0.047965$ then $r_m(\omega_1) > r_m(\omega_2)$ with a small difference between them and r_i which is in the middle. As $t'' = n$, a realization risk does not exists but the reinvestment risk exists with an increase (decrease) of total revenue and yield at maturity when the market rate increases (decreases).

Note

In Examples 9.8 to 9.10 when $2=t' \le t''<n=10$, the rates of return in the middle, between those achieved for $t''=2$ and $t''=10$, have been obtained. By varying t'' continuously from the time 2 to 10, the rate $r_a(\omega_1)$ increases from 0.0200 to 0.0519, whereas the rate $r_a(\omega_2)$ decreases from 0.0816 to 0.0480. Then it is plausible that, as $r_a(\omega_1)$ and $r_a(\omega_2)$ are continuous functions of t'', we can settle on a *critical time* $\hat{t}$ of investment ($2<\hat{t}<10$) for which $r_a(\omega_1)=r_a(\omega_2)$, so that two opposite effects of a market rate's change exactly compensate one another. Then, for this critical time $\hat{t}$ we obtains:

$$r_a(\omega_1) = r_a(\omega_2) = r_a(\omega_0) = 0.05 = r_i \quad \text{(certain rate).}$$

In such a way the risk rate is removed.

9.3.2. *Preliminaries to classic immunization*

In section 9.3.1 we dealt with rate risk and critical time $\hat{t}$ of investment, which allows the removal of such a risk by suitable methods. Now we address processes, called *classic immunization*, that we also call *semi-deterministic* because all elements of involved operations are fixed except for the market interest rate, which is exposed to random changes.

We will begin with the critical time calculation which removes risk rate in a particular context. We will give some theorems concerning semi-deterministic immunization, distinguishing between problem of *cover of single liability* and *cover of multiple liabilities* problems[15].

The market term structure, if not flat-yield, will be identified by temporal changes of intensity $\delta(x,u)$ as defined in Chapter 2, where x is the time of agreement or valuation and u is the current time (see section 7.5.3 for other characteristic quantities of term structure).

In classic immunization we usually take the hypothesis of *additive shifts* of rates, i.e. of random changes Y_k, from x to t, of the instantaneous intensities corresponding to them, whose result is $Z(x,t)=\sum_k Y_k$. Therefore, with $x<t<y$

15 For a thorough analysis on such subjects, see Devolder (1993) and De Felice Moriconi (1991).

$$\delta(t,y) = \delta(x,y) + Z(x,t) \tag{9.28}$$ [16]

However, for simplicity we will proceed under the assumption of only additive shifts in the considered time interval.

9.3.3. *The optimal time of realization*

In section 9.3.1 we have seen that, in the case of only random additive shifts, for continuity in the interval of financial cash-inflows operation J a critical time $\hat{t}$ exists, such that the random change of value (and thus of the fulfilled rate of return) due to additive shifts, vanishes. Now we look for the calculation of this $\hat{t}$.

It is not restrictive, and it simplifies symbols, to put the time origin in the instant of J valuation and of rate (or intensity $\delta(0,u)$) agreement. Moreover, let us assume that in the J interval only one additive shift on $\delta(0,u)$ of random size Y occurs in the market at time t', before times $\{t_k\}$ (k=1,...,n), set in chronological order, where the inflows of J, components of vector $a = \{a_k\}$, are cashed. Thus, the intensity $\delta(0,u)$ from 0 to t' and $\delta(t',u)$ from t' to t_n are in force in the market, linked by

$$\delta(t',u) = \delta(0,u) + Y \quad , \quad 0 < t' < t_1 < ... < t_k < ... < t_n \;\; ; u > t' \tag{9.29}$$

Let us denote by $V(T,\underline{a};Y)$ (where the 3rd variable represents the size of a possible shift) the value in $T \leq t_n$ of total revenue due to $\underline{a}$, obtained adding reinvestment revenue and realization revenue. Thus, this value depends on random shift size. Lacking shift, it results in

$$V(T,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(0,u)du} \tag{9.30}$$

On the other hand, if the additive shift Y occurs at $t' < t_1$, according to (9.29) the total revenue due to $\underline{a}$ at T is given (by distinguishing reinvestment and realization components) by

$$V(T,\underline{a};Y) = \sum_{k:t_k \leq T} a_k e^{\int_{t_k}^{T} \delta(t',u)du} + \sum_{k:t_k > T} a_k e^{-\int_T^{t_k} \delta(t',u)du} =$$

$$= \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(t',u)du} = \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(0,u)du} e^{-Y(t_k - T)}$$

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16 In the case of flat-yield structure, unless additive shifts, (9.28) becomes: $\delta_t = \delta_x + Z(x,t)$, where δ_u is the intensity agreed at u.

Thus

$$V(T,\underline{a};Y) = \frac{1}{v(0,T)} \sum_{k=1}^{n} a_k v(0,t_k) e^{-Y(t_k - T)} \tag{9.30'}$$

where $v(0,t) = e^{-\int_0^t \delta(0,u)du}$ is the price at 0 of an unitary zero coupon bond (UZCB) having maturity at t, valued according to $\delta(0,u)$ (see (7.42)).

Since the second derivative with respect to Y of $V(T,\underline{a};Y)$ for every Y is positive, the function $f(Y) = V(T,\underline{a};Y)$ has the absolute minimum point at Y=0 (then $V(T,\underline{a};Y) \geq V(T,\underline{a};0)$ for every Y if the first derivative of $f(Y)$ vanishes at 0. Then we obtain the immunization. However, this sentence is true if T is chosen equal to the duration of J. In fact, due to

$$\left[\frac{\partial}{\partial Y} V(T,\underline{a};Y)\right]_{Y=0} = \frac{1}{v(0,T)} \sum_{k=1}^{n} a_k v(0,t_k) = 0$$

it follows that

$$T = \frac{\sum_{k=1}^{n} t_k a_k v(0,t_k)}{\sum_{k=1}^{n} a_k v(0,t_k)} = D_J(0)$$

Then we conclude: $\hat{t} = D_J(0)$, i.e., *the critical time for immunizing against interest rate risk is the duration of J valued at* 0. Moreover, $\hat{t}$ is the only solution to the problem.

Example 9.11

Carrying out Examples 9.8, 9.9 and 9.10, on the basis of data and events specified in Example 9.8, except for t'', let us verify that, putting the investment time equal to duration, we obtain immunization.

Let us buy the bond at 0 and redeem it at par in a maturity of 10 years, par value 100, rate $r(0) = r_i = 0.05 = 5\%$. The duration at 0, according to (9.9), is worth D = 8.107822. Let us calculate the economic results obtainable under the various states of Ω.

$$F_\omega(8.107822) = 5\ \{(1.05)[1+r(\omega)]^6 + s_{\overline{7}|r(\omega)}\}[1+r(\omega)]^{0.107822}$$

- if $\omega = \omega_1$: $Y = +0.01$, $r(\omega_1) = 0.06$, $F_{\omega_1}(8.107822) = 49.73$;
- if $\omega = \omega_2$: $Y = -0.01$, $r(\omega_2) = 0.04$, $F_{\omega_2}(8.107822) = 46.33$;

$P_\omega(8.107822) = \{5\ a_{\overline{2}|r(\omega)} + 100\ [1+r(\omega)]^{-2}\}[1+r(\omega)]^{0.107822}$

- if $\omega = \omega_1$: $Y = +0.01$, $r((\omega_1) = 0.06$, $P_{\omega_1}(8.107822) = 98.79$
- if $\omega = \omega_2$: $Y = -0.01$, $r((\omega_2) = 0.04$, $P_{\omega_2}(8.107822) = 102.32$;

The seller's total revenue at $t_1 = 8.107822$ is

$S_\omega(8.107822) = F_\omega(8.107822) + P_\omega(8.107822)$. Then

- if $\omega = \omega_1$: $Y = +0.01$, $r((\omega_1) = 0.06$, $S_{\omega_1}(8.107822) = 148.52$;
- if $\omega = \omega_2$: $Y = -0.01$, $r((\omega_2) = 0.04$, $S_{\omega_2}(8.107822) = 148.65$.

The yield in advance $r_a(\omega)$ depends on state, as it is the solution of

$100\ [1+r_a(\omega)]^{8.107822} = S_\omega(8.107822)$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.0500$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.0501$

To conclude: $S_{\omega_1}(8.107822) \cong S_{\omega_2}(8.107822)$ and $r_a(\omega_1) \cong r_a(\omega_2) \cong 0.05$. Therefore, we obtain immunization against rate risk using an investment the time length of which is its duration = 8.107822.

9.3.4. *The meaning of classical immunization*

Let us proceed, step by step, to analyze in depth the immunization with respect to yield shifts under increasing generalization, summarizing the characteristic features of a theory which would need a wider treatment.

For the sake of simplicity, let us use 0 for the valuation time where the intensity $\delta(0,u)$ identifying the structure is agreed. We refer to operation O giving a vector $\underline{a} = (a_1,..., a_n)$ of cash-inflows (also called *assets*) and a vector $\underline{b} = (b_1,..., b_n)$ of cash-outflows (also called *liabilities*). It is not restrictive to assume that $\underline{a}$ and $\underline{b}$ have the same tickler $\underline{t} = (t_1,..., t_n)$, under the constraints $\{a_h \geq 0\}$,$\{b_h \geq 0\}$, because $\underline{t}$ can be obtained by the union of $\{a_h > 0\}$ and $\{b_h > 0\}$ ticklers[17]. Denoting by $V(0,\underline{a};0)$ the value at 0 of assets and by $V(0,\underline{b};0)$ that of liabilities, if $V(0,\underline{a};0) = V(0,\underline{b};0)$ results, we can tell that the flows $\underline{a}$ and $\underline{b}$ are *in equilibrium*. This equality is also called a *budget constraint*. Moreover, by definition flows $\underline{a}$ and $\underline{b}$ are immunized if, with only one additive shift Y (positive or negative, and with small size) at the time $t'<t_1<...<t_n$, $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$ holds. This weak inequality assures *the cover by* $\underline{a}$

17 In such a case, if compensations between assets and liabilities are allowed, then at each maturity t_h we cannot have net receipts a_h-b_h and net outlays b_h-a_h both positive.

of the liabilities $\underline{b}$[18]. Denoting by $\underline{s} = (s_1,..., s_n)$, where $s_h = a_h - b_h$, the *net flows* vector and by $V(0,\underline{s};0)$ its value at 0, the equilibrium implies: $V(0,\underline{s};0) = 0$ and we have immunization if furthermore $V(0,\underline{s};Y) \geq 0$. In other words, immunization implies that the function $f(Y) = V(0,\underline{s};Y)$ has a local minimum point at $Y = 0$.

9.3.5. *Single liability cover*

We have immunization against random additive shift following the Fisher-Weil theorem (1971) if the revenue due to a "portfolio" at the end of the period of its management is, in case of an additive shift, not lower than that obtainable without a shift. It is easy to prove that to keep the bond up to maturity, on reinvesting the encashments, generally does not give immunization (see Example 9.10).

Let us state the version of the Fisher-Weil theorem that works on present values and gives the immunization conditions in asset portfolio management to cover only one liability (or, which is the same, a financial target which implies future outlays) under any term structure.

Theorem B (Fisher-Weil). Given the intensity $\delta(0,u)$ summarizing the structure at 0, let b be the amount of a payment scheduled at time $T>0$ and $\underline{a} = (a_1,..., a_n)$ be an asset flow at positive times $t_1<...<t_n$. Assume the value at 0 of $\underline{a}$ is equal to that of b according to $\delta(0,u)$, i.e., the following budget constraint is valid:

$$V(0,\underline{a};0) = V(0,b;0) \tag{9.31}$$

If at t', where $0<t'<t_1$, a random additive shift Y according to (9.29) occurs, then for the values calculated under the new intensity

$$V(t',\underline{a};Y) \geq V(t',b;Y) \tag{9.32}$$

results, if and only if the duration of $\underline{a}$ calculated at 0 equals maturity T of the liability.

Proof. Using

$$\rho(\underline{a},b;0) = V(0,\underline{a};0)/V(0,b;0) =$$

---

18 It would be more convenient to use $t' = 0$ for an immediate comparison with the equilibrium case. However, this is not needed. We can observe that

$$V(0,\underline{a};Y) = e^{-\int_0^{t'}\delta(0,u)du} V(t',\underline{a};Y),\ V(0,\underline{b};Y) = e^{-\int_0^{t'}\delta(0,u)du} V(t',\underline{b};Y);$$

then $V(t',\underline{a};Y) \geq V(t',\underline{b};Y)$ implies $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$, and vice versa. It must be highlighted that in the times following t' the discounts have carried out using the intensity $\delta(t',u)$.

$$= \frac{\sum_{k=1}^{n} a_k e^{-\int_0^{t_k} \delta(0,u)du}}{b\, e^{-\int_0^{T} \delta(0,u)du}} = \frac{1}{b}\sum_{k=1}^{n} a_k e^{\int_{t_k}^{T} \delta(0,u)du} \tag{9.33}$$

because of (9.31) ρ($\underline{a}$,b;0) =1 results. After shift Y at t', ρ($\underline{a}$,b;0) is modified in

$$g(Y) = \rho(\underline{a},b;t',Y) = V(t',\underline{a};Y)/V(t',b;Y) = \frac{\sum_{k=1}^{n} a_k e^{-\int_0^{t'} \delta(0,u)du}\, e^{-\int_{t'}^{t_k} \delta(t',u)du}}{b\, e^{-\int_0^{t'} \delta(0,u)du}\, e^{-\int_{t'}^{T} \delta(t',u)du}} \tag{9.34}$$

thus, due to (9.29)

$$g(Y) = \rho(\underline{a},b;t',Y) = \frac{1}{b}\sum_{k=1}^{n} a_k e^{\int_{t_k}^{T} \delta(0,u)du}\, e^{Y(T-t_k)} \tag{9.34'}$$

By calculating the first and second derivative of $g(Y)$ we obtain

$$g'(Y) = \frac{1}{b}\sum_{k=1}^{n} (T-t_k) a_k e^{\int_{t_k}^{T} \delta(0,u)du}\, e^{Y(T-t_k)} \tag{9.35}$$

$$g''(Y) = \frac{1}{b}\sum_{k=1}^{n} (T-t_k)^2 a_k e^{\int_{t_k}^{T} \delta(0,u)du}\, e^{Y(T-t_k)} \tag{9.36}$$

We obtain: $g''(Y) > 0$, $\forall Y$, then (9.34') is a convex function. If and only if $g'(0)=0$, $g(Y)$ holds the minimum point at Y=0 where its value is 1. Therefore, around Y=0 it results in $g(Y) = \rho(\underline{a},b;t',Y) \geq 1$, i.e. (9.32) holds. However, owing to (7.42) and (9.35), $g'(0)=0$ is equivalent to

$$\frac{\sum_{k=1}^{n} (T-t_k) a_k v(0,t_k)}{b\, v(0,T)} = 0$$

Taking into account the budget constraint in (9.31), written as $\sum_{k=1}^{n} a_k v(0,t_k) = b\, v(0,T)$, the equation $g'(0)=0$ is also equivalent to

$$D := \frac{\sum_{k=1}^{n} t_k a_k v(0,t_k)}{\sum_{k=1}^{n} a_k v(0,t_k)} = T$$

Summarizing the reasoning, the budget constraint in (9.31) between $\underline{a}$ and $\underline{b}$ signifies, if the term rates structure remains unchanged, the suitability of receipts $\underline{a}$

under the tickler $\underline{t} = \{t_1, ..., t_n\}$, $0<t_1<...<t_n$, for covering outlay (or target) b at time T, accumulating or discounting by law $v(0,t)$. Under a random additive shift, the cover is still assured provided that T equals $D_0(\underline{a})$, i.e. the duration of $\underline{a}$ at 0[19]. Immunization gives a guarantee of yield at the minimum assured rate $\{b/V(0,\underline{a};0)-1\}$. Theorem B can be applied to the selection of immune portfolios in order to obtain a *single liability cover.*

The operational meaning of Theorem B is as follows. To obtain immunization, we should build a portfolio of assets, the duration of which in 0 equals T. This is always possible, because of the duration's mixing property (see section 9.1.4) and the associative property of the averages considered here (see section 2.5.2).

In fact, let us assume that in 0 the market gives two bond packages (that without loss of generality we can assume to be of the ZCB type). Let each bond of such packages be the redemption values U_1 and U_2 at maturities t_1 and t_2, ($t_1<T<t_2$), respectively. If $T=t_1$ or $T=t_2$ occurs, the immunization problem would be trivially solved, choosing only one of the packages. The market financial law should be identified by spot prices $\{v(0,u)\}$, $(0\leq u\leq t_2)$. We can settle the portfolio $\underline{a} = (a_1, a_2)$ with tickler $\underline{t} = (t_1, t_2)$ to cover the liability b (or to assure the target b) in T, by calculating the shares (i.e. the numbers α_1, α_2 of the bonds of two packages) to make up $\underline{a}$ so as to satisfy the budget constraint on values at 0 and the constraints on $\underline{a}$ duration at 0. Using $V(0,b) = b\ v(0,T)$, is sufficient to solve the linear system

$$\begin{cases} \alpha_1 U_1 v(0,t_1)+\alpha_2 U_2 v(0,t_2)=V(0,b;0) \\ t_1\alpha_1 U_1 v(0,t_1)+t_2\alpha_2 U_2 v(0,t_2)=T\ V(0,b;0) \end{cases} \tag{9.37}$$

If linear independence between such equations holds, we obtain the following only solution

$$\alpha_1=\frac{V(0,b;0)(t_2-T)}{U_1 v(0,t_1)(t_2-t_1)}\ ,\ \alpha_2=\frac{V(0,b;0)(T-t_1)}{U_2 v(0,t_2)(t_2-t_1)} \tag{9.38}$$

If N types of ZCB subject to law $\{v(0,u)\}$ are available in the market, having par values U_1, U_2 , ..., U_n , is sufficient to put them into two subgroups and, owing to the mixing property, to obtain two portfolios having face value amounts U_1^*, U_2^* and durations t_1, t_2 to substitute into (9.37)[20].

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19 This condition can also be written as equality between T-t' and the duration $D_{t'}(\underline{a})$ valued at t'. In fact, the duration is a mean of the times and, denoting by $D_{t'}$ and D_0 the durations calculated in t' and in 0, we obtain: $D_{t'} = D_0 - t'$.

20 If bonds are not ZCB, we consider that each coupon bond is equivalent to a group of ZCB, the face value of which equals the coupons or the redemption value.

Exercise 9.3

Let us use two types of ZCB, called $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A}$ has redemption values \$1,000 at maturity 6; $\mathcal{B}$ has redemption value \$500 at maturity 9. Let us calculate the numerical shares of $\mathcal{A}$ and $\mathcal{B}$ to obtain the cover of \$98,000 at time 7.25 (=7y+3m) if the financial market law is settled by intensity $\delta(0.u)$=0.06-0.002u. Let us verify the immunization by examples.

A. According to given data, we obtain:

$U_1 = 1000$; $U_2 = 500$; $t_1 = 6$; $t_2 = 9$; $T = 7.25$; $v(0,u) = e^{-\int_0^u [0.06-0.002z]dz}$ and then $v(0,6) = e^{-0.324} = 0.723250$; $v(0,9) = e^{-0.459} = 0.631915$; $v(0,7.25) = e^{-0.382}$ = 0.682197.

Applying (9.38) we obtain

$$\alpha_1 = \frac{98000 \bullet 0.682197 \bullet 1.75}{1000 \bullet 0.723250 \bullet 3} = 53.921726 \cong 54$$

$$\alpha_2 = \frac{98000 \bullet 0.682197 \bullet 1.25}{500 \bullet 0.631915 \bullet 3} = 88.164856 \cong 88$$

Let us verify the budget constraint in terms of present values at 0.

On 1st bond:	53.921716·1000·0.723250	=	38,998.89
On 2nd bond:	88.164856·500·0.631915	=	27,956.36
Asset present value		=	66,855.25
Liability present value	98,000·0.682197	=	66,855.25

Let us assume that at time 5 a random additive shift occurs with the following possible events

$\Delta = +0.01$ i.e. $\delta_+(5,u) = 0.07 - 0.002\,u$

$\Delta = -0{,}01$ i.e. $\delta_-(5,u) = 0.05 - 0.002\,u$

Thus, the new spot prices at 0 are:

if $\Delta = +0.01$:

$$v^+(0,6) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^6 [0.07-0.002z]dz} = 0.759572 \bullet 0.942707 = 0.716054$$

$$v^+(0,9) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^9 [0.07-0.002z]dz} = 0.759572 \bullet 0.799315 = 0.607137$$

$$v^+(0,7.25) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^{7.25} [0.07-0.002z]dz} = 0.759572 \bullet 0.878150 = 0.667018;$$

if Δ = -0.01:

$$v^{-}(0,6) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^6 [0.05-0.002z]dz} = 0.759572 \bullet 0.961751 = 0.730519$$

$$v^{-}(0,9) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^9 [0.05-0.002z]dz} = 0.759572 \bullet 0.865888 = 0.657704$$

$$v^{-}(0,7.25) =$$

$$= e^{-\int_0^5 [0,06-0,002z]dz} e^{-\int_5^{7,25} [0,05-0,002z]dz} = 0.759572 \bullet 0.918569 = 0.697719$$

Let us verify immunization with respect to given additive shifts:

if Δ = +0.01:

on 1st bond:	53,921716·1000·0,716054	= 38,610.86
on 2nd bond:	88,164856·500·0,607137	= 26,764.07
present value of assets		= 65,374.93
present value of liabilities	98000·0,667018	= 65,367.76

If Δ = -0,01:

on 1st bond:	53,921716·1000·0,730519	= 39,390.84
on 2nd bond:	88,164856·500·0,657704	= 28,993.19
present value of assets		= 68,384.03
present value of liabilities	98,000·0.697719	= 68,376.46

If both Δ = +0,01 and Δ = -0,01: asset present value ≥ liability present value.

9.3.6. *Multiple liability cover*

The immunization problem with regard to single liability cover can be generalized into that of multiple liabilities cover, i.e. with reference to many outlays (or financial obligations). Then we assumes that the operator must deal to pay many debts $\underline{b}$ (*liabilities*), spread over time, by means of many receipts due to credits $\underline{a}$ (*assets*). Such a process is called: *Asset-Liability Management* (ALM).

Let us consider an initial balance statement in terms of the present value of assets $\underline{a} = (a_1,..., a_n)$, $a_h \geq 0$, and of liabilities $\underline{b} = (b_1,..., b_n)$, $b_h \geq 0$, according to the market rate in force at time 0. $\underline{t} = (t_1,..., t_n)$ $(0<t_1< ...<t_n)$ is the common[21] tickler of $\underline{a}$ and $\underline{b}$. However, under what conditions does the initial equilibrium not change into unfavorable imbalance under a subsequent change of the market rates' structure?

21 As already seen, this coincidence is not restrictive if we refer to the union of $\underline{a}>\underline{0}$ and $\underline{b}>\underline{0}$ ticklers.

It is evident that an easy solution is obtained using an asset portfolio "devoted" to a given liabilities vector, that is: $a_h = b_h$, $\forall t_h$. In such a case each receipt corresponds to an outlay with the same amount and maturity. Then the former exactly covers the latter without residual debts or credits. However, such a situation, i.e. a sufficient condition of immunization, is quite unusual.

For situations when equality does not occur between distributions of cash inflows and outflows, a rule, with regard to the rate risk of insurance companies under flat-yield-curve hypothesis for the market rates, was first given by Redington (1952). Bearing in mind Redington's rule, let us assume a balance statement at 0 between assets and liabilities, without shift, given by a budget constraint

$$V(0,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\delta t_k} = \sum_{k=1}^{n} b_k e^{-\delta t_k} = V(0,\underline{b};0) \qquad (9.39)$$

where $V(0,\underline{a};0)$ and $V(0,\underline{b};0)$ are the values of $\underline{a}$ and $\underline{b}$ at 0 without shift and δ is the intensity in force at time 0. Still denoting by $\underline{s} = (s_1,..., s_n)$, where $s_h = a_h - b_h$, the vector of net flows, (9.39) is equivalent to $V(0,\underline{s};0)$=0, which means the fairness of the whole operation the valued according δ. If an additive shift occurs, the following theorem holds

Theorem C (Redington). Let us assume that at 0 the constant intensity δ and (9.39) holds in the market and that an additive shift from δ to $\delta+Y$, with random $|Y|$ sufficiently small occurs just after 0[22]. Thus, according to previous definitions about $\underline{a}$, $\underline{b}$, $\underline{t}$, a sufficient condition to realize immunization, i.e.

$$V(0,\underline{s};Y) = V(0,\underline{a};Y) - V(0,\underline{b};Y) \geq 0 \qquad (9.40)$$

– where the values at 0 are calculated in the hypothesis of shift Y – is that both

$$\sum_{k=1}^{n} t_k a_k e^{-\delta t_k} = \sum_{k=1}^{n} t_k b_k e^{-\delta t_k} \qquad (9.41)$$

and

$$\sum_{k=1}^{n} t_k^2 a_k e^{-\delta t_k} > \sum_{k=1}^{n} t_k^2 b_k e^{-\delta t_k} \qquad (9.42)$$

hold.

Proof. Equation (9.41) signifies equality between the first derivatives of $\underline{a}$ and $\underline{b}$ in Y=0, i.e. $V'(0,\underline{a};0) = V'(0,\underline{b};0)$. Equation (9.42) signifies inequality between their second derivatives in Y=0, i.e. $V''(0,\underline{a};0) > V''(0,\underline{b};0)$. This implies that

$$V'(0,\underline{s};0) = 0 \ ; \quad V''(0,\underline{s};0) > 0. \qquad (9.43)$$

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22 This specification, given for the sake of simplicity, is not basic: these results also hold with a shift in some time after 0 but before t_1.

The truth of the system in (9.43) is, as well known, a sufficient condition in order that $V(0,\underline{s};Y)=\sum_{k=1}^{n} s_k e^{-(\delta+Y)t_k}$ has a relative minimum into Y=0, so with $|Y|$ sufficiently small, (9.40) holds[23].

Recalling (9.13') and (9.16) and taking into account the budget constraint in (9.39), we can observe that (9.41) leads to equality

$$D(\underline{a})=D(\underline{b}) \tag{9.41'}$$

which is the well known necessary Redington condition for immunization with regard to ALM. Moreover, still owing to the budget constraint, (9.42) leads to inequality

$$\gamma_\delta(\underline{a})>\gamma_\delta(\underline{b}) \tag{9.42'}$$

Therefore, the immunization condition for multiple liability cover can meaningfully be formulated requiring that at time 0 the duration of assets are equal to that of liabilities and the convexity of assets are larger than that of liabilities (inequality satisfied, of course, in case of single liability cover and in the Fisher-Weil theorem).

Under the two hypotheses of budget constraint and equality of durations, the inequality condition in (9.42') between asset and liabiliy convexities implies the following meaning of immunization: a market rate decrease (a market rate increase) leads to an increase (a decrease) of the value of the assets which is larger (smaller) than that of the liabilities. Then in both shift cases we obtain a net margin increase.

We must still observe that (9.42') implies

$$\sigma^2(\underline{a})>\sigma^2(\underline{b}) \tag{9.42''}$$

where $\sigma^2(\underline{a})$ and $\sigma^2(\underline{b})$ are the variances of $\underline{a}$ and $\underline{b}$, i.e. the central second moments of distributions ($\underline{t}$&$\underline{a}$) and ($\underline{t}$&$\underline{b}$). To prove this statement, it is sufficient to recall the equalities $\sigma^2=D^2-D^{(2)}$ and equation (9.41').

Both observations can be generalized to the case of variable rates under a term structure and possible additive shifts. In relation to this argument let us now give a theorem generalizing the Redington condition under financial law following $\delta(x,u)$ intensity.

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23 It is evident proof of Theorem C can be obtained by the Taylor expansion up to 2nd order of $V(0,\underline{s};Y)$ with starting point Y=0.

Theorem D (generalization of Redington theorem). Let $\delta(0,u)$ be the intensity current at time 0 on the market. Given two cash-flows, the former with assets $\underline{a} = (a_1,..., a_n)$, $(a_h \geq 0)$, the latter with liabilities $\underline{b} = (b_1,..., b_n)$, $(b_h \geq 0)$, both with tickler $\underline{t} = (t_1,..., t_n)$, $0<t_1< ...<t_n$. Let us assume that $\underline{a}$ and $\underline{b}$ are balanced under $\delta(0,u)$, or that the budget constraint

$$V(0,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\int_0^{t_k} \delta(0,u)du} = \sum_{k=1}^{n} b_k e^{-\int_0^{t_k} \delta(0,u)du} = V(0,\underline{b};0) \quad (9.44)$$

holds. In addition, we suppose that $\delta(0,u)$ has at t' just after 0[24] an additive infinitesimal shift Y according to (9.29) with $t' = 0^+$ for simplicity. Then

$$V(t',\underline{a};Y) \geq V(t',\underline{b};Y) \quad (9.45)$$

(equivalent to $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$ and implying immunization against shift Y) holds, if, valuing with the use of $\delta(0,u)$, equality (9.41') between $\underline{a}$ and $\underline{b}$ durations at 0, i.e.

$$\sum_{k=1}^{n} t_k a_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0) = \sum_{k=1}^{n} t_k b_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,b;0),$$

is verified, as well as the inequality

$$D^{(2)}(\underline{a}) > D^{(2)}(\underline{b}) \quad (9.46)$$

between $\underline{a}$ and $\underline{b}$ 2nd order durations in 0, i.e.

$$\sum_{k=1}^{n} t_k^2 a_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0) > \sum_{k=1}^{n} t_k^2 b_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0)$$

is valid.

Proof. With reference to net amounts $\underline{s} = \underline{a} - \underline{b}$, let us denote by

$$D(\underline{s}) = \sum_{k=1}^{n} t_k s_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{s};0)$$

the $\underline{s}$ duration in 0. We obtain:

$$D(\underline{s}) = D(\underline{a}) - D(\underline{b}) = \left(\frac{\partial}{\partial Y} V(0,\underline{s};Y) \right)_{Y=0}.$$

In addition, let us denote by

$$D^{(2)}(\underline{s}) = \sum_{k=1}^{n} t_k^2 s_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{s};0)$$

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24 See also footnote 22.

the $\underline{s}$ 2nd order duration, resulting in:

$$D^{(2)}(\underline{s}) = D^{(2)}(\underline{a}) - D^{(2)}(\underline{b}) = \left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y) \right)_{Y=0}$$

Let us consider the Taylor expansion of $V(0, \underline{s}; Y)$, starting by Y=0, up to 1st order and using the 2nd order remainder. We obtain, with η included between 0 and Y,

$$V(0,\underline{s};Y) = V(0,\underline{s};0) + \left(\frac{\partial}{\partial Y} V(0,\underline{s};Y) \right)_{Y=0} Y + \frac{1}{2} \left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y) \right)_{Y=\eta} Y^2 \quad (9.47)$$

Thus condition (9.41') is equivalent to $\left(\frac{\partial}{\partial Y} V(0,\underline{s};Y) \right)_{Y=0} = 0$; moreover, the condition in (9.46) is equivalent to $\left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y) \right)_{Y=\eta} Y^2 > 0$ provided that $|Y|$ is sufficiently small. Therefore, (9.47) implies the sufficiency of given conditions in order that (9.45) holds.

Owing to the budget constraint and (9.41'), inequality (9.46) is equivalent to inequality (9.42") between the central second moments.

The operative meaning of Theorem D consists of portfolio selection of assets $\underline{a}$ to cover liabilities $\underline{b}$, immunized with respect to rate risk related to the chance of additive shift. For the stated reasons regarding Theorem B, it is not restrictive, for the sake of simplicity to limit ourselves to the case of two assets and two liabilities. Let the assets be ZCB having unit value U_1 at maturity t_1 and U_2 at maturity $t_2 > t_1$; the liabilities are b_1 at maturity T_1 and b_2 at maturity $T_2 > T_1$. We have to calculate the shares, i.e. the numbers α_1 and α_2 of the asset bond in order to satisfy the budget constraint and the 1st order condition on the durations that are necessary for immunization. Let us agree the unit price $v(0,u) = e^{-\int_0^u \delta(0,z)dz}$ depending on intensity $\delta(0,u)$ and then calculate the value $V(0,\underline{b};0) = \sum_{k=1}^{2} b_k v(0,T_k)$, depending on rates at 0, and the duration $D(\underline{b}) = \sum_{k=1}^{2} T_k b_k v(0,T_k) / \sum_{k=1}^{2} b_k v(0,T_k)$ of liabilities. Then the asset bonds shares are obtained resolving the linear system

$$\begin{cases} \alpha_1 U_1 v(0,t_1) + \alpha_2 U_2 v(0,t_2) = V(0,\underline{b};0) \\ t_1\alpha_1 U_1 v(0,t_1) + t_2\alpha_2 U_2 v(0,t_2) = D(\underline{b})\ V(0,\underline{b};0) \end{cases} \tag{9.48}$$

which generalizes system (9.37), as well as its solution

$$\alpha_1 = \frac{V(0,\underline{b};0)(t_2 - D(\underline{b}))}{U_1 v(0,t_1)(t_2 - t_1)} \ , \ \alpha_2 = \frac{V(0,\underline{b};0)(D(\underline{b}) - t_1)}{U_2 v(0,t_2)(t_2 - t_1)} \quad ^{25} \tag{9.49}$$

generalizes solution (9.38). In particular, $\underline{b}$ duration takes the place of maturity T of the only b in system (9.37).

In the case of $N = N_1+N_2$ asset bonds, it is sufficient to consider two subgroups N_1 , N_2 substituting their durations for t_1 and t_2.

Exercise 9.4

Let us consider a portfolio, having liabilities of 50,000 at time 5 and 40,000 at time 7, to cover by shares of two packages of ZCB, the former with U_1=1,000 at maturity 3, the latter with U_2= 800 at maturity 9. We assume that in the market the intensity is $\delta(0,u)$=0,06-0,001u. Let us carry out the immunization and check that it is obtained, applying the Theorem D rules with a check of condition (9.46) on 2nd order durations.

A. According to cash-flow distribution and given intensity, we obtain:

- discount factor from u to 0: $v(0,u) = e^{-\int_0^u (0.06-0.001z)dz} = e^{-(0.06u-0.001u^2/2)}$;
- liability value: $V(0,\underline{b};0) = 50000\ e^{-0.2875} + 40000\ e^{-0.3955} = 64440.56$;
- liability duration: $D(\underline{b}) = 5 \cdot 50000 \cdot e^{-0.2875} + 7 \cdot 40000 \cdot e^{-0.3955} = 5.8359$.

The unknowns of the resulting system (9.48) are the real numbers α_1 and α_2 of ZCB shares, which make up the assets. Since

$$t_1 = 3 \quad ; \quad U_1 = 1000\ ; \quad v(0,t_1) = e^{-(0.06\cdot 3 - 0.001\cdot 4.5)} = 0.839037$$
$$t_2 = 9 \quad ; \quad U_2 = \ 800\ ; \quad v(0,t_2) = e^{-(0.06\cdot 9 - 0.001\cdot 40.5)} = 0.606834$$

the matrix of the coefficients and the constant terms of system (9.48) is given by

$$\begin{Vmatrix} 839.037 & 485.467 & 64,440.56 \\ 25,17.111 & 4,369.203 & 376,068.66 \end{Vmatrix}$$

25 We can observe that: $(\alpha_1>0)\cap(\alpha_2>0) \Leftrightarrow (t_1<D(\underline{b})<t_2)$.

Therefore, owing to (9.49), the shares are

$$\alpha_1 = \frac{64,440.56\ (9-5.8359)}{839.037\ (9-3)} = 40.50206\ ; \alpha_2 = \frac{64,440.56\ (5.8359-3)}{485.467\ (9-3)} = 62.73995$$

The total par value to obtain for the two assets is:

par value (1) = 40,502.06 ; par value (2) = 50,191.96

With such amounts the budget constraint is verified, because

$$V(0,\underline{a};0) = \alpha_1 U_1 v(0,t_1) + \alpha_2 U_2 v(0,t_2) = 40.50206 \cdot 839.037 + 62.73995 \cdot 485.467 =$$
$$= 64440.55 = V(0,\underline{b};0)$$

The equality between durations is also verified. Thus,

$$D(\underline{a}) = (40.50206 \cdot 2517.111 + 62.73995 \cdot 4369.203)/64440.56 = 5.8359 = D(\underline{b})$$

We must now evaluate the 2nd order durations to verify if the immunization sufficient condition is satisfied. We obtain:

$$D^{(2)}(\underline{a}) = (3^2 \cdot 40.50206 \cdot 839.037 + 9^2 \cdot 62.73995 \cdot 485.467)/64440.56 = 43.031249$$
$$D^{(2)}(\underline{b}) = (5^2 \cdot 50000 \cdot e^{-0.2875} + 7^2 \cdot 40000 \cdot e^{-0.3955})/64440.56 = 35.031098$$

Regarding the central second moments, i.e. the variances, of ($\underline{t'}\&\underline{a}$) and ($\underline{t''}\&\underline{b}$) we obtain:

$$\sigma^2(\underline{a}) = D^{(2)}(\underline{a}) - \left(D(\underline{a})\right)^2 = 8.973520 \quad ; \quad \sigma^2(\underline{b}) = D^{(2)}(\underline{b}) - \left(D(\underline{b})\right)^2 = 0.973369$$

Therefore, the immunization condition is satisfied. We can verify that the value of $\underline{s}$ is 0 with a relative minimum if the intensity is the given $\delta(0,u) = 0.06\text{-}0.001u$, valuing under shift $|Y| = 0.005$. For the sake of simplicity, we assume that the shift occurs in 0^+ only after valuation but this hypothesis is not restrictive: the conclusions also hold with any shift before 3. Valuing after shift, we obtain:

$$\delta(0^+,z)\text{=}0.06\text{-}0.001z \quad ; \quad v(0^+,u) = e^{-\int_0^u (0.06+Y-0.001z)dz}$$

The statements are: $\omega_1 = (Y = +0.005)$; $\omega_2 = (Y = -0.005)$.

– if $\omega = \omega_1$: $v(0^+,u) = e^{-\int_0^u (0.065-0.001z)dz} = e^{-(0.065u-0.001u^2/2)}$

– if $\omega = \omega_2$: $v(0^+,u) = e^{-\int_0^u (0.055-0.001z)dz} = e^{-(0.055u-0.001u^2/2)}$

The values at 0 under shift in 0^+ are:

– if $\omega = \omega_1$:

$V(0,\underline{a};+0.005) =$

$= 40.50206 \cdot 1000 \cdot e^{-(0.065 \cdot 3 - 0.001 \cdot 4.5)} + 62.73995 \cdot 800 \cdot e^{-(0.065 \cdot 9 - 0.001 \cdot 40.5)} =$

$= 40502.06 \cdot e^{-0.1905} + 50191.96 \cdot e^{-0.5445} = 62594.76;$

$V(0,\underline{b};+0.005) = 50000 \cdot e^{-(0.065 \cdot 5 - 0.001 \cdot 12.5)} + 40000 \cdot e^{-(0.065 \cdot 7 - 0.001 \cdot 24.5)} =$

$= 50000 \cdot e^{-0.3125} + 40000 \cdot e^{-0.4305} = 62588.14;$

$V(0,\underline{s};+0.005) = 62594.76 - 62588.14 = +6.62;$

– if $\omega = \omega_2$:

$V(0,\underline{a};-0.005) =$

$= 40502.06 \cdot e^{-0.1605} + 50191.96 \cdot e^{-0.4545} = 66356.44;$

$V(0,\underline{b};-0.005) = 50000 \cdot e^{-(0.055 \cdot 5 - 0.001 \cdot 12.5)} + 40000 \cdot e^{-(0.055 \cdot 7 - 0.001 \cdot 24.5)} =$

$= 50000 \cdot e^{-0.2625} + 40000 \cdot e^{-0.3605} = 66349.42;$

$V(0,\underline{s};-0.005) = 66356.44 - 66349.42 = +7.02.$

Thus the immunization is checked. Let us verify the different changes of asset and liability values depending on a shift, implying immunization:

– if $\omega = \omega_1$ (δ increases):

(assets) $V(0,\underline{a};+0.005) - V(0,\underline{a};0) = 62594.76 - 64440.56 = -1845.80$

(liabilities) $V(0,\underline{b};+0.005) - V(0,\underline{b};0) = 62588.14 - 64440.56 = -1852.42$

The decrease of the value of the assets is less than the decrease of the value of the liabilities:

– if $\omega = \omega_2$ (δ decreases):

(assets) $V(0,\underline{a};-0.005) - V(0,\underline{a};0) = 66356.44 - 64440.56 = +1915.88$

(liabilities) $V(0,\underline{b};-0.005) - V(0,\underline{b};0) = 66349.42 - 64440.56 = +1908.86$

The increase in the value of the assets is greater than the increase in the value of the liabilities.

We gave the conditions for semi-deterministic immunization of rate risk in several hypotheses, but always with reference to one additive random shift. In the case of several additive shifts, we can carry out subsequent immunizations.

Shiu (1990) generalized the Redington scheme, not only referring to a non-flat rate structure but to *non-additive shifts Y(u) with u*>0 as well. With regards to this extension, we can prove that to obtain immunization the conditions in (9.45) and (9.46) are needed jointly with other inequality constraints.

However, we do not dwell here on these generalizations and stochastic extensions of the immunization, leaving such questions to be discussed in specialized papers.

Part II

Stochastic Models

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Chapter 10

Basic Probabilistic Tools for Finance

In this chapter, the reader will find a short summary of the basic probability tools useful for understanding of the following chapters. A more detailed version including proofs can be found in Janssen and Manca (2006).

We will focus our attention on stochastic processes in discrete time and continuous time defined by sequences of random variables.

10.1. The sample space

In order to model finance problems, the basic concrete notion in probability theory is that of the *random experiment*, that is to say an experiment for which we cannot predict in advance the outcome. With each random experiment, we can associate the *elementary events* ω, which often represent the time evolution of the values of an asset on a stock exchange on a time interval $[0,T]$. The set of all these events Ω is called the *sample space*. Some other subsets of Ω will represent possible *events*.

Let us consider the following examples.

Example 10.1 A bank is to invest in some shares, so the bank looks at the history of the value of different shares. In this case, the sample space is the set of all non-negative real numbers $\mathbb{R}^+$.

To be useful, the set of all possible events must have some properties of stability so that we can generate new events such as:

(i) *the complement* A^c : $A^c = \{\omega \in \Omega : \omega \notin A\}$ *;* (10.1)

(ii) *the union* $A \bigcup B$ *:* $A \bigcup B = \{\omega : \omega \in A \text{ or} \omega \in B \}$ *;* (10.2)

(iii) *the intersection* $A \bigcap B$: $A \bigcap B = \{\omega : \omega \in A, \omega \in B\}$. (10.3)

More generally, if $(A_n, n \geq 1)$ represents a sequence of events, we can also consider the following events:

$$\bigcup_{n\geq 1} A_n, \ \bigcap_{n\geq 1} A_n \tag{10.4}$$

representing respectively the *union* and the *intersection* of all the events of the given sequence. The first of these two events occurs if and only if at least one of these events occurs and the second if and only if all the events of the given sequence occur. The set Ω is called the *certain event* and the set $\varnothing$ the *empty event*. Two events *A* and *B* are said to be *disjoint* or *mutually exclusive* if and only if

$$A \bigcap B = \varnothing . \tag{10.5}$$

Event A *implies* event *B* if and only if

$$A \subset B . \tag{10.6}$$

In Example 10.1, the event "the value of the share is between \$50 and \$80" is given by the set [50,80].

10.2. Probability space

Given a sample space Ω, the set of all possible events will be noted by $\Im$, assumed to have the structure of an σ*-field* or an σ*-algebra.*

Definition 10.2 The family $\Im$ of subsets of Ω is called a σ-field or a σ-algebra if and only if the following conditions are satisfied:

(i) $\Omega, \varnothing$ *belong* to $\Im$*;*

(ii) Ω *is stable under a denumerable intersection:*

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcap_{n\geq 1} A_n \in \Im, \tag{10.7}$$

(iii) $\Im$ *is stable for the complement set operation:*

$$A \in \Im \Rightarrow A^c \in \Im, \tag{10.8}$$

(*with* $A^c = \Omega - A$).

Using the well-known de Morgan's laws of set theory, it is easy to prove that a σ-*algebra* $\Im$ is also stable under a denumerable union:

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcup_{n\geq 1} A_n \in \Im. \tag{10.9}$$

Any couple $(\Omega, \Im)$ where $\Im$ is an σ-algebra is called a *measurable space*.

The next definition concerning the concept of *probability measure* or simply *probability* is an idealization of the concept of the *frequency* of an event.

Let us consider a random experiment called *E* with which the couple $(\Omega, \Im)$ is associated; if set *A* belongs to $\Im$ and if we can repeat experiment *E n* times under the same environmental conditions, we can count how many times *A* occurs. If *n*(*A*) represents the number of occurrences, the *frequency* of the event *A* is defined as

$$f(A) = \frac{n(A)}{n}. \tag{10.10}$$

In general, this number tends to become stable for large values of *n*.

The notion of frequency satisfies the following elementary properties:

(i) $\left(A, B \in \Im, A \cap B = \varnothing\right) \Rightarrow f(A \cup B) = f(A) + f(B),$ (10.11)

(ii) $f(\Omega) = 1$, (10.12)

(iii) $A, B \in \Im, \Rightarrow f(A \cup B) = f(A) + f(B) - f(A \cap B),$ (10.13)

(iv) $A \in \Im \Rightarrow f(A^c) = 1 - f(A).$ (10.14)

In order to have a useful mathematical model for the theoretical idealization of the notion of frequency, we now introduce the following definition.

Definition 10.3

a) *The triplet* $(\Omega, \Im, P)$ *is called a probability space if* Ω *is a non-void set of elements,* $\Im$ *a* σ*-algebra of subsets of* Ω *and P an application from* $\Im$ *to* [0,1] *such that:*

(i)
$$(A_n, n \geq 1), A_n \in \Im, n \geq 1 : \left(i \neq j \Rightarrow A_i \bigcap A_j = \phi\right)$$
$$\Rightarrow P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n=1}^{\infty} P(A_n) \ (\sigma\text{ - } additivity\ of\ P), \tag{10.15}$$

(ii) $$P(\Omega) = 1. \tag{10.16}$$

b) *The application P satisfying conditions* (10.15) *and* (10.16) *is called a probability measure or simply probability.*

Remark 10.1

Relation (10.17) assigns the value 1 for the probability of the entire sample space Ω. There may exist events A' which are strictly subsets of Ω such that

$$P\left(A'\right) = 1. \tag{10.17}$$

In this case, we say that *A* is *almost sure* or that the statement defining *A* is true *almost surely* (in short a.s.) or holds for almost all ω.

From axioms (10.15) and (10.16), we can deduce the following properties.

Property 10.1

(i) If $A, B \in \Im$, then

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B). \tag{10.18}$$

(ii) If $A \in \Im$, then

$$P(A^c) = 1 - P(A). \tag{10.19}$$

(iii) $$P(\varnothing) = 0. \tag{10.20}$$

(iv) If $(B_n, n \geq 1)$ is a sequence of disjoint elements of $\Im$ forming a partition of Ω, then for all *A* belonging to $\Im$,

$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n) . \tag{10.21}$$

(v) *Continuity property of P*: if $(A_n, n \geq 1)$ is an increasing (decreasing) sequence of elements of $\Im$, then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \lim_n P(A_n) ; \left(P\left(\bigcap_{n\geq 1} A_n\right) = \lim_n P(A_n)\right). \tag{10.22}$$

(vi) *Boole's inequality* asserts that if $(A_n, n \geq 1)$ is a sequence of events, then

$$P\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} P(A_n). \tag{10.23}$$

Example 10.2

a) *The discrete case*

When the sample space Ω is *finite* or *denumerable*, we can set

$$\Omega = \left\{\omega_1, ..., \omega_j, ...\right\} \tag{10.24}$$

and select for $\Im$ the set of all the subsets of Ω, represented by 2^{Ω}.

Any probability measure *P* can be defined with the following sequence:

$$(p_j, j \geq 1),\ p_j \geq 0, j \geq 1,\ \sum_{j\geq 1} p_j = 1 \tag{10.25}$$

so that

$$P\left(\left\{w_j\right\}\right) = p_j, j \geq 1. \tag{10.26}$$

On the probability space $(\Omega, 2^{\Omega}, P)$, the probability assigned for an arbitrary event $A = \{\omega_{k_1}, ..., \omega_{k_l}\}, k_j \geq 1, j = 1, ..., l, k_i \neq k_j$ if $i \neq j$ is given by

$$P(A) = \sum_{j=1}^{l} p_{k_j}. \tag{10.27}$$

b) *The continuous case*

Let Ω be the real set $\mathbb{R}$; it can be proven (Halmos (1974)) that there exists a minimal σ-algebra generated by the set of intervals:

$$\beta = \{(a,b),[a,b],[a,b),(a,b],a,b \in \mathbb{R}, a \leq b\}. \tag{10.28}$$

It is called the *Borel σ-algebra* represented by β and the elements of β are called *Borel sets*.

Given a probability measure P on (Ω, β), we can define the real function F, called the distribution function related to P, as follows.

Definition 10.4 *The function F from $\mathbb{R}$ to $[0,1]$ defined by:*

$$P((-\infty, x]) = F(x), x \in \mathbb{R} \tag{10.29}$$

is called the distribution function related to the probability measure P.

From this definition and the basic properties of P, we easily deduce that:

$$\begin{aligned} &P((a,b]) = F(b) - F(a),\ P((a,b)) = F(b-) - F(a), \\ &P([a,b)) = F(b-) - F(a-),\ P([a,b]) = F(b) - F(a-). \end{aligned} \tag{10.30}$$

Moreover, from (10.29), any function F from $\mathbb{R}$ to $[0,1]$ is a distribution function (in short d.f.) if and only if it is a non-decreasing function satisfying the following conditions:

– F is right continuous at every point x_0,

$$\lim_{x \uparrow x_0} F(x) = F(x_0), \tag{10.31}$$

– and moreover

$$\lim_{x \to +\infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0 . \tag{10.32}$$

If function F is derivable on $\mathbb{R}$ with f as the derivative, we have

$$F(x) = \int_{-\infty}^{x} f(y)dy, x \in \mathbb{R}. \tag{10.33}$$

Function f is called the density function associated with the d.f. F and in the case of the existence of such a Lebesgue integrable function on $\mathbb{R}$, F is said to be *absolutely continuous*.

From the definition of the concept of integral, we can give the intuitive interpretation of f as follows; given the small positive real number Δx, we have:

$$P\left(\left\{x, x + \Delta x\right\}\right) \approx f(x)\Delta x . \tag{10.34}$$

Using the Lebesgue Stieltjes integral, it can be seen that it is possible to define a probability measure P on $(\mathbb{R}, \beta)$ starting from a d.f. F on $\mathbb{R}$ by the following definition of P:

$$P(A) = \int_{A} dF(x), \forall A \in \Im. \tag{10.35}$$

In the absolutely continuous case, we obtain

$$P(A) = \int_{A} f(y)dy. \tag{10.36}$$

10.3. Random variables

Let us suppose the probability space $(\Omega, \Im, P)$ and the measurable space (E, ψ) are given.

Definition 10.5 *A random variable* (*in short r.v.*) *with values in E is an application X from* Ω *to E such that*

$$\forall B \in \psi : X^{-1}(B) \in \Im , \tag{10.37}$$

where $X^{1}(B)$ is called the inverse image of the set B defined by

$$X^{-1}(B)=\{\omega : X(\omega)\in B\}, X^{-1}(B)\in \Im. \tag{10.38}$$

Particular cases

a) If $(E,\psi)=(\mathbb{R},\beta)$, X is called a *real random variable*.

b) If $(E,\psi)=(\overline{\mathbb{R}},\overline{\beta})$, where $\overline{\mathbb{R}}$ is the *extended real line* defined by $\mathbb{R}\bigcup\{+\infty\}\bigcup\{-\infty\}$ and $\overline{\beta}$ the *extended Borel σ-field* of $\overline{\mathbb{R}}$, that is, the minimal σ-field containing all the elements of β and the extended intervals

$$\begin{aligned}&[-\infty,a),(-\infty,a],[-\infty,a],(-\infty,a),\\&[a,+\infty),(a,+\infty],[a,+\infty],(a,+\infty),\ a\in\mathbb{R},\end{aligned} \tag{10.39}$$

X is called a *real extended value random variable*.

c) If $E=\mathbb{R}^{n}(n>1)$ with the product σ-field $\beta^{(n)}$ of β, X is called an *n-dimensional real random variable*.

d) If $E=\overline{\mathbb{R}}^{(n)}$ (n>1) with the product σ-field $\overline{\beta}^{(n)}$ of $\overline{\beta}$, X is called an *extended n-dimensional real random variable.*

A r.v. X is called *discrete* or *continuous* according to the fact that X takes a value in a set at most denumerable or non-denumerable.

Remark 10.2 In *measure theory*, the only difference is that condition (10.17) is no longer required and in this case the definition of a r.v. given above gives the notion of a *measurable function*. In particular, a measurable function from $(\mathbb{R},\beta)$ to $(\mathbb{R},\beta)$ is called a *Borel function*.

Let X be a real r.v. and let us consider, for any real x, the following subset of Ω: $\{\omega : X(\omega)\le x\}$.

Given that, from relation (10.38),

$$\{\omega : X(\omega)\le x\}=X^{-1}((-\infty,x]), \tag{10.40}$$

it is clear from relation (10.37) that this set belongs to the σ-algebra $\Im$.

Conversely, it can be proved that the condition

$$\{\omega : X(\omega) \le x\} \in \Im, \tag{10.41}$$

valid for every x belonging to a dense subset of $\mathbb{R}$, is sufficient for X being a real r.v. defined on Ω.

The probability measure P on $(\Omega, \Im)$ induces a probability measure μ on $(\mathbb{R}, \beta)$ defined as

$$\forall B \in \beta : \mu(B) = P\left(\{\omega : X(\omega) \in B\}\right). \tag{10.42}$$

We say that μ is the induced probability measure on $(\mathbb{R}, \beta)$, called the *probability distribution* of the r.v. X.

Introducing the distribution function related to μ, we obtain the next definition.

Definition 10.6 *The distribution function of the r.v. X, represented by* F_X, *is the function from* $\mathbb{R} \to [0,1]$ *defined by*

$$F_X(x) = \mu\left((-\infty, x]\right) = P\left(\{\omega : X(\omega) \le x\}\right). \tag{10.43}$$

In short, we write

$$F_X(x) = P\left(X \le x\right). \tag{10.44}$$

This last definition can be extended to the multi-dimensional case with r.v. X being an n-dimensional real vector: $X = (X_1, ..., X_n)$, a measurable application from $(\Omega, \Im, P)$ to $(\mathbb{R}^n, \beta^n)$.

Definition 10.7 *The distribution function of the r.v.* $X = (X_1, ..., X_n)$, *represented by* F_X, *is the function from* $\mathbb{R}^n$ *to* $[0,1]$ *defined by*

$$F_X(x_1, ..., x_n) = P\left(\{\omega : X_1(\omega) \le x_1, ..., X_n(\omega) \le x_n\}\right). \tag{10.45}$$

In short, we write

$$F_X(x_1, ..., x_n) = P(X_1 \le x_1, ..., X_n \le x_n). \tag{10.46}$$

Each component X_i (i=1,…,n) is itself a one-dimensional real r.v. whose d.f., called the *marginal d.f.*, is given by

$$F_{X_i}(x_i) = F_X(+\infty,...,+\infty,x_i,+\infty,...,+\infty). \quad (10.47)$$

The concept of random variable is *stable* under many mathematical operations; thus, any Borel function of a r.v. X is also an r.v.

Moreover, if X and Y are two r.v., so are

$$\inf\{X,Y\},\sup\{X,Y\},X+Y,X-Y,X\cdot Y,\frac{X}{Y}, \quad (10.48)$$

provided, in the last case, that Y does not vanish.

Concerning the convergence properties, we must mention the property that, if $(X_n, n \geq 1)$ is a *convergent* sequence of r.v. – that is, for all $\omega \in \Omega$, the sequence $(X_n(\omega))$ converges to $X(\omega)$ – then the limit X is also a r.v. on Ω. This convergence, which may be called the *sure convergence*, can be weakened to give the concept of an a.s. *convergence* of the given sequence.

Definition 10.8 *The sequence* $(X_n(\omega))$ *converges a.s. to* $X(\omega)$ *if*

$$P\left(\{\omega : \lim X_n(\omega) = X(\omega)\}\right) = 1. \quad (10.49)$$

This last notion means that the possible set where the given sequence does not converge is a *null set*, that is, a set N belonging to $\Im$ such that

$$P(N) = 0. \quad (10.50)$$

In general, let us note that, given a null set, it is not true that every subset of it belongs to $\Im$ but of course if it belongs to $\Im$, it is clearly a null set (see relation (10.26)).

To avoid unnecessary complications, we will assume from now on that any considered probability space is *complete*, This means that all the subsets of a null set also belong to $\Im$ and thus that their probability is zero.

10.4. Expectation and independence

Let us consider a complete measurable space $(\Omega, \Im, \mu)$ and a real measurable variable *X* defined on Ω. Using the concept of an integral, it is possible to define the *expectation* of *X* represented by

$$E(X) = \int_{\Omega} XdP\left(= \int XdP\right), \tag{10.51}$$

provided that this integral exists. The calculation of the integral

$$\int_{\Omega} XdP\left(= \int XdP\right) \tag{10.52}$$

can be done using the induced measure μ on $(\mathbb{R}, \beta)$, defined by relation (10.42) and then using the d.f. *F* of *X*.

Indeed, we can write

$$E(X)\left(= \int_{\Omega} XdP\right) = \int_{R} Xd\mu, \tag{10.53}$$

and if F_X is the d.f. of *X*, it can be shown that

$$E(X) = \int_{R} xdF_X(x), \tag{10.54}$$

this last integral being a Lebesgue Stieltjes integral.

Moreover, if F_X is absolutely continuous with f_X as the density, we obtain

$$E(X) = \int_{-\infty}^{+\infty} xf_x(x)dx. \tag{10.55}$$

If *g* is a Borel function, we also have (see for example Chung (2000), Royden (1963), Loeve (1963))

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x)dF_X \tag{10.56}$$

and with a density for X

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x) dx . \tag{10.57}$$

The most important properties of the expectation are given in the next proposition.

Proposition 10.1

(i) *Linearity property of the expectation: if X and Y are two integrable r.v. and a, b two real numbers, then the r.v. aX+bY is also integrable and*

$$E(aX + bY) = aE(X) + bE(Y). \tag{10.58}$$

(ii) *If* $(A_n, n \geq 1)$ *is a partition of* Ω*, then*

$$E(X) = \sum_{n=1}^{\infty} \int_{A_n} X dP . \tag{10.59}$$

(iii) *The expectation of a non-negative r.v. is non-negative.*

(iv) *If X and Y are integrable r.v., then*

$$X \leq Y \Rightarrow E(X) \leq E(Y). \tag{10.60}$$

(v) *If X is integrable, then so is* $|X|$ *and*

$$|E(X)| \leq E|X| . \tag{10.61}$$

(vi) *Dominated convergence theorem (Lebesgue): if* $(X_n, n \geq 1)$ *is a sequence of r.v. converging a.s. to the integrable r.v. X, then all r.v.* X_n *are integrable and moreover*

$$\lim E(X_n) = E(\lim X_n) \; (= E(X)) . \tag{10.62}$$

(vii) *Monotone convergence theorem (Lebesgue): if* $(X_n, n \geq 1)$ *is a non-decreasing sequence of non-negative r.v, then relation* (10.62) *is still true provided that* $+\infty$ *is a possible value for each member.*

(viii) *If the sequence of integrable r.v.* $(X_n, n \geq 1)$ *is such that*

$$\sum_{n=1}^{\infty} E\left(\left|X_n\right|\right) < \infty, \tag{10.63}$$

then the random series $\sum_{n=1}^{\infty} X_n$ *converges absolutely a.s. and moreover*

$$\sum_{n=1}^{\infty} E(X_n) = E\left(\sum_{n=1}^{\infty} X_n\right) (= E(X)), \tag{10.64}$$

where the r.v. is defined as the sum of the convergent series.

Given a r.v. *X*, *moments* are special cases of expectation.

Definition 10.8 *If a is a real number and r a positive real number, then the expectation*

$$E\left(\left|X - a\right|^r\right) \tag{10.65}$$

is called the absolute moment of X of order r, centered on a.

The moments are said to be centered moments of order *r* if $a=E(X)$. In particular, for $r=2$, we obtain the *variance* of *X* represented by σ^2 $(\mathrm{var}(X))$,

$$\sigma^2 = E\left(\left|X - m\right|^2\right). \tag{10.66}$$

Remark 10.3 From the linearity of the expectation (see relation (10.58)), it is easy to prove that

$$\sigma^2 = E(X^2) - (E(X))^2, \tag{10.67}$$

and so

$$\sigma^2 \leq E(X^2), \tag{10.68}$$

and more generally, it can be proven that the variance is the smallest moment of order 2 regardless of what *a* is.

The last fundamental concept we will now introduce in this section is that of *stochastic independence* or, more simply, *independence*.

Definition 10.9 *The events* $A_1,...,A_n,(n>1)$ *are stochastically independent or independent if and only if*

$$\forall m=2,...,n, \forall n_k=1,...,n: n_1 \neq n_2 \neq \cdots \neq n_k : P\left(\bigcap_{k=1}^{m} A_{n_k}\right) = \prod_{k=1}^{m} P(A_{n_k}) . \quad (10.69)$$

For *n=2*, relation (10.69) reduces to

$$P(A_1 \cap A_2) = P(A_1)P(A_2) . \quad (10.70)$$

Let us note that piecewise independence of the events $A_1,...,A_n,(n>1)$ does not necessarily imply the independence of these sets and thus does not imply the stochastic independence of these *n* events.

Definition 10.10

(i) *The n real r.vs.* $X_1,X_2,...,X_n$ *defined on the probability space* $(\Omega,\Im,P)$ *are said to be stochastically independent, or simply independent, if and only if for any Borel sets* $B_1,B_2,...,B_n$, *we have*

$$P\left(\bigcap_{k=1}^{n} \{\omega : X_k(\omega) \in B_k\}\right) = \prod_{k=1}^{n} P\left(\{\omega : X_k(\omega) \in B_k\}\right) . \quad (10.71)$$

(ii) *For an infinite family of r.vs., independence means that the members of every finite subfamily are independent. It is clear that if* $X_1,X_2,...,X_n$ *are independent, so are the r.vs.* $X_{i_1},...,X_{i_k}$ *with*

$$i_1 \neq \cdots \neq i_k,\ i_k = 1,...,n, k = 2,...,n .$$

From relation (10.71), we find that

$$P(X_1 \le x_1,...,X_n \le x_n) = P(X_1 \le x_1) \cdots P(X_n \le x_n), \forall (x_1,...,x_n) \in \mathbb{R}^n . \quad (10.72)$$

If the functions $F_X, F_{X_1},...,F_{X_n}$ are the distribution functions of r.v. $X=(X_1,...,X_n), X_1,...,X_n$, we can write the preceding relation under the form

$$F_X(x_1,....,x_n) = F_{X_1}(x_1) \cdot \cdots \cdot F_{X_n}(x_n), \forall (x_1,...,x_n) \in \mathbb{R}^n . \quad (10.73)$$

It can be shown that this last condition is also sufficient for the independence of $X=(X_1,...,X_n), X_1,...,X_n$. If these d.f. have densities $f_X, f_{X_1},...,f_{X_n}$, relation (10.73) is equivalent to

$$f_X(x_1,\dots,x_n)=f_{X_1}(x_1)\cdots f_{X_n}(x_n), \forall(x_1,...,x_n)\in\mathbb{R}^n. \tag{10.74}$$

In case of the integrability of n real r.vs. $X_1, X_2, ..., X_n$, a direct consequence of relation (10.72) is that we have a very important property for the expectation of the product of n independent r.vs.:

$$E\left(\prod_{k=1}^{n}X_k\right)=\prod_{k=1}^{n}E(X_k). \tag{10.75}$$

The notion of independence gives the possibility to prove the result called the *strong law of large numbers* which says that if $(X_n, n\geq 1)$ is a sequence of integrable independent and identically distributed r.vs., then

$$\frac{1}{n}\sum_{k=1}^{n}X_k \xrightarrow{a.s.} E(X). \tag{10.76}$$

The next section will present the most useful distribution functions for stochastic modeling.

10.5. Main distribution probabilities

Here we shall restrict ourselves to presenting the principal distribution probabilities related to real random variables.

10.5.1. *The binomial distribution*

Let us consider a random experiment E such that only two results are possible: a "success" (S) with probability p and a "failure" (F) with probability q=1-p. If n independent trials are made in exactly the same experimental environment, the total number of trials in which the event S occurs may be represented by a r.v. X whose distribution $(p_i, i=0,...,n)$ with

$$p_i=P(X=i),\ i=1,...,n \tag{10.77}$$

is called a *binomial distribution* with parameters (n,p).

From the basic axioms of probability theory previously stated, it is easy to prove that

$$p_i = \binom{n}{i} p^i q^{n-i}, \; i = 0,\ldots,n, \tag{10.78}$$

a result from which we get

$$E(X) = np, \operatorname{var}(X) = npq. \tag{10.79}$$

The *characteristic function* and the *generating function,* when they exist, of X respectively defined by

$$\begin{aligned} \varphi_X(t) &= E(e^{itX}), \\ g_X(t) &= E(e^{tX}) \end{aligned} \tag{10.80}$$

are given by

$$\begin{aligned} \varphi_X(t) &= (pe^{it} + q)^n, \\ g_X(t) &= (pe^{t} + q)^n. \end{aligned} \tag{10.81}$$

This distribution is currently used in the financial model of Cox, Ross and Rubinstein (1979), developed in Chapter 5.

10.5.2. *The Poisson distribution*

If X is an r.v. with values in $\mathbb{N}$ such that the probability distribution is given by

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, i = 0,1,\ldots \tag{10.82}$$

where λ is a strictly positive constant, then X is called a *Poisson variable with parameter* λ. This is one of the most important distributions for all applications. For example, if we consider an insurance company looking at the total number of claims in one year, this variable may often be considered as a Poisson variable.

Basic parameters of this Poisson distribution are given here:

$$E(X) = \lambda, \ \text{var}(X) = \lambda,$$
$$\varphi_X(t) = e^{\lambda(e^{it}-1)}, \ g_X(t) = e^{\lambda(e^t-1)}. \tag{10.83}$$

A remarkable result is that the Poisson distribution is the limit of a binomial distribution of parameters (*n,p*) if n tends to $+\infty$ and *p* to 0, so that np converges to λ.

The Poisson distribution is often used for the occurrence of rare events, for example, in credit risk presented in Chapter 19.

10.5.3. *The normal (or Laplace Gauss) distribution*

The real r.v. *X* has a normal (or Laplace Gauss) distribution of parameters $(\mu,\sigma^2), \mu \in \mathbb{R}, \sigma^2 > 0$, if its density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}. \tag{10.84}$$

From now on, we will use the notation $X \prec N(\mu,\sigma^2)$.

The main parameters of this distribution are

$$E(X) = \mu, \ \text{var}(X) = \sigma^2,$$
$$\varphi_X(t) = \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right), \ g_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right). \tag{10.85}$$

If $\mu = 0, \ \sigma^2 = 1$, the distribution of *X* is called a *reduced* or *standard normal distribution*. In fact, if *X* has a normal distribution $(\mu,\sigma^2), \mu \in R, \sigma^2 > 0$, then from (10.85), the reduced r.v. *Y* defined by

$$Y = \frac{X-\mu}{\sigma} \tag{10.86}$$

has a standard normal distribution with mean 0 and variance 1.

Let Φ be the distribution function of the standard normal distribution; it is possible to express the distribution function of any normal r.v. X with parameters $(\mu,\sigma^2), \mu \in \mathbb{R}, \sigma^2 > 0$ as follows:

$$F_X(x) = P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right). \tag{10.87}$$

Also, from the numerical point of view, it suffices to know numerical values for the standard distribution.

From relation (10.87), we also deduce that

$$f_X(x) = \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right), \tag{10.88}$$

where of course from (10.84)

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{10.89}$$

From the definition of Φ, we have

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,\ x \in \mathbb{R} \tag{10.90}$$

and so

$$\Phi(-x) = 1 - \Phi(x), x > 0, \tag{10.91}$$

and consequently, for X normally distributed with parameters (0,1), we obtain

$$P\left(|X| \leq x\right) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1,\ x > 0. \tag{10.92}$$

In particular, let us mention the following numerical results:

$$\begin{aligned}
&P\left(|X-m|\le \frac{2}{3}\sigma\right)=0.4972(\approx 50\%),\\
&P\left(|X-m|\le \sigma\right)=0.6826(\approx 68\%),\\
&P\left(|X-m|\le 2\sigma\right)=0.9544(\approx 95\%),\\
&P\left(|X-m|\le 3\sigma\right)=0.9974(\approx 99\%).
\end{aligned} \tag{10.93}$$

Remark 10.4 *Numerical calculation of the d.f.* Φ

For applications in finance, for example the Black-Scholes (1973) model for option pricing (see Chapter 5), we will need the following numerical approximation method for calculating Φ *with seven decimal places* instead of the four given by the standard statistical tables:

$$\begin{aligned}
&1)\ x>0\ :\\
&\Phi(x)\approx 1-\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(b_1c+\cdots+b_5c^5),\\
&c=\frac{1}{1+px},\\
&p=0.2316419,\ b_1=0.319381530,\\
&b_2=-0.356563782,\ b_3=1.781477937,\\
&b_4=-1.821255978,\ b_5=1.330274429,\\
&2)\ x<0\ :\\
&\Phi(x)=1-\Phi(-x).
\end{aligned} \tag{10.94}$$

The normal distribution is one of the most commonly used distributions, by virtue of the *central limit theorem* which says that if $(X_n, n\ge 1)$ is a sequence of independent identically distributed (in short IID) r.vs. with mean *m* and variance σ^2, then the sequence of r.vs. defined by

$$\frac{S_n-nm}{\sigma\sqrt{n}} \tag{10.95}$$

with

$$S_n = X_1 + \cdots + X_n, \quad n > 0 \tag{10.96}$$

converges in law to a standard normal distribution.

This means that the sequence of the distribution functions of the variables defined by (10.93) converges to Φ.

This theorem was used by the Nobel Prize winner H. Markowitz (1959) to justify that the return of a diversified portfolio of assets has a normal distribution. As a particular case of the central limit theorem, let us mention *de Moivre's theorem*, starting with

$$X_n = \begin{cases} 1, \text{ with prob. } p, \\ 0, \text{ with prob. } 1-p, \end{cases} \tag{10.97}$$

so that, for each *n,* the r.v. defined by relation (10.94) has a binomial distribution with parameters (*n,p*).

By applying the central limit theorem, we obtain the following result:

$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow[n \to +\infty]{law} N(0,1), \tag{10.98}$$

called de Moivre's result.

10.5.4. *The log-normal distribution*

Though the normal distribution is the most frequently used, it is nevertheless true that it could not be used for example to model the time evolution of a financial asset like a share or a bond, as the minimal value of these assets is 0 and so the support of their d.f. is half of the real line $[0, +\infty)$. One possible solution is to consider the *truncated normal distribution*, defined by setting all the probability mass of the normal distribution on the negative half-real line on the positive one; however, then all the interesting properties of the normal distribution are lost.

Also, in order to have a better approach to some financial market data, we have to introduce the *log-normal distribution*. The real non-negative r.v. X has a *lognormal distribution* with parameters μ,σ – which we will write as $X \prec LN(\mu,\sigma)$ – if the r.v. log*X* has a normal distribution with parameters μ,σ^2. Consequently, the density function of *X* is given by

$$f_X(x) = \begin{cases} 0, x \leq 0, \\ \dfrac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}, x > 0. \end{cases} \tag{10.99}$$

Indeed, we can write

$$P(X \leq x) = P(\log X \leq \log x), \tag{10.100}$$

and so

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\log x} e^{\frac{(t-\mu)^2}{2\sigma^2}} dt \left(= \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right), \tag{10.101}$$

and after the change of variable t=log$x$, we obtain relation (10.99).

Let us note that relation (10.101) is the most useful for the calculation of the d.f. of *X* with the help of the normal d.f.

For the density function, we can also write

$$f_X(x) = \frac{1}{\sigma x} \Phi\left(\frac{\log x - \mu}{\sigma}\right). \tag{10.102}$$

The basic parameters of this distribution are given by

$$\begin{aligned} &E(X) = e^{\mu+\frac{\sigma^2}{2}}, \\ &\operatorname{var}(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \\ &E\left(X^r\right) = e^{r\left(\mu+\frac{\sigma^2}{2}\right)}. \end{aligned} \tag{10.103}$$

Let us mention that the lognormal distribution has no generating function and that the characteristic function has no explicit form. When $\sigma < 0.3$, some authors recommend a normal approximation with parameters (μ, σ^2).

The normal distribution is *stable* under the addition of independent r.vs.; this property means that the sum of *n* independent normal r.vs. is still normal. That is no longer the case with the lognormal distribution which is stable under *multiplication*, which means that for two independent lognormal r.vs. X_1, X_2, we have

$$X_i \prec LN(\mu_i, \sigma_i), i = 1,2 \Rightarrow X_1 \times X_2 \prec LN\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right). \quad (10.104)$$

10.5.5. *The negative exponential distribution*

The non-negative r.v. X has a *negative exponential distribution* (or simply *exponential distribution*) of parameter λ if its density function is given by

$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0, \quad (10.105)$$

where λ is a strictly positive real number.

By integration, we obtain the explicit form of the exponential distribution function

$$F_X(x) = 1 - e^{-\lambda x}, x \geq 0. \quad (10.106)$$

Of course, F_X is zero for negative values of x.

The basic parameters are

$$\begin{aligned} &E(X) = \frac{1}{\lambda}, \ \operatorname{var} X = \frac{1}{\lambda^2}, \\ &\varphi_X(t) = \frac{1}{1 - i\frac{t}{\lambda}}, \ g_X(t) = \frac{1}{1 - \frac{t}{\lambda}}, t < \lambda. \end{aligned} \quad (10.107)$$

In fact, this distribution is the first to be used in reliability theory.

10.5.6. *The multidimensional normal distribution*

Let us consider an n-dimensional real r.v. X represented as a column vector of its n components $X = (X_1,...,X_n)'$. Its d.f. is given by:

$$F_X(x_1,...,x_n) = P(X_1 \le x_1,...,X_n \le x_n) . \tag{10.108}$$

If the density function of X exists, the relations between the d.f. and the density function are:

$$\begin{aligned} f_X(x_1,...,x_n) &= \frac{\partial^n F_X}{\partial x_1 ... \partial x_n}(x_1,...,x_n), \\ F_X(x_1,...,x_n) &= \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_X(\xi_1,...,\xi_n) d\xi_1,...,d\xi_n. \end{aligned} \tag{10.109}$$

For the principal parameters we will use the following notation:

$$\begin{aligned} &E(X_k) = \mu_k, k = 1,...,n, \\ &E((X_k - \mu_k)(X_l - \mu_l)) = \sigma_{kl}, k,l = 1,...,n, \\ &E((X_k - \mu_k))^2 = \sigma_k^2, k = 1,...,n, \\ &\rho_{kl} = \frac{E((X_k - \mu_k)(X_l - \mu_l))}{\sqrt{E((X_k - \mu_k)^2)E((X_k - \mu_k)^2)}} \left(= \frac{\sigma_{kl}}{\sigma_k \sigma_l} \right), k,l = 1,...,n. \end{aligned} \tag{10.110}$$

The parameters σ_{kl} are called the *covariances* between the r.v. X_k and X_l, and the parameters ρ_{kl}, the *correlation coefficients* between the r.v. X_k and X_l.

It is well known that the correlation coefficient ρ_{kl} measures a certain linear dependence between the two r.v. X_k and X_l. More precisely, if it is equal to 0, there is no such dependence and the two variables are called *uncorrelated*; for the values +1 and –1 this dependence is certain.

With matrix notation, the following $n \times n$ matrix

$$\mathbf{\Sigma}_X = \left[\sigma_{ij} \right] \tag{10.111}$$

is called the *variance-covariance* matrix of X.

The *characteristic function* of *X* is defined as:

$$\varphi_X(t_1,...,t_n) = E\left(e^{i(t_1X_1+..+t_nX_n)_1}\right)\left(=E\left(e^{i\mathbf{t}'X}\right)\right). \tag{10.112}$$

Let $\boldsymbol{\mu},\boldsymbol{\Sigma}$ be an *n*-dimensional real vector and an $n\times n$ positive definite matrix, respectively. The *n*-dimensional real r.v. *X* has a *non-degenerated n-dimensional normal distribution* with parameters $\boldsymbol{\mu},\boldsymbol{\Sigma}$ if its density function is given by:

$$f_X(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det\boldsymbol{\Sigma}}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}, \mathbf{x}\in\mathbb{R}^n. \tag{10.113}$$

Then, it can be shown by integration that parameters $\boldsymbol{\mu},\boldsymbol{\Sigma}$ are indeed respectively the *mean vector* and the *variance-covariance matrix* of *X*.

As usual, we will use the notation: $X \prec N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$.

The characteristic function of *X* is given by:

$$\varphi_X(\mathbf{t}) = e^{i\boldsymbol{\mu}'\mathbf{t}-\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}. \tag{10.114}$$

The main fundamental properties of the *n*-dimensional normal distribution are:

– every subset of *k* r.vs. of the set $\{X_1,\ldots,X_n\}$ also has a *k*-dimensional distribution which is also normal;

– the multi-dimensional normal distribution is *stable* under linear transformations of *X*;

– the multi-dimensional normal distribution is *stable* for addition of r.vs., which means that if $X_k \prec N_n(\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k), k=1,...,m$ and if these *m* random vectors are independent, then

$$X_1+\cdots+X_m \prec N_n(\boldsymbol{\mu}_1+\cdots+\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_1+\cdots+\boldsymbol{\Sigma}_m). \tag{10.115}$$

Particular case: *the two-dimensional normal distribution*

In this case, we have:

$$\boldsymbol{\mu} = (\mu_1, \mu_2)', \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2},$$
$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \dfrac{1}{\sigma_1^2} & -\dfrac{\rho}{\varsigma_1 \sigma_2} \\ -\dfrac{\rho}{\varsigma_1 \sigma_2} & \dfrac{1}{\sigma_2^2} \end{bmatrix}, \det \boldsymbol{\Sigma} = \sigma_1 \sigma_2 \sqrt{1-\rho^2}. \tag{10.116}$$

From the first main fundamental properties of the n-dimensional normal distribution given above, we have:

$$X_k \prec N_1(\mu_k, \sigma_k^2), k = 1,2. \tag{10.117}$$

For the special degenerated case of $|\rho| = 1$, it can be proved that:

$$\rho = 1 : \frac{X_2 - \mu_2}{\sigma_2} = \frac{X_1 - \mu_1}{\sigma_1},$$
$$\rho = -1 : \frac{X_2 - \mu_2}{\sigma_2} = -\frac{X_1 - \mu_1}{\sigma_1}, \tag{10.118}$$

meaning that in this case, all the probability mass in the plane lies on a straight line so the two r.vs. X_1,X_2 are perfectly dependent with probability 1.

To conclude this section, let us recall the well-known property stating that two independent r.vs. are uncorrelated, but the converse is not true except for the normal distribution.

10.6. Conditioning

Let us begin to briefly recall the concept of *conditional probability*. Let $(\Omega, \Im, P)$ be a probability space and let A, B be elements of $\Im$, and let us observe the number of occurrences of event A whenever B has already been observed in a sequence of n trials of our experiment. We shall call this number $n\left(A|B\right)$.

In terms of the frequency of events defined by relation (10.11), we have:

$$n\left(A|B\right)=\frac{n(A\cap B)}{n(B)}, \tag{10.119}$$

provided that $n(B)$ is not 0.

Dividing by n the two members of relation (10.119), we obtain:

$$\frac{n\left(A|B\right)}{n}=\frac{\frac{n(A\cap B)}{n}}{\frac{n(B)}{n}}. \tag{10.120}$$

In terms of frequencies, we obtain:

$$f\left(A|B\right)=\frac{f(A\cap B)}{f(B)}. \tag{10.121}$$

From the experimental interpretation of the concept of probability of an event seen in section 10.2, we can now define the *conditional probability of A given B* as:

$$P\left(A|B\right)=\frac{P(A\cap B)}{P(B)}, P(B)>0. \tag{10.122}$$

If events A and B are independent, from relation (10.122), we obtain:

$$P\left(A|B\right)=P(A), \tag{10.123}$$

meaning that, in the case of independence, the conditional probability of set A does not depend on the given set B.

As the independence of sets A and B is equivalent to the independence of sets A and B^c, we also have:

$$P\left(A|B^c\right)=P(A). \tag{10.124}$$

The notion of conditional probability is very useful for calculating probabilities of a product of *dependent* events A and B not satisfying relation (4.39). Indeed, from relations (10.122) and (10.124), we can write:

$$P(A \cap B) = P(A)P\left(A \middle| B\right) = P(B)P\left(B \middle| A\right). \tag{10.125}$$

More generally, for n events $A_1,...,A_n$, we obtain the "theorem of compound probability":

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1)P\left(A_2 \middle| A_1\right)\cdots P\left(A_n \middle| A_1 \cap A_2 ... \cap A_{n-1}\right), \tag{10.126}$$

a relation expanding relation (10.125).

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1)...P(A_n) \tag{10.127}$$

is true in the case of the independence of the n considered events.

If event B is fixed and of strictly positive probability, relation (10.122) provides the way of defining a *new* probability measure on $(\Omega, \Im)$ denoted P_B as follows:

$$P_B(A) = \frac{P(A \cap B)}{P(B)}, \forall A \in \Im. \tag{10.128}$$

P_B is in fact a probability measure as it is easy to verify that it satisfies conditions (10.16) and (10.17), and so P_B is called the *conditional probability measure given B.*

The integral with respect to this measure is called the conditional expectation E_B relative to P_B.

From relation (10.128) and since $P_B(B)$=1, we thus obtain for any integrable r.v. Y:

$$E_B(Y) = \int_{\Omega} Y(\omega) dP_B = \frac{1}{P(B)} \int_{B} Y(\omega) dP. \tag{10.129}$$

We can now extend this definition to *arbitrary* sub-σ-algebras instead of the simple case of $\left\{\varnothing, B, B^c, \Omega\right\}$ using an extension of property (10.129) as a definition with the help of the Radon Nikodym theorem (Halmos (1974)).

Definition 10.11 *If $\Im_1$ is a sub-σ-algebra of $\Im$, the conditional expectation of the integrable r.v. Y given $\Im_1$, denoted by $E_{\Im_1}(Y)$ or $E\left(Y \middle| \Im_1\right)$, is any r.v. of the equivalence class such that:*

(i) $E_{\Im_1}(Y)$ is $\Im_1$-*measurable,*

(ii) $\int_B E_{\Im_1}(Y)(\omega)dP = \int_B Y(\omega)dP, B \in \Im_1.$ (10.130)

In fact, the class of equivalence contains all the r.vs. a.s. equally satisfying relation (10.130).

Remark 10.5 Taking $B = \Omega$ in relation (10.130), we obtain:

$$E(E_{\Im_1}Y)) = E(Y). \tag{10.131}$$

Particular cases

(i) $\Im_1$ *is generated by r.v. X.*

This case means that $\Im_1$ is the sub-σ-algebra of $\Im$ generated by all the inverse images of *X*, and we will use as notation:

$$E_{\Im_1}(Y) = E\left(Y \middle| X\right), \tag{10.132}$$

where this conditional expectation is called the *conditional expectation of Y given X.*

(ii) $\Im_1$ *is generated by n r.vs.* $X_1,\ldots,X_n$.

This case means that $\Im_1$ is the sub-σ-algebra of $\Im$ generated by all the inverse images of $X_1,\ldots,X_n$ and we will use as notation:

$$E_{\Im_1}(Y) = E(Y | X_1,\ldots,X_n), \tag{10.133}$$

where this conditional expectation is called the *conditional expectation of Y given* $X_1,\ldots,X_n$.

In this latter case, it can be shown (Loeve (1977)) that there exists a version $\varphi(X_1,\ldots,X_n)$ of the conditional expectation so that φ is a Borel function from $\mathbb{R}^n$ to $\mathbb{R}$, and as such it follows that $E(Y | X_1,\ldots,X_n)$ is constant on each set belonging to $\Im_1$ for which $X_1(\omega) = x_1,\ldots,X_n(\omega) = x_n$, for instance.

This justifies the abuse of notation

$$E\left(Y \middle| X_1(\omega)=x_1,...,X_n(\omega)=x_n\right)=\varphi(x_1,...,x_n) \tag{10.134}$$

representing the value of this conditional expectation on all the ωs belonging to the set $\{\omega : X_1(\omega)=x_1,...,X_n(\omega)=x_n\}$.

Taking $B=\Omega$ in relation (10.130), we obtain:

$$\begin{aligned} &E(Y) \\ &=\int_{R_n} E\left(Y \middle| X_1(\omega)=x_1,...,X_n(\omega)=x_n\right)dP(X_1(\omega)\le x_1,...,X_n(\omega)\le x_n) \end{aligned} \tag{10.135}$$

a result often used in the sequel to evaluate the mean of an r.v. using its conditional expectation with respect to some given event.

(iii) If $\Im_1=\{\varnothing,\Omega\}$, we obtain $E(Y|\Im_1)=E(Y)$ and if $\Im_1=\{\varnothing,B,B^c,\Omega\}$, then $E(Y|\Im_1)=E(Y|B)$ on *B* and $E(Y|\Im_1)=E(Y|B^c)$ on B^c.

(iv) Taking r.v. *Y* as the indicator of the event *A*, that is to say:

$$1_{A(\omega)}=\begin{cases}1,\omega\in A,\\ 0,\omega\notin A,\end{cases} \tag{10.136}$$

the conditional expectation becomes the *conditional probability of A given* $\Im_1$ denoted as follows:

$$P(A|\Im_1)=E(1_A(\omega)|\Im_1) \tag{10.137}$$

and then relation (10.130) becomes:

$$\int_B P\left(A \middle| \Im_1(\omega)\right)dP=P(A\cap B),B\in\Im_1. \tag{10.138}$$

Letting $B=\Omega$ in this final relation, we obtain:

$$E\left(P\left(A \middle| \Im_1\right)\right)=P(A), \tag{10.139}$$

a property extending the *theorem of total probability* .

If, moreover, A is independent of $\Im_1$, that is to say, if for all B belonging to $\Im_1$:

$$P(A \cap B) = P(A)P(B), \tag{10.140}$$

then we see from relation (10.137) that:

$$P\left(A \middle| \Im_1\right)(\omega) = P(A), \omega \in \Omega. \tag{10.141}$$

Similarly, if r.v. Y is independent of $\Im_1$, that is to say if for each event B belonging to $\Im_1$ and each set A belonging to the σ-algebra generated by the inverse images of Y, denoted by $\sigma(Y)$, relation (10.140) is true, then from relation (10.130), we have:

$$E\left(Y \middle| \Im_1\right) = E(Y). \tag{10.142}$$

Indeed, from relation (10.140), we can write that:

$$\begin{aligned} \int_B E_{\Im_1}(Y)(\omega)dP &= \int_B Y(\omega)dP, B \in \Im_1, \\ &= E\left(Y1_B\right), \\ &= E(Y)P(B), \\ &= \int_B E(Y)dP, \end{aligned} \tag{10.143}$$

and so, relation (10.142) is proved.

In particular, if $\Im_1$ is generated by the r.vs. $X_1,\dots,X_n$, then the independence between Y and $\Im_1$ implies that:

$$E\left(Y \middle| X_1,\dots,X_n\right) = E(Y). \tag{10.144}$$

Relations (10.142) and (10.144) allow us to have a better understanding of the *intuitive meaning of conditioning* and its importance in finance.

Under independence assumptions, conditioning has absolutely no impact, for example, on the expectation or the probability; on the contrary, dependence implies that the results with or without conditioning will be different, meaning that we can interpret conditioning as given *additional information* useful to obtain more precise results in the case of dependence of an asset.

The properties of expectation, stated in section 10.4, are also properties of conditional expectation, true a.s., but there are supplementary properties which are very important in stochastic modeling. They are given in the next proposition.

Proposition 10.2 (Supplementary properties of conditional expectation) *On the probability space* $(\Omega, \Im, P)$*, we have the following properties:*

(i) *If r.v. X is* $\Im_1$*-measurable, then*

$$E(X|\Im_1) = X, a.s.. \tag{10.145}$$

(ii) *If X is a r.v. and Y* $\Im_1$*-measurable, then*

$$E(XY|\Im_1) = YE(X|\Im_1), a.s.. \tag{10.146}$$

This property means that $\Im_1$*-measurable r.vs. are like constants for the classical expectation.*

(iii) *Since from relation (10.145) we have* $E_{\Im}(Y) = Y$*, taking* $Y = E_{\Im_1}(Y)$*, we see that:*

$$E_{\Im}(E_{\Im_1}(Y)) = E_{\Im_1}(Y) \tag{10.147}$$

and of course since:

$$E_{\Im_1}(E_{\Im}(Y)) = E_{\Im_1}(Y), \tag{10.148}$$

combining these last two relations, we obtain:

$$E_{\Im}(E_{\Im_1}(Y)) = E_{\Im_1}E_{\Im}((Y)) = E_{\Im_1}(Y). \tag{10.149}$$

This last result may be generalized as follows.

Proposition 10.3 (Smoothing property of conditional expectation) *Let* $\Im_1$, $\Im_2$ *be two sub-*σ*-algebras of* $\Im$ *such that* $\Im_1 \subset \Im_2$*; then it is true that:*

$$E_{\Im_2}(E_{\Im_1}(Y)) = E_{\Im_1}(E_{\Im_2}(Y)) = E_{\Im_1}(Y), \tag{10.150}$$

a property called the smoothing property in Loeve (*1977*).

A particular case of relation (10.150) is for example:

$$E\left(E\left(Y\middle|X_1,...,X_n\right)\middle|X_1\right)=E\left(E\left(Y\middle|X_1\right)\middle|X_1,...,X_n\right)=E\left(Y\middle|X_1\right). \quad (10.151)$$

This type of property is very useful for calculating probabilities using conditioning and will often be used in the following chapters.

Here is an example illustrating sums of a random number of r.vs. with the *Wald identities*.

Example 10.3 (Wald's identities) Let $(X_n, n \geq 1)$ be a sequence of IID real r.v.s and N a non-negative r.v. with integer values independent of the given sequence. The r.v. defined by:

$$S_N = \sum_{n=1}^{N} X_n \quad (10.152)$$

is called a *sum of a random number of random variables* and the problem to be solved is the calculation of the mean and the variance of this sum assuming that the r.vs. X_n have a variance.

From relation (10.150), we have:

$$E(S_N) = E\left(E\left(S_N\middle|N\right)\right) \quad (10.153)$$

and as, from the independence assumptions:

$$E\left(S_N\middle|N\right) = NE(X), \quad (10.154)$$

we also have:

$$E(S_N) = E(N)E(X), \quad (10.155)$$

called the *first Wald's identity*.

For the variance of S_N, it is possible to show that (see for example Janssen and Manca (2007))

$$\text{var}(S_N) = E(N)\,\text{var}(X) + \text{var}(N)(E(X))^2 \quad (10.156)$$

called the *second Wald's identity*.

In the particular case of an n-dimensional real r.v. $X=(X_1,...,X_n)$, we can now introduce the very useful definition of the *conditional distribution function of X given* $\Im_1$ defined as follows:

$$\begin{aligned} F_{\Im_1}(x_1,...,x_n,\omega) &= P\left(X_1 \le x_1,...,X_n \le x_n \middle| \Im_1\right) \\ &= Q\left(\left\{\omega' : X_1(\omega') \le x_1,...,X_n(\omega') \le x_n\right\},\omega\right). \end{aligned} \tag{10.157}$$

Another useful definition concerns an extension of the concept of the independence of random variables for the definition of *conditional independence of the n variables* $X_1,\ldots,X_n$. For all $(x_1,\ldots,x_n)$ belonging to $\mathbb{R}^n$, we have the following identity:

$$F_{\Im_1}(x_1,...,x_n,\omega) = \prod_{k=1}^{n} F_{\Im}(x_k,\omega), \tag{10.158}$$

where of course we have:

$$F_{\Im}(x_k,\omega) = P\left(X_k \le x_k \middle| \Im_1\right) \tag{10.159}$$

according to definition (10.157) with $n=1$.

Example 10.4 On the probability space $(\Omega,\Im,P)$, let (X,Y) be a two-dimensional real r.v. whose d.f. is given by

$$F(x,y) = P(X \le x, Y \le y). \tag{10.160}$$

As $\mathbb{R}^2$ is a complete separable metric space, there exist regular conditional probabilities given the sub-σ-algebras $\sigma(X)$ or $\sigma(Y)$, and so the related conditional d.f. denoted by:

$$F_{X|Y}\left(x \middle| \{\omega : Y = y\}\right),\ F_{Y|X}\left(y \middle| \{\omega : X = x\}\right) \tag{10.161}$$

also exists.

If, moreover, the d.f. F has a density f, we can also introduce the concept of *conditional density* for functions $F_{X|Y}$, $F_{Y|X}$ and F_X, giving at the same time an intuitive interpretation of conditioning in this special case.

We know that for every fixed (*x,y*):

$$f(x,y)\Delta x\Delta y + o(x,y,\Delta x,\Delta y) = P(x < X \le x+\Delta x, y < Y \le y+\Delta y), \quad (10.162)$$

where $o(x,y,\Delta x,\Delta y) \to 0$ for $(\Delta x,\Delta y) \to (0,0)$, and similarly for the marginal density function of *X*:

$$f_X(x)\Delta x + \overline{o}(x,\Delta x) = P(x < X \le x+\Delta x), \quad (10.163)$$

where $\overline{o}(x,\Delta x) \to 0$ for $\Delta x \to 0$ with of course:

$$f_X(x) = \int_R f(x,y)dy\,. \quad (10.164)$$

Using formula (10.122), we thus obtain:

$$P\left(y < Y \le y+\Delta y \middle| x < X \le x+\Delta x\right) = \frac{f(x,y)\Delta x\Delta y + o(x,y,\Delta x,\Delta y)}{f_X(x)\Delta x + \overline{o}(x,\Delta x)}. \quad (10.165)$$

Letting Δx tend to 0, we obtain:

$$\lim_{\Delta x \to 0} P\left(y < Y \le y+\Delta y \middle| x < X \le x+\Delta x\right) = \frac{f(x,y)}{f_X(x)}\Delta y\,. \quad (10.166)$$

This relation shows that the function $f_{Y|X}$ defined by:

$$f_{Y|X}\left(y|x\right) = \frac{f(x,y)}{f_X(x)} \quad (10.167)$$

is the *conditional density of Y, given X*. Similarly, the *conditional density of X, given Y* is given by:

$$f_{X|Y}\left(x|y\right) = \frac{f(x,y)}{f_Y(y)}\,. \quad (10.168)$$

Consequently, for any Borel subsets A and B of $\mathbb{R}$, we have:

$$P\left(X \in A \middle| Y(\omega)=y\right)=\int_A f_{X|Y}\left(x \middle| y\right)dx=\frac{1}{f_Y(y)}\int_A f(x,y)dx,$$
$$P((X,Y)\in A\cap B)=\int_{A\cap B} f(x,y)dxdy=\int_B\left(\int_A f_{X|Y}\left(x\middle|y\right)dx\right)f_Y(y)dy. \tag{10.169}$$

The last equalities show that the density of (X,Y) can also be characterized by one marginal d.f. and the associated conditional density, as from relations (10.166) and (10.169):

$$f=f_X\times f_{Y|X}=f_Y\times f_{X|Y}. \tag{10.170}$$

It is possible that *conditional means* exist; if so, they are given by the following relations:

$$E\left(X\middle|Y=y\right)=\int_{\mathbb{R}} f\left(x\middle|y\right)dx,\ E\left(Y\middle|X=x\right)=\int_{\mathbb{R}} f\left(y\middle|x\right)dy. \tag{10.171}$$

The conditional mean of X (respectively Y) given $Y=y$ (respectively $X=x$) can be seen as a function of the real variable y (respectively x) called *the regression curve of X (respectively Y) given Y (respectively X)*.

The two regression curves will generally not coincide and not be straight lines except if the two r.vs. X and Y are independent because, in this case, we obtain from relations (10.166) and (10.168) that:

$$f_{X|Y}=f_X,\ f_{Y|X}=f_Y \tag{10.172}$$

and so:

$$E\left(X\middle|Y\right)=E(X),\ E\left(Y\middle|X\right)=E(Y), \tag{10.173}$$

proving that the two regression curves are straight lines parallel to the axes passing through the "center of gravity" $(E(X), E(Y))$ of the probability mass in $\mathbb{R}^2$.

In the special case of a non-degenerated normal distribution for (X,Y) with vector mean (m_1,m_2) and variance covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \tag{10.174}$$

it can be shown that the two conditional distributions are also normal with parameters:

$$\begin{aligned} Y \middle| X \prec N_2\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2) \right), \\ X \middle| Y \prec N_2\left(\mu_1 + \frac{1}{\rho}\frac{\sigma_1}{\sigma_2}(y-\mu_2), \sigma_2^2(1-\rho^2) \right). \end{aligned} \tag{10.175}$$

Thus, the two regression curves are linear.

10.7. Stochastic processes

In this section, we shall always consider a *complete* probability space $\left(\Omega, \Im, \mathrm{P}\right)$ with a *filtration F*.

Let us recall that a probability space $\left(\Omega, \Im, \mathrm{P}\right)$ is *complete* if every subset of an event of probability 0 is measurable, i.e. in the σ-algebra $\Im$, and so also of probability 0.

Definition 10.12 *F is a filtration on the considered basic probability space if F is a family of* $\left(\Im_t, t \in T\right)$ *of sub-*σ*-algebras of* $\Im$*, the index set T being either the natural set* $\{0,1,...,n,...\}$ *or the positive half real line* $[0,\infty)$ *such that:*

$$\begin{aligned} &\text{(i)} \quad s < t \Rightarrow \Im_s \subset \Im_t, \\ &\text{(ii)} \quad \Im_t = \bigcap_{u>t} \Im_u, \\ &\text{(iii)} \quad \Im_0 \textit{ contains all subsets with probability } 0. \end{aligned} \tag{10.176}$$

Assumption (ii) is called the *right continuity property* of filtration *F*.

Any filtration satisfying these three assumptions is called a filtration *satisfying the usual assumptions*.

The concept of filtration can be interpreted as a family of amounts of information so that $\Im_t$ gives all the observable events at time *t*.

Definition 10.13 *The quadruplet* $\left(\left(\Omega, \Im, \mathrm{P}, \left(\Im_t, t \in T\right)\right)\right.$ *is called a filtered probability space.*

Definition 10.14 *A r.v.* $\tau : \Omega \mapsto T$ *is a stopping time if:*

$$\forall t \in T: \ \left\{\omega : \tau(\omega) \le t\right\} \in \Im_t. \tag{10.177}$$

The interpretation is the following: the available information at time *t* allows for the possibility to observe the event given in (10.177) and to decide for example if the future observations will be stopped after time *t,* or not.

We have the following proposition:

Proposition 10.4 *The r.v.* τ *is a stopping time if and only if*

$$\left\{\omega : \tau(\omega) < t\right\} \in \Im_t, \ \forall t \in T. \tag{10.178}$$

Definition 10.5 *A stochastic process (or simply process) with values in the measurable space* $\left(E, \aleph\right)$ *is a family of r.vs.:*

$$\left\{X_t, t \in T\right\} \tag{10.179}$$

where for all t:

$$X_t : \Omega \mapsto E, \ \left(\Im, \aleph\right)\text{-measurable.}$$

This means, in particular, that for every subset *B* of the σ-algebra $\aleph$, the set

$$X_t^{-1}\left(B\right) = \left\{\omega : X_t(\omega) \in B\right\} \tag{10.180}$$

belongs to the σ-algebra $\Im$.

Remark 10.6 If $\left(E, \aleph\right) = \left(\mathbb{R}, \beta\right)$, the process is called a *real stochastic process* with values in $\mathbb{R}$; if $\left(E, \aleph\right) = \left(\mathbb{R}^n, \beta^n\right)$, it is called a *real multidimensional process* with values in $\mathbb{R}^n$.

If *T* is the natural set $\left\{0, 1, \ldots, n, \ldots\right\}$, the process *X* is called a *discrete time stochastic process* or a *random sequence*; if *T* is the positive half of the real line $\left[0, \infty\right)$, the process *X* is called a *continuous time stochastic process.*

Definition 10.16 The stochastic process *x* is adapted to the filtration *f* if, for all *t*, the r.v. X_t is $\Im_t$-measurable. This means that, for all $t \in T$:

$$X_t^{-1}(B) = \{\omega : X_t(\omega) \in B\} \in \Im_t, \forall B \in \aleph. \tag{10.181}$$

Definition 10.17 *Two processes x and y are indistinguishable if a.s., for all* $t \in T$ *:*

$$X_t = Y_t. \tag{10.182}$$

This means that:

$$\mathrm{P}(X_t = Y_t, \forall t \in T) = 1. \tag{10.183}$$

Definition 10.18 *The process X (or Y) is a modification of the process Y (or X) if a.s., for all* $t \in T$:

$$X_t = Y_t,\ a.s. \tag{10.184}$$

This means that:

$$\mathrm{P}(X_t = Y_t., \forall t \in T) = 1 \tag{10.185}$$

for all $t \in T$.

Definition 10.19 *For every stochastic process x, the function from t to e,*

$$t \mapsto X_t(\omega) \tag{10.186}$$

defined for each $\omega \in \Omega$, *is called a trajectory or sample path of the process.*

It must be clearly understood that the "modern" study of stochastic processes is concerned with the study of the properties of these trajectories.

For example, we can affirm that if two processes *X* and *Y* are indistinguishable, then there exists a set *N* belonging to $\Im$ of probability 0 such that:

$$\forall \omega \notin N : X_t(\omega) = Y_t(\omega), \forall t \in T. \tag{10.187}$$

In other words, for each ω element of the set $\Omega - N$, the two functions $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are equal.

As the basic probability space is complete, the neglected set *N* belongs to $\Im_t$, for all $t \in T$.

Definition 10.20 *A real stochastic process x is càdlàg if a.s. the trajectories of x are right continuous and have left limits at every point t.*

Definition 10.21 *If x is a real stochastic process and a set* $\Lambda \in \beta$*, then the r.v. defined by:*

$$T(\omega) = \inf\left\{t > 0 : X_t(\omega) \in \Lambda\right\} \tag{10.188}$$

is called the hitting time of Λ *by process X.*

It is easily shown that the *properties* of stopping and hitting times are as follows (see Protter (1990)):

(i) if *X* is càdlàg, adapted and $\Lambda \in \beta$, then the hitting time related to Λ is a stopping time;

(ii) if *S* and *T* are two stopping times, then the following r.v.:

$$S \wedge T\left(= \min\left\{S,T\right\}\right),\ S \vee T\left(= \max\left\{S,T\right\}\right), S+T,\ \alpha S(\alpha > 1) \tag{10.189}$$

are also stopping times.

Definition 10.22 *If T is a stopping time, the* σ*-algebra* $\Im_T$ *defined by:*

$$\Im_T = \left\{\Lambda \in \Im : \Lambda \bigcap \left\{\omega : T(\omega) \le t\right\} \in \Im_t, \forall t \ge 0\right\} \tag{10.190}$$

is called the stopping time σ*-algebra.*

In fact, the σ-algebra $\Im_T$ represents the information of all observable sets up to stopping time *T*. We can also say that $\Im_T$ is the smallest stopping time containing all the events related to the r.v. $X_{T(\omega)}(\omega)$ for all the adapted càdlàg processes *X* or generated by these r.v.

We also have for two stopping times *S* and *T*:

(i) $S \le T$ a.s. $\Rightarrow \Im_S \subset \Im_T$, (10.191)

(ii) $\Im_S \bigcap \Im_T, = \Im_{S \wedge T}$. (10.192)

10.8. Martingales

In this section, we shall briefly present some topics related to the most well-known category of stochastic processes called *martingales*.

Let X be a real stochastic process defined on the filtered complete probability space $\left(\Omega, \Im, P, \left(\Im_t, t \in T\right)\right)$.

Definition 10.23 *The process x is called a* $(\Im_t)$ *-martingale if:*

$$\text{(i) } \forall t \geq 0, \exists E\left(X_t\right), \tag{10.193}$$

$$\text{(ii) } s < t \Rightarrow E\left(X_t \mid \Im_s\right) = X_s, \text{a.s.} \tag{10.194}$$

The latter equality is called the *martingale property* or the *martingale equality*.

Definition 10.24 *The process X is called a super-martingale (respectively sub-martingale) if:*

$$\text{(i) } \forall t \geq 0, \exists E\left(X_t\right), \tag{10.195}$$

$$\text{(ii) } s < t \Rightarrow E\left(X_t \mid \Im_s\right) \leq (\geq) X_s, \text{a.s.} \tag{10.196}$$

The martingale concept is interesting; indeed, as the best estimator at time *s* (*s*>*t*) for the value of X_t, as given by the conditional expectation appearing in relation (8.2), the martingale equality means that *the best predicted value* is simply the observed value of the process at the time of predicting *s*.

In finance the martingale is frequently used (see Janssen and Skiadas (1995)) to model the concept of an *efficient financial market*.

Definition 10.25 *The martingale X is closed if:*

$$\begin{aligned} &\exists\, Y : \\ &\text{(i)} \mathrm{E}\left(\left|Y\right|\right) < \infty, \\ &\text{(ii)} \forall t \in \left[0, \infty\right) : \mathrm{E}\left(Y \middle| \Im_t\right) = X_t, \text{a.s..} \end{aligned} \tag{10.197}$$

It is possible to prove the following result (see for example Protter (1990)).

Proposition 10.5

(i) *If X is a supermartingale, then the function* $t \mapsto E\left(X_t\right)$ *is right continuous if and only if there exists a unique modification Y of X such that Y is càdlàg.*

(ii) *If X is a martingale then, up to a modification, the function* $t \mapsto E\left(X_t\right)$ *is right continuous.*

It follows that every martingale, such that the function $t \mapsto E\left(X_t\right)$ is right continuous, is càdlàg.

The two most important results about martingales are *the martingale convergence theorem* and the *optional sampling (or Doob's) theorem.*

Before giving these results, we still need a final technical definition.

Definition 10.25 (Meyer (1966)) *A family* $\left(\xi_u, u \in A\right)$ *where A is an infinite index set is uniformly integrable if:*

$$\lim_{n\to\infty} \sup_{\alpha} \int_{\{\omega:|\xi_\alpha(\omega)|\geq n\}} \left|\xi_\alpha(\omega)\right| d\mathrm{P}(\omega) = 0. \tag{10.198}$$

Proposition 10.6 *Let x be a super-martingale in such a way that the function* $t \mapsto E\left(X_t\right)$ *is right continuous such that:*

$$\sup_{t\in[0,\infty)} E\left(\left|X_t\right|\right) < \infty; \tag{10.199}$$

then, there exists a r.v. Y such that:

$$\begin{aligned} &(\mathrm{i}) E\left(\left|Y\right|\right), \\ &(\mathrm{ii}) Y = \lim_{t\to\infty} X_t, \quad \text{a.s.} \end{aligned} \tag{10.200}$$

Moreover, if X is a martingale closed by r.v. Z, then r.v. Y also closes X and:

$$Y = E\left(Z \middle| \Im_\infty\right), \tag{10.201}$$

where

$$\mathfrak{I}_\infty = \sigma\left(\bigcup_{0 \le t < \infty} \mathfrak{I}_t\right). \tag{10.202}$$

With the aid of the concept of uniform integrability, we can obtain the following corollary.

Corollary 10.1

(i) *Let X be a right continuous martingale and uniformly integrable; then the following limit:*

$$Y = \lim_{t \to \infty} X_t \tag{10.203}$$

exists a.s.; moreover $Y \in L^1$ *and the r.v. Y closes the martingale X.*

(ii) *Let X be a right continuous martingale; then* $X = (X_t, t \ge 0)$ *is uniformly integrable if and only if*

$$Y = \lim_{t \to \infty} X_t \tag{10.204}$$

exists a.s., $Y \in L^1$*, and* $\left(X_t, t \in [0, \infty]\right)$ *is a martingale with, a.s.:*

$$X_\infty = Y. \tag{10.205}$$

Now, an interesting question is: what happens if we observe a martingale *X* at two stopping times *S,T* (*S*<*T*, a.s.)? The solution is given by the *optional sampling theorem*, also called *Doob's theorem.*

Proposition 10.7 (The optional sampling theorem or Doob's theorem) *Let X be a right continuous martingale closed by* X_∞ *and let S and T be two stopping times so that a.s. S < T; then the r.v.* $X_S, X_T \in L^1$ *and:*

$$X_S = E\left(X_T \middle| \mathfrak{I}_S\right), \text{ a.s.} \tag{10.206}$$

This important theorem means that if we restrict the random observation time set to $\{S, T\}$, then the restriction of the martingale to this set is *still* a martingale provided that *S* and *T* are two stopping times with of course *S*<*T*, a.s.

This result is interesting for the concept of *stopped process.*

Definition 10.26 *Let X be a stochastic process and T a stopping time. The stopped stochastic process* X^T *is defined by:*

$$X^T = \left(X_t^T, t \in [0, \infty] \right) \tag{10.207}$$

where:

$$\begin{aligned} & X_t^T(\omega) = X_{t \wedge T}(\omega), \\ & \text{with } t \wedge T = \inf\{t, T\}. \end{aligned} \tag{10.208}$$

From this definition, it follows that if process *X* is adapted and càdlàg, then so is the stopped process X^T. This is due to the fact that $t \wedge T$ is also a stopping time and moreover:

$$X_t^T = X_t 1_{\{t<T\}} + X_T 1_{\{t \geq T\}}. \tag{10.209}$$

This leads to the last result we want to mention.

Proposition 10.8 *Let x be a right continuous uniformly integrable martingale; then the stopped process* $X^T = \left(X_{t \wedge T}, t \in [0, \infty] \right)$ *has the same properties with respect to the filtration* $\left(\Im_t, t \in [0, \infty] \right)$.

10.9. Brownian motion

There are many types of stochastic process and some of them will be extensively studied in the following chapters, such as renewal processes, random walks, Markov chains, semi-Markov and Markov processes and their main extensions.

Figure 10.1 shows a typical sample path for models in finance.

To obtain such trajectories, it is necessary to introduce a specific stochastic process called the Brownian motion.

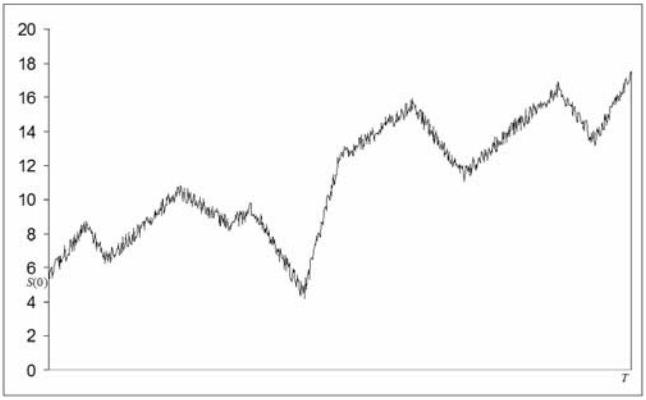

Figure 10.1. *Sample of a Brownian motion*

We will work on a basic complete filtered probability space satisfying the usual assumptions and noted $\left(\Omega, \Im, P, \left(\Im_t, t \in [0,\infty)\right)\right)$.

Definition 10.27 *The real stochastic process* $B = \left(B_t, t \in [0,\infty)\right)$ *will be called a Brownian motion or Brownian or Wiener process with trend* μ *and variance* σ^2 *provided that:*

(i) *B is adapted to the basic filtration,*

(ii) *B has independent increments, i.e. that:*

$$\begin{aligned}&\forall s,t\ (0 \le s < t)\ :\ P\left(B_t - B_s \in A \middle| \Im_S\right) = P\left(B_t - B_s \in A\right),\\&\forall Borel\ set\ B,\end{aligned} \tag{10.210}$$

(iii) *B* has *stationary increments*, *i.e.*:

$$\begin{aligned}&\forall s,t\ (0 \le s < t):\ B_t - B_s\ \text{has a normal distribution}\\&N(\mu(t-s), \sigma^2(t-s)),\end{aligned} \tag{10.211}$$

(iv) $P\left(B_0 = x\right) = 1,\ (x \in \mathbb{R}).$ (10.212)

If, moreover, we have:

$$\mu = 0,\ \sigma^2 = 1,\ x = 0, \tag{10.213}$$

then the Brownian motion is said to be *standard*.

Let us now give the most important properties of the standard Brownian motion.

Property 10.2 *If B is a Brownian motion, then there exists a modification of B, the process B*, such that B* has, a.s., continuous trajectories.*

Property10.3 *If B is a standard Brownian motion, then B is a martingale.*

Property 10.4 *If B is a standard Brownian motion, then the process Q where*

$$Q = \left(B_t^2 - t, t \in \left[0, \infty \right) \right) \tag{10.214}$$

is a martingale.

Remark 10.8 It can also be proved that both Properties 10.3 and 10.4 characterize a standard Brownian motion.

Property 10.5 *If B is a standard Brownian motion, then for almost all* ω*, the trajectory* $\omega \mapsto B_t(\omega)$ *is not of bounded variation on every closed interval* $\left[a, b \right]$.

This explains why it is necessary for models in finance and in insurance to define a new type of integral, called the *Itô* or *stochastic integral*, if we want to integrate with respect to *B* (see for example Protter (1990)). This will be done in section 13.3.

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Chapter 11

Markov Chains

This section briefly presents some fundamental results concerning the theory of Markov chains with a finite number of states. These results will be used in the following chapter. We will use the usual terminology introduced by Chung (1960) and Parzen (1962).

11.1. Definitions

Let us consider an economic or physical system S with m possible states, represented by the set I :

$$I = \{1, \ 2, \ldots, \ m\} . \tag{11.1}$$

Let the system S evolve randomly in discrete time $(t = 0, \ 1, \ 2, \ldots, \ n, \ldots)$, and let J_n be the r.v. representing the state of the system S at time n .

Definition 11.1 *The random sequence* $(J_n, \quad n \in \mathrm{N})$ *is a Markov chain if and only if for all* $j_0, \ j_1, \ldots, j_n \in I$:

$$P(J_n = j_n \mid J_0 = j_0, \ J_1 = j_1, \ldots, \ J_{n-1} = j_{n-1}) = P(J_n = j_n \mid J_{n-1} = j_{n-1}) \tag{11.2}$$

(*provided this probability has meaning*).

Definition 11.2 *A Markov chain* $(J_n, \quad n \geq 0)$ *is homogenous if and only if probabilities* (1.2) *do not depend on* n *and are non-homogenous in the other cases.*

For the moment, we will only consider the homogenous case for which we write:

$$P(J_n = j \mid J_{n-1} = i) = p_{ij}, \tag{11.3}$$

and we introduce matrix **P** defined as:

$$\mathbf{P} = [\, p_{ij} \,]. \tag{11.4}$$

The elements of matrix **P** have the following properties:

(i) $p_{ij} \geq 0, \text{ for all } i, j \in I,$ (11.5)

(ii) $\sum_{j \in I} p_{ij} = 1, \text{ for all } i \in I.$ (11.6)

A matrix **P** satisfying these two conditions is called a *Markov matrix* or a *transition matrix*.

With every transition matrix, we can associate a *transition graph* where vertices represent states. There exists an *arc* between vertices *i* and *j* if and only if $p_{ij} > 0$.

To fully define the evolution of a Markov chain, it is also necessary to fix an *initial distribution* for state J_0, i.e. a vector

$$\mathbf{p} = (\, p_1, \ldots, p_m \,), \tag{11.7}$$

such that:

$$p_i \geq 0, \qquad i \in I, \tag{11.8}$$

$$\sum_{i \in I} p_i = 1. \tag{11.9}$$

For all i, p_i represents the *initial probability* of starting from i:

$$p_i = P(J_0 = i). \tag{11.10}$$

For the rest of this chapter we will consider homogenous Markov chains as being characterized by the couple $(\mathbf{p}, \mathbf{P})$.

If $J_n = i$ a.s., that is, if the system starts with probability equal to 1 from state i, then the components of vector **p** will be:

$$p_j = \delta_{ij}. \tag{11.11}$$

We now introduce the *transition probabilities of order* $p_{ij}^{(n)}$, defined as:

$$p_{ij}^{(n)} = P(J_{\nu+n} = j \mid J_\nu = i). \tag{11.12}$$

From the Markov property (11.2), it is clear that conditioning with respect to $J_{\nu+1}$, and we obtain

$$p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}. \tag{11.13}$$

Using the following matrix notation:

$$\mathbf{P}^{(2)} = \left[p_{ij}^{(2)} \right], \tag{11.14}$$

we find that relation (11.13) is equivalent to

$$\mathbf{P}^{(2)} = \mathbf{P}^2. \tag{11.15}$$

Using induction, it is easy to prove that if

$$\mathbf{P}^{(n)} = \left[p_{ij}^{(n)} \right], \tag{11.16}$$

then we obtain for all $n \geq 1$:

$$\mathbf{P}^{(n)} = \mathbf{P}^n. \tag{11.17}$$

Note that (11.17) implies that the transition probability matrix in n steps is equal to the nth power of matrix **P**.

For the marginal distributions related to J_n, we define for $i \in I$ and $n \geq 0$:

$$p_i(n) = P(J_n = i). \tag{11.18}$$

These probabilities may be calculated as follows:

$$p_i(n) = \sum_j p_j p_{ji}^{(n)}, \qquad i \in I. \tag{11.19}$$

If we write:

$$p_{ji}^{(0)} = \delta_{ji} \text{ or } \mathbf{P}^{(0)} = \mathbf{I}, \tag{11.20}$$

then relation (11.19) is true for all $n \geq 0$.

If:

$$\mathbf{p}(n) = (\, p_1(n), \ldots, p_m(n)\,), \tag{11.21}$$

then relation (11.19) can be expressed, using matrix notation, as:

$$\mathbf{p}(n) = \mathbf{p}\mathbf{P}^n. \tag{11.22}$$

Definition 11.3 *A Markov matrix* **P** *is regular if there exists a positive integer* k*, such that all the elements of matrix* $\mathbf{P}^{(k)}$ *are strictly positive.*

From relation (11.17), **P** is regular if and only if there exists an integer $k > 0$ such that all the elements of the *k*th power of **P** are strictly positive.

Example 11.1

(i) If:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} \tag{11.23}$$

we have:

$$\mathbf{P}^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix} \tag{11.24}$$

so that **P** is regular.

The transition graph associated with **P** is given in Figure 11.1.

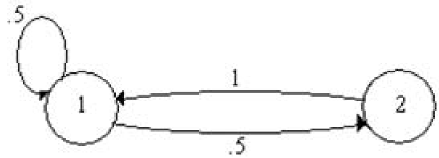

Figure 11.1. *Transition graph of matrix (11.23)*

(ii) If:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0.75 & 0.25 \end{bmatrix}, \tag{11.25}$$

P is not regular, because for any integer k,

$$p_{12}^{(k)} = 0. \tag{11.26}$$

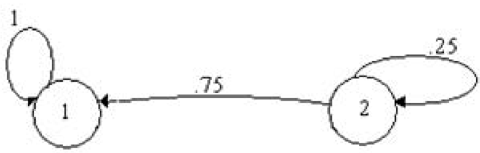

Figure 11.2. *Transition graph for matrix (11.25)*

The transition graph in this case is depicted in Figure 11.2.

The same is true for the matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{11.27}$$

(iii) Any matrix **P** whose elements are all strictly positive is regular.

For example:

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \qquad \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}. \tag{11.28}$$

11.2. State classification

Let $i \in I$, and let $d(i)$ be the greatest common divisor of the set of integers n, such that

$$p_{ii}^{(n)} > 0. \tag{11.29}$$

Definition 11.4 *If $d(i) > 1$, the state i is said to be periodic with period $d(i)$. If $d(i) = 1$, then state i is aperiodic.*

Clearly, if $p_{ii} > 0$, then i is aperiodic. However, the converse is not necessarily true.

Remark 11.1 If **P** is regular, then all the states are aperiodic.

Definition 11.5 *A Markov chain whose states are all aperiodic is called an aperiodic Markov chain.*

From now on, we will have only Markov chains of this type.

Definition 11.6 *A state i is said to lead to state j (written $i \rhd j$) if and only if there exists a positive integer n such that*

$$p_{ij}^{n} > 0. \tag{11.30}$$

i C j means that i does not lead to j.

Definition 11.7 *States i and j are said to communicate if and only if $i \rhd j$ and $j \rhd i$, or if $j = i$. We write $i \lhd\rhd j$.*

Definition 11.8 *A state i is said to be essential if and only if it communicates with every state it leads to; otherwise it is called inessential.*

Relation $\lhd\rhd$ defines an equivalence relation over the state space I resulting in a partition of I. The equivalence class containing state i is represented by $C(i)$.

Definition 11.9 *A Markov chain is said to be irreducible if and only if there exists only one equivalence class.*

Clearly, if **P** is regular, the Markov chain is both irreducible and aperiodic. Such a Markov chain is said to be *ergodic*.

It is easy to show that if the state i is essential (inessential), then all the elements of class $C(i)$ are essential (inessential) (see Chung (1960)).

We can thus speak of essential and inessential classes.

Definition 11.10 *A subset E of the state space I is said to be closed if and only if:*

$$\sum_{j \in E} p_{ij} = 1 \text{, for all } i \in E. \tag{11.31}$$

It can be shown that every essential class is minimally closed; see Chung (1960).

Definition 11.11 *For given states i and j, with $J_0 = i$, we can define the r.v. τ_{ij} called the first passage time to state j as follows:*

$$\tau_{ij} = \begin{cases} n & \text{if } J_\nu \neq j, \quad 0 < \nu < n, \quad J_n = j, \\ \infty & \text{if } J_\nu \neq j, \quad \text{for all } \nu > 0. \end{cases} \tag{11.32}$$

τ_{ij} is said to be the *hitting time* of the singleton $\{ j \}$, starting from state i at time 0.

Assuming:

$$f_{ij}^{(n)} = P\left(\tau_{ij} = n \mid J_0 = i \right), \qquad n \in \mathbb{N}_0 \tag{11.33}$$

and

$$f_{ij} = P\left(\tau_{ij} < \infty \mid J_0 = i \right), \tag{11.34}$$

we can see that for $n > 0$:

$$f_{ij}^{(n)} = P\left(J_n = j, \quad J_\nu \neq j, \quad 0 < \nu < n \mid J_0 = i \right), \tag{11.35}$$

$$= \sum_{S'_{n,i,j}} \prod_{k=0}^{n-1} p_{\alpha_k \alpha_{k+1}}, \tag{11.36}$$

where the summation set $S'_{n,i,j}$ is defined as:

$$S'_{n,i,j} = \{(\alpha_0, \alpha_1, \ldots, \alpha_n) : \alpha_0 = i, \alpha_n = j, \alpha_k \in I, \alpha_k \neq j, k = 1, \ldots, n-1\}. \tag{11.37}$$

We also have:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}, \tag{11.38}$$

$$1 - f_{ij} = P(\tau_{ij} = \infty \mid J_0 = i). \tag{11.39}$$

Elements $f_{ij}^{(n)}$ can readily be calculated by induction, using the following relations:

$$p_{ij} = f_{ij}^{(1)}, \tag{11.40}$$

$$p_{ij}^{(n)} = \sum_{\nu=1}^{n-1} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)} + f_{ij}^{(n)}, \qquad n \geq 2. \tag{11.41}$$

Let:

$$m_{ij} = E(\tau_{ij} \mid J_0 = i), \tag{11.42}$$

with the possibility of an infinite mean. The value of m_{ij} is given by:

$$m_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)} - \infty(1 - f_{ij}).^{(*)} \tag{11.43}$$

If $i = j$, then m_{ij} is called the *first passage time mean* or the *mean recurrence* time of state i.

For every j, we define the sequence of successive return times to state j $\left(r_n^{(j)}, \quad n \geq a\right)$ as follows:

(*) Using the following conventions: $\infty + a = \infty, \quad a \in \mathbb{R}, \quad \infty \cdot a = \infty, \quad (a > 0)$, and in this particular case, $\infty \cdot 0 = 0$.

$$r_0^{(j)} = 0, \tag{11.44}$$

$$r_n^{(j)} = \sup_k \left\{ k \in \mathrm{N}_{0,} \quad k > r_{n-1}^{(j)}, \;\; J_\nu \neq j, \;\; r_{n-1}^{(j)} < \nu < k \right\}, \quad n > 0. \tag{11.45}$$

Using the Markov property and supposing $J_0 = j$, the sequence of return times to state j is a renewal sequence with the r.v.

$$r_n^{(j)} - r_{n-1}^{(j)}, \qquad n \geq 1 \tag{11.46}$$

is a sequence of independent r.v. all distributed according to τ_{jj}.

If $J_0 = i, \;\; i \neq j,$ then the first time of hitting j is

$$r_1^{(j)} = \tau_{ij}, \tag{11.47}$$

and

$$r_n^{(j)} - r_{n-1}^{(j)} \sim \tau_{jj}, \qquad n > 1. \tag{11.48}$$

Definition 11.12 *A state i is*

$$i \quad \text{transient} \quad \Leftrightarrow \quad f_{ii} < 1, \tag{11.49}$$

$$i \quad \text{recurrent} \quad \Leftrightarrow \quad f_{ii} = 1. \tag{11.50}$$

A recurrent state i is said to be *zero* (*positive*) if $m_{ii} = \infty \quad (m_{ii} < \infty)$. It can be shown that if $m_{ii} < \infty$, then we can only have positive recurrent states.

This classification leads to the decomposition theorem (see Chung (1960)).

Proposition 11.1 (*Decomposition theorem*) *The state space I of any Markov chain can be decomposed into* r $(r \geq 1)$ *subsets* $C_1, \ldots, C_r$ *forming a partition, such that each subset* C_i *is one and only one of the following types*:

(i) an essential recurrent positive closed set;

(ii) an inessential transient non-closed set.

Remark 11.2

(1) If an inessential class reduces to a singleton $\{i\}$, there are two possibilities:

a) there exists a positive integer *N* such that:

$$0 < p_{ii}^N < 1. \tag{11.51}$$

b) the N in a) does not exist. In this case, state i is said to be a *non-return state*.

(2) If singleton $\{i\}$ forms an essential class, then

$$p_{ii} = 1 \tag{11.52}$$

and state i is said to be an *absorbing state*.

(3) If $m = \infty$, there may be two other types of classes in the decomposition theorems:

a) essential transient closed;

b) essential recurrent non-closed classes.

Other works on Markov chains give the following necessary and sufficient conditions for recurrence and transience.

Proposition 11.2

(i) State *i* is transient if and only if

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty. \tag{11.53}$$

In this case, for all $k \in I$:

$$\sum_{n=1}^{\infty} p_{ki}^{(n)} < \infty, \tag{11.54}$$

and in particular:

$$\lim_{n\to\infty} p_{ki}^{(n)} = 0, \quad \forall k \in I. \tag{11.55}$$

(ii) State *i* is recurrent if and only if

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty. \tag{11.56}$$

In this case:

$$k \lhd\rhd i \Rightarrow \sum_{n=1}^{\infty} p_{ki}^{(n)} = \infty, \tag{11.57}$$

and

$$k \,\mathsf{C}\, i \Rightarrow \sum_{n=1}^{\infty} p_{ki}^{(n)} = 0. \tag{11.58}$$

11.3. Occupation times

For any state j, and for $n \in \mathbb{N}_0$, we define the r.v. $N_j(n)$ as the number of times state j is occupied in the first n transitions:

$$N_j(n) = \#\{k \in \{1,\ldots,n\} : J_k = j\}. \tag{11.59}$$

By definition, the r.v. $N_j(n)$ is called the *occupation time of state j in the first n transitions.*

The r.v.

$$N_j(\infty) = \lim_{n\to\infty} N_j(n) \tag{11.60}$$

is called the *total occupation time of state j.*

For any state j and $n \in \mathbb{N}_0$ let us define:

$$Z_j(n) = \begin{cases} 1 & \text{if } J_n = j, \\ 0 & \text{if } J_n \neq j. \end{cases} \tag{11.61}$$

We may write:

$$N_j(n) = \sum_{\nu=1}^{n} Z_j(\nu). \tag{11.62}$$

We have from relation (11.34):

$$\mathrm{P}\big(N_j(\infty) > 0 \mid J_0 = i\big) = f_{ij}. \tag{11.63}$$

Let g_{ij} be the conditional probability of an infinite number of visits to state j, starting with $J_0 = i$; that is:

$$g_{ij} = P\left(N_j(\infty) = \infty \mid J_0 = i \right). \tag{11.64}$$

It can be shown that:

$$g_{ii} = \lim_{n\to\infty} f_{ii}^{(n)}, \tag{11.65}$$

$$g_{ij} = f_{ij} \cdot g_{jj}, \tag{11.66}$$

$$g_{ii} = 1 \Leftrightarrow f_{ii} = 1 \Leftrightarrow i \text{ is recurrent,} \tag{11.67}$$

$$g_{ii} = 0 \Leftrightarrow f_{ii} < 1 \Leftrightarrow i \text{ is transient.} \tag{11.68}$$

Results (11.67) and (11.68) can be interpreted as showing that system S will visit a recurrent state an infinite number of times, and that it will visit a transient state a finite number of times.

11.4. Absorption probabilities

Proposition 11.3

(i) If *i* is recurrent and if $j \in C(i)$, then $f_{ij} = 1$.

(ii) If *i* is recurrent and if $j \notin C(i)$, then $f_{ij} = 0$.

Proposition 11.4 *Let T be the set of all transient states of I, and let C be a recurrent class.*

For all $j,\ k \in C$,

$$f_{ij} = f_{ik}. \tag{11.69}$$

Labeling this common value as f_{iC}, *the probabilities* $\left(f_{i,C},\ i \in T \right)$ *satisfy the linear system*:

$$f_{i,C} = \sum_{k\in T} p_{ik} f_{k,C} + \sum_{k\in C} p_{ik}, \qquad i \in T. \tag{11.70}$$

Remark 11.3 Parzen (1962) proved that under the assumption of Proposition 11.4, the linear system (11.70) has a unique solution. This shows, in particular, that if there is only one irreducible class C, then for all $i \in T$:

$$f_{i,C} = 1. \tag{11.71}$$

Definition 11.13 *The probability* $f_{i,C}$ *introduced in Proposition 11.4 is called absorption probability in class C, starting from state i.*

If class C is recurrent:

$$f_{i,C} = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{if } i \text{ is recurrent, } \ i \notin C. \end{cases} \tag{11.72}$$

11.5. Asymptotic behavior

Consider an irreducible aperiodic Markov chain which is positive recurrent.

Suppose that the following limit exists:

$$\lim_{n\to\infty} p_j(n) = \pi_j, \quad j \in I \tag{11.73}$$

starting with $J_0 = i$.

The relation

$$p_j(n+1) = \sum_{k\in I} p_k(n) p_{kj} \tag{11.74}$$

becomes:

$$p_{ij}^{(n+1)} = \sum_{k\in I} p_{ik}^{(n)} p_{kj}, \tag{11.75}$$

because

$$p_j(n) = p_{ij}^{(n)}. \tag{11.76}$$

Since the state space I is finite, we obtain from (11.73) and (11.75):

$$\pi_j = \sum_{k\in I} \pi_k p_{kj} , \tag{11.77}$$

and from (11.76):

$$\sum_{i\in I} \pi_i = 1. \tag{11.78}$$

The result:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \tag{11.79}$$

is called an *ergodic result*, since the value of the limit in (11.79) is independent of the initial state i.

From result (11.79) and (11.19), we see that for any initial distribution **p**:

$$\lim_{n\to\infty} p_i(n) = \lim_{n\to\infty} \sum_j p_j p_{ji}^{(n)}, \tag{11.80}$$

$$= \sum_j p_j \pi_i , \tag{11.81}$$

so that:

$$\lim_{n\to\infty} p_i(n) = \pi_i . \tag{11.82}$$

This shows that the asymptotic behavior of a Markov chain is given by the existence (or non-existence) of the limit of matrix $\mathbf{P}^n$.

A standard result concerning the asymptotic behavior of $\mathbf{P}^n$ is given in the next proposition. The proof can be found in Chung (1960), Parzen (1962) or Feller (1957).

Proposition 11.5 *For any aperiodic Markov chain of transition matrix* **P** *and having a finite number of states, we have*:

a) if state j is recurrent (necessarily positive), then

(i) $i \in C(j) \Rightarrow \lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{m_{jj}}$, (11.83)

(ii) i is recurrent and $\notin C(j) \Rightarrow \lim_{n\to\infty} p_{ij}^{(n)} = 0$, (11.84)

(iii) i is transient and $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{f_{i,C(j)}}{m_{jj}}$. (11.85)

b) If j is transient, then for all $i \in I$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0. \tag{11.86}$$

Remark 11.4 Result (ii) of part a) is trivial since in this case:

$$p_{ij}^{(n)} = 0 \text{ for all positive } n.$$

From Proposition 11.5, the following corollaries can be deduced.

Corollary 11.1 (*Irreducible case*) *If the Markov chain of transition matrix* **P** *is irreducible, then for all* $i, j \in I$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \tag{11.87}$$

with

$$\pi_j = \frac{1}{m_{jj}}. \tag{11.88}$$

It follows that for all j:

$$\pi_j > 0. \tag{11.89}$$

If we use Remark 11.4 in the particular case where we have only one recurrent class and where the states are transient (the *uni-reducible* case), then we have the following corollary.

Corollary 11.2 (*Uni-reducible case*) *If the Markov chain of transition matrix* **P** *has one essential class C* (*necessarily recurrent positive*) *and T as transient set, then we have*:

(i) for all $i, \quad j \in C$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \tag{11.90}$$

with $\left\{\pi_j, j \in C\right\}$ *being the unique solution of the system*:

$$\pi_j = \sum_{i\in C} \pi_i p_{ij}, \tag{11.91}$$

$$\sum_{j\in C} \pi_j = 1; \tag{11.92}$$

(ii) for all $j \in T$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0 \text{ for all } i \in I; \tag{11.93}$$

(iii) for all $j \in C$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \text{ for all } i \in T. \tag{11.94}$$

Remark 11.5 Relations (11.91) and (11.92) are true because the set C of recurrent states can be seen as a Markov sub-chain of the initial chain.

If the ℓ transient states belong to the set $\{1,\ldots,\ell\}$, using a permutation of the set I, if necessary, then matrix **P** takes the following form:

$$\begin{array}{c} \\ 1 \\ \vdots \\ \ell \\ \ell+1 \\ \vdots \\ m \end{array} \begin{array}{c} 1 \;\cdots\; \ell \quad \ell+1 \;\cdots\; m \\ \left[\begin{array}{cc} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{O} & \mathbf{P}_{22} \end{array}\right] \end{array} = \mathbf{P}. \tag{11.95}$$

This proves that the sub-matrix $\mathbf{P}_{22}$ is itself a Markov transition matrix.

Let us now consider a Markov chain of matrix **P**. The general case is given by a partition of I:

$$I = T \cup C_1 \cup \cdots \cup C_r, \tag{11.96}$$

where T is the set of transient states and $C_1,\ldots,C_r$ the r positive recurrent classes.

By reorganizing the order of the elements of I, we can always suppose that

$$T = \{1,\ldots,\ell\}, \tag{11.97}$$
$$C_1 = \{\ell+1,\ldots,\ell+\nu_1\}, \tag{11.98}$$
$$C_2 = \{\ell+\nu_1+1,\ldots,\ell+\nu_1+\nu_2\}, \tag{11.99}$$
$$\vdots$$
$$C_r = \left\{\ell + \sum_{j=1}^{r-1} \nu_j + 1,\ldots, \; m\right\}, \tag{11.100}$$

where ν_j is the number of elements in C_j, $(j = 1,\ldots,r)$ and

$$\ell + \sum_{j=1}^{r} \nu_j = m. \tag{11.101}$$

This results from the following block partition of matrix **P**:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{\ell\times\ell} & \mathbf{P}_{\ell\times\nu_1} & \mathbf{P}_{\ell\times\nu_2} & \cdots & \mathbf{P}_{\ell\times\nu_r} \\ \mathbf{0} & \mathbf{P}_{\nu_1\times\nu_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{\nu_2\times\nu_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_{\nu_r\times\nu_r} \end{bmatrix} \tag{11.102}$$

where, for $j = 1,\ldots, r$:

– $\mathbf{P}_{\ell\times\ell}$ is the transition sub-matrix for T ;

– $\mathbf{P}_{\ell\times\nu_j}$ is the transition sub-matrix from T to C_j ;

– $\mathbf{P}_{\nu_j\times\nu_j}$ is the transition sub-matrix for the class C_j .

From Proposition 11.1, we obtain the following corollary.

Corollary 11.3 *For a general Markov chain of matrix* **P**, *given by* (11.102), *we have*:

(i) *for all* $i \in I$ *and all* $j \in T$ *:* $\lim_{n\to\infty} p_{ij}^{(n)} = 0;$ (11.103)

(ii) *for all* $j \in C_\nu \quad (\nu = 1,\ldots, r)$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \begin{cases} \pi_j & \text{if } i \in C_\nu, \\ 0 & \text{if } i \in C_{\nu'} \quad \nu' \neq \nu, \\ f_{i,C_\nu}\pi_j^\nu & \text{if } i \in T, \end{cases} \tag{11.104}$$

moreover, for all $\nu = 1,\ldots, r$:

$$\sum_{j\in C_\nu} \pi_j^\nu = 1. \tag{11.105}$$

This last result allows us to calculate the limit values quite simply.

For $\left(\pi_j^\nu,\ j \in C_\nu\right),\quad \nu = 1,\ldots, r$, it suffices to solve the linear systems for each fixed ν :

$$\begin{cases} \pi_j^\nu = \sum_{k \in C_\nu} \pi_k^\nu p_{kj}, & j \in C_\nu, \\ \sum_{i \in C_\nu} \pi_i^\nu = 1. \end{cases} \tag{11.106}$$

Indeed, since each C_ν is itself a space set of an irreducible Markov chain of matrix $\mathbf{P}_{\nu\times\nu}$, the above relations are none other than (11.77) and (11.78).

For the absorption probabilities $\left(f_{i,C_\nu}, \quad i \in T\right), \quad \nu = 1,\ldots, \quad r$, it suffices to solve the following linear system for each fixed ν. Using Proposition 11.4, we have:

$$f_{i,C_\nu} = \sum_{k \in T} p_{ik} f_{i,C_\nu} + \sum_{k \in C_\nu} p_{ik}, \qquad i \in T. \tag{5.35}$$

An algorithm, given in De Dominicis and Manca (1984b) and very useful for the classification of the states of a Markov chain, is fully developed in Janssen and Manca (2006).

11.6. Examples

Markov chains appear in many practical problems in fields such as operations research, business, social sciences, etc.

To give an idea of this potential, we will present some simple examples followed by a fully developed case study in the domain of social insurance.

11.6.1. *A management problem in an insurance company*

A car insurance company classifies its customers in three groups:
- G_0: those having no accidents during the year;
- G_1: those having one accident during the year;
- G_2: those having more than one accident during the year.

The statistics department of the company observes that the annual transition between the three groups can be represented by a Markov chain with state space $\{G_0, G_1, G_2\}$ and transition matrix **P**:

$$\mathbf{P} = \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0 & 0.80 & 0.20 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11.108}$$

We assume that the company produces 50,000 new contracts per year and wants to know the distribution of these contracts for the next four years.

After one year, we have, on average:

– in group G_0 : $50,000 \times .85 = 42,500$;

– in group G_1 : $50,000 \times .10 = 5,000$;

– in group G_2 : $50,000 \times .05 = 2,500$.

These results are simply the elements of the first row of **P**, multiplied by 50,000. After two years, multiplying the elements of the first row of $\mathbf{P}^{(2)}$ by 50,000, we obtain:

– in group G_0 : $36,125$;

– in group G_1 : $8,250$;

– in group G_2 : $5,625$.

A similar calculation gives:

	After three years	After four years
G_0	30,706	26,100
G_1	10,213	11,241
G_3	9,081	12,659

To find the type of the Markov chain with transition matrix (11.108), the simple graph of possible transitions given in Figure 11.3 shows that class $\{1,\ 2\}$ is transient and class $\{3\}$ is absorbing. Thus, using Corollary 11.2 we obtain the limit matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11.109}$$

The limit matrix can be interpreted as showing that regardless of the initial composition of the group the customers will finish by having at least two accidents.

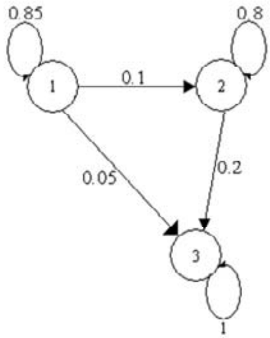

Figure 11.3. *Transition graph of matrix (11.108)*

Remark 11.6 If we want to know the situation after one or two changes, we can use relation (1.19) with $n = 1,2,3$ and with **p** given by:

$$\mathbf{p} = (0.26, 0.60, 0.14). \tag{11.110}$$

We obtain the following results:

$$\begin{array}{lll} p_1^{(1)} = 0.257 & p_2^{(1)} = 0.597 & p_3^{(1)} = 0.146 \\ p_1^{(2)} = 0.255 & p_2^{(2)} = 0.594 & p_3^{(2)} = 0.151 \\ p_1^{(3)} = 0.254 & p_2^{(3)} = 0.590 & p_3^{(3)} = 0.156. \end{array}$$

These results show that the convergence of $\mathbf{p}^{(n)}$ to π is relatively fast.

11.6.2. *A case study in social insurance (Janssen (1966))*

To calculate insurance or pension premiums for occupational diseases such as silicosis, we need to calculate the average (mean) degree of disability at pre-assigned time periods. Let us suppose we retain m degrees of disability:

$S_1, \ldots, S_m$, the last being 100% and including the pension paid out at death.

Let us suppose, as Yntema (1962) did, that an insurance policy holder can go from degree S_i to degree S_j with a probability p_{ij}. This strong assumption leads to the construction of a Markov chain model in which the $m \times m$ matrix:

$$\mathbf{P} = [\, p_{ij} \,] \tag{11.111}$$

is the transition matrix related to the degree of disability.

For individuals starting at time 0 with S_i as the degree of disability, the mean degree of disability after the *n*th transition is:

$$\bar{S}_i(n) = \sum_{j=1}^{m} p_{ij}^{(n)} S_j. \tag{11.112}$$

To study the financial equilibrium of the funds, we must calculate the limiting value of $\bar{S}_i(n)$:

$$\bar{S}_i = \lim_{n\to\infty} \bar{S}_i(n), \tag{11.113}$$

or

$$\bar{S}_i = \lim_{n\to\infty} \sum_{j=1}^{m} p_{ij}^{(n)} S_j. \tag{11.114}$$

This value can be found by applying Corollary 11.3 for $i = 1, \ldots, m$.

Numerical example

Using real-life data for silicosis, Yntema (1962) began with the following intermediate degrees of disability:

$$S_1 = 10\%$$
$$S_2 = 30\%$$
$$S_3 = 50\%$$
$$S_4 = 70\%$$
$$S_5 = 100\%$$

Using real observations recorded in the Netherlands, he considered the following transition matrix **P**:

$$\mathbf{P} = \begin{bmatrix} 0.90 & 0.10 & 0 & 0 & 0 \\ 0 & 0.95 & 0.05 & 0 & 0 \\ 0 & 0 & 0.90 & 0.05 & 0.05 \\ 0 & 0 & 0 & 0.90 & 0.10 \\ 0 & 0 & 0.05 & 0.05 & 0.90 \end{bmatrix}; \tag{11.115}$$

the transition graph associated with matrix (11.115) being given in Figure 11.4. This immediately shows that:

(i) all states are aperiodic;

(ii) the set $\{S_3, S_4, S_5\}$ is an essential class (positive recurrent);

(iii) the singletons $\{1\}$ and $\{2\}$ are two inessential transient classes.

Thus a uni-reducible Markov chain can be associated with matrix **P**. We can thus apply Corollary 11.2. It follows from relation (11.114) that:

$$\overline{S_i} = \lim_{n\to\infty} \sum_{j=3}^{5} \pi_j S_j , \tag{11.116}$$

where (π_3, π_4, π_5) is the unique solution of the linear system:

$$\begin{array}{rcrcrcr} \pi_3 & = & 0.9\cdot\pi_3 & + & 0\cdot\pi_4 & + & 0.05\cdot\pi_5, \\ \pi_5 & = & 0.05\cdot\pi_3 & + & 0.9\cdot\pi_4 & + & 0.05\cdot\pi_5, \\ \pi_4 & = & 0.05\cdot\pi_3 & + & 0.05\cdot\pi_4 & + & 0.9\cdot\pi_5, \\ 1 & = & \pi_3 & + & \pi_4 & + & \pi_5. \end{array} \tag{11.117}$$

The solution is:

$$\pi_3 = \frac{2}{9}, \quad \pi_4 = \frac{3}{9}, \quad \pi_5 = \frac{4}{9}. \tag{11.118}$$

Therefore:

$$\overline{S_i} = \left(\frac{2}{9}50 + \frac{3}{9}70 + \frac{4}{9}100 \right)\% \tag{11.119}$$

or

$$\overline{S_i} = 79\% \tag{11.120}$$

which is the result obtained by Yntema.

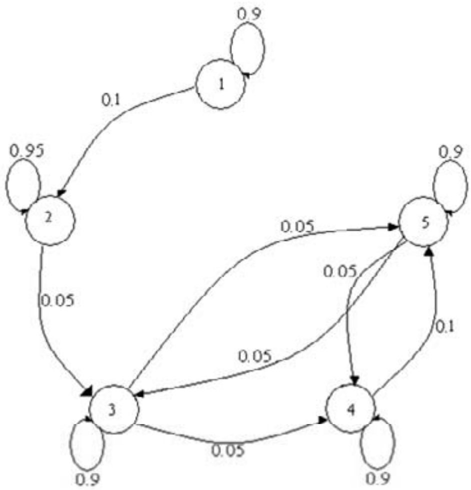

Figure 11.4. *Transition graph of matrix (11.115)*

The last result proves that the mean degree of disability is, at the limit, independent of the initial state *i*.

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Chapter 12

Semi-Markov Processes

12.1. Positive (J-X) processes

Let us consider a *physical* or *economic system* called S with m possible states, m being a finite natural number.

For simplicity, we will note by I the set of all possible states:

$$I = \{1, \ldots, m\} \tag{12.1}$$

as we did in Chapter 11.

At time 0, system S starts from an initial state represented by the r.v. J_0, stays a non-negative random length of time X_1 in this state, and then goes into another state J_1 for a non-negative length of time X_2 before going into J_2, etc.

So we have a two-dimensional stochastic process in discrete time called a *positive* (*J-X*) *process*:

$$(J - X) = ((J_n, X_n), n \geq 0) \tag{12.2}$$

assuming

$$X_0 = 0, \ a.s. \tag{12.3}$$

where the sequence $(J_n, n \geq 0)$ gives the successive *states* of S in time and the sequence $(X_n, n \geq 0)$ gives the successive *sojourn times*.

More precisely, X_n is the time spent by S in state J_{n-1} ($n>0$).

Times at which transitions occur are given by the sequence $(T_n, n \geq 0)$ where:

$$T_0 = 0,\ T_1 = X_1, \ldots, T_n = \sum_{r=1}^{n} X_r \tag{12.4}$$

and so

$$X_n = T_n - T_{n-1},\ n \geq 1\,. \tag{12.5}$$

12.2. Semi-Markov and extended semi-Markov chains

On the complete probability space $(\Omega, \Im, P)$, the stochastic dynamic evolution of the considered (J-X) process will be determined by the following assumptions:

$$P(X_0=0)=1, \text{ a.s.},$$
$$P(J_0=i)=p_i,\ i=1,\ldots,m \text{ with } \sum_{i=1}^{m} p_i = 1\,, \tag{12.6}$$

for all $n>0, j=1,\ldots,m,$ we have:

$$P(J_n = j, X_n \leq x \,|\, (J_k, X_k), k = 0,\ldots,n-1) = Q_{J_{n-1}j}(x),\ a.s. \tag{12.7}$$

where any function Q_{ij} ($i,j=1,\ldots,m$) is a non-decreasing real function null on $\mathbb{R}^+$ such that if

$$p_{ij} = \lim_{x \to +\infty} Q_{ij}(x),\ i, j \in I\,, \tag{12.8}$$

then:

$$\sum_{j=1}^{m} p_{ij} = 1,\ i \in I\,. \tag{12.9}$$

With matrix notation, we will write:

$$\mathbf{Q} = \left[Q_{ij} \right],\ \mathbf{P} = \left[p_{ij} \right](= \mathbf{Q}(\infty)),\ \mathbf{p} = (p_1, \ldots, p_m)\,. \tag{12.10}$$

This leads to the following definitions.

Definition 12.1 *Every matrix* $m \times m$ *Q of non-decreasing functions null on* $\mathbb{R}^+$ *satisfying properties (12.8) and (12.9) is called a semi-Markov matrix or a semi-Markov kernel.*

Definition 12.2 Every couple (p,Q) where Q is a semi-Markov kernel and *p* a vector of initial probabilities defines a positive (*J,X*) process

$(J,X) = ((J_n, X_n), n \geq 0)$ *with* $I \times \mathbb{R}^+$

as state space, also called a semi-Markov chain (*SMC*).

Sometimes, it is useful that the random variables $X_n, n \geq 0$ take their values in $\mathbb{R}$ instead of $\mathbb{R}^+$, in which case, we need the next two definitions.

Definition 12.3 *Every matrix* $m \times m$ Q *of non-decreasing functions satisfying properties* (12.8) *and* (12.9) *is called an extended semi-Markov matrix or an extended semi-Markov kernel.*

Definition 12.4 *Every couple* (**p,Q**) *where* **Q** *is an extended semi-Markov kernel and* **p** *a vector of initial probabilities defines a* (*J,X*) *process* $(J,X) = ((J_n, X_n),\ n \geq 0)$ *with* $I \times \mathbb{R}$ *as state space, also called an extended semi-Markov chain (ESMC).*

Let us return to the main condition (12.7); its meaning is clear. For example if we assume that we observe for a certain fixed *n* that $J_{n-1} = i$, then the basic relation (12.7) gives us the value of the following conditional probability:

$$P(J_n = j, X_n \leq x \mid (J_k, X_k), k = 0, ..., n-1, J_{n-1} = i) = Q_{ij}(x). \tag{12.11}$$

That is, the knowledge of the value of J_{n-1} suffices to give the conditional probabilistic evolution of the future of the process whatever the values the other past variables might be.

According to Kingman (1972), the event $\{\omega : J_{n-1}(\omega) = i\}$ is *regenerative* in the sense that the observation of this event gives the complete evolution of the process in the future as it could evolve from $n = 0$ with *i* as the initial state.

(*J-X*) processes will be fully developed in section 12.4.

Remark 12.1 The second member of the semi-Markov characterization property (12.7) does not explicitly depend on *n*; also we can be precise that we are now studying *homogenous* semi-Markov chains in opposition with the *non-homogenous* case where this dependence with respect to *n* is valid.

12.3. Primary properties

We will start by studying the marginal stochastic processes $(J_n, n \geq 0)$, $(X_n, n \geq 0)$ called the *J-process* and the *X-process* respectively.

The J-process

From properties of the conditional expectation, the process $(J_n, n \geq 0)$ satisfies the following property:

$$P(J_n = j | (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(+\infty). \tag{12.12}$$

Using the smoothing property (see property (10.150)) of conditional expectation, we obtain

$$P(J_n = j | (J_k), k = 0,...,n-1) = E(Q_{J_{n-1}j}(+\infty) | (J_k), k = 0,...,n-1), \tag{12.13}$$

and as the r.v. $Q_{J_{n-1}j}(+\infty)$ is $(J_k, k = 0,...n-1)$,k=0,...,n-1)-measurable, we finally obtain from relation (12.8) that:

$$P(J_n = j | (J_k), k = 0,...,n-1) = p_{J_{n-1}j}. \tag{12.14}$$

Since relation (12.9) implies that matrix **P** is a Markov matrix, we have thus proved the following result.

Proposition 12.1 *The J-process is a homogenous Markov chain with P as its transition matrix.*

That is the reason why this *J*-process is called the *embedded Markov chain* of the considered SMC in which the r.v. J_n represents the state of the system *S* just after the *n*th transition.

From results of Corollary 11.1, it follows that in the ergodic case there exists one and only one stationary distribution of probability $\boldsymbol{\pi} = (\pi_1, ..., \pi_m)$ satisfying:

$$\begin{aligned} &\pi_i = \sum_{j=1}^{m} \pi_j p_{ji}, j = 1,...,m, \\ &\sum_{i=1}^{m} \pi_i = 1 \end{aligned} \tag{12.15}$$

such that

$$\lim_{n\to\infty} P(J_n = j \mid J_0 = i)(= \lim_{n\to\infty} p_{ij}^{(n)}) = \pi_j, i, j \in I, \tag{12.16}$$

where we know from relation (11.22) that

$$\left[p_{ij}^{(n)} \right] = \mathbf{P}^n . \tag{12.17}$$

The X-process

Here, the situation is entirely different for the fact that the distribution of X_n depends on J_{n-1}. Nevertheless, we have an interesting property of *conditional independence*, but before giving this property we must introduce some definitions.

Definition 12.5 *The two following conditional probability distributions:*

$$\begin{aligned} F_{J_{n-1}J_n}(x) &= P(X_n \le x \mid J_{n-1}, J_n), \\ H_{J_{n-1}}(x) &= P(X_n \le x \mid J_{n-1}) \end{aligned} \tag{12.18}$$

are respectively called the conditional and unconditional distributions of the sojourn time X_n.

From the general properties of conditioning recalled in section 10.2, we successively obtain

$$\begin{aligned} F_{J_{n-1}J_n}(x) &= E\left(P(X_n \le x \mid (J_k, X_k), k \le n-1, J_n) \mid J_{n-1}, J_n \right), \\ &= E\left(\frac{Q_{J_{n-1}J_n}(x)}{p_{J_{n-1}J_n}} \mid J_{n-1}, J_n \right), \\ &= \frac{Q_{J_{n-1}J_n}(x)}{p_{J_{n-1}J_n}}, \end{aligned} \tag{12.19}$$

provided that $p_{J_{n-1}J_n}$ is strictly positive. If not, we can arbitrarily give to (12.19) for example the value $U_1(x)$ defined as

$$U_1(x) = \begin{cases} 0, x < 0, \\ 1, x \ge 0. \end{cases} \tag{12.20}$$

Moreover, from the smoothing property, we also have:

$$\begin{aligned} H_{J_{n-1}}(x)(=P(X_n \le x | J_{n-1})) &= E(F_{J_{n-1}J_n}(x) | J_{n-1}), \\ &= \sum_{j=1}^{m} p_{J_{n-1}J_n} F_{J_{n-1}J_n}(x). \end{aligned} \tag{12.21}$$

We have thus proved the following proposition.

Proposition 12.2 *As a function of the semi-kernel* **Q**, *the conditional and unconditional distributions of the sojourn time X_n are given by:*

$$F_{ij}(x)(=P(X_n \le x | J_{n-1} = i, J_n = j)) = \begin{cases} \dfrac{Q_{ij}(x)}{p_{ij}}, p_{ij} > 0, \\ U_1(x), p_{ij} = 0, \end{cases} \tag{12.22}$$

$$H_i(x)(=P(X_n \le x | J_{n-1} = i)) = \sum_{j=1}^{m} p_{ij} F_{ij}(x).$$

Remark 12.2

(a) From relation (12.22), we can also express kernel **Q** as a function of F_{ij}, $i,j=1,...,m$:

$$Q_{ij}(x) = p_{ij} F_{ij}(x), i, j \in I, x \in \mathbb{R}^+. \tag{12.23}$$

So, every SMC can also be characterized by the triple (**p**,**P**,**F**) instead of the couple (**p**,**Q**) where the $m \times m$ matrix **F** is defined as $\mathbf{F} = \left[F_{ij} \right]$, and where the functions $F_{ij}, i, j = 1, \ldots, m$ are distribution functions on support $\mathbb{R}^+$.

(b) We can also introduce the *means* related to these conditional and unconditional distribution functions.

When they exist we will note:

$$\begin{aligned} \beta_{ij} &= \int_R x dF_{ij}(x), i, j = 1, ..., m, \\ \eta_i &= \int_R x dH_i(x), i = 1, ..., m \end{aligned} \tag{12.24}$$

and relation (12.22) leads to the relation:

$$\eta_i = \sum_{j=1}^{m} p_{ij}\beta_{ij} \ . \tag{12.25}$$

The quantities β_{ij} , i,j = 1,...,m and η_i ,I = 1,...,m are respectively called the *conditional* and *unconditional means* of the sojourn times.

We can now give the property of *conditional independence*.

Proposition 12.3 *For each integer k, if $n_1,n_2,...,n_k$ are k positive integers such that $n_1<n_2<\cdots<n_k$ and $x_{n_1},...,x_{n_k}$ k are real numbers. We have:*

$$\begin{aligned} &P\left(X_{n_1} \le x_{n_1},...,X_{n_k} \le x_{n_k} \middle| J_{n_1-1}, J_{n_1},...,J_{n_k-1}, J_{n_k}\right) \\ &= F_{J_{n_1-1}J_{n_1}}(x_{n_1})...F_{J_{n_k-1}J_{n_k}}(x_{n_k}), \end{aligned} \tag{12.26}$$

that is, k random variables $X_{n_1},...,X_{n_k}$ are conditionally independent given $J_{n_1-1}, J_{n_1},...,J_{n_k-1}, J_{n_k}$.

The T-process

By relation (12.4), the sequence $(T_n, n \ge 0)$ represents successive *renewal epochs*, that is, times at which transitions occur.

By analogy with renewal theory, we have the following definition.

Definition 12.6 *The two-dimensional process $((J_n,T_n), n \ge 0)$ is called the Markov renewal process of kernel* **Q**.

Before giving the expression of the marginal distribution of the random vector (J_n,T_n) with values in $I \times \mathbb{R}^+$, given that $J_0=i$, let us define the *marginal distribution* of the (J,T) process $((J_n,T_n), n \ge 0)$:

$$Q_{ij}^n(t) = P(J_n = j, T_n \le t \,|\, J_0 = i), \ i,j \in I, \ n \ge 0, \ t \ge 0 \ . \tag{12.27}$$

With $\mathbf{A} = \left[A_{ij}\right]$ and $\mathbf{B} = \left[B_{ij}\right]$, two $m \times m$ matrices of integrable functions, we associate a new matrix $\mathbf{A} \bullet \mathbf{B}$ whose general element $(\mathbf{A} \bullet \mathbf{B})_{ij}$ is the function of t defined by:

$$(\mathbf{A} \bullet \mathbf{B})_{ij}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} A_{kj}(t-y)\, dB_{ik}(y). \tag{12.28}$$

It can be easily seen that this type of product, called the *convolution product for matrices*, is *associative* but not always commutative.

In the particular case of $A=B$, we set:

$$\begin{aligned}&\mathbf{A}\bullet\mathbf{A}=\mathbf{A}^{(2)},\ldots,\mathbf{A}\bullet\cdots\bullet\mathbf{A}=\mathbf{A}^{(n)}\left(=\left[A_{ij}^{(n)}\right]\right),\\&\mathbf{A}^{(0)}=(\delta_{ij}U_0),\mathbf{A}^{(1)}=\mathbf{A}.\end{aligned}\tag{12.29}$$

If all the functions $A_{ij},B_{ij},i,j=1,\ldots,m,$ vanish at $-\infty$, we can also use an integration by parts to express (12.28) as follows:

$$(\mathbf{A}\bullet\mathbf{B})_{ij}(t)=\sum_{k=1}^{m}\int_{\mathbb{R}}B_{ik}(t-y)dA_{kj}(y)\tag{12.30}$$

and moreover if **A**=**B**, we obtain:

$$(\mathbf{A}\bullet\mathbf{B})_{ij}(t)=\sum_{k=1}^{m}\int_{\mathbb{R}}A_{ik}(t-y)dA_{kj}(y)\ .\tag{12.31}$$

Proposition 12.4 *For all* $n\geq 0$*, we have:*

$$Q_{ij}^{n}=Q_{ij}^{(n)}\ .\tag{12.32}$$

Moreover, we also have:

$$\lim_{t\to\infty}Q^{(n)}(t)=P^{n}\ .\tag{12.33}$$

12.4. Examples

Semi-Markov theory is one of the most productive subjects of stochastic processes to generate applications in real-life problems, particularly in the following fields: economics, manpower models, insurance, finance (more recently), reliability, simulation, queuing, branching processes, medicine (including survival data), social sciences, language modeling, seismic risk analysis, biology, computer science, chromatography and fluid mechanics.

Important results in such fields may be found in Janssen (1986), Janssen and Limnios (1999), and Janssen and Manca (2006 and 2007).

Let us give three examples in the fields of insurance and reliability.

Example 12.1 *The claim process in insurance*

Let us consider an insurance company covering m types of risks or having m different types of customers for the same risk forming the set $I=\{1,...,m\}$.

For example, in automobile insurance, we can distinguish three types of drivers: *good, average* and *bad* and so I is a space consisting of three states: 1 for good, 2 for average and 3 for bad.

Now, let $(X_n, n \geq 1)$ represent the sequence of successive observed *claim amounts*, $(Y_n, n \geq 1)$ the sequence of interarrivals between two successive claims and $(J_n, n \geq 1)$ successive *types of observed risks*.

In the traditional model of risk theory called the Cramer Lundberg model (1909, 1955), it is assumed with that there is only one type of risk and the claim arrival process is a *Poisson process* parameter λ; later, Andersen (1967) extends this model to an arbitrary *renewal process* and moreover in these two traditional models, the process of claim amounts is a renewal process independent of the claim arrival process.

The consideration of an SMC for the two-dimensional processes $((J_n, X_n), n \geq 0)$ and/or $((J_n, Y_n), n \geq 0)$ provides the possibility to introduce a certain dependence between the successive claim amounts. This model was first developed by Janssen (1969b, 1970, 1977) along the lines of Miller's work (1962) and since then has led to many extensions; see for example Asmussen (2000).

Example 12.2 *Occupational illness insurance*

This problem is related to occupational illness insurance with the possibility of leading to partial or permanent disability. In this case, the amount of the incapacitation allowance depends on the degree of disability recognized in the policyholder by the occupational health doctor, in general on an annual basis, because this degree is a function of an occupational illness which can take its course.

Considering as in the example in section 11.6.2 this invalidity degree as a stochastic process $(J_n, n \geq 0)$ where J_n represents the value of this degree when the illness really takes its course, and we must then introduce the r.v. X_n representing the time between two successive transitions from J_{n-1} to J_n.

In practice, these transitions can be observed with periodic medical inspections.

The assumption that the *J-X* process is an SMC extends the Markov model of Chapter 11 and is fully discussed in Janssen and Manca (2006).

Example 12.3 *Reliability*

There are many semi-Markov models in reliability theory; see for example Osaki (1985) and more recently Limnios and Oprisan (2001), (2003).

Let us consider a *reliability system* S that can be at any time t in one of the m states of $I=\{1,...,m\}$.

The stochastic process of the successive states of S is represented by $S=\left(S_t,t\geq 0\right)$.

The state space I is partitioned into two sets U and D so that

$$I=U\bigcup D,\ U\bigcap D=\varnothing,\ U\neq\varnothing,\ D\neq\varnothing. \tag{12.34}$$

The interpretation of these two sets is the following: the subset U contains all "good" states, in which the system is working and the subset D of all "bad" states, in which the system is not working well or has failed.

The indicators used in reliability theory are the following:

(i) the *reliability function* R gives the probability that the system was always working from time 0 to time t:

$$R(t)=P\left(S_u\in U,\forall u\in\left[0,t\right]\right), \tag{12.35}$$

(ii) the *pointwise availability function* A gives the probability that the system is working at time t whatever happens on $(0,t)$:

$$A(t)=P\left(S_t\in U\right), \tag{12.36}$$

(iii) the *maintainability function* M gives the probability that the system, being in D on $[0,t)$, will leave set D at time t:

$$M(t)=P\left(S_u\in D,u\in\left[0,t\right),S_t\in U\right). \tag{12.37}$$

12.5. Markov renewal processes, semi-Markov and associated counting processes

Let us consider an SMC of kernel **Q**; we then have the following definitions.

Definition 12.7 *The two-dimensional process* $(J,T)=((J_n,T_n),n\geq 0)$ *where* T_n *is given by relation* (12.4) *is called a Markov renewal sequence or Markov renewal process.*

Cinlar (1969) also gives the term *Markov additive process*. It is justified by the fact that, using relation (12.5), we obtain:

$$\begin{aligned}&P(J_{n+1}=j,T_{n+1}\leq x\,|(J_k,T_k),k=0,\ldots,n)=\\&P(J_{n+1}=j,X_{n+1}\leq x-T_n\,|(J_k,T_k),k=0,\ldots,n)=\mathrm{Q}_{J_n j}(x-T_n).\end{aligned}\tag{12.38}$$

This last equality shows that the (J,T) process is a Markov process with $I\times\mathbb{R}^+$ as state space and having the "additive property":

$$T_{n+1}=T_n+X_{n+1}\,.\tag{12.39}$$

Let us state that according to the main definitions of Chapter 11, Definition 11.4, and always in the case of positive (J,X) chains, the random variables $T_n,(n\geq 0)$ are from now on called *Markov renewal times* or simply *renewal times*, the random variables $X_n,(n\geq 1)$ *interarrival* or *sojourn times* and the random variables $J_n,(n\geq 0)$ the *state variables*.

We will now define the *counting processes* associated with any Markov renewal process (MRP) as we did in the special case of renewal theory.

For any fixed time t, the r.v. $N(t)$ represents the *total number of jumps or transitions of the* (J,X) *process on* $(0,t]$, including possible transitions from any state towards itself (virtual transitions), assuming transitions are observable.

We have:

$$N(t)>t\Leftrightarrow T_n\leq t\,.\tag{12.40}$$

However here, we can be more precise and only count the total number of passages in a fixed state I always in $(0,t]$ represented by the r.v. $N_i(t)$.

Clearly, we can write:

$$N(t) = \sum_{i=1}^{m} N_i(t), t \geq 0 . \tag{12.41}$$

Definition 12.8 *With each Markov renewal process, the following* m+1 *stochastic processes are associated respectively with values in* $\mathbb{N}$ *:*

(i) the N-process ($N(t)$, $t \geq 0$);

(ii) the N_i-process ($N_i(t)$, $t \geq 0$), i=1,…,m,

respectively called the associated total counting process and the associated partial counting processes with of course:

$$N(0)=0,\ N_i(0)=0,\ i=1,\dots,m. \tag{12.42}$$

It is now easy to introduce the notion of a *semi-Markov process* by considering at time t, the state entered at the last transition before or at t, that is, $J_{N(t)}$.

Definition 12.9 *With each Markov renewal process, we associate the following stochastic Z-process with values in I:*

$$Z=(Z(t),\ t \geq 0), \tag{12.43}$$

with:

$$Z(t)=J_{N(t)}. \tag{12.44}$$

This process will be called the associated semi-Markov process or simply the semi-Markov process (*SMP*) *of kernel* **Q**.

Remark 12.3

1) We will often use counting variables including the initial renewal, that is:

$$\begin{aligned} N'(t) &= N(t) + 1, \\ N_i'(t) &= N_i(t) + \delta_{iJ_0} . \end{aligned} \tag{12.45}$$

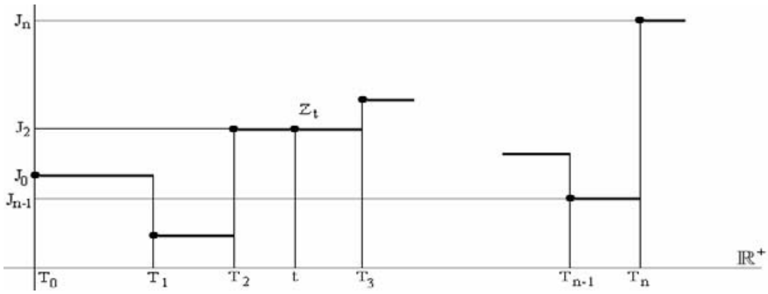

Figure 12.1. *A trajectory of an SMP*

2) Figure 12.1 gives a typical trajectory of MRP and SMP.

3) It is now clear that we can immediately consider an MRP defined by kernel **Q** without speaking explicitly of the basic (J,X) process with the same kernel **Q**, because the basic property (12.11) is equivalent to (12.38).

12.6. Particular cases of MRP

We will devote this section to particular cases of MRP having the advantage to lead to some explicit results.

12.6.1. *Renewal processes and Markov chains*

For the sake of completeness, let us first state that with $m = 1$, that is, that the observed system has only one possible state, the kernel **Q** has only one element, say the d.f. **F**, and the process $(X_n, n > 0)$ is a *renewal process*.

Secondly, to obtain *Markov chains* studied in Chapter 11, it suffices to choose for matrix **F** the following special degenerating case:

$$F_{ij} = U_1, \forall i, j \in I \tag{12.46}$$

and of course an arbitrary Markov matrix **P**.

This means that all r.v. X_n have a.s. the value 1, and so the single random component is the (J_n) process, which is, from relation (12.15), a homogenous MC of transition matrix **P**.

12.6.2. *MRP of zero order (Pyke (1962))*

There are two types of such processes.

12.6.2.1. *First type of zero order MRP*

This type is defined by the following semi-Markov kernel

$$\mathbf{Q} = \left[p_i F_i \right], \tag{12.47}$$

so that:

$$p_{ij} = p_i, F_{ij} = F_i, j \in I. \tag{12.48}$$

Naturally, we assume that for every i belonging to I, p_i is strictly positive.

In this present case, we discover that the r.vs. $J_n, n \geq 0$ are independent and identically distributed and moreover that the conditional interarrival distributions do not depend on the state to be reached, so that, by relation (12.22),

$$H_i = F_i, i \in I. \tag{12.49}$$

Moreover, since:

$$P(X_n \leq x \,|\, (J_k, X_k), k \leq n-1, J_n) = F_{J_{n-1}}(x), \tag{12.50}$$

we obtain:

$$P(X_n \leq x \,|\, (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x). \tag{12.51}$$

Introducing the d.f. F defined as

$$F = \sum_{j=1}^{m} p_j F_j, \tag{12.52}$$

the preceding equality shows that, for an MRP of zero order of the first type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F.

12.6.2.2. *Second type of zero order MRP*

This type is defined by the following semi-Markov kernel

$$\mathbf{Q} = \left[p_i F_j \right], \tag{12.53}$$

so that:

$$p_{ij} = p_i, F_{ij} = F_j, \; i, j \in I. \tag{12.54}$$

Here too, we suppose that for every i belonging to I, p_i is strictly positive.

Once again, the r.v. $J_n, n \geq 0$ are independent and equi-distributed and moreover the conditional interarrival distributions do not depend on the state to be *left*, so that, by relation (12.22)

$$H_i = \sum_{j=1}^{m} p_j F_j (= F), i \in I. \tag{12.55}$$

Moreover, since:

$$P(X_n \leq x \,|\, (J_k, X_k), k \leq n-1, J_n) = F_{J_n}(x), \tag{12.56}$$

we obtain

$$P(X_n \leq x \,|\, (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x) = F(x). \tag{12.57}$$

The preceding equality shows that, for an MRP of zero order of the second type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F as in the first type.

The basic reason for these similar results is that these two types of MRP are the *reverse*s (timewise) of each other.

12.6.3. *Continuous Markov processes*

These processes are defined by the following particular semi-Markov kernel

$$\mathbf{Q}(x) = \left[p_{ij} \left(1 - e^{-\lambda_i x} \right) \right], x \geq 0, \tag{12.58}$$

where $\mathbf{P} = \left[p_{ij} \right]$ is a stochastic matrix and where parameters $\lambda_i, i \in I$ are strictly positive.

The standard case corresponds to that in which $p_{ii} = 0, i \in I$ (see Chung (1960)). From relation (12.58), we obtain:

$$F_{ij}(x) = 1 - e^{-\lambda_i x}. \tag{12.59}$$

Thus, the d.f. of sojourn time in state *i* has an exponential distribution depending uniquely upon the occupied state *i*, such that both the excess and age processes also have the same distribution.

For $m = 1$, we obtain the usual Poisson process of parameter λ.

12.7. Markov renewal functions

Let us consider an MRP of kernel **Q** and to avoid trivialities, we will assume that:

$$\sup_{i,j} Q_{ij}(0) < 1, \tag{12.60}$$

where the functions Q_{ij} are defined by relation (12.7).

If the initial state J_0 is *i*, let us define the r.v. $T_n\left(i \middle| i\right), n \geq 1$, as the times (possibly infinite) of *successive returns* to state *i*, also called *successive entrance times* into $\{i\}$.

From the regenerative property of MRP, whenever the process enters into state *i*, say at time *t*, the evolution of the process on $\left[t, \infty\right)$ is probabilistically the same as if we had started at time 0 in the same state *i*.

It follows that the process $\left(T_n\left(i \middle| i\right), n \geq 0\right)$ with:

$$T_0\left(i \middle| i\right) = 0 \tag{12.61}$$

is a renewal process that could possibly be defective.

From now on, the r.v. $T_n\left(i \middle| i\right)$ will be called the *nth return time to state i*.

More generally, let us also fix state j, different from the state i already fixed; we can also define the *nth return* or *entrance time to state j*, but *starting from i as the initial state*. This time, possibly infinite as well, will be represented by $\left(T_n\left(j|i\right), n \geq 0\right)$, also using the convention that

$$T_0\left(j|i\right) = 0. \tag{12.62}$$

Now, the sequence $\left(T_n\left(j|i\right), n \geq 0\right)$ is a delayed renewal process with values in $\mathbb{R}^+$.

It is thus defined by two d.fs.: G_{ij} being that of $T_1\left(j|i\right)$ and G_{jj} that of $T_2\left(j|i\right) - T_1\left(j|i\right)$, so that:

$$\begin{aligned} G_{ij}(t) &= P\left(T_1\left(j|i\right) \leq t\right), \\ G_{jj}(t) &= P\left(T_n\left(j|i\right) - T_{n-1}\left(j|i\right) \leq t\right), \ n \geq 2. \end{aligned} \tag{12.63}$$

Of course, the d.f. G_{jj} suffices to define the renewal process $\left(T_n\left(j|j\right), \ n \geq 0\right)$.

Remark 12.4 From the preceding definitions, we can also write that:

$$\begin{aligned} & G_{ij}(t) = P\left(N_j(t) > 0 | J_0 = i\right); i, j \in I, \\ & P\left(T_1\left(j|i\right) = +\infty\right) = 1 - G_{ij}(+\infty) \end{aligned} \tag{12.64}$$

and for the mean of the $T_n\left(i|i\right), n \geq 1$, possibly infinite, we obtain:

$$\mu_{ij} = E\left(T_1\left(j|i\right)\right) = \int_0^\infty t dG_{ij}(t), \tag{12.65}$$

with the usual convention that

$$0 \cdot (+\infty) = 0. \tag{12.66}$$

The means $\mu_{ij}, i, j \in I$ are called the *first entrance* or *average return times*.

It is possible to show that the functions $G_{ij}, i, j \in I$ satisfy the following relationships:

$$G_{ij}(t) = \sum_{k=1}^{m} G_{kj} \bullet Q_{ik}(t) + (1 - G_{jj}) \bullet Q_{ij}(t), i, j \in I, t \geq 0. \tag{12.67}$$

We will now define by A_{ij} and R_{ij} the associated *renewal functions*

$$A_{ij}(t) = E(N_j(t)|J_0 = i),$$
$$R_{ij}(t) = E(N_j^{'}(t)|J_0 = i) \tag{12.68}$$

and by relations (12.45) we have:

$$R_{ij}(t) = \delta_{ij}U_0(t) + A_{ij}(t). \tag{12.69}$$

Using classical results of renewal theory (see Janssen and Manca (2006)) we obtain:

$$R_{jj}(t) = \sum_{n=0}^{\infty} G_{jj}^{(n)}(t), j \in I,$$
$$R_{ij}(t) = G_{ij} \bullet R_{jj}(t), \tag{12.70}$$

or equivalently, we have:

$$R_{ij}(t) = \delta_{ij}U_0(t) + G_{ij} \bullet \sum_{n=0}^{\infty} G_{jj}^{(n)}(t), i, j \in I. \tag{12.71}$$

Proposition 12.5 *Assumption* $m < \infty$ *implies that:*

(i) at least one of the renewal processes $\left(T_n\left(j|j\right), n \geq 0\right), j \in I$ is not defective;

(ii) for all i belonging to I, there exists a state s such that

$$\lim_n T_n\left(s|i\right) = +\infty, \text{ a.s.}; \tag{12.72}$$

(iii) for the r.v. T_n defined by relation (12.4), given that $J_0=i$ whatever i is, we have a.s. that

$$\lim_n T_n = +\infty. \tag{12.73}$$

The following relations will express the renewal functions R_{ij}, $i, j \in I$ in function of the kernel **Q** instead of the m^2 functions G_{ij}.

Proposition 12.6 *For every i and j of I, we have:*

$$R_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \tag{12.74}$$

Using matrix notation with:

$$\mathbf{R} = \left[R_{ij} \right], \tag{12.75}$$

relation (12.74) takes the form:

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}. \tag{12.76}$$

Let us now introduce the *L-S transform of matrices.*

For any matrix of suitable functions A_{ij} from $\mathbb{R}^{+}$ to $\mathbb{R}$ represented by

$$\mathbf{A} = \left[A_{ij} \right] \tag{12.77}$$

we will represent its L-S transform by:

$$\overline{\mathbf{A}} = \left[\overline{A}_{ij} \right] \tag{12.78}$$

with

$$\overline{A}_{ij}(s) = \int_{0}^{\infty} e^{-st} dA_{ij}(t). \tag{12.79}$$

Doing so for matrix $\mathbf{R}$, we obtain the matrix form of relation (12.76),

$$\overline{\mathbf{R}}(s) = \sum_{n=0}^{\infty} \left(\overline{\mathbf{Q}}(s) \right)^{n}. \tag{12.80}$$

From this last relation, a simple algebraic argument shows that, for any $s>0$, relations

$$\overline{\mathbf{R}}(s)(\mathbf{I} - \overline{\mathbf{Q}}(s)) = (\mathbf{I} - \overline{\mathbf{Q}}(s)\overline{\mathbf{R}}(s) = \mathbf{I} \tag{12.81}$$

hold and so we also have:

$$\bar{\mathbf{R}}(s) = (\mathbf{I} - \bar{\mathbf{Q}}(s))^{-1}. \tag{12.82}$$

We have thus proved the following proposition.

Proposition 12.7 *The Markov renewal matrix* $\mathbf{R}$ *is given by*

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}, \tag{12.83}$$

the series being convergent in $\mathbb{R}^+$*, and the inverse existing for all positive s.*

The knowledge of the Markov renewal matrix $\mathbf{R}$ or its L-S transform $\bar{\mathbf{R}}$ leads to useful expressions for d.f. of the first entrance times.

12.8. The Markov renewal equation

This section will extend the basic results related to the renewal equation developed in section 11.4 to the Markov renewal case.

Let us consider an MRP of kernel $\mathbf{Q}$.

From relation (12.74), we obtain:

$$\begin{aligned} R_{ij}(t) &= \delta_{ij} U_0(t) + \sum_{n=1}^{\infty} Q_{ij}^{(n)}(t) \\ &= \delta_{ij} U_0(t) + \left(Q \bullet R\right)_{ij}(t). \end{aligned} \tag{12.84}$$

Using matrix notation with:

$$\mathbf{I}(t) = \left[\delta_{ij} U_0(t)\right], \tag{12.85}$$

relations (12.84) take the form:

$$\mathbf{R}(t) = \mathbf{I}(t) + \mathbf{Q} \bullet \mathbf{R}(t). \tag{12.86}$$

This integral matrix equation is called the *Markov renewal equation* for $\mathbf{R}$.

To obtain the corresponding matrix integral equation for the matrix

$$\mathbf{H} = \left[H_{ij} \right], \tag{12.87}$$

we know, from relation (12.76), that

$$\mathbf{R}(t) = \mathbf{I}(t) + \mathbf{H}(t). \tag{12.88}$$

Inserting this expression of $\mathbf{R}(t)$ in relation (12.86), we obtain:

$$\mathbf{H}(t) = \mathbf{Q}(t) + \mathbf{Q} \bullet \mathbf{H}(t) \tag{12.89}$$

which is the *Markov renewal equation for* **H**.

For $m=1$, this last equation gives the traditional renewal equation.

In fact, the Markov renewal equation (12.86) is a particular case of the matrix integral equation of the type:

$$\mathbf{f} = \mathbf{g} + \mathbf{Q} \bullet \mathbf{f}, \tag{12.90}$$

called an integral equation of *Markov renewal type* (MRT), where

$$\mathbf{f} = \left(f_1, ..., f_m \right)', \mathbf{g} = \left(g_1, ..., g_m \right)' \tag{12.91}$$

are two column vectors of functions having all their components in B, the set of single-variable measurable functions, bounded on finite intervals, or to B^+ if all their components are non-negative.

Proposition 12.7 *The Markov integral equation of MRT,*

$$\mathbf{f} = \mathbf{g} + \mathbf{Q} \bullet \mathbf{f} \tag{12.92}$$

with **f**, **g** *belonging to* B^+, *has the unique solution:*

$$\mathbf{f} = \mathbf{R} \bullet \mathbf{g}. \tag{12.93}$$

12.9. Asymptotic behavior of an MRP

We will give asymptotic results, first for the Markov renewal functions and then for solutions to integral equations of an MRT. To conclude, we will apply these results to transition probabilities of an SMP.

We know that the renewal function R_{ij}, i, j belonging to I, is associated with the delayed renewal process, possibly transient, characterized by the couple (G_{ij}, G_{jj}) d.f. on $\mathbb{R}^+$.

Let us recall that μ_{ij} represents the mean, possibly infinite, of the d.f. G_{ij}.

Proposition 12.8 *For all i, j of I, we have:*

$$\text{(i)} \lim_{t\to\infty} \frac{R_{ij}(t)}{t} = \frac{1}{\mu_{jj}}, \tag{12.94}$$

$$\text{(ii)} \lim_{t\to\infty} \frac{R_{ij}(t) - R_{ij}(t-\tau)}{\tau} = \frac{\tau}{\mu_{jj}}, \text{ for every fixed } \tau. \tag{12.95}$$

The next proposition, due to Barlow (1962), is a useful complement to the last proposition as it gives a method for computing the values of the mean return times $\mu_{jj}, j \in I$, in the ergodic case.

Proposition 12.9 *For an ergodic MRP, the mean return times satisfy the following linear system:*

$$\mu_{ij} = \sum_{k \neq j} p_{ik} \mu_{kj} + \eta_i, i = 1, \ldots, m. \tag{12.96}$$

In particular, for i=j, we have $\mu_{jj} = \frac{1}{\pi_j} \sum_k \pi_k \eta_k, j = 1, \ldots, m,$ (12.97)

where the $\eta_i, i \in I$ *are defined by relation* (12.25), *and where* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ *is the unique stationary distribution of the imbedded Markov chain.*

Remark 12.5 In a similar manner, Barlow (1962) proved that if $\mu_{ij}^{(2)}, i, j \in I$ is the second order moment related to the d.f. G_{ij}, then:

$$\mu_{ij}^{(2)} = \eta_i^{(2)} + \sum_{k \neq j} p_{ik} (\mu_{ik}^{(2)} + 2b_{ik}\mu_{kj}) \tag{12.98}$$

and in particular for $i = j$:

$$\mu_{jj}^{(2)} = \frac{1}{\pi_j} (\sum_k \pi_k \eta_k^{(2)} + 2\sum_{k \neq j} \sum_l \pi_l p_{lk} b_k \mu_{kj}) \tag{12.99}$$

with

$$\eta_k^{(2)} = \int_{[0,\infty)} x^2 dH_k(x), k \in I, \tag{12.100}$$

provided that these quantities are finite.

12.10. Asymptotic behavior of SMP

12.10.1. *Irreducible case*

Let us consider the SMP ($Z(t)$, $t \geq 0$) associated with the MRP of kernel **Q** and defined by relation (12.43).

Starting with $Z(0) = i$, it is important for the applications to know the probability of being in state j at time t, that is:

$$\phi_{ij}(t) = P\left(Z(t) = j \middle| Z(0) = i\right). \tag{12.101}$$

A simple probabilistic argument using the regenerative property of the MRP gives the system satisfied by these probabilities as a function of the kernel **Q**

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_k \int_0^t \phi_{kj}(t - y) dQ_{ik}(y), \ i, j \in I. \tag{12.102}$$

It is also possible to express the transition probabilities of the SMP with the aid of the first passage time distributions $G_{ij}, i, j \in I$:

$$\phi_{ij}(t) = \phi_{jj} \bullet G_{ij}(t) + \delta_{ij}(1 - H_i(t)), \ i, j \in I. \tag{12.103}$$

If we fix the value j in relation (12.102), we see that the m relations for i=1,...,m form a Markov renewal type equation (MRE) of form (12.92).

Applying Proposition 12.7, we immediately obtain the following proposition.

Proposition 12.10 *The matrix of transition probabilities*

$$\mathbf{\Phi} = \left[\phi_{ij}\right] \tag{12.104}$$

is given by

$$\mathbf{\Phi} = \mathbf{R} \bullet (\mathbf{I} - \mathbf{H}) \tag{12.105}$$

with

$$\mathbf{H} = \left[\delta_{ij} H_i\right]. \tag{12.106}$$

So, instead of relation (12.103), we can now write:

$$\phi_{ij}(t) = \int_{[0,t]} (1 - H_j(t-y)) dR_{ij}(y). \tag{12.107}$$

Remark 12.6 *Probabilistic interpretation of relation* (12.107).

This interpretation is analogous to that of the renewal density given in Chapter 11.

Remark 12.7 The "infinitesimal" quantity $dR_{ij}(y)$ ($=r_{ij}(y)dy$, if $r_{ij}(y)$ is the density of function R_{ij}, if it exists) represents the probability that there is a Markov renewal into state j in the time interval $(y,y+dy)$, starting at time 0 in state i.

Of course, the factor $(1 - H_j(t-y))$ represents the probability of not leaving state j before a time interval of length $t-y$.

The behavior of transition probabilities of matrix (12.104) will be given in the next proposition.

Proposition 12.11 *If* $Z = \left(Z(t), t \geq 0\right)$ *is the SMP associated with an ergodic MRP of kernel* $\mathbf{Q}$*; then:*

$$\lim_{t\to\infty} \phi_{ij}(t) = \frac{\pi_j \eta_j}{\sum_k \pi_k \eta_k}, \; i,j \in I. \tag{12.108}$$

Remark 12.8

(i) As the limit in relation (12.108) does not depend on *i*, Proposition 12.11 establishes an *ergodic property* stating that:

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi_j,$$
$$\Pi_j=\frac{\pi_j\eta_j}{\sum_k \pi_k\eta_k}. \tag{12.109}$$

(ii) As the number m of states is finite, it is clear that $(\Pi_j, j\in I)$ is a probability distribution. Moreover, as $\pi_j>0$ for all j (see relation (11.89)), we also have

$$\Pi_j>0, j\in I. \tag{12.110}$$

So, asymptotically, every state is reachable with a strictly positive probability.

(iii) In general, we have:

$$\lim_{n\to\infty}p_{ij}^{(n)}\neq\lim_{t\to\infty}\phi_{ij}(t) \tag{12.111}$$

since of course

$$\pi_j\neq\Pi_j, j\in I. \tag{12.112}$$

This shows that the limiting probabilities for the embedded Markov chain are not, in general, the same as taking limiting probabilities for the SMP.

From Propositions 12.11 and 12.9, we immediately obtain the following corollary.

Corollary 12.1 *For an ergodic MRP, we have:*

$$\Pi_j=\frac{\pi_j}{\mu_{jj}}. \tag{12.113}$$

This result states that the limiting probability of being in state j for the SMP is the ratio of the mean sojourn time in state j to the mean return time of j.

This intuitive result also shows how the different return times and sojourn times have a crucial role in explaining why we have relation (12.113) as, indeed, for the imbedded MC, these times have no influence.

12.10.2. *Non-irreducible case*

It is often the case that stochastic models used for applications need non-irreducible MRP, as, for example, in the presence of an absorbing state, i.e. a state j such that

$$p_{jj} = 1. \tag{12.114}$$

We will now see that the asymptotic behavior is easily deduced from the irreducible case studied above.

12.10.2.1. *Uni-reducible case*

As for Markov chains, this is simply the case in which the imbedded MC is uni-reducible so that there exist l ($l<m$) transient states, and so that the other m-l states form a recurrent class C.

We always suppose aperiodicity both for the imbedded MC and for the considered MRP.

Let $T=\{1,...,l\}$ be the set of transient states $(T = I - C)$. From Corollary 11.2 we know that:

$$\lim_{t\to\infty} \phi_{ij}(t) = 0,\ i,j \in T. \tag{12.115}$$

Moreover, from Proposition 12.11 and relation (12.103):

$$\lim_{t\to\infty} \phi_{ij}(t) = G_{ij}(\infty)\frac{\pi_j \eta_j}{\sum_{k=l+1}^{m} \pi_k \eta_k},\ \ i,j \in C, \tag{12.116}$$

where $(\pi_{l+1},...,\pi_m)$ represents the unique stationary probability distribution of the sub-Markov chain with C as state space.

Since

$$G_{ij}(\infty) = f_{ij}, \tag{12.117}$$

we obtain

$$G_{ij}(\infty) = f_{i,C}, \tag{12.118}$$

where $f_{i,C}$ is the probability that the system, starting in state i will be absorbing by the recurrent class C.

As there is only one essential class, we know that for all states i of I:

$$f_{i,C}=1, \tag{12.119}$$

thus proving the following proposition.

Proposition 12.12 *For any periodic uni-reducible MRP, we have:*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi_j^{'}, j\in I, \tag{12.120}$$

where

$$\Pi_j^{'}=\begin{cases}0, & j\in T,\\ \dfrac{\pi_j\eta_j}{\sum_{k=l+1}^{m}\pi_k\eta_k}, & j\in C.\end{cases} \tag{12.121}$$

Here too, as the limit in (12.121) is independent of the initial state i, this result gives an *ergodic property*.

12.10.2.2. *General case*

For any aperiodical MRP, there exists a unique partition of the state space I:

$$I=T\cup C_1\cup\cdots\cup C_r,\ r<m, \tag{12.122}$$

where T represents the set of transient states and $C_\nu,\nu=1,...,r$ represents the ν th essential class necessarily formed of positive recurrent.

From Chapter 11, we know that the system will finally enter one of the essential classes and will then stay in it forever. Thus, a slight modification of the last proposition leads to the next result.

Proposition 12.13 *For any aperiodic MRP, we have:*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi_{ij}^{'},i,j\in I, \tag{12.123}$$

with, for any $j\in C_\nu,\nu=1,...,r$ *:*

$$\Pi'_{ij} = \begin{cases} \Pi'^{\nu}_{j}, & i \in C_\nu, \nu = 1,...,r, \\ 0, & i \in C_{\nu'}, \nu \neq \nu', \nu' = 1,...,r, \\ f_{i,C_\nu} \Pi'^{\nu}_{j}, & i \in T \end{cases} \tag{12.124}$$

where $(\Pi'^{\nu}_{j}, j \in C_\nu)$ *is the only stationary distribution of the sub-SMP with* C_ν *as state space, that is:*

$$\Pi'_{j} = \frac{\pi_j \eta_j}{\sum_{k \in C_\nu} \pi_k \eta_k}, \tag{12.125}$$

where $\left(\pi^{\nu}_{k}, k \in C_\nu\right)$ *is the unique stationary distribution of the sub-Markov chain with* C_ν *as the state space and* $\left(f_{i,C_\nu}, i \in T\right)$ *is the unique solution of the linear system*

$$y_i - \sum_{j \in T} p_{ij} y_j = \sum_{j \in C_\nu} p_{ij}, i \in T. \tag{12.126}$$

Note that, in this proposition, the ergodic property is lost; this is due to the presence of the quantities f_{i,C_ν} in relation (12.124).

12.11. Non-homogenous Markov and semi-Markov processes

To finish this chapter, let us recall the basic definitions and results for the *non-homogenous* case for which time itself has influence on the transition probabilities. Due to the importance of its applications, in particular within insurance, we carefully develop some special cases such as non-homogenous Markov processes.

12.11.1. *General definitions*

To begin, we present the general definition of non-homogenous semi-Markov processes (NHSMP) including, as particular cases, non-homogenous Markov processes (NHMP) in continuous time, non-homogenous Markov chains (NHMC) in discrete time and non-homogenous renewal processes (NHRP).

We follow the original presentation given by Janssen and De Dominicis (1984).

12.11.1.1. *Completely non-homogenous semi-Markov processes*

As usual, let us consider a system *S* having *m* possible states constituting the set $I=\{1,...,m\}$ defined on the probability space $(\Omega,\Im,P)$.

Definition 12.10 *The two-dimensional process in discrete time* $((J_n,X_n),n\geq 0)$ *with values in* $I\times\mathbb{R}^+$ *such that:*

$$\begin{aligned}&J_0=i,X_0=0,a.s.,i\in I,\\&P(J_n=j,X_n\leq x\big|(J_k,X_k),k\leq n-1)={}^{(n-1)}Q_{J_{n-1}j}(T_{n-1},T_{n-1}+x),\\&j\in I,x\in\mathbb{R}^+,\\&T_0=0,T_n=\sum_{k=0}^{n}X_k,a.s.\end{aligned}\tag{12.127}$$

is called a completely non-homogenous semi-Markov chain (*CNHSMC*) *of kernel* $\mathbf{Q}(s,t)=\left({}^{(n-1)}\mathbf{Q}(s,t),n\geq 1\right)$.

Consequently, the past influences the evolution of the process by the presence of T_{n-1} and *n* in (12.127).

Definition 12.11 *The sequence* $\mathbf{Q}=\left({}^{(n-1)}\mathbf{Q}(s,t),n\geq 1\right)$ *of* $m\times m$ *matrices of measurable functions of* $\mathbb{N}_0\times\mathbb{R}^+\times\mathbb{R}^+\mapsto[0,1]$ *where*:

$${}^{(n-1)}\mathbf{Q}(s,t)=\left[{}^{(n-1)}Q_{ij}(s,t)\right]\tag{12.128}$$

and satisfying the following conditions:

$$\begin{aligned}&\text{(i) }\forall n>0,\forall i,j\in I,\forall t,s\in\mathbb{R}^+:t\leq s\Rightarrow{}^{(n-1)}Q_{ij}(s,t)=0,\\&\text{(ii) }\forall n>0,\forall i\in I,\forall s\in\mathbb{R}^+:\sum_{j=1}^{n}{}^{(n-1)}Q_{ij}(s,\infty)=1,\\&\text{with }{}^{(n-1)}Q_{ij}(s,\infty)=\lim_{t\to\infty}{}^{(n-1)}Q_{ij}(s,t),\end{aligned}\tag{12.129}$$

is called a completely non-homogenous semi-Markov (*CNHSM*) *kernel*.

Clearly, for all fixed *s*, ${}^{(n-1)}Q_{ij}(s,.)$ is a mass function, zero for $t\leq s$.

Definition 12.12 *For all* $i, j \in I, n \in \mathbb{N}_0, s, t \in \mathbb{R}^+$, *functions* ${}^{(n-1)}p_{ij}(s)$, ${}^{(n-1)}H_{ij}(s,t)$, ${}^{(n-1)}F_{ij}(s,t)$ *are defined as follows:*

$$
\begin{aligned}
&{}^{(n-1)}p_{ij}(s) = {}^{(n-1)}Q_{ij}(s,\infty),\\
&{}^{(n-1)}H_{ij}(s,t) = \sum_j {}^{(n-1)}Q_{ij}(s,t),\\
&{}^{(n-1)}F_{ij}(s,t) = \begin{cases} U_1(s)U_1(t), & {}^{(n-1)}p_{ij}(s) = 0,\\ \dfrac{{}^{(n-1)}Q_{ij}(s,t)}{{}^{(n-1)}p_{ij}(s)}, & {}^{(n-1)}p_{ij}(s) > 0. \end{cases}
\end{aligned}
\tag{12.130}
$$

Working as in section 12.3, it is easy to prove that we still have the following probabilistic meaning:

$$
\begin{aligned}
&{}^{(n-1)}p_{ij}(s) = P(J_n = j \mid J_{n-1} = i, T_{n-1} = s),\\
&{}^{(n-1)}H_i(s,t) = P(X_n \le t - s \mid J_{n-1} = i, T_{n-1} = s)(= P(T_n \le t \mid J_{n-1} = i, T_{n-1} = s)),\\
&{}^{(n-1)}F_{ij}(s,t) = P(X_n \le t - s \mid J_{n-1} = i, J_n = j, T_{n-1} = s)\\
&\qquad (= P(T_n \le t \mid J_{n-1} = i, J_n = j, T_{n-1} = s)).
\end{aligned}
\tag{12.131}
$$

In matrix notation, using the element by element product (Scott product) defined as:

$$
\begin{aligned}
&\mathbf{A} \cdot \mathbf{B} = \left[a_{ij} b_{ij} \right],\\
&\mathbf{A} = \left[a_{ij} \right], \mathbf{B} = \left[b_{ij} \right],
\end{aligned}
\tag{12.132}
$$

we will write:

$$
\begin{aligned}
&{}^{(n-1)}\mathbf{F}(s,t) = \left[{}^{(n-1)}F_{ij}(s,t) \right],\\
&{}^{(n-1)}\mathbf{P}(s) = \left[{}^{(n-1)}p_{ij}(s) \right],\\
&{}^{(n-1)}\mathbf{Q}(s,t) = {}^{(n-1)}\mathbf{P}(s) \cdot {}^{(n-1)}\mathbf{F}(s,t).
\end{aligned}
\tag{12.133}
$$

We can now give the following definitions similar to the traditional or homogenous semi-Markov theory presented in section 12.5.

Definition 12.13 *The counting process* $(N(t), t \geq 0)$ *defined as*

$$N(t) = \sup_n \{n : T_n \leq t\} \tag{12.134}$$

is called the associated counting process with the CNHSM kernel **Q**.

Definition 12.14 *The process* $\left((J_n, T_n), n \geq 0\right)$ *is called a completely non-homogenous Markov additive process or Markov renewal process* (*CNHMAP or CNHMRP*).

Definition 12.15 *The process* $Z = (Z(t), t \geq 0)$ *defined as*

$$Z(t) = \begin{cases} J_{N(t)}, N(t) < \infty, \\ \theta, N(t) < \infty, \end{cases} \tag{12.135}$$

where θ *is a new state added to I, is called the completely non-homogenous semi-Markov process* (*CNHSMP*) *of kernel* **Q**.

Definition 12.16 *The random variable L defined as*

$$L = \inf \{t : Z(t) = \theta\} \tag{12.136}$$

is called the lifetime of the CNHSMP Z.

Definition 12.17 *The associated counting process* $(N(t), t \geq 0)$ *or the CNHSMP Z=* $(Z(t), t \geq 0)$ *of kernel* **Q** *is explosive if and only if*

$$L = \infty, a.s. \tag{12.137}$$

and non-explosive if and only if

$$L < \infty, a.s. \tag{12.138}$$

For very general counting processes, De Vylder and Haezendonck (1980) have given necessary and sufficient conditions for non-explosion. Here, in general, we always assume non-explosive processes.

For the two-dimensional process $((J_n, T_n), n \geq 0)$, we have the following result:

$$P\left(J_1=j,T_1\le t\middle|J_0=i,T_0=0\right)={}^{(0)}Q_{ij}(0,t)(=Q_{ij}^{(1)}(t)),$$
$$P\left(J_2=j,T_2\le t\middle|J_0=i,T_0=0\right)=\sum_k\int_0^t {}^{(1)}Q_{kj}(x,t)\,{}^{(0)}Q_{ij}(0,dx)(=Q_{ij}^{(2)}(t)), \tag{12.139}$$

and in general

$$P\left(J_n=j,T_n\le t\middle|J_0=i,T_0=0\right)=\sum_k\int_0^t {}^{(n-1)}Q_{kj}(x,t)\,{}^{(1)}Q_{ij}(dx)$$
$$=(Q_{ij}^{(n)}(t)),n>1). \tag{12.140}$$

Using matrix notation, we may write for two $m\times m$ matrices of mass functions $\mathbf{A}(t)$, $\mathbf{B}(t)$:

$$\int_0^t \mathbf{A}(t)d\mathbf{B}(t)=\left[\sum_{k=1}^n\int_0^t B_{kj}(z)dA_{ik}(z)\right], \tag{12.141}$$

and so relations (12.140) can be written in matrix form:

$$\mathbf{Q}^{(n)}(t)=\int_0^t \mathbf{Q}^{(n-1)}(z,t)d\mathbf{Q}^{(1)}(x),n>1,$$
with
$$\mathbf{Q}^{(n)}(t)=\left[Q_{ij}^{(n)}(t)\right],n\ge 1,$$
$$\mathbf{Q}^{(1)}(t)=\left[{}^{(0)}Q_{ij}(0,t)\right]. \tag{12.142}$$

In the particular class of traditional SMP, relation (12.142) gives the *n*-fold convolution of the SM kernel **Q**.

Another very important distribution is the marginal distribution of the *Z* process as it gives the state occupied by the system *S* at time *t*.

Let us introduce the following probabilities:

$${}^{(n)}\phi_{ij}(s,t)=P\left(Z(t)=j\middle|Z(0)=i,N(s-)<N(s),N(s)=n\right),i,j\in I,n\ge 0. \tag{12.143}$$

The conditioning means that $T_n=s$ and that there exists a transition at time *s* such that the new state occupied after the transition is *i*.

Clearly, these probabilities satisfy the following relations:

$$ {}^{(n)}\phi_{ij}(s,t) = \delta_{ij}(1 - {}^{(n)}H_i(s,t)) + \sum_{k\in I}\int_s^t {}^{(n)}\phi_{kj}(u,t)\,{}^{(n-1)}Q_{ik}(s,du), i,j \in I. \qquad (12.144) $$

From relation (12.127), it is clear that we have:

$$ P(J_n = j \,|\, (J_k, T_k), k \le n-1) = {}^{(n-1)}p_{J_{n-1}j}(T_{n-1}), a.s. \qquad (12.145) $$

It follows that the process $(J_n, n \ge 0)$ can be viewed as a *conditional multiple Markov chain*; this means that, given the sequence $(T_n, n \ge 0)$, each transition from $J_{n-1} \to J_n$ obeys a Markov chain of kernel ${}^{(n-1)}P(T_{n-1})$.

Definition 12.18 *The conditional multiple Markov chain* $(J_n, n \ge 0)$ *is called the embedded multiple MC.*

12.11.1.2. *Special cases*

Let us point out that Definition 12.18 is quite general as indeed it is non-homogenous both for the time *s* and for the number of transitions *n*, this last one giving the possibility to model *epidemiological* phenomena such as AIDS (see in Janssen and Manca (2006) the example of *Polya processes* and semi-Markov extensions).

This extreme generality gives importance to the following particular cases.

(i) Non-homogenous Markov additive process and semi-Markov process

Writing $\mathbf{Q}=\left({}^{(n-1)}\mathbf{Q}(s,t), n \ge 1\right)$, we have as first special case:

$$ {}^{(n-1)}\mathbf{Q}(s,t) = \mathbf{Q}(s,t), n \ge 1, s < t, \qquad (12.146) $$

that is **Q** independent of *n*, then the kernel **Q** is called a non-homogenous semi-Markov kernel (NHSMK) defining a non-homogenous Markov additive process (NHMAP) $((J_n, T_n), n \ge 0)$ and a non-homogenous semi-Markov process (NHSMP) $Z = (Z(t), t \ge 0)$.

This family was introduced in a different way by Hoem (1972).

It is clear that relation (12.146) means that the sequences

$$ {}^{(n-1)}\mathbf{F}(s,t) = \mathbf{F}(s,t),\ {}^{(n-1)}\mathbf{P}(s) = \mathbf{P}(s) \quad \forall n \geq 0 \tag{12.147} $$

are independent of n or equivalently that

$$ \mathbf{Q}(s,t) = \mathbf{P}(s) \cdot \mathbf{F}(s,t). \tag{12.148} $$

Let us point out that, in this case, relation (12.144) becomes:

$$ \phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_k \int_s^t \phi_{ij}(u,t) Q_{ik}(s,du), i,j \in I. \tag{12.149} $$

If moreover, we have

$$ \mathbf{P}(s) = \mathbf{P}, s \geq 0, \tag{12.150} $$

then the kernel **Q** is called a *partially non-homogenous semi-Markov kernel* (PNHSMK) defining a *partially non-homogenous Markov additive process* (PNHMAP) $\left((J_n, T_n), n \geq 0\right)$ and a partially non-homogenous semi-Markov process (PNHSMP) $Z = (Z(t), t \geq 0)$.

This family was introduced in a different way by Hoem (1972).

(ii) Non-homogenous MC

If the sequences ${}^{(n-1)}\mathbf{P}(s), \forall s \geq 0$ are independent of *s*, then $\left(J_n, n \geq 0\right)$ is a traditional NHMC.

Homogenous Markov additive process

A PNHSMK **Q** such that

$$ F(s,t) = F(t-s), s,t \geq 0, t-s \geq 0, \tag{12.151} $$

is of course a traditional homogenous SM kernel as in section 12.2.

(iii) Non-homogenous renewal process

For m=1, the CNHMRP of kernel **Q** is given by

$$ \mathbf{Q}(s,t) = ({}^{(n-1)}\mathbf{F}(s,t)), s,t > 0, t-s \geq 0 \tag{12.152} $$

and characterizes the sequence $(X_n, n \geq 0)$ with, as in (12.127),

$$\begin{aligned} &X_0 = 0, a.s., \\ &P(X_n \le x \big| X_k, k \le n-1) = {}^{(n-1)}F(T_{n-1}, T_{n-1} + x), x \in \mathbb{R}^+, \\ &T_0 = 0, T_n = \sum_{k=0}^{n} X_k, a.s. \end{aligned} \tag{12.153}$$

In this case, the process $(X_n, n \ge 0)$ is called a *completely non-homogenous dependent renewal process* (CNHDRP) of kernel **Q**.

If, moreover,

$$^{(n-1)}F(s,t) = {}^{(n-1)}F(t-s), s,t > 0, t-s \ge 0, n \ge 1, \tag{12.154}$$

it follows that

$$\begin{aligned} &X_0 = 0, a.s., \\ &P(X_n \le x \big| X_k, k \le n-1) = {}^{(n-1)}F(x), x \in \mathbb{R}^+, n \ge 1 \end{aligned} \tag{12.155}$$

and so the process $(X_n, n \ge 0)$ is a sequence of independent t r.v. called a *completely non-homogenous renewal process* (CNHRP) of kernel F.

Remark 12.9 In the non-homogenous case, it is much more difficult to obtain asymptotic results (see, for example, Benevento (1986) and Thorisson (1986) for interesting theoretical results). That is not so dramatic as we can say that non-homogenous models are used for modeling *transient* situations and not *asymptotic* ones, and that is why we personally think that all attention must be given to the construction of numerical methods, for example, to be able to solve the non-homogenous integral equations system (12.149) (see Janssen and Manca (2007)).

However, let us mention that, for the particular case of non-homogenous Markov chains, there exist more asymptotic results (see, for example, Isaacson and Madsen (1976)).

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Chapter 13

Stochastic or Itô Calculus

This chapter presents the basic results concerning the *Itô calculus* also called *stochastic calculus*, one of the main tools used in stochastic finance particularly for building stochastic models used in option theory, developed in Chapter 14 and in bond evaluation, developed in Chapter 15.

13.1. Problem of stochastic integration

In traditional analysis, it is well known that the Riemann-Stieltjes integral noted

$$\int_a^b f d\alpha \tag{13.1}$$

is well defined if for example f is continuous and α of bounded variation on $[a,b]$, or inversely if α is continuous and f of bounded variation on $[a,b]$. From integration by parts, we obtain:

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df. \tag{13.2}$$

Let us work now on a filtered probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ on which we define two adapted stochastic processes:

$$f = \left(f(t), t \geq 0\right), X = \left(X(t), t \geq 0\right) \tag{13.3}$$

where the process f has its trajectories a.s. of bounded variation and the process X has its trajectories a.s. continuous on $[0,t]$.

For each trajectory ω, it is still possible to integrate "à la Riemann-Stieltjes" to obtain a new random variable Y

$$Y = \int_0^t f(s)dX(s) \tag{13.4}$$

or

$$Y(w) = \int_0^t f(s,\omega)dX(s,\omega) . \tag{13.5}$$

The process f is called the *integrand* process and the process X the *integrator* process.

So, if process f has its trajectories a.s. of bounded variation and process X has its trajectories a.s. continued on $[0,T]$, the stochastic process $Y = (Y(t), t \in [0,T])$ is also represented by $\int fdX$ or:

$$\int fdX = \left\{ \int_0^t f(s,\omega)dX(s,\omega), t \in [0,T] \right\} \tag{13.6}$$

Nevertheless, this approach of stochastic integration is completely unsatisfactory if, for example, we are considering a standard Brownian motion, as defined in Chapter 10, $W = (W(t), t \geq 0)$ as indeed, we cannot define the following integral

$$\int_0^t W(s,\omega)dW(s,\omega) \tag{13.7}$$

as these trajectories of a Brownian motion are p.s. not of bounded variation on any interval [*0, t*]. That is why it is necessary to construct a new theory of integration called *the stochastic* or *Itô integration*.

In particular, we will see that in this new theory, the "natural" result in traditional analysis:

$$\int_0^t W(s,\omega)dW(s,\omega) = \frac{1}{2}\left[W(t,\omega)\right]^2 ; \tag{13.8}$$

is here false!

More generally, the traditional formula of derivation and differentiation will no longer be systematically true.

13.2. Stochastic integration of simple predictable processes and semi-martingales

Let $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ be a filtered complete probability space, T a stopping time and $\Im_T$ the σ-algebra of all the events anterior to T and introduce the following definitions.

Definition 13.1 *A stochastic process*

$$H = \left(H_t, t \geq 0\right) \tag{13.9}$$

is predictable simple if $H = \left(H_t, t \geq 0\right)$ *if:*

(i) $H_t = H_{-1}1_{\{0\}}(t) + \sum_{i=0}^{n} H_i 1_{\left(T_i, T_{i+1}\right]}$,

(ii) $T_0 = 0, \left(T_i, i = 1,...,n\right)$ is an increasing sequence of a.s. finite stopping times, (13.10)

(iii) $\forall i = 1,...,n: \ \left|H_i\right| < \infty, p.s., \ \mathrm{H}_i \in \Im_{t_i}$.

Definition 13.2 *On* $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$*, the set of all predictable simple stochastic processes is called* S *and* S_u *if it is topologized with the uniform convergence in* (t,ω).

The basic idea is to define, with a given integrator process *X*, the stochastic integral process noted $\int_0^\infty HdX$ of a simple predictable process *H*

$$\int_0^\infty HdX = \int_0^\infty H_s(\omega)dX_s(\omega) \tag{13.11}$$

eventually with the completion of *H* for $t > T_{n+1}$ as

$$H_t(\omega) = 0, p.s. \ \forall t > T_{n+1} \tag{13.12}$$

as the *operator* $I_X : S \to L^0$, this last set being the set of all r.v. with the convergence in probability, defined by relation (13.10) such that this operator has the following properties:

(i) I_X is linear:

$$H_1, H_2 \in S : \int_0^\infty (H_1 + H_2)dX = \int_0^\infty H_1 dX + \int_0^\infty H_2 dX, \tag{13.13}$$

(ii) I_X is continuous:

$$\left(H_n\right) \overset{c.u.}{\to} H \Rightarrow \int_0^\infty H_n dX \overset{c.pr.}{\to} \int_0^\infty HdX . \tag{13.14}$$

We see that the continuity property is well related to the two modes of convergence introduced before: the uniform convergence on *S* and the convergence in probability on L^0.

To define now the operator I_X for simple predictable processes, we will follow the traditional definition as follows.

Definition 13.3 *The operator* $I_X : S \to L^0$*, is defined as follows:*

$$I_X(H) = H_{-1}1\{0\} + \sum_{i=0}^{n} H_i(X_{T_{i+1}} - X_{T_i}) . \tag{13.15}$$

The new problem now is to see what the "good" integrator processes are so that this definition has a meaning and satisfies properties (13.13) and (13.14).

As from Definition 13.3, it is clear that the linearity property is always fulfilled for simple predictable processes. To see for what classes of process it remains true, it suffices to obtain property (13.14), justifying the introduction of a large class of stochastic processes called *semi-martingales*.

Definition 13.4 *The stochastic process X is a total semi-martingale if:*

(i) X is càdlàg;

(ii) X is adapted;

(iii) *operator* $I_X : S \to L^0$ is continuous.

For the restriction of the integration on the interval [*0,t*], we give the next definition.

Definition 13.5 *The stochastic process X is a total semi-martingale if for all* $t \in [0,\infty)$, *the stopped process at t,* X^t *defined by*

$$X_s^t = \begin{cases} X_s, s < t, \\ X_t, s \geq t. \end{cases} \tag{13.16}$$

is a total semi-martingale.

It is now possible to prove that this class of stochastic processes is good enough for stochastic integration with the following theorem proved by Protter (1990).

Proposition 13.1

(i) Every adapted càdlàg process of bounded variation on all compacts is a semi-martingale.

(ii) Every càdlàg square integrable martingale g is a semi-martingale.

(iii) Every standard Brownian motion is a semi-martingale.

Proof Let us prove (ii) and (iii).

(ii) From Definition 13.3 and relation (13.15), we obtain:

$$E\left[\left(I_X(H)\right)^2\right] = E\left[\left(\sum_{i=0}^{n} H_i\left(X_{T_{i+1}} - X_{T_i}\right)\right)^2\right]. \tag{13.17}$$

As the double products have a zero expectation, we obtain:

$$E\left[\left(I_X(H)\right)^2\right]=E\left[\left(\sum_{i=0}^{n}H_i^2\left(X_{T_{i+1}}-X_{T_i}\right)^2\right)\right] \tag{13.18}$$

and so

$$E\left[\left(I_X(H)\right)^2\right]\leq \sup_i\left|H_i^2\right|E\left[\left(\sum_{i=0}^{n}\left(X_{T_{i+1}}-X_{T_i}\right)^2\right)\right]. \tag{13.19}$$

Using the smoothing property of conditional expectation and the stopping time theorem of Doob (see Chapter 10), we can successively write:

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[E\left[X_{T_i}X_{T_{i+1}}\right]\middle|\Im_{T_i}\right], \tag{13.20}$$

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[X_{T_i}E\left[X_{T_{i+1}}\right]\middle|\Im_{T_i}\right], \tag{13.21}$$

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[X_{T_i}{}^2\right], \tag{13.22}$$

and so from relation (13.15):

$$E\left[\left(I_X(H)\right)^2\right]\leq \sup_i\left|H_i^2\right|\sum_{i=0}^{n}\left(E\left[X_{T_{i+1}}^2\right]+E\left[X_{T_i}^2\right]-2E\left[X_{T_i}^2\right]\right) \tag{13.23}$$

or:

$$E\left[\left(I_X(H)\right)^2\right]\leq \sup_i\left|H_i^2\right|\sum_{i=0}^{n}\left(E\left[X_{T_{i+1}}^2\right]-E\left[X_{T_i}^2\right]\right). \tag{13.24}$$

This last result finally gives:

$$E\left[\left(I_X(H)\right)^2\right]\leq \sup_i\left|H_i^2\right|\left(E\left[X_{T_{i+1}}^2\right]-E\left[X_0^2\right]\right) \tag{13.25}$$

which proves the continuity property of operator I_X.

(iii) This result is a direct consequence of the property that every standard Brownian motion is a square integrable and càdlàg martingale (see Chapter 10) with trajectories a.s. continuous.

13.3. General definition of the stochastic integral

Let us now go to the last step of stochastic integration, that is, to define this concept for more general processes than the predictable simple processes. To do so, we must introduce a class of stochastic processes we can obtain using an adequate convergence using a technique similar to the construction of real numbers from the rational numbers or the construction of the integral of measurable functions starting from the integral of simple functions.

The basic idea, fully developed in Protter (1990), is always the same one. Firstly, we define a larger class of integrable functions on which the initial class is *dense*. Secondly, we approach each element of the new class with a sequence of elements of the initial class using an *adequate* mode of convergence, i.e. the punctual convergence in number theory, the uniform convergence in traditional integration and here the *uniform convergence in probability on every compact set.*

Definition 13.6 *(The uniform convergence in probability on every compact set) A sequence of stochastic processes* $\left(H^n, n \geq 1\right)$ *where* $H^n = \left(H_t^n, t > 0\right)$ *converges uniformly in probability on the compacts towards the process* $H = \left(H_t, t \geq 0\right)$ *if, for all t>0, we have:*

$$\sup_{0 \leq s \leq t} \left| H_n^s - H_s \right| \xrightarrow{pr} 0 . \tag{13.26}$$

So, we now have four basic spaces of topologized stochastic processes:

D: the space of càdlàg simple adapted processes;

L: the space of adapted càdlàg processes;

S_u : the space of predictable simple processes with the uniform convergence;

L^0 : the space of finite random variables with the convergence in probability.

The spaces of stochastic processes *D*, *L* and *S* with the *uniform* convergence in probability on the compacts are noted respectively $D_{ucp}, L_{ucp}, S_{ucp}$.

We have now the following result.

Proposition 13.2 (Protter (1990)) *With the uniform convergence in probability on the compacts, space S of predictable simple processes is dense on L.*

This result leads to the extension of the definition of stochastic integral from *S* to *L*.

Firstly, let us recall that the application

$$I_X : S_u \mapsto L^0 \tag{13.27}$$

defined from relation (13.10) is written in the following form:

$$I_X(H) = \int_0^\infty H_s dX_s \tag{13.28}$$

and with the stopped process X^t:

$$I_{X^t}(H) = \int_0^t H_s dX_s \tag{13.29}$$

For a given stochastic process *H,* this last relation defines a new stochastic process J_X:

$$J_X(H)_t = I_{X^t}(H) \tag{13.30}$$

such that for each process $H = \left(H_t, t \geq 0\right)$, the corresponding associated process is $\left(\int_0^t H_s dX_s,\, t \geq 0\right)$ and so

$$J_X(H))_t = I_{X^t}(H)\,. \tag{13.31}$$

Protter (1990) proved the two following propositions.

Proposition 13.3 *If process X is a semi-martingale, then the application*

$$J_X : S_{ucp} \mapsto D_{ucp} \tag{13.32}$$

is continuous.

Proposition 13.4 *The continuous linear operator* $J_X : S_{ucp} \mapsto D_{ucp}$ *can be extended to a continuous linear operator* $J_X : L_{ucp} \mapsto D_{ucp}$.

This last proposition is a special case of the fundamental result that every linear operator on a sub-vector space can be extended in a unique way to the whole vector space.

Definition 13.7 *If X is a semi-martingale, the continuous linear application:*

$$J_X : L_{ucp} \mapsto D_{ucp} \tag{13.33}$$

is called a stochastic integral.

Of course, we will use the same notations as for simple processes:

$$I_X(H) = \int_0^{\infty} H_s dX_s \tag{13.34}$$

$$I_{X^t}(H) = \int_0^t H_s dX_s,$$

$$(J_X(H))_t = I_{X^t}(H) \tag{13.35}$$

Thus, the main conclusion is that it is possible to define the stochastic integral on $[0,t]$ *for every adapted càdlàg process as integrand process and for every semi-martingale integrator process*.

Example 13.1 Let us consider a standard Brownian motion $B = (B_t \geq 0)$ on the filtered probability space $(\Omega, \Im, (\Im_t, t \geq 0), P)$.

From Proposition 13.1, process B is a semi-martingale and moreover continuous (see Chapter 10); it follows that the following stochastic integral $\int_0^t B_s dB_s$ exists.

To calculate its value, let us introduce the following sequence of nested partitions $(\Pi_n, n \geq 1)$ of [0,*t*] such that the sequence of these norms $(\nu_n, n \geq 1)$ tends to 0.

For every partition Π_n, we introduce the following simple function B^n defined as follows:

$$B_s^n = \sum_{k=0}^{n-1} B_{t_k} 1_{(t_k, t_{k+1}]}, \tag{13.36}$$

with

$$\Pi_n = (t_0, ..., t_k, ..., t_n), t_0 = 0, t_n = t. \tag{13.37}$$

From the definition of the stochastic integral of simple functions, we obtain:

$$\int_0^t B_s^n dB_s = \sum_{k=0}^{n-1} B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right). \tag{13.38}$$

Using the theorem of the approximation of every continuous function by a uniformly convergent sequence of step functions, we have on $[0,t]$:

$$B^n \overset{ucp}{\rightarrow} B \tag{13.39}$$

and so:

$$\int_0^t B_s dB_s = \lim_{\nu_n \to 0} \sum_{k=0}^{n-1} B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right). \tag{13.40}$$

As

$$\sum_{k=0}^{n-1} B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right) = \frac{1}{2}\sum_{k=0}^{n-1}\left\{\left(B_{t_{k+1}} + B_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right) - \left(B_{t_{k+1}} - B_{t_k}\right)\right\}^2, \tag{13.41}$$

or even

$$\sum_{k=0}^{n-1} B_{k}\left(B_{k+1} - B_{k}\right) = \frac{1}{2}B_t^2 - \frac{1}{2}\sum_{k=0}^{n-1}\left(B_{k+1} - B_{k}\right)^2, \tag{13.42}$$

we obtain:

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}\lim_{\nu_n \to 0}\sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2 \tag{13.43}$$

The final result comes from the application of the next proposition showing that the second term of this last relation tends towards $t/2$ and so:

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t. \tag{13.44}$$

This last result illustrates well the fact that the traditional formula of differential analysis is, in general, no more true for the Itô calculus; here, in result (13.44), there is a supplementary term $-t/2$ with respect to the traditional formula, called the *drift*.

Let us now prove the previous result.

Proposition 13.5 *For any standard Brownian motion we have, with the convergence in probability:*

$$\lim_{\nu_n \to 0} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_{k+1}})^2 = t. \tag{13.45}$$

Proof With

$$\Pi_n = (t_0, ..., t_k, ..., t_n), t_0 = 0, t_n = t,$$

let us define:

$$\sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 = S_{n.}. \tag{13.46}$$

From the identity

$$S_n - t = \sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right], \tag{13.47}$$

we obtain:

$$E[S_n - t]^2 = E\left[\left(\sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right] \right)^2 \right], \tag{13.48}$$

using the property that a standard Brownian motion has independent increments (see Chapter 10):

$$E[(S_n - t)^2] = E\left[\sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right]^2 \right]. \tag{13.49}$$

Consequently, it follows that:

$$E[(S_n - t)^2] = E\left[\sum_{k=0}^{n-1} \left[\left(\frac{B_{t_{k+1}} - B_{t_{k+1}}}{\sqrt{t_{k+1} - t_k}} \right)^2 - 1 \right]^2 (t_{k+1} - t_k)^2 \right]. \tag{13.50}$$

Let us now introduce the r.v.

$$Y_k = \frac{B_{t_{k+1}} - B_{t_k}}{\sqrt{t_{k+1} - t_k}} \tag{13.51}$$

having a $N(0,1)$ distribution (see Chapter 1) to write relation (13.50) in the form:

$$\begin{aligned} E[(S_n - t)^2] &= E\left[\sum_{k=0}^{n-1}\left[Y_k^2 - 1\right]^2 (t_{k+1} - t_k)^2\right], \\ &= \sum_{k=0}^{n-1} E\left[Y_k^2 - 1\right]^2 (t_{k+1} - t_k)^2. \end{aligned} \tag{13.52}$$

As the r.v. Y_k have the same distribution, we also obtain:

$$E[(S_n - t)^2] = E\left[(Y_1^2 - 1)^2\right]\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2. \tag{13.53}$$

From the following inequality:

$$\sum_{k=0}^{n-1}\left(t_{k+1} - t_k\right)^2 \le (b-a)\nu_n \tag{13.54}$$

we obtain:

$$E[(S_n - t)^2] \le E\left[(Y_1^2 - 1)^2\right](b-a)\nu_n, \tag{13.55}$$

and so the result for $\nu_n \to 0$. □

Remark 13.1 The last proposition also shows that effectively the trajectories of a standard Brownian motion are not, a.s., of bounded variation on any compact of the real set.

Indeed, from the a.s. continuity of the trajectories on $[0,t]$, there is on this interval a subdivision $\Pi_n = \left(t_0, \ldots, t_k, \ldots, t_n\right), t_0 = 0, t_n = t$ of sufficiently small norm such that:

$$\left|B_{t_{k+1}} - B_{t_{k+1}}\right| < 1, \forall k = 0, \ldots, n-1 \tag{13.56}$$

and so:

$$\sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2 \leq \sup_k \left|B_{t_{k+1}} - B_{t_k}\right| \sum_{k=0}^{n-1}\left|B_{t_{k+1}} - B_{t_k}\right|. \tag{13.57}$$

This last relation proves that if the trajectories of a standard Brownian motion were a.s. of bounded variation on $[0,t]$, then the first member will tend to 0, which is in contradiction with Proposition 13.5.

13.4. Itô's formula

The fact that the rules of traditional differential calculus are no longer true for stochastic calculus implies finding a new tool of differentiation and integration. This tool was created by Itô (1944) who proved a lemma called *Itô's lemma* whose main result is called Itô's *formula*.

This formula became a very important basic tool for stochastic calculus and particularly in stochastic finance.

13.4.1. *Quadratic variation of a semi-martingale*

Let us recall that we use the following notations:

$$\begin{aligned} \int_0^t H_s dX_s &= \int_{[0,t]} H_s dX_s, \\ \int_{0+}^t H_s dX_s &= \int_{(0,t]} H_s dX_s \end{aligned} \tag{13.58}$$

and so:

$$\int_0^t H_s dX_s = H_0 X_0 + \int_{0+}^t H_s dX_s. \tag{13.59}$$

Definition 13.8 *If X and Y are two semi-martingales, then*:

(i) the quadratic variation of X or bracket of X noted:

$$[X,X] = \left([X,X]_t, t \geq 0\right) \tag{13.60}$$

is the stochastic process

$$[X,X]_t = X_t^2 - 2\int_0^t X_{t\_}\, dX_s,$$
$$(X_{0\_} = 0), \tag{13.61}$$

(ii) the quadratic covariation process of X and Y or bracket of X and Y is the stochastic process noted

$$[X,Y] = \left([X,Y]_t, t \geq 0\right) \tag{13.62}$$

where

$$[X,Y]_t = X_t Y_t - \int_0^t X_{s-}\, dY_s - \int_0^t Y_{s-}\, dX_s. \tag{13.63}$$

Protter (1990) proved some interesting properties of these new processes and the most important ones for us are presented in the next proposition.

Proposition 13.6

(i) The process $[X,X]$ is càdlàg, non-decreasing and adapted.

(ii) The process $[X,Y]$ is càdlàg, t bilinear and symmetric and:

$$[X,Y]_t = \frac{1}{2}\left([X+Y, X+Y]_t - [X,X]_t - [Y,Y]_t\right). \tag{13.64}$$

(iii) For every sequence of partitions of stopping times:

$$T_0^n = 0, T_1^n, \ldots, T_k^n, \ldots, T_n^n = t, \tag{13.65}$$

if norm tends a.s. to 0, then:

$$X_0^2 + \sum_{k=0}^{n-1}\left(X^{T_{k+1}^n} - X^{T_k^n}\right)^2 \underset{ucp}{\rightarrow} [X,X]. \tag{13.66}$$

(iv) X and Y being two semi-martingale, so is the process $[X, X]$.

(v) Integration by parts asserts that:

$$X_t Y_t = \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X,Y]_t. \tag{13.67}$$

(vi) If it is a process of class D, then the jump process of Y, denoted $\Delta Y = (\Delta Y_t, t \geq 0)$, is defined as

$$\Delta Y_t = Y_t - Y_{t-}. \tag{13.68}$$

Then, for X=Y, we have:

$$\Delta[X,X]_t = (\Delta X_t)^2, \tag{13.69}$$

it follows the non-decreasing property of [X,X] and its decomposition in

$$[X,X]_t = [X,X]_t^c + \sum_{0 \leq s \leq t} (\Delta X_s)^2,$$

or (13.70)

$$[X,X]_t = [X,X]_t^c + X_0^2 + \sum_{0 < s \leq t} (\Delta X_s)^2,$$

the first term representing the "continuous" part of [X,X] .

Remark 13.2 From (ii) and Proposition (13.5), it follows that for every standard Brownian motion:

$$[B,B]_t = t. \tag{13.71}$$

13.4.2. *Itô's formula*

In traditional differential calculus, it is well-known that the *fundamental theorem* asserts that for any integrable function f on [0,t], we have:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt. \tag{13.72}$$

From stochastic calculus, the problem is as follows: with a semi-martingale process X as integrator process, we seek the additional term, if it exists, such that we can extend the preceding result (13.72) to obtain the following extension:

$$f(X_t) - f(X_0) = \int_{0+}^{t} f'(X_{s-})dX_s + \ldots. \tag{13.73}$$

For any function f of class $C_{\mathbb{R}}^2$, the solution is given by the two next propositions.

Proposition 13.7 (general Itô formula) *If X is a semi-martingale and f a function of class* $C^2_{\mathbb{R}}$*, then the composed process* $f(X)=\left(f(X_t),t\geq 0\right)$ *is also a semi-martingale and moreover:*

$$\begin{aligned}f(X_t)-f(X_0)&=\int_{0+}^{t} f'(X_{s-})dX_s+\frac{1}{2}\int_{0+}^{t} f'(X_{s-})d\left[X,X\right]_s^c\\&+\sum_{0<s\leq t}\left\{f(X_s)-f(X_{s-})-f'(X_s)\Delta X_s\right\}.\end{aligned} \tag{13.74}$$

Proposition 13.8 (Itô formula: continuous case) *If X is a continuous semi-martingale and f a function of class* $C^2_{\mathbb{R}}$*, then the composed process* $f(X)=\left(f(X_t),t\geq 0\right)$ *is also a semi-martingale and moreover:*

$$f(X_t)-f(X_0)=\int_{0+}^{t} f'(X_{s-})dX_s+\frac{1}{2}\int_{0+}^{t} f'(X_{s-})d\left[X,X\right]_s. \tag{13.75}$$

Proof Relation (13.75) is a direct consequence of result (13.74) as the continuity assumption on *X* implies that:

$$\forall s\geq 0: X_s=X_{s-}, \Delta X_s=0 \tag{13.76}$$

Remark 13.3 It is possible to show that (see Protter (1990)) the first supplementary term in the general Itô's formula is nothing other than:

$$\frac{1}{2}\left[f'(X),X\right]_t^c \tag{13.77}$$

and so we can put the Itô formula in the form:

$$\begin{aligned}f(X_t)-f(X_0)&=\int_{0+}^{t} f'(X_{s-})dX_s+\frac{1}{2}\left[f'(X),X\right]_t^c\\&+\sum_{0<s\leq t}\left\{f(X_s)-f(X_{s-})-f'(X_s)\Delta X_s\right\}.\end{aligned} \tag{13.78}$$

13.5. Stochastic integral with standard Brownian motion as integrator process

Main applications in finance begin with stochastic integrals with a standard Brownian motion as integrator process; thus, we will now particularize the general preceding results to this special case to obtain results that are more precise.

13.5.1. *Case of predictable simple processes*

On the probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$, let us consider:

– a simple predictable process defined on [0, *t*]:

$$H_s = H_k, t_k < s \leq t_{k+1}, k=0,...,n-1, \tag{13.79}$$

$\left(t_0 = 0, t_1, ..., t_n = t\right)$ being a partition of [0,*t*];

– *B*, a standard Brownian motion.

From the construction of the stochastic integral, we know that:

$$\int_0^t H_s dB_s = \sum_{k=0}^{n-1} H_k (B_{t_{k+1}} - B_{t_k}) . \tag{13.80}$$

Consequently, the mean and variance of the stochastic integral are given by:

(i) *mean*:

$$E\left[\int_0^t H_s dB_s\right] = \sum_{k=0}^{n-1} E\left[H_k \left(B_{t_{k+1}} - B_{t_k})\right)\right], \tag{13.81}$$

and as the process *H* is adapted and *B* with independent increments, we obtain:

$$E\left[\int_0^t H_s dB_s\right] = \sum_{k=0}^{n-1} E\left[H_k\right] E\left[B_{t_{k+1}} - B_{t_k}\right] \tag{13.82}$$

and finally:

$$E\left[\int_0^t H_s dB_s\right] = 0. \tag{13.83}$$

(ii) variance

As from result (13.83):

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = E\left[\left(\sum_{k=0}^{n-1} H_k (B_{t_{k+1}} - B_{t_k})\right)^2\right], \tag{13.84}$$

we obtain:

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = E\left[\left(\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} H_k H_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})\right)\right], \tag{13.85}$$

or:

$$\begin{aligned}\operatorname{var}\left(\int_0^t H_s dB_s\right) &= \sum_{k=0}^{n-1} E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] \\ &+2E\left[\left(\sum_{k<l} H_k H_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})\right)\right],\end{aligned} \tag{13.86}$$

using the “smoothing property” from Chapter 10, we obtain:

$$E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] = E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2 \middle| \Im_{t_k}\right], \tag{13.87}$$

and so from the fact that H is adapted to the given filtration and B with independent increments such that:

$$E\left[B_{t_{k+1}} - B_{t_k}\right] = t_{k+1} - t_k, \tag{13.88}$$

we obtain:

$$E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] = E\left[H_k^2\right](t_{k+1} - t_k),\ k = 0,\ldots,n-1. \tag{13.89}$$

Using analog reasoning, we also have that all the double products in relation (13.86) have a zero expectation so that finally:

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = \sum_{k=0}^{n-1} E\left[H_k^2\right](t_{k+1} - t_k). \tag{13.90}$$

To summarize, we have the following basic results:

$$\begin{aligned}&E\left[\int_0^t H_s dB_s\right] = \int_0^t H_s dE\left[B_s\right] = 0, \\ &\operatorname{var}\left[\int_0^t H_s dB_s\right] = E\left(\int_0^t H_s dB_s\right)^2 = E\left(\int_0^t H_s^2 ds\right) = \int_0^t E\left[H_s^2\right] ds.\end{aligned} \tag{13.91}$$

Similarly, we can prove the following proposition.

Proposition 13.9 *Under the above assumptions and if moreover the process H is square integrable, then the following process*

$$\left(\int_0^t H_s dB_s, t \geq 0\right) \tag{13.92}$$

is a square integrable $(\Im_t)$*-martingale with a.s. continuous trajectories and moreover the process*

$$\left(\left(\int_0^t H_s dB_s\right)^2 - \int_0^t H_s^2 ds, t \geq 0\right) \tag{13.93}$$

is a $(\Im_t)$*-martingale with a.s. continuous trajectories.*

Let us also mention the following property: *if X and Y are two simple predictable square integrable processes, then*

$$E\left[\int_0^t X_s dB_s \int_0^t Y_s dB_s\right] = E\left[\int_0^t X_s Y_s ds\right] = \int_0^t E\left[X_s Y_s\right] ds. \tag{13.94}$$

13.5.2. *Extension to general integrand processes*

As we know from the preceding section, we will use uniform convergence in probability to extend the preceding results to the class *D* of square integrable adapted càdlàg processes.

For such a process *X*, there exists a sequence adapted simple square integrable processes $(H_n, n \geq 0)$ ucp converging to *X* such that in particular:

$$\int_0^t X_s dB_s = \lim_{L^2} \int_0^t H_s^n dB_s. \tag{13.95}$$

From this result, it follows that all the properties of section 13.5.1 remain valid in this general case.

13.6. Stochastic differentiation

13.6.1. *Definition*

On the probability space $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),P\right)$, let us consider an adapted standard Brownian motion *B* and two sufficiently smooth adapted processes *a* and *b*.

Definition 13.9 *The stochastic process*

$$\xi=\left(\xi(t),t\geq 0\right) \tag{13.96}$$

has as stochastic differential on [0, *T*]

$$d\xi(t)=a(t)dt+b(t)dB(t) \tag{13.97}$$

if and only if:

$$\begin{aligned}&\forall t_1,t_2:0\leq t_1<t_2\leq T:\\&\xi(t_2)-\xi(t_1)=\int_{t_1}^{t_2}a(t)dt+\int_{t_1}^{t_2}b(t)dB(t).\end{aligned} \tag{13.98}$$

13.6.2. *Examples*

1) Result (13.44) gives:

$$\int_0^t B_s dB_s=\frac{1}{2}B_t^2-\frac{1}{2}t. \tag{13.99}$$

Consequently, we also have:

$$\int_{t_1}^{t_2} B_s dB_s=\frac{1}{2}(B_{t_2}^2-B_{t_1}^2)-\frac{1}{2}(t_2-t_1) \tag{13.100}$$

and from our new definition, it follows that:

$$dB_t^2=dt+2B_t dB_t. \tag{13.101}$$

2) From the definition of the stochastic integral, we know that:

$$\int_{t_1}^{t_2} t dB_t=\lim_n\sum_{k=1}^{n-1}t_{n,k}\left[B_{t_{n,k+1}}-B_{t_{n,k}}\right], \tag{13.102}$$

$\left(t_{n,1}=t_1,\ldots,t_{n,k},\ldots,t_{n,n}=t_2\right)$ being a subdivision of order *n* of the interval $\left[t_1,t_2\right]$.

Moreover, from the definition of the traditional Lebesgue integral, we obtain:

$$\int_{t_1}^{t_2} B_t dt = \lim_n \sum_{k=0}^{n-1} B_{t_{n,k+1}}(t_{n,k+1}-t_{n,k}). \tag{13.103}$$

Adding member-to-member relations (13.102) and (13.103), we obtain:

$$\int_{t_1}^{t_2} B_t dt + \int_{t_1}^{t_2} tdB_t + = \lim_n \sum_{k=1}^{n-1}\left[t_{n,k+1}B_{t_{n,k+1}} - t_{n,k}B_{t_{n,k}}\right] \tag{13.104}$$

and so:

$$\int_{t_1}^{t_2} B_t dt + \int_{t_1}^{t_2} tdB_t = t_2 B_{t_2} - t_1 B_{t_1} \tag{13.105}$$

or in terms of stochastic differential:

$$d(tB_t) = B_t dt + tdB_t, \tag{13.106}$$

this formula also being different from the one of the traditional calculus.

13.7. Back to Itô’s formula

Using now the concept of stochastic differential, we will have a supplementary look to Itô’s formula.

13.7.1. *Stochastic differential of a product*

On the probability space $\left(\Omega,\Im,\left(\Im_t, t\geq 0\right),P\right)$, let us consider an adapted standard Brownian motion *B* and four càdlàg adapted processes a_1, a_2, b_1, b_2 of class *D* and sufficiently smooth defining the two following stochastic differentials:

$$d\xi_i(t) = a_i(t)dt + b_i(t)dB(t), i = 1,2. \tag{13.107}$$

Then, we have as next result.

Proposition 13.10 (A. Friedman (1975)) *The process* $\xi_1\xi_2$ *is differentiable (in Itô's sense) and*

$$d\left(\xi_1(t)\xi_2(t)\right) = \xi_1(t)d\xi_2(t) + \xi_2(t)d\xi_1(t) + b_1(t)b_2(t)dt. \tag{13.108}$$

Examples

1) With $\xi_1(t) = \xi_2(t) = B(t)$, we find back this known result (see relation (13.101)):

$$d\left(B^2(t)\right) = 2B(t)dB(t) + dt \; . \tag{13.109}$$

2) Similarly, we can find result (13.106) concerning

$$d(tB(t)) = tdB(t) + B(t)dt, \tag{13.110}$$

with

$$\begin{aligned} &\xi_1(t) = t \Rightarrow a_1(t) = 1, b_1(t) = 0, \\ &\xi_2(t) = B(t) \Rightarrow a_1(t) = 0, b_1(t) = 1. \end{aligned} \tag{13.111}$$

13.7.2. *Itô's formula with time dependence*

For our applications, the main result is *Itô's lemma* or the *Itô formula*, which is equivalent to the rule of derivatives for composed functions in traditional differential calculus, but now with a function *f* of two variables.

Starting with

$$d\xi(t) = a(t)dt + b(t)dB(t), \tag{13.112}$$

let *f* be a function of two non-negative real variables *x, t* such that

$$f \in C^0_{\mathbb{R}\times\mathbb{R}^+}, f_x, f_{xx}, f_t \in C^0_{\mathbb{R}\times\mathbb{R}^+}. \tag{13.113}$$

Then *the composed stochastic process*

$$\left(f(\xi(t),t), t \geq 0\right) \tag{13.114}$$

is also Itô differentiable and its stochastic differential is given by:

$$d\left(f(\xi(t),t)\right)=$$
$$\left[\frac{\partial f}{\partial x}(\xi(t),t)a(t)+\frac{\partial f}{\partial t}(\xi(t),t)+\frac{1}{2}\frac{\partial^2}{\partial^2 x^2}f(\xi(t),t)b^2(t)\right]dt \tag{13.115}$$
$$+\frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t).$$

Remark 13.4 Compared with traditional differential calculus, we know that in this case, we should have:

$$d\left(f(\xi(t),t)\right)=\left[\frac{\partial f}{\partial x}(\xi(t),t)a(t)+\frac{\partial f}{\partial t}(\xi(t),t)\right]dt$$
$$+\frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t). \tag{13.116}$$

Therefore, the difference between relations (13.115) and (13.116) is the *supplementary term*

$$\frac{1}{2}\frac{\partial^2}{\partial^2 x^2}f(\xi(t),t)b^2(t) \tag{13.117}$$

appearing in (13.115) and which is zero if and only if in two cases:

1) f is a linear function of x,

2) b is identically equal to 0.

Example 13.1

1) For ξ given by:

$$d\xi(t)=dB(t),$$
$$\xi(0)=0. \tag{13.118}$$

Using notation (13.112), we obtain:

$$a(t) = 0,\ b(t) = 1. \tag{13.119}$$

With the aid of Itô's formula, the value of $de^{B(t)}$ is thus given by

$$de^{B(t)} = \frac{1}{2}e^{B(t)}dt + e^{B(t)}dB(t). \tag{13.120}$$

As we can see, the first term is the supplementary term with respect to the traditional formula and is called the *drift*.

13.7.3. *Interpretation of Itô's formula*

Itô's formula simply means that the composed stochastic process

$$\left((f(\xi(t),t) - f(\xi(0),0),t \geq 0\right) \tag{13.121}$$

is stochastically equivalent to the following stochastic process:

$$\begin{aligned}&\left(\int_0^t \left[f_t(\xi(s),s)ds + f_x(\xi(s),s)a(s) + \frac{1}{2}f_{xx}(\xi(s),s)b^2(s)\right]ds\right)\\&+\int_0^t f_x(\xi(s),s)b(s)dB(s),t \geq 0).\end{aligned} \tag{13.122}$$

13.7.4. *Other extensions of Itô's formula*

13.7.4.1. *First extension*

It is possible to extend Itô's formula in the following way. Let $\boldsymbol{\xi} = \left(\boldsymbol{\xi}(t),t \geq 0\right)$ be an m-dimensional stochastic process:

$$\boldsymbol{\xi}(t) = \left(\xi_1(t),\ldots,\xi_n(t)\right)' \tag{13.123}$$

with every component having a stochastic differential given by:

$$d\xi_i(t) = a_i(t)dt + b_i(t)dB(t), i = 1,\ldots,m\ . \tag{13.124}$$

Then, it can be shown that the stochastic differential of the one-dimensional process:

$$\left(f\left(\boldsymbol{\xi}(t),t\right),t \geq 0\right), \tag{13.125}$$

with f being a real function of m+1 variables:

$$f(\mathbf{x},t)=f(x_1,\ldots,x_n,t) \tag{13.126}$$

satisfying the following assumptions:

$$f\in C^0_{\mathbb{R}^m\times\mathbb{R}^+}, f_{x_i}, i=1,...,m, f_{x_ix_j}, i,j=1,...,m, f_t\in C^0_{\mathbb{R}^m\times\mathbb{R}^+} \tag{13.127}$$

exists and is given by

$$\begin{aligned}&d\left(f\left(\xi(t),t\right)\right)=\\&\left[\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\xi(t),t\right)a_i(t)+\frac{\partial f}{\partial t}\left(\xi(t),t\right)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}f\left(\xi(t),t\right)b_i(t)b_j(t)\right]dt\\&+\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\xi(t),t\right)b_i(t)dB(t)\end{aligned} \tag{13.128}$$

Here, the supplementary time is given by

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}f\left(\xi(t),t\right)b_i(t)b_j(t) \tag{13.129}$$

13.7.4.2. *Second extension*

The second possible extension also starts with an m-dimensional stochastic process $\xi(t)=\left(\xi_1(t),\ldots,\xi_n(t)\right)'$ such that its dynamics are governed by the following stochastic differential:

$$d\xi(t)=\mathbf{a}(t)dt+\mathbf{b}(t)d\mathbf{B}(t), i=1,\ldots,m \tag{13.130}$$

a being a m-dimensional random vector of class L or D and **b** a stochastic matrix *mxn* whose elements are stochastic processes of class L and **B** a n-vector of n independent standard Brownian motions.

As in the preceding section, we are interested in the stochastic differential of the one-dimensional process:

$$\left(f\left(\xi(t),t\right),t\geq 0\right), \tag{13.131}$$

with f being a real function of m+1 variables:

$$f(\mathbf{x},t)=f(x_1,\ldots,x_n,t) \tag{13.132}$$

satisfying the following assumptions:

$$f\in C^0_{\mathbb{R}^m\times\mathbb{R}^+}, f_{x_i}, i=1,\ldots,m, f_{x_ix_j}, i,j=1,\ldots,m, f_t\in C^0_{\mathbb{R}^m\times\mathbb{R}^+}. \tag{13.133}$$

Under these assumptions, it is still possible to show the composed stochastic process $\left(f\left(\xi(t),t\right),t\geq 0\right)$ is Itô differentiable and that its stochastic differential is given by:

$$\begin{aligned}
&d\left(f\left(\xi(t),t\right)\right)=\\
&\left[\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\xi(t),t\right)a_i(t)+\frac{\partial f}{\partial t}\left(\xi(t),t\right)+\frac{1}{2}\sum_{i,j=1}^{n}\sigma_{ij}(t)\frac{\partial^2}{\partial x_i\partial x_j}f\left(\xi(t),t\right)\right]dt\\
&+\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_i}\left(\xi(t),t\right)b_{ij}(t)dB_j(t)\\
&\sigma_{ij}(t)=\frac{1}{2}\left(bb'(t)\right)_{ij}
\end{aligned} \tag{13.134}$$

Using matrix notation, we can rewrite this last expression in the form:

$$\begin{aligned}
&d\left(f\left(\xi(t),t\right)\right)=\frac{\partial f}{\partial t}\left(\xi(t),t\right)dt+\mathrm{grad}f(t)d\xi(t)+\frac{1}{2}\mathrm{tr}(\mathbf{bb}')(t)\mathbf{f}_{\mathbf{xx}}(t)dt,\\
&\mathbf{f}_{\mathbf{xx}}(t)=\left[\frac{\partial^2 f}{\partial x_i\partial x_j}(t)\right].
\end{aligned} \tag{13.135}$$

Here, the supplementary time is given by

$$\frac{1}{2}\mathrm{tr}(\mathbf{bb}')(t)\mathbf{f}_{\mathbf{xx}}(t)dt \tag{13.136}$$

13.7.4.3. *Third extension*

The last extension we will present now is related to the case of vector **B** whose components are *n dependent* standard Brownian motions.

This means that:

$$\forall i,j,\forall s,t(s<t):E\left[(B_i(t)-B_i(s))(B_i(t)-B_i(s))\right]=\rho_{ij}(t-s). \tag{13.137}$$

The matrix $\mathbf{Q}=\left[\rho_{ij}\right]$ is called the *correlation matrix* of the vector Brownian motion $\mathbf{B}=\left(\mathbf{B}(t),t\geq 0\right)$.

If $\mathbf{Q}=\mathbf{I}$ and $\mathbf{B}(0)=0$. the vector Brownian motion $\mathbf{B}=\left(\mathbf{B}(t),t\geq 0\right)$ is called *standard*.

In the case of a n-dimensional Brownian motion and with the same assumptions of the function f as above, Itô's formula becomes:

$$\begin{gathered}
d\left(f\left(\xi(t),t\right)\right)=\frac{\partial f}{\partial t}\left(\xi(t),t\right)dt+\mathrm{grad}f(t)d\xi(t)+\frac{1}{2}\mathrm{tr}(\mathbf{bQb}')(t)\mathbf{f}_{\mathbf{xx}}(t)dt,\\
\mathbf{f}_{\mathbf{xx}}(t)=\left[\frac{\partial^2 f}{\partial x_i\partial x_j}(t)\right].
\end{gathered} \tag{13.138}$$

13.7.4.4. *Exercises*

1) Prove the following results:

$$\begin{gathered}
dB^n(t)=nB^{n-1}(t)dB(t)+\frac{1}{2}n(n-1)B^{n-2}(t)dt,\\
da^{B(t)}=a^{B(t)}\ln a dB+\frac{1}{2}a^{B(t)}\ln^2 a dt, a>0.
\end{gathered} \tag{13.139}$$

2) (i) Prove that:

$$\int_0^t sdB(s)=tB(t)-\int_0^t B(s)ds. \tag{13.140}$$

(ii) Generalize to the following case (partial validity of the traditional integration by parts formula)

$$\int_0^t f(s)dB(s)=f(t)B(t)-\int_0^t B(s)df(s), \tag{13.141}$$

f being a deterministic function with bounded variation.

3) Let **B** an *n*-dimensional standard Brownian motion and consider the following one-dimensional process

$$R = \left(R(t), t \geq 0\right),$$
$$R(t) = \sqrt{\sum_{k:=1}^{n} B_k^2(t)}. \tag{13.142}$$

called the *Bessel process of order n*.

Prove that:

$$dR = \frac{1}{R}\sum_{i=1}^{n} B_i(t)dB(t) + \frac{n-1}{2R}dt. \tag{13.143}$$

4) Calculate $E\left[e^{B(t)}\right]$.

Solution

The integral form of the Itô's formula leads to

$$e^{B(t)} - 1 = \int_0^t e^{B(t)}dB(s) + \frac{1}{2}\int_0^t e^{B(t)}ds. \tag{13.144}$$

Then, if:

$$X(t) = E\left[e^{B(t)}\right], \tag{13.145}$$

we get:

$$X(t) - 1 = \frac{1}{2}\int_0^t X(s)ds. \tag{13.146}$$

By derivation, we obtain:

$$X'(t) = \frac{1}{2}X(t). \tag{13.147}$$

Moreover, as $X(0)=1$, the traditional differential equation has as unique solution:

$$X(t) = e^{\frac{t}{2}}. \tag{13.148}$$

5) a and b being two deterministic functions of bounded variation, calculate the mean and the variance of the process X defined by

$$dX(t) = a(t)dt + b(t)dB(t), \tag{13.149}$$

B being a standard Brownian motion.

6) If the stochastic process $\lambda = (\lambda(t), t \geq 0)$ has the following stochastic differential:

$$d\lambda(t) = a(t)dt + b(t)dB(t), \tag{13.150}$$

calculate Itô's differential of $e^{\lambda(t)}$

Answer

$$de^{\lambda(t)} = e^{\lambda(t)}\left[\left(a(t) + \frac{1}{2}b^2(t)\right)dt + b(t)dB(t)\right]. \tag{13.151}$$

13.8. Stochastic differential equations

13.8.1. *Existence and unicity general theorem (Gikhman and Skorokhod (1969))*

The problem is, in the deterministic case, as follows: given the following stochastic differential:

$$\begin{aligned} &d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ &\xi(0) = \xi_0, a.s. \end{aligned} \tag{13.152}$$

$B = (B(t), t \geq 0)$ being a standard Brownian motion on the complete filtered probability space $(\Omega, \Im, (\Im_t, t \geq 0), P)$, find, if possible, a stochastic process

$$\xi = (\xi(t), t \in [0,T]) \tag{13.153}$$

satisfying in the interval $[0, T]$ relations (13.152), under minimal assumptions on the two functions μ, σ from $\mathbb{R} \times [0,T] \mapsto \mathbb{R}$.

Relation (13.152) is called a stochastic differential equation (SDE). Gikhman and Skorokhod (1969) proved a general *theorem of existence and unicity* also given, in a more modern form, by Protter (1990).

Under a relatively simple form, the main result is as follows.

Proposition 13.11 (general theorem of existence and unicity) *Let us consider the following SDE:*

$$\begin{aligned} &d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ &\xi(0) = \xi_0, a.s. \end{aligned} \tag{13.154}$$

under the following assumptions:

(i) the functions μ, σ are measurable functions from $\mathbb{R} \times [0,T] \mapsto \mathbb{R}$ verifying a Lipschitz condition in the first variable:

$$\begin{aligned} &\forall (x_1,t),(x_2,t) \in \mathbb{R} \times [0,T]: \\ &|\mu(x_1,t) - \mu(x_2,t)| \le \bar{K}|x_1 - x_2|, \\ &|\sigma(x_1,t) - \sigma(x_2,t)| \le \bar{K}|x_1 - x_2|, \end{aligned} \tag{13.155}$$

$\bar{K}$ *being a positve constant;*

(ii) *on* $\mathbb{R} \times [0,T]$, *the functions* μ, σ *are linearly bounded:*

$$|\mu(x,t)| \le K(1+|x|), |\sigma(x,t)| \le K(1+|x|), \tag{13.156}$$

K being a positve constant;

(iii) *the r.v.* ξ_0 belongs to $L^2(\Omega, \Im, P)$ and is independent *of the* σ*-algebra* $\sigma(B(t), t \in [0,T])$, *then, there exists a solution belonging for all* $t \in [0,T]$, *to* $L^2(\Omega, \Im, P)$, *continuous and a.s. unique on* $[0,T]$.

Remark 13.5

1) The initial condition:

$$\xi(0) = x_0, \in R \tag{13.157}$$

naturally satisfies assumption (iii).

2) This theorem can be extended in the case of a SDE on $[s, s+T]$, with as initial condition:

$$\xi(s) = \xi_s, \tag{13.158}$$

the r.v. now independent of the σ-algebra $\sigma\left(B(s+\tau)-B(s),\tau\in[0,T]\right)$ and belonging to $L^2(\Omega,\Im,P)$.

3) It is also possible to prove that:

$$E\left[\sup_{[0,T]}|\xi(t)|^2\right]\leq C\left(1+E\left[\xi_0^2\right]\right), \tag{13.159}$$

C being a constant depending only on K and T.

In Proposition 8.1, the coefficients μ,σ are deterministic functions but it is possible to extend it in the stochastic case. Then, formally, we have:

$$\mu(x,t)=\mu(x,t,\omega),\sigma(x,t)=\sigma(x,t,\omega),\forall x\in\mathbb{R},\forall t\in[0,T]. \tag{13.160}$$

The initial condition (13.157) becomes:

$$\xi(0)=\varphi(0), \tag{13.161}$$

where

$$\varphi=\left(\varphi(t),t\in[0,T]\right) \tag{13.162}$$

is the given initial process.

The extension of Proposition 8.1 is now given.

Proposition 13.12 (case of random coefficients) *For the SDE:*

$$\begin{aligned}&d\xi(t)=d\varphi(t)+\mu(\xi(t),t)dt+\sigma(\xi(t),t)dB(t),\\&\xi(0)=\varphi(0),\end{aligned} \tag{13.163}$$

where:

(i) the processes μ,σ are measurable as functions from $\mathbb{R}\times[0,T]\times\Omega\mapsto\mathbb{R}$, adapted and lipschitzian in the first variable, i.e. with probability 1:

$$\begin{aligned}&\forall(x_1,t),(x_2,t)\in R\times[0,T]:\\&|\mu(x_1,t)-\mu(x_2,t)|\leq\bar{K}|x_1-x_2|,\\&|\sigma(x_1,t)-\sigma(x_2,t)|\leq\bar{K}|x_1-x_2|,\end{aligned} \tag{13.164}$$

$\overline{K}$ being a positive constant;

(ii) the processes μ,σ are measurable as functions from $\mathbb{R}\times[0,T]\times\Omega\mapsto\mathbb{R}$, satisfy a.s. the following condition:

$$|\mu(x,t)|^2+|\sigma(x,t)|^2\leq K^2(1+x^2), \tag{13.165}$$

K being a positive constant;

(iii) the process $\varphi=(\varphi(t),t\in[0,T])$ is of bounded variation, adapted and such that

$$E\left[\sup_{[0,T]}|\varphi(t)|^2\right]<\infty \tag{13.166}$$

then, there is a solution belonging for $t\in[0,T]$, to $L^2(\Omega,\Im,P)$; moreover, if ξ_1,ξ_2 are two solutions, then they are stochastically equivalent, i.e.:

$$P[\xi_1(t)=\xi_2(t)]=1,\forall t\in[0,T]. \tag{13.167}$$

Finally, if the process φ is continuous a.s. on $[0,T]$, then there exists a.s. unicity on $[0,T]$:

$$P\left[\sup_{[0,T]}\{t:|\xi_1(t)-\xi_2(t)|\}>0\right]=0. \tag{13.168}$$

Remark 13.6 This theorem can be extended in the case of a SDE on $[s,s+T]$.

The proofs of these two fundamental propositions use the method of *successive approximations* used in the deterministic case under the name of *Piccard method*: on $[0,T]$, we begin to use the following very rough approximation:

$$\xi_0(t)=\xi_0 \tag{13.169}$$

and, by induction, on constructs on $[0,T]$, the following sequence of stochastic processes $\xi_n=(\xi_n(t),n>0)$ is defined by

$$\xi_{n+1}(t)=\xi_0+\int_0^t\mu(\xi_n(s),s)ds+\int_0^t\sigma(\xi_n(s),s)dB(s). \tag{13.170}$$

Then, it is possible to show (see, for example, Friedman (1975)) that the sequence $\xi_n = (\xi_n(t), n > 0)$ converges uniformly a.s. on $[0,T]$ towards the stochastic process $\xi = (\xi(t), 0 \le t \le T)$, which is a solution of the considered SDE (13.163). Using assumption (13.164), Friedman (1975) also proved the a.s. unicity.

13.8.2. *Solution of stochastic differential equations*

Let us consider the following general SDE

$$\begin{aligned} d\xi(t) &= d\varphi(t) + \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ \xi(0) &= \varphi(0), \end{aligned} \tag{13.171}$$

where $B = (B(t), t \ge 0)$ is a standard Brownian motion *on* $(\Omega, \Im, (\Im_t, t \ge 0), P)$.

The general procedure to find the process $\xi = (\xi(t), t \in [0,T])$ solution of this SDE under the assumptions of Proposition 13.12 is to try to put this SDE in its *canonical form*, that is to say

$$\begin{aligned} d\xi(t) &= a(t)dt + b(t)dB(t), \\ \xi(0) &= \xi_0, \end{aligned} \tag{13.172}$$

with known a and b functions or stochastic processes. If so, the unique solution of the considered SDE takes the form:

$$\xi(t) = \xi_0 + \int_0^t a(s)ds + \int_0^t b(s)dB(s). \tag{13.173}$$

More generally, we can look for a transformation f in two variables x and t, monotone in t satisfying the assumptions of Itô's lemma and such that:

$$df(\xi(t),t) = \overline{A}(t)dt + \overline{B}(t)dB(t) \tag{13.174}$$

In this case, we obtain:

$$f(\xi(t),t) = f(\xi(0),0) + \int_0^t \overline{A}(s)ds + \int_0^t \overline{B}(s)dB(s) \tag{13.175}$$

where we find by inverse transformation in variable x the form of $\xi(t), t \in [0,T]$.

13.9. Diffusion processes

Let us consider the SDE:

$$d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t),\\ \xi(0) = \xi_0, \tag{13.176}$$

under the assumptions of Proposition 13.12.

The solution $\xi = \left(\xi(t), t \in [0,T]\right)$ of this SDE is called a *diffusion process or Itô process*.

Let *s* and *t* be such that: $0 \le s < t \le T$ and suppose that $\xi(s) = x$.

From the theorem of existence and unicity, we know that on the interval $[s,T]$ there exists only one process solution, noted $\xi_{x,s}$, of the SDE (13.176) such that

$$\xi_{x,s}(s) = x . \tag{13.177}$$

So it is clear that, setting $x = \xi(t)$, we have the Markov property for the ξ-process in continuous time, which is of course generally non-homogenous.

More precisely, we have the following propositions.

Proposition 13.13 *Under the assumptions of Proposition 13.12 and if, for each t,* $\Im_t$ *represents the* σ*-algebra generated by* ξ_0 *and the set* $\left(B(s), s \le t\right)$*, then the a.s. unique stochastic process solution of (13.176), satisfies a.s.:*

$$P\left[\xi(t) \in A \middle| \Im_s\right] = P\left[\xi(t) \in A \middle| \xi(s)\right] (= p(s,\xi(s),t,A)) \tag{13.178}$$

for all t>s and for all Borel set A.

Proposition 13.14 *The function of* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \beta \mapsto [0,1]$ *defined by relation (13.178) satisfies the following properties:*

(i) for all fixed *s, x, t, p*(*s, x, t*) is a probability measure on $\mathbb{R}$;

(ii) for all fixed *s, t, A, p*(*s, t, A*) is Borel-measurable;

(iii) the function *p* satisfies the *Chapman-Kolmogorov* equations:

$$\forall 0 \le s < t < \tau, x \in R, A \in \beta :\\ \int_R p(s,x,t,dy)p(t,y,\tau,A) = p(s,x,\tau,A). \tag{13.179}$$

(iv) the process solution $\xi = \left(\xi(s), s \geq 0\right)$ is a *Feller process*; i.e. for all continuous bounded function of $\mathbb{R} \mapsto \mathbb{R}$, the application

$$(s,x) \mapsto \int f(y)\, p(s,x,s+t,dy) \tag{13.180}$$

is continuous.

(v) the process solution $\xi = \left(\xi(s), s \geq 0\right)$ satisfies the *strong Markov property*, i.e. condition (13.178) but where *s* and *t* are replaced by stopping times.

Remark 13.7

a) If the drift and the diffusion coefficient are continuous functions, it can be shown that:

(i)

$$\begin{aligned} &\forall \varepsilon > 0, t \geq 0, x \in R: \\ &\lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|>\varepsilon} p(t,x,t+h,dy) = 0, \end{aligned} \tag{13.181}$$

(ii)

$$\begin{aligned} &\forall \varepsilon > 0, t \geq 0, x \in R: \\ &a) \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|<\varepsilon} (y-x)\, p(t,x,t+h,dy) = \mu(x,t), \\ &b) \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|<\varepsilon} (y-x)^2\, p(t,x,t+h,dy) = \sigma^2(x,t), \end{aligned} \tag{13.182}$$

For the applications of such processes in finance, it is interesting to give the interpretations of these last properties:

1) the probability for the process $\xi = \left(\xi(s), s \geq 0\right)$ to have a jump of amplitude more then ε between *t* and *t+h* is $o(h)$. Consequently, the process $\xi = \left(\xi(s), s \geq 0\right)$ is continuous in probability;

2) properties *a* and *b* can be rewritten as follows:

$$\begin{aligned} &a) E\left[\xi(t+h) - \xi(t) \middle| \xi(t) = x\right] = \mu(x,t)h + o(h), \\ &b) E\left[\left|\xi(t+h) - \xi(t)\right|^2 \middle| \xi(t) = x\right] = \sigma^2(x,t)h + o(h). \end{aligned} \tag{13.183}$$

Consequently, drift μ gives the rate of the conditional mean of the increment of the diffusion process on the infinitesimal time $(t,t+h)$ interval and the square of the diffusion coefficient of diffusion σ, the conditional variance of this increment as the square of the mean is of order $o(h)$.

b) If the function p has a density p', then it is a solution of *the partial differential equation of Fokker-Planck:*

$$\frac{\partial p'}{\partial t}+\frac{\partial}{\partial x}(\sigma(x,t)p')-\frac{1}{2}\frac{\partial^2}{\partial x^2}(\mu(x,t)p')=0. \tag{13.184}$$

Example 13.2 For the Ornstein-Uhlenbeck-Vasicek process defined by the SDE (see later in section 15.3.1)

$$\begin{aligned} &d\xi(t)=a(b-\xi(t))dt+\sigma dB(t),\\ &\xi(0)=\xi_0. \end{aligned} \tag{13.185}$$

it can be shown that:

$$p'(s,x,t,y)=\frac{1}{\sqrt{2\pi V_t}}e^{-\frac{1}{2V_t}(x-M_t)^2}, \tag{13.186}$$

M_t,V_t representing respectively the mean and variance of $\xi(t)$ whose explicit forms will be given in Chapter 15.

Chapter 14

Option Theory

14.1. Introduction

During the last 30 years, financial innovation has generalized the systematic use of new financial instruments called *derivative instruments* such as *options* and *swaps*, mainly used for hedging but also, sometimes, used as speculative tools. This matter is now essential in mathematical finance and will be fully developed here following the presentation of Janssen and Manca (2007).

However, we will also develop some main results concerning *exotic options* and foreign currency options with the presentation of the *Garman-Kohlagen formula* and some important results on *American options.*

The first basic derivative instruments are now called *plain vanilla options*: the two types of such options are now defined.

Definition 14.1 *A* call option (*respectively* put option) *is a contract giving the right to buy* (*respectively to sell*) *a financial asset*, *called an underlying asset, for a fixed price*, *called an exercise price*, *at the end of the contract time, called maturity time, also laid down in the contract.*

If we can exercise the option at any time before maturity, this type of option is said to be of an *American type*; if we can exercise it only at maturity, the option is said to be of a *European type*.

We will use the following notation: K for the exercise price, T for the maturity time and S for the value of the underlying asset at maturity.

The "gain" of the holder of a European option at maturity time T is represented by the following graph.

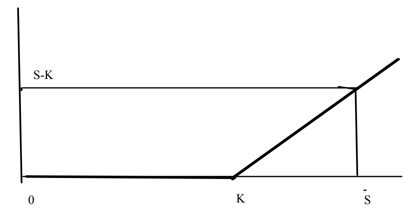

Figure 14.1. *Call option: holder's gain at maturity*

For the holder of a put, this graph becomes the following.

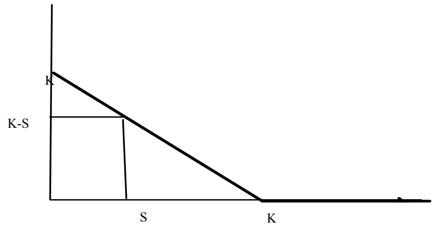

Figure 14.2. *Put option: holder's gain at maturity*

Of course, to obtain the "net gain", we must estimate the cost of the option, often called *option premium*, and furthermore *transaction costs* and *taxes*.

Let us represent respectively by C and P the premiums of call and put options.

So, we obtain, without taking into account transaction costs and taxes, the following two graphical representations.

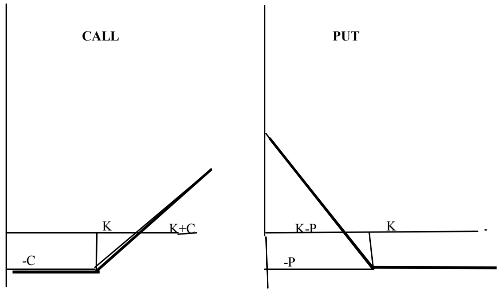

Figure 14.3. *Call and put options: net gains at maturity for the holder*

We will now cover the main problem for plain vanilla options, that is, the *pricing of such optional products*. We have to give within an economic-financial theory framework, the values of premiums C and P as a function of the maturity T and the value S of the asset at time 0.

More generally, as the holder of an option can sell his option on the option market at any time t, $0<t<T$, it is also necessary to give the "fair" value of the option at this time t knowing that the underlying asset has, at this time, the value $S = S(t)$, the fair market value represented by

$$C(S,\tau) \tag{14.1}$$

where

$$\tau = T - t \tag{14.2}$$

represents the maturity calculated at time t.

Sometimes, it is also useful to represent the call value as a function of the time $C(S, t)$.

To discuss this pricing problem, it is absolutely necessary to give assumptions about the stochastic process

$$S = \left(S(t), 0 \le t \le T\right). \tag{14.3}$$

Concerning the economic-financial theory framework, we adopt the assumption of *efficient market,* meaning that all the information available at time t is reflected in the values of the assets and so, transactions having an abnormal high profitability are not possible.

More precisely, an efficient market satisfies the following assumptions:

1) absence of transaction costs;

2) possibility of short sales;

3) availability of all information to all the economic agents;

4) perfect divisibility of assets;

5) continuous time financial market.

Furthermore, the market is *complete*, meaning that there exists zero-coupon bonds without risk for all possible maturities.

A zero-coupon bond is merely an asset giving the right to receive €1 and time $t+\tau$ for the payment of the sum B at time t.

Let us note that the word "information" used in point 3 can have different interpretations: weak, semi-strong or strong, depending on if it is based on past prices, on all public information or finally on all possible information that the agent can find.

According to Fama (1965), the efficient assumption justifies the "random walk" model in discrete time, saying that if $\Delta R_i(s)$ represents the increment of an asset i between s and $s+1$, we have:

$$\Delta R_i(s) = \mu_i + \varepsilon_i(s), \tag{14.4}$$

μ_i being a constant and $(\varepsilon_i(s))$ a sequence of uncorrelated r.v. of mean 0, sometimes called *errors*.

If we add the assumptions of equality of variances and of normality of the sequence $(\varepsilon_i(s))$, we obtain a special case of the traditional random walk.

Even if the efficiency assumption seems to be natural, some empirical studies show that this is not always the case, particularly, since some agents can have access to preferential information in principle forbidden by the legal authority control.

Nevertheless, should such agents use the pertinent information, it will be quickly noticed by those markets and balance between agents will be restored.

This possibility, also called the case of *asymmetric information*, was studied by Spencer, Akerlof and Stiglitz, who were awarded the Nobel Prize in Economics in 2001.

We feel that the efficiency assumption seems quite normal for the long term, i.e. with a large enough time unit, as it does not always seem to be true locally, i.e. with a short time unit. Indeed, deregulation of markets where investors want to secure very small benefits in a short time but in a lot of transactions plainly explains the intense activity of, for example, the currency markets receiving very small benefits.

That is why models for *asymmetric information* should always be short term models rejecting the *Absence Of Arbitrage* (AOA) assumption, that is, making money without any investment otherwise known as a "free lunch".

To be complete, let us note that it is now possible to construct models without the AOA assumption but with assumptions on the time asset evolution and a selection of different possible scenarios, so that the investor can predict what will happen if such scenarios occur (see Janssen, Manca and Di Biase (1997) and Jousseaume (1995)).

In this chapter, we will give the two most commonly used traditional models in option theory: the Cox, Ross, Rubinstein model in discrete time and the Black-Scholes model in continuous time.

14.2. The Cox, Ross, Rubinstein (CRR) or binomial model

The model we will present here has the advantage of being quite simple in a financial world not always open to the use of sophisticated mathematical tools such as those used by Black and Scholes in 1973 to obtain their famous formula. Thus, the CRR model, though coming later, was very good for the use of the Black-Scholes formula since, in the limit, the CRR model provides this formula again.

Moreover, the CRR model has still its own utility for financial institutions using discrete time models even with a short time period.

14.2.1. *One-period model*

To begin with, let us consider a model with only one time period, from time 0 to time 1; the time unit can be chosen as the user wishes: a quarter, a month, a week, a day, an hour, etc.

The basic assumption concerning the stochastic evolution of the underlying asset is that, starting from value $S(0)=S_0$ at time 0, it can only obtain two values at the end of the time period: $uS_0 (u>1)$ if there is an up movement or $dS_0 (0<d<1)$ in the case of a down movement, parameter u and d assumed to be known for the moment.

The probability measure is thus defined by the probability q of an up movement and to avoid trivialities, we will assume that:

$$0<q<1. \tag{14.5}$$

The next figure shows the two possible trajectories with of course

$$p=1-q. \tag{14.6}$$

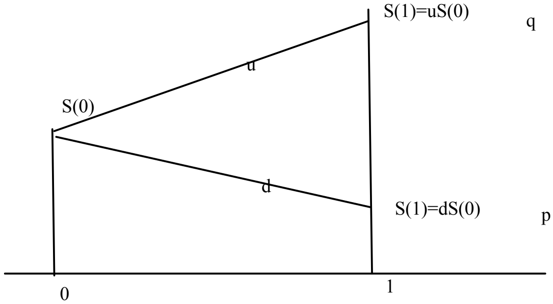

Figure 14.4. *One-period binomial model*

If we prefer to work with the percentages x and y respectively of gain and loss, we can express u and d as follows:

$$u = \left(1 + \frac{x}{100}\right), d = \left(1 - \frac{y}{100}\right). \tag{14.7}$$

We also suppose that there is no dividend repartition during the period.

Let us now consider an investor wishing to buy a European call at time 0 with maturity 1 and with K as exercise price.

The problem is thus to fix the *premium* of this call, which the investor has to pay at time 0 to buy this call, knowing the value S_0 of the underlying asset at time 0.

14.2.1.1. *The arbitrage model*

If the investor wants to buy a call, it is clear that he anticipates an up movement of the call so that exercising the call at the end of the period will be advantageous for him, and of course for the seller of the call the reverse will happen.

Nevertheless, the investor would take as little risk as possible knowing that he has always the possibility to invest on the non-risky market giving a fixed interest rate i per period.

In order to build a theory taking into account the apparently contradictory points of view, modern financial theory is based on the AOA principle meaning that there is no possibility to gain money without any investment, that is, there is no possibility of getting a *free lunch*.

This principle implies that the parameters *d*, *u* and *i* of the model must satisfy the following inequalities:

$$d < 1 + i < u\,. \tag{14.8}$$

Indeed, let us suppose for example that the first inequality is wrong. In this case the investment in the asset is always better than that on the non-risky market. If at time 0 we borrow the sum S_0 from the bank to by a share, at the end of the period obtaining the investment on assets, a free lunch of at least the amount $\left(d-(1+i)\right)S_0$ always exists.

Similarly, if the right inequality is false, we can sell the asset at time 0 to get it to the seller at time 1 and so, the minimum value of the free lunch is, in this case, $(1+i-u)S_0$, *so that in both cases* the AOA principle is not satisfied.

The seller of a call option, for example, has the obligation to sell the shares if the holder of the call exercises his right, he must be able to do it whatever the value of the considered share is; that is why we have to introduce the important concept of *hedging*.

To do so, let us consider a portfolio in which at time 0 we have Δ shares and an amount *B* of money invested at the non-risky rate *i* per period.

B may be negative in case of a loan given by the bank.

Under the AOA assumption, the investment in the call must follow the same random evolution as the considered portfolio so that we have the following relations for $t = 1$:

$$\begin{aligned} C_u(1) &= uS_0 + (1+i)B_0, \\ C_d(1) &= dS_0 + (1+i)B_0, \end{aligned} \tag{14.9}$$

where

$$\begin{aligned} C_u(1) &= \max\{0, uS_0 - K\}, \\ C_d(1) &= \max\{0, dS_0 - K\}. \end{aligned} \tag{14.10}$$

System (14.9) is a linear system with two unknown values Δ, B.

The unique solution is given by:

$$\Delta = \frac{C_u(1) - C_d(1)}{(u-d)S_0},$$
$$B = \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{14.11}$$

Now, as stated above, from the AOA assumption, the value of the call at $t = 0$, denoted for the moment by $C\left(S_0, 0\right)$, is equal to the initial value of the portfolio so that:

$$C(S_0,1) = S_0\Delta + B_0,$$
$$C(S_0,1) = S_0\frac{C_u(1) - C_d(1)}{(u-d)S_0} + \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{14.12}$$

We can also write this value in the following form:

$$C(S_0,0) = \frac{1}{1+i}\left[\overline{q}C_u(1) + (1-\overline{q})C_d(1)\right],$$
$$\overline{q} = \frac{1+i-d}{u-d}. \tag{14.13}$$

This last expression shows that the value of the call at the beginning of the period can be seen as the *present value of the expected value of the "gain" at the end of the period.* However, this expectation is calculated under a new probability measure defined by $\overline{q}$, called *risk neutral measure* in opposition to the initial measure defined by *q*, and called the *historical* or *physical measure*.

From assumption (14.8), this risk neutral measure is *unique* and moreover independent of *q*, that is, on the historical measure.

This shows that whatever the investor has as anticipation about the price of the considered underlying asset, using this model, he will always get the same result as another investor.

However, it must be clear that this risk neutral measure only gives an easy way to calculate the "fair" value of the call, but if we want to calculate probabilities of events, such as for example the probability of exercising the call at the end of the period, then it is the historical measure that must be used.

14.2.1.2. *Numerical example*

Let us consider the following data:

$$S_0 = 80, K = 80, u = 1.5, d = 0.5, i = 3\%. \tag{14.14}$$

It follows from the model that:

$$\begin{aligned} C_u(1) &= \max\{0.80 \times 1.5 - 80\} = 40, \\ C_d(1) &= \max\{0.80 \times 0.5 - 80\} = 0. \end{aligned} \tag{14.15}$$

The value of $\overline{q}$ is obtained, i.e.

$$\overline{q} = \frac{1.03 - 0.5}{1.5 - 0.5} = 0.53 \tag{14.16}$$

and so we obtain the option value

$$C_{fin}(80,0) = \frac{1}{1.03}\left[\overline{q} \times 40 + (1 - \overline{q}) \times 0\right] = 20.5825. \tag{14.17}$$

14.2.2. *Multi-period model*

14.2.2.1. *Case of two periods*

The two following figures show how the model with two periods works.

Here we have to evaluate not only the value of the call at the origin but also at the intermediary time $t = 1$.

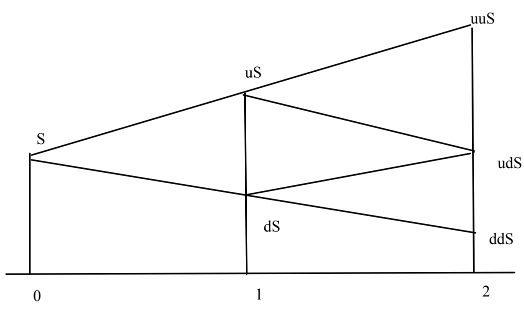

Figure 14.5. *Two-period model: scenarios for the underlying asset*

Using the notation $C(S,t)$, $t = 0,1,2$ in which the second variable represents the time, here 0, 1 or 2, the first variable is the value of the underlying asset at this considered time.

Here too, as in the case of only one period, the call values will be assessed with the risk neutral measure as the present values at time t of the "gains" at maturity $t = 2$, i.e.:

$$E_{\overline{q}}\left(C(S,2)\right). \tag{14.18}$$

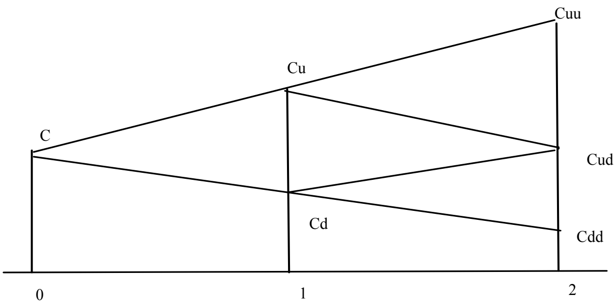

Figure.14.6. *Two-period model: values of the call*

For example, we obtain for $t = 0$:

$$C(S_0,0) = \frac{1}{(1+i)^2}\begin{bmatrix} \overline{q}^2 \max\left\{0,u^2S_0 - K\right\} + 2\overline{q}(1-\overline{q}) \\ \cdot\max\left\{0,udS_0 - K\right\} + (1\text{-}\overline{q})^2 \max\left\{0,d^2S_0 - K\right\}\end{bmatrix}. \tag{14.19}$$

Remark 14.1 Using a backward reasoning from $t = 2$ to $t = 1$ and from $t = 1$ to $t =$ 0, it is also possible to obtain this last result:

$$\begin{aligned} C(uS_0,1) &= \frac{1}{1+i}\left[\overline{q}C\ (u^2S_0,2) + (1-\overline{q})C(udS_0,2)\right], \\ C(dS_0,1) &= \frac{1}{1+i}\left[\overline{q}C(udS_0,2) + (1-\overline{q})C(d^2S_0,2)\right], \\ C(S_0,0) &= \frac{1}{1+i}\left[\overline{q}C(uS_0,1) + (1-\overline{q})C(dS_0,1)\right]. \end{aligned} \tag{14.20}$$

Substituting the first two values in the last equality given above, we rediscover relation (14.19).

14.2.2.2. *Case of n periods*

$C_{u^jd^{n-j}}(S_0,n)$ represents the call value at $t = n$ if the underlying asset has had j up movements and n-j down movements and with an initial value of the underlying asset of $S(0)$, that is:

$$C_{u^j d^{n-j}}(S_0, n) = \max\left\{0, u^j d^{n-j} S_0 - K\right\}, j = 0,1,...,n. \tag{14.21}$$

A straightforward extension of the case of two periods gives the following result:

$$C(S_0, 0) = \frac{1}{(1+i)^n} \sum_{j=0}^{n} \binom{n}{j} \overline{q}^j (1-\overline{q})^{n-j} C_{u^j d^{n-j}}(n) \tag{14.22}$$

and similar results for intermediary time values.

From the calculational point of view, Cox and Rubinstein introduce the minimum number of up movements *a* so that the call will be "in the money", which will mean that the holder has the advantage to exercise his option; clearly, this integer is given by:

$$a = \min\left\{j \in N : u^j d^{n-j} S_0 > K\right\}. \tag{14.23}$$

Of course, if *a* is strictly larger than *n*, the call will always finish "out of the money" so that the call value at $t = n$ is zero.

From relation (14.23), we obtain:

$$u^j d^{n-j} S_0 = K \Leftrightarrow a = \left\lfloor \frac{\log K S_0^{-1} d^{-n}}{\log u d^{-1}} \right\rfloor + 1, \tag{14.24}$$

$\lfloor x \rfloor$ representing the larger integer smaller than or equal to the real *x*.

From section 10.1, we know that if *X* is an r.v. having a binomial distribution with parameters (n, q), we have:

$$P\left(X > a-1\right) = \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} (= \overline{B}(n, q; a)). \tag{14.25}$$

As we have (see Cox, Rubinstein (1985)):

$$\overline{q} < \frac{1+i}{u} < 1, \tag{14.26}$$

it follows that the quantity $\overline{q}'$ defined here below is such that $0 < \overline{q}' < 1$ and so the call value can be written in the form:

$$C_{fin}(S_0,0) = S_0\overline{B}(n,\overline{q}';a) - \frac{K}{(1+i)^n}\overline{B}(n,\overline{q};a),$$
$$\overline{q} = \frac{1+i-d}{u-d}, \overline{q}' = \frac{u}{1+i}q. \tag{14.27}$$

In conclusion, the binomial distribution is sufficient to calculate the call values.

14.2.2.3. *Numerical example*

Coming back to the preceding example for which

$$S_0 = 80, K = 80, u = 1.5, d = 0.5, i = 3\%, \tag{14.28}$$

and $\overline{q} = 0.53$ but now for *n=2*, we obtain:

$$\overline{q}' = \frac{1.5}{1.03} \times 0.6 = 0.7718 \tag{14.29}$$

and consequently

$$C(80,0) = 26.4775. \tag{14.30}$$

14.3. The Black-Scholes formula as the limit of the binomial model

14.3.1. *The log-normality of the underlying asset*

Since nowadays financial markets operate in continuous time, we will study the asymptotical behavior of CRR formula (14.27) to obtain the value of a call at time 0 and of maturity *T*.

To begin with, we will work with a discrete time scale on [*0,T*] with a unit time period *h* defined by *n = T/h* or more precisely $n = \lfloor T/h \rfloor$.

Moreover, if *i* represents the annual interest rate, the rate for a period of length *h* called $\hat{\imath}$ is defined by the relation:

$$(1+\hat{\imath})^n = (1+i)^T, \tag{14.31}$$

so that

$$\hat{\imath} = (1+i)^{T/n} - 1. \tag{14.32}$$

If J_n represents the r.v. giving the number of ascending movements of the underlying asset, we know that:

$$J_n \prec B(n,q) \tag{14.33}$$

and so, starting from S_0, the value of the underlying asset at the end of period n is given by

$$S(n) = u^{J_{,n}} d^{n-J_{,n}} S_0. \tag{14.34}$$

It follows that

$$\log \frac{S(n)}{S_0} = J_n \log \frac{u}{d} + n \log d. \tag{14.35}$$

The results of the binomial distribution (see section 10.5.1) imply that

$$\begin{aligned} &E\left(\log \frac{S(n)}{S_0} \right) = \hat{\mu} n, \\ &\operatorname{var}\left(\log \frac{S(n)}{S_0} \right) = \hat{\sigma}^2 n, \\ &\hat{\mu} = q\hat{\sigma}^2 + \log d, \\ &\hat{\sigma}^2 = q(1-q)\left(\log \frac{u}{d} \right)^2 . \end{aligned} \tag{14.36}$$

To obtain a limit behavior, for every fixed n, we must introduce a dependence of u, d and q with respect to $n = \lfloor T/h \rfloor$ so that

$$\begin{aligned} &\lim_{n\to\infty} \hat{\mu}(n) n = \alpha T, \\ &\lim_{n\to\infty} \hat{\sigma}^2(n) n = \sigma^2 T, \end{aligned} \tag{14.37}$$

α, σ being constant values as parameters of the limit model. As an example, Cox and Rubinstein (1985) select the values

$$u = e^{\sigma\sqrt{T/n}}, d = \frac{1}{u}(= e^{-\sigma\sqrt{T/n}}),$$
$$q = \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\sigma}\sqrt{T/n}. \tag{14.38}$$

This choice leads to the values:

$$\hat{\mu}(n)n = \alpha T,$$
$$\hat{\sigma}^2(n)n = \left(\sigma^2 - \alpha^2 \frac{T}{n}\right)T. \tag{14.39}$$

Using a version of the central limit theorem for independent but non-identically distributed r.vs., the authors show that $S(n)/S_0$ converges in law to a lognormal distribution for $n \to \infty$. More precisely, we have:

$$P\left(\frac{\log\frac{S(n)}{S_0} - \hat{\mu}(n)n}{\hat{\sigma}\sqrt{n}} \leq x\right) \to \Phi(x), \tag{14.40}$$

Φ, being as defined in section 10.3, is the distribution function of the reduced normal distribution provided that the following condition is satisfied:

$$\frac{q\left|\log u - \hat{\mu}\right|^3 + (1-q)\left|\log u - \hat{\mu}\right|^3}{\hat{\sigma}^3\sqrt{n}} \xrightarrow[n\to\infty]{} 0. \tag{14.41}$$

This condition is equivalent to

$$\frac{(1-q)^2 + q^2}{\sqrt{nq(1-q)}} \to 0 \tag{14.42}$$

which is true from assumption (14.38).

This result and the definition given in section 10.4, gives the next proposition.

Proposition 14.1 (*Cox and Rubinstein* (*1985*)) *Under assumption (14.38), the limit law of the underlying asset is a lognormal law with parameters* $(\alpha T, \sigma^2 T)$ *or*

$$P\left(\frac{\log\frac{S(T)}{S_0}-\alpha T}{\sigma\sqrt{T}}\le x\right)=\Phi(x). \tag{14.43}$$

In particular, it follows that:

$$\begin{aligned} &E\left(\frac{S(T)}{S_0}\right)=e^{\alpha T+\frac{\sigma^2}{2}T},\\ &\operatorname{var}\left(\frac{S(T)}{S_0}\right)=e^{2\alpha T+\sigma^2 T}(e^{\sigma^2 T}-1). \end{aligned} \tag{14.44}$$

14.3.2. *The Black-Scholes formula*

Starting from result (14.25) and using Proposition 14.1 under the risk neutral measure, Cox and Rubinstein (1985) proved that the asymptotical value of the call is given by the famous Black and Scholes (1973) formula:

$$\begin{aligned} &C(S,T)=S\Phi(x)-K(1+i)^{-T}\Phi(x-\sigma\sqrt{T}),\\ &x=\frac{\ln\left(S/K(1+i)^{-T}\right)}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}. \end{aligned} \tag{14.45}$$

Here, we note the call using the maturity as a second variable and S representing the value of the underlying asset at time 0.

The interpretation of the Black and Scholes formula can be given in the concept of a hedging portfolio.

Indeed, we already know that in the CRR model, the value of the call takes the form:

$$C=S\Delta+B, \tag{14.46}$$

Δ representing the proportion of assets in the portfolio and B the quantity invested on the non-risky rate at $t=0$.

From result (14.46), at the limit, we obtain:

$$\begin{aligned} &\Delta = \Phi(x), \\ &B = -K(1+i)^{-T}\,\Phi(x - \sigma\sqrt{T}). \end{aligned} \tag{14.47}$$

So, under the assumption of an efficient market, the hedging portfolio is also known in continuous time.

Remark 14.2 This hedging portfolio must of course, at least theoretically, be rebalanced at every time s on $[0, T]$. Rewriting the Black and Scholes formula in order to calculate the call at time s, the underlying asset having the value S, we obtain:

$$\begin{aligned} &\Delta = \Phi(x), B = -K(1+i)^{-(T-s)}\Phi\left(x - \sigma\sqrt{T-s}\right), \\ &x = \frac{\ln\left(S / K(1+i)^{-(T-s)}\right)}{\sigma\sqrt{T-s}} + \frac{1}{2}\sigma\sqrt{T-s}. \end{aligned} \tag{14.48}$$

Of course, a continuous rebalancing and even a portfolio with frequent time changes are not possible due to the costs of transaction.

14.4. The Black-Scholes continuous time model

14.4.1. *The model*

In fact, Black and Scholes used a continuous time model for the underlying asset introduced by Samuelson (1965).

On a complete filtered probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ (see Definition 10.13) the stochastic process

$$S = \left(S(t), t \geq 0\right) \tag{14.49}$$

will now represent the time evolution of the underlying asset.

The basic assumption is that the stochastic dynamic of the S-process is given by

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \\ &S(0) = S_0, \end{aligned} \tag{14.50}$$

where the process $B=(B(t),t\in[0,T])$ is a standard Brownian process (see section 10.9 which is adapted to the considered filtration).

14.4.2. *The solution of the Black-Scholes-Samuelson model*

Let us go back to model (14.50). Using the Itô formula of Chapter 13 for *lnS*(*t*), we obtain:

$$d\ln S(t)=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma dB(t) \tag{14.51}$$

and so by integration:

$$\ln S(t)-\ln S_0=\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B(t). \tag{14.52}$$

As, for every fixed *t*, *B*(*t*) has a normal distribution with parameters (0*, t*) – *t* for the variance – (see Chapter 13), this last result shows that the r.v. $S(t)/S_0$ has a lognormal distribution with parameters $\left(\left(\mu-\frac{\sigma^2}{2}\right)t,\sigma^2 t\right)$ and so:

$$\begin{aligned}&E\left(\log\frac{S(t)}{S_0}\right)=\left(\mu-\frac{\sigma^2}{2}\right)t,\\&\operatorname{var}\left(\log\frac{S(t)}{S_0}\right)=\sigma^2 t.\end{aligned} \tag{14.53}$$

Of course, from result (14.52), we obtain the explicit form of the trajectories of the *S*-process:

$$S(t)=S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)t}e^{\sigma B(t)}. \tag{14.54}$$

This process is called a *geometric Brownian motion.*

The fact of having the lognormality confirms the CRR process at the limit as, indeed, a lot of empirical studies show that, for an efficient market, stock prices are well adjusted with such a distribution.

From properties of the lognormal distribution, we obtain:

$$E\left(\frac{S(t)}{S_0}\right) = e^{\mu t},$$
$$\operatorname{var}\left(\frac{S(t)}{S_0}\right) = e^{2\mu t}(e^{\sigma^2 t} - 1). \tag{14.55}$$

So, we see that the mean value of the asset at time t is given as if the initial amount S_0 was invested at the non-risky instantaneous interest rate μ and that its value is above or below S_0 following the "hazard" variations modeled with the Brownian motion.

From the second result of (14.55), it is also clear that the expectations of large gains – and losses! – are better for large values of σ; that is why σ is called the *volatility* of the considered asset.

It follows that a market with high volatility will attract *risk lover* investors and not *risk adverse* investors.

From the explicit form, it is not difficult to simulate trajectories of the S-process. The next figure shows a typical form.

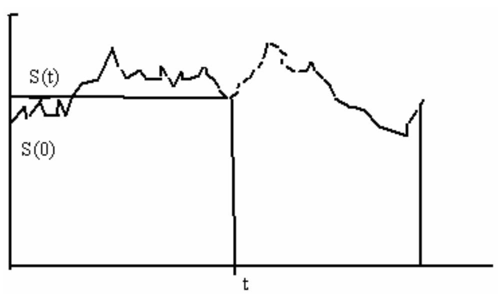

Figure 14.7. *A typical trajectory*

14.4.3. *Pricing the call with the Black-Scholes-Samuelson model*

14.4.3.1. *The hedging portfolio*

The problem consists of pricing the value of a European call of maturity T and exercise price K at every time t belonging to $[0, T]$ as a function of t or the maturity at time *t*, $\tau = T - t$, and of the value of the asset at time *t*, $S = S(t)$, knowing that the non-risky instantaneous interest rate is *r*, so that if *i* is the non-risky annual rate, we have:

$$e^r = 1 + i\,. \tag{14.56}$$

We will use the notations $C(S, t)$ or, more frequently, $C(S,\tau)$.

As in the CRR model, we introduce a portfolio *P* containing, at every time *t* of a call and a proportion α, which may be negative, of shares of the underlying asset.

The stochastic differential of $P(t)$ is given by:

$$dP(t) = dC(S,t) + \alpha dS(t) \tag{14.57}$$

or, from relation (14.50):

$$dP(t) = dC(S,t) + \alpha\mu S(t)dt + \alpha\sigma S(t)dB(t). \tag{14.58}$$

Using Itô's formula, in a correct form as proved by Bartels (1995) of the first initial form given by Black and Scholes (1973), we obtain:

$$\begin{aligned} dP(t) &= \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt \\ &+ \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right]dB(t). \end{aligned} \tag{14.59}$$

Now, using the principle of AOA, this variation must be identical to that of the same amount invested at the non-risky interest, that is:

$$rP(t)dt = r\left[C(S,t) + \alpha S\right]dt. \tag{14.60}$$

So, we obtain the following relation:

$$rP(t)dt = dP(t), \tag{14.61}$$

$$\begin{aligned} & r\left[C(S,t) + \alpha S\right]dt = \\ & \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt \\ & + \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right]dB(t). \end{aligned} \tag{14.62}$$

By identification, we obtain:

$$
\begin{aligned}
& r\left[C(S,t)+\alpha S\right]dt - \\
& \left[\frac{\partial C}{\partial S}(S,t)\mu S+\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2+\alpha\mu S(t)\right]dt=0, \\
& \left[\alpha\sigma S(t)+\frac{\partial C}{\partial S}(S,t)\sigma S\right]=0.
\end{aligned}
\tag{14.63}
$$

From the last equality, we obtain:

$$
\alpha=-\frac{\partial C}{\partial S}(S,t). \tag{14.64}
$$

Substituting this value in the first equality of (14.63), we obtain after simplification:

$$
r\left[C(S,t)-\frac{\partial C}{\partial S}(S,t)S\right]-\left[\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2\right]=0, \tag{14.65}
$$

or finally

$$
-rC(S,t)+r\frac{\partial C}{\partial S}(S,t)S+\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2=0, \tag{14.66}
$$

a *linear partial differential equation of order* 2 for the unknown function $C(S, t)$ with as initial condition

$$
C(S,t)=\begin{cases} 0, t\in\left[0,T\right), \\ \max\left\{0,S-K\right\}, t=T \end{cases}. \tag{14.67}
$$

Using results from the heat equation in physics, for which an explicit solution is given in terms of a Green function, Black and Scholes (1973) obtained the following explicit form for the call value:

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\right], \quad (14.68)$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$
$$S = S(t).$$

Remark 14.3

(i) Using relation (14.61), we obtain relation (14.45) for t = 0 or $\tau = T$. The interpretation is, of course, already given in section 14.3.2.

(ii) The differentiation in relation (14.57) is correct only if we assume that the supplementary terms produced by Itô's calculus (see relation (13.108)) are zero. In fact, this assumption is equivalent to assuming that the used portfolio strategy is *self financing*; this means that each rebalancing of the portfolio has no cost.

14.4.3.2. *The risk neutral measure and the martingale property*

As for the CRR model, it is possible to construct another probability measure *Q* on $(\Omega, \Im, (\Im_t))$, called the *risk neutral measure*, such that the value of the call given by formula (14.68) is simply the expectation value of the present value of the "gain" at maturity time *T*.

Using a change of probability measure for going from *P* to *Q*, with the famous Girsanov theorem (see for example Gihman and Skorohod (1975) and Chapter 15) it can be shown that the new measure *Q,* which moreover is unique*,* can be defined by replacing in the stochastic differential equation (14.50) the trend μ by *r*.

Doing so, the explicit form of *S*(*t*) given by relation (14.54) becomes:

$$S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t} e^{\sigma B'(t)} \quad (14.69)$$

where process B' is an adapted standard Brownian motion and the value of *C* can be calculated as the present value of the expectation of the final "gain" of the call at time *T*:

$$C(S,t) = e^{-r(T-t)} E_Q\left(\sup\{S(T) - K, 0\}\right). \quad (14.70)$$

The risk neutral measure gives another important property for the process of present values of the asset values on [0, *T*]:

$$\left\{e^{-rt}S(t), t\in\left[0,T\right]\right\} \tag{14.71}$$

Indeed, under Q, this process is a martingale, so that (see section 10.8) for all s and t such that $s<t$:

$$E\left(e^{-rt}S(t)\middle|\Im_s\right)\ (s<t)=S(s). \tag{14.72}$$

This means that at every time s, the best statistical estimation of $S(t)$ is given by the observed value at time s, a result consistent with the assumption of an efficient market.

From relation (14.72), we obtain in particular:

$$E\left(e^{-rt}S(t)\right)=S_0. \tag{14.73}$$

So, on average, the present value of the asset at any time t equals its value at time 0.

To conclude, we see that the knowledge of the risk neutral measure avoids the resolution of the partial differential equation and replaces it by the calculation of an expectation, which is in general easier, as it only uses the marginal distribution of $S(T)$.

However, we must add that, for more complicated derivative products, it may be more interesting, from the numerical point of view, to solve this partial differential equation with the finite difference method, and particularly in the case of American options.

14.4.3.3. *The call put parity relation*

From section 14.1, we know that the value of a put at maturity time T and exercise price K is given by:

$$P\left(S(T),K\right)=\max\left\{0,K-S(T)\right\}. \tag{14.74}$$

As for the call, we have:

$$C\left(S(T),K\right)=\max\left\{0,S(T)-K\right\}, \tag{14.75}$$

and so, we obtain:

$$C\left(S(T),K\right)-P\left(S(T),K\right)=S(T)-K. \tag{14.76}$$

And so, for the expectations:

$$E\left(C\left(S(T),K\right)\right)-E\left(P\left(S(T),K\right)\right)=E\left(S(T)\right)-K. \tag{14.77}$$

Using the principle of mathematical expectation for pricing the call and put, we obtain:

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0)=E\left(S(T)\right)-K. \tag{14.78}$$

We call this relation the *general call put parity relation* as it gives the value of the put knowing the value of the call and vice versa.

Now, under the assumption of an efficient market, we can use property (14.73) to get

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0)=S_0e^{rT}-K \tag{14.79}$$

and so the put value is given by:

$$P(S_0,0)=C(S_0,0)-S_0+e^{-rT}K. \tag{14.80}$$

Remark 14.4 We can interpret this relation as follows: assume a portfolio having at time 0 a share of value S_0, a put on the same asset with maturity T and an exercise price K, and a sold call with the same maturity and exercise price; the value of the portfolio at time T is always K, whatever the value of $S(T)$ is.

From the call put parity relation, we easily obtain the value of a put having the same maturity time T and exercise price K as for the call:

$$P(S,t)=C(S,t)-S+e^{-r(T-t)}K, \tag{14.81}$$

and using the Black and Scholes result, we obtain:

$$\begin{aligned}
&P(S,t)=Ke^{-r(T-t)}\Phi(-d_2)-S\Phi(-d_1),\\
&d_1=\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right],\\
&d_2=d_1-\sigma\sqrt{T-t},\\
&S=S(t).
\end{aligned} \tag{14.82}$$

14.5. Exercises on option pricing

Exercise 14.1 Let us consider a portfolio with Δ shares of unit price €1,000 and an amount b invested at the non-risky interest rate of 4% per period.

1) What is the price C of a European call having €1,050 as the exercise price, of maturity two periods if, per period, the share increases by a quarter of its value with probability 0.75 and decreases by a third of its value with probability 0.25? What are the intermediate values of the call?

2) What is the composition of the hedging portfolio at time 0?

3) If the maturity has a value of 2 weeks and the period is the day, give an estimation of the volatility and the trend of the considered asset.

Solution:

1)

$$C_{uu} = 512.5, C_{ud} = C_{dd} = 0,$$
$$C_u = 315.38, C_d = 0,$$
$$C = 194.08.$$

2)

$$C = \Delta S + B \text{ where:}$$
$$\Delta = \frac{C_u - C_d}{S(u-d)} = 54.07\% \text{ (part of the asset),}$$
$$B = \frac{uC_d - dC_u}{(u-d)} = -346.57F \text{ (loan at the non-risky rate from the bank).}$$

3) We know that:

$$1{,}000\times\frac{5}{4}=1{,}000\times e^{\mu\frac{t}{n}+\sigma\sqrt{\frac{t}{n}}},$$

$$1{,}000\times\frac{2}{3}=1{,}000\times e^{\mu\frac{t}{n}-\sigma\sqrt{\frac{t}{n}}},$$

or:

$t=14\ days,$

$n=1\ day,$

so:

$$\frac{5}{4}=e^{\mu 14+\sigma\sqrt{14}}\Rightarrow 14\mu+\sqrt{14}\sigma=\ln\frac{5}{4},$$

$$\frac{2}{3}=e^{\mu 14-\sigma\sqrt{14}}\Rightarrow 14\mu-\sqrt{14}\sigma=\ln\frac{5}{4}.$$

Finally,we obtain:

$$\mu=\frac{1}{28}0.2231436=0.0079694,$$

$$\mu_{year}=360\times 0.0079694=2.868994,$$

$$\sigma=\frac{1}{2\sqrt{14}}0.2231436=0.0298188,$$

$$\sigma_{year}=\sqrt{360}\times 0.0298188=0.565772.$$

14.6. The Greek parameters

14.6.1. *Introduction*

The technical management of the trader of options, particularly by the brokers, uses the *Greek parameters* to measure the impacts of small variations of parameters involved in formulas (4.20) and (4.34) for the pricing of options:

$$S,\sigma,\tau,r,K\,.$$

The delta coefficient

This is an indicator concerning the influence of small variations ΔS of the asset price defined as follows:

$$C(S+\Delta S,t)\approx C(S,t)+\Delta(\Delta S),$$
$$\Delta=\frac{\partial C}{\partial S}(S,t). \tag{14.83}$$

This parameter is often used to cancel the variations of the asset value in the hedging portfolio.

The gamma coefficient

This is defined as:

$$\gamma=\frac{\partial^2 C}{\partial S^2}(S,t) \tag{14.84}$$

and so it may be seen as the *delta of the delta.*

It gives a measure of the acceleration of the variation of the call and a refinement of the measure of the variation of the call using the Taylor formula of order 2:

$$C(S+\Delta S,t)\approx C(S,t)+\Delta\Delta t+\frac{1}{2}\gamma\Delta t^2. \tag{14.85}$$

The theta coefficient

It gives the dependence of *C* with respect to the maturity $\tau(=T-t)$, and so also from time *t*:

$$\theta=-\frac{\partial C}{\partial t}\left(=\frac{\partial C}{\partial \tau}\right). \tag{14.86}$$

It follows the first order approximation:

$$C(S,t+\Delta t)\approx C(S,t)-\theta\Delta t. \tag{14.87}$$

For the maturity variations $\tau=T-t$, we obtain:

$$C(S,\tau+\Delta\tau)\approx C(S,\tau)+\theta\Delta\tau. \tag{14.88}$$

The elasticity coefficient

Recall the economic definition of this coefficient which gives:

$$e(S,t) = \frac{\partial C}{\partial S}(S,t) \times \frac{S}{C(S,t)} \tag{14.89}$$

and so:

$$\frac{\Delta C}{C}\left(= \frac{C(S+\Delta S,t) - C(S,t)}{C(S,t)} \right) \approx e(S,t)\frac{\Delta S}{S}. \tag{14.90}$$

The vega coefficient

This is the indicator concerning the measure of small variations of the volatility σ and so:

$$\upsilon = \frac{\partial C}{\partial \sigma}(S,t). \tag{14.91}$$

Thus, we have approximately for small variations $\Delta\sigma$,

$$C(S+\Delta S,t) \approx C(S,t) + \upsilon \Delta o. \tag{14.92}$$

The rho coefficient

This concerns the non-risky instantaneous rate *r* and so:

$$\rho = \frac{\partial C}{\partial r}(S,t). \tag{14.93}$$

14.6.2. *Values of the Greek parameters*

The following table gives the values of the Greek parameters first for the call and then for the put.

I. For the calls:

1) delta $\left(=\frac{\partial C}{\partial S}\right)=\Phi(d_1)>0$

2) gamma $\left(=\frac{\partial \Delta}{\partial S}\right)=\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}>0$

3) vega $\left(=\frac{\partial C}{\partial \sigma}\right)=S\sqrt{\tau}\Phi'(d_1)>0$

4) rho $\left(=\frac{\partial C}{\partial r}\right)=K\tau e^{-r\tau}\Phi(d_2)>0$

5) theta$(=\frac{\partial C}{\partial \tau})=rKe^{-r\tau}\Phi(d_2)+\frac{\sigma S}{2\sqrt{\tau}}\Phi'(d_1)>0$

6) $\frac{\partial C}{\partial K}=-e^{-r\tau}\Phi(d_2)<0$

II. For the puts:

1) delta $\left(=\frac{\partial P}{\partial S}\right)=(\Phi(d_1)-1)=-\Phi(-d_1)(=\Delta_C-1)<0$

2) gamma $\left(=\frac{\partial \Delta}{\partial S}\right)=\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}(=gamma_C)>0$

3) vega $\left(=\frac{\partial P}{\partial \sigma}\right)=S\sqrt{\tau}\Phi'(d_1)(=v\acute{e}ga_C)>0$

4) rho $\left(=\frac{\partial P}{\partial r}\right)=-\tau Ke^{-r\tau}\Phi(-d_2)=\tau Ke^{-r\tau}\left[\Phi(d_2)-1\right](=rh\hat{o}_C-\tau Ke^{-r\tau})<0$

5) theta $\left(=\frac{\partial P}{\partial \tau}\right)=\frac{\sigma S}{2\sqrt{\tau}}\Phi'(d_1)-rKe^{-r\tau}\left[1-\Phi(d_2)\right](=\theta_C-rKe^{-r\tau})$

6) $\frac{\partial P}{\partial K}=e^{-r\tau}(-\Phi(d_2)+1)=e^{-r\tau}\Phi(-d_2)\left(=\frac{\partial P}{\partial K}_C+e^{-r\tau}\right)>0$

These values give interesting results concerning the influence of the considered parameters of the call and put values.

For example, we deduce that the call and put values are increasing functions of the volatility, and the call increases as S increases but the put decreases as S increases.

14.6.3. *Exercises*

Exercise 14.2 Let us consider an asset of value €1,700 and having a weekly variance of 0.000433.

(i) What is the value of a call of exercise price €1,750 with maturity 30 weeks under a non-risky rate of 6%?

(ii) Under the anticipation of a rise of €100 of the underlying asset and of a rise of 0.000018 of the weekly variance, what will be the consequences of the call and put values?

Solutions

(i) The values of the parameters necessary to calculate the call value using the Black and Scholes formula are:

$$\sigma^2_{week} = 0.00043 \Rightarrow \sigma^2_{year} = 52 \times 0.00043 = 0.2236,\ \sigma_{year} = 0.47286,$$
$$\tau = 30\ weeks = 0.576923\ year,\ K = 1750, S = 1700,$$
$$i = 6\% \Rightarrow r = \ln(1+i) = 0.05827.$$

It follows that:

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S}{K} + \tau\left(r + \frac{\sigma^2}{2}\right)\right] \Rightarrow d_1 = 0.09760272,$$
$$\Phi(d_1) = 0.5388762,$$
$$\Rightarrow d_2 = d_1 - \sigma\sqrt{\tau} = -0.01637096, \Phi(d_2) = 0.4934692,$$
$$C(S,\tau) = S\Phi(d_1) - Ke^{-r\tau} = 81.07 Euro.$$

Using call put parity relation; we obtain for the put value

$$P = Ke^{-r\tau} + C - S \Rightarrow P = 73.07 Euro.$$

(ii) Rise of the underlying asset. We know that:

$$C(S + \Delta S, \tau) = C(S,\tau) + \frac{\partial C}{\partial S}(S,\tau)\Delta S,$$
$$\frac{\partial C}{\partial S}(S,\tau) = \Phi(d_1),$$
$$\text{so}:$$
$$C(1700 + 100, \tau) = 81.07 + 100 \times 0.5388762 = 135.95 Euro.$$

For the put, we obtain:

$$P(S+\Delta S,\tau) = Ke^{-r\tau} + C(S+\Delta S) - (S+\Delta S) = 27.1 Euro.$$

(iii) Rise of the volatility. The value of the new weekly variance is now given by:

$$0.000433 + 0.00018 = 0.000613$$

and, so the new yearly variance and volatility are given by

$$\sqrt{0.031876} = 0.1785385,$$

and consequently, the variation of the yearly volatility is given by:

$$\Delta\sigma = 0.1785385 - 0.1500533 = 0.284852.$$

As the increase in volatility comes after that of the asset value, we have

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau) = C(S+\Delta S,\sigma,\tau) + \frac{\partial C}{\partial \sigma}\Delta\sigma,$$

with:

$$\frac{\partial C}{\partial \sigma} = \sqrt{\tau}\Phi'(d_1).$$

However:

$$\Phi'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} = 0.39704658,$$

and so:

$$\frac{\partial C}{\partial \sigma} = 542.84.$$

Finally, we obtain:

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau) = C(S+\Delta S,\sigma,\tau) + \frac{\partial C}{\partial \sigma}\Delta\sigma = 150.41F.$$

For the variation for the put, we use the call put parity relation and so:

$$P(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma+\Delta\sigma,\tau)+Ke^{-r\tau}-(S+\Delta S)=42.56F.$$

Exercise 14.3 For the following data, calculate the values of the call, the put and the Greek parameters

$$S=100, K=98, \tau=30 \text{ days}, \ \sigma_{week}=0{,}01664, i=8\%.$$

Solution

Yearly vol.	0.12	
Maturity	0.08219	
R=ln(1+i)	0.076962	
Results	Call	Put
Price	3.04721	0.42926
Delta	0.7847	–0.2153
Vega	8.3826	8.3826
Theta	11.924	4.334
Gamma	0.08499	0.08499
Rho	6.199	–1.805

Table 14.1. *Example option calculation*

14.7. The impact of dividend repartition

If, between t and T, the asset distributes N dividends of amounts $D_1,\ldots,D_N$ at times:

$$(0<t<)t_1<t_2<\cdots<t_N(<T), \tag{14.94}$$

the impact of the value of a European call is the following: as the buyer of the call cannot receive these dividends, it suffices to calculate the present value at time t of these dividends and to subtract the sum from the asset value at time t so that the call value is now:

$$C(S,\tau;D_1,\ldots,D_N)=C\left(S-\sum_{j=1}^{N}D_je^{-r\tau_j},\tau\right), \tag{14.95}$$

$$\tau_j=t_j-t, j=1,\ldots,N.$$

Of course, the most usual case is N=1.

If we assume that the distribution of dividends is given with a continuous payout at rate D per unit of time,

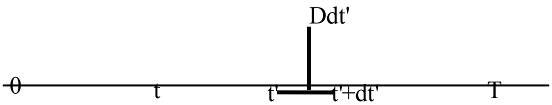

Figure 14.8. *Continuous "payout"*

the capitalized value is $e^{D\tau}$ and so the value of the call is given by:

$$C(S,\tau;D) = C(Se^{-Dt},\tau). \tag{14.96}$$

14.8. Estimation of the volatility

14.8.1 *Historic method*

This method is based on the data of the underlying asset evolution in the past, for example the n daily values

$$\left(\mathrm{S}_0,\mathrm{S}_1,\ldots,\mathrm{S}_\mathrm{n}\right). \tag{14.97}$$

Let us consider the following sample of the consecutive ratios:

$$\left(R_1,\ldots,R_n\right) = \left(\frac{S_1}{S_0},\ldots,\frac{S_n}{S_{n-1}}\right). \tag{14.98}$$

From the lognormal distribution property, we have:

$$\frac{\ln R_t - \left(\mu - \dfrac{\sigma^2}{2}\right)}{\sigma} \succ N(0,1), \tag{14.99}$$

with $R_t = \dfrac{S_t}{S_{t-1}}, t = 1,\ldots,n.$

It follows that the random sample $\left(\ln R_1,\ldots,\ln R_n\right)$ can be seen as extracted from a normal population $\left(\mu',\sigma^2\right)$ with:

$$\mu' = \mu - \frac{\sigma^2}{2}. \tag{14.100}$$

The traditional results of mathematical statistics give as best estimators:

$$\begin{aligned}\hat{\mu}' &= \frac{1}{n}\sum_{k=1}^{n}\ln\frac{R_k}{R_{k-1}},\\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}} - \hat{\mu}'\right)^2.\end{aligned} \tag{14.101}$$

To obtain an unbiased estimator of the variance, we have to use:

$$\hat{\hat{\sigma}}^2 = \frac{n}{n-1}\hat{\sigma}^2 \tag{14.102}$$

or:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}}\right)^2 - \frac{n}{n-1}(\hat{\mu}')^2. \tag{14.103}$$

Example 14.1 On the basis of a sample of 27 weekly values of an asset starting from the initial value of €26.375, the following *weekly* estimations are found:

$$\begin{aligned}\hat{\mu} &= 0.016732\\ \hat{\sigma}^2 &= 0.005216.\end{aligned}$$

Consequently, as the parameters of the Black and Scholes model must be evaluated on a yearly basis, we obtain

$$\begin{aligned}\hat{\mu}_{year} &= 52\times 0.016732 = 0.870064 \cong 0.87,\\ \hat{\sigma}^2_{year} &= 52\times 0.005216 = 0.271232,\\ \hat{\sigma}_{year} &= \sqrt{0.271232} = 0.520799 \cong 0.52.\end{aligned}$$

14.8.2. *Implicit volatility method*

This method assumes that the Black and Scholes calibrates the market values of the observed calls well.

Theoretically, an inversion of the Black and Scholes formula gives the value of the volatility σ.

On the basis of several observations of the calls for the same underlying asset, we can use the least square statistical method to refine the estimation.

Example 14.2 Using the data of Exercise 14.3, we assume that we have an observed value of the call 3.04715, but without knowing the volatility.

The next table gives the results using a step by step approximation method.

Weekly vol.	Annual vol.	Call value
0.02	0.144	3.26
0.015	0.1081	2.95
0.017	0.1225	3.069
0.016	0.1153	3.008
0.0165	0.1189	3.038
0.01664	0.1199	3.04713

Table 14.2. *Volatility calculation*

So, we find the correct volatility value to be 0.12.

Remark 14.5 The main difficulty is to select the historical data.

The set must not be too long or too short in order to avoid disrupted periods introducing strong biases in the results.

Moreover, we always work with the assumption of a constant volatility that we will overtake in section 14.10.

14.9. Black-Scholes on the market

14.9.1. *Empirical studies*

Since the opening of the CBOT in Chicago in 1972, numerous studies have been carried out for testing the results of the Black and Scholes formula.

In the case of efficient markets, the conclusions are as follows:

(i) the non-risky interest rate has little influence on the option values;

(ii) the Black and Scholes formula *underestimates* the market values for calls with short maturity times, for calls "deep out of the money" ($S/K<0.75$) and for calls with weak volatility;

(iii) the Black and Scholes formula *overestimates* the market values for calls "deep in the money" ($S/K<1.25$) and for calls with high volatility. The put values are often underestimated particularly in the out of the money ($S>>K$) case;

(iv) the puts are often underestimated particularly when they are out of money ($S<<K$).

14.9.2. *Smile effect*

If we calculate the volatility values with the implicit method in different times, in general, the results show that the volatility is *not constant*, thus invalidating one of the basic assumptions of the considered Black and Scholes model.

The graph of the volatility as a function of the exercise price often gives a graph with a convex curve, a result commonly called the "*smile effect*".

However, sometimes, concave functions are also observed.

Although, theoretically, volatilities for the pricing of calls and puts are identical, in practice, some differences are observed; they are assigned to differences of "bid-offer spread" and to the methodology of the implicit method used at different times.

The fact that it is important to consider option pricing models with non-constant volatility is one of the approaches of the next model.

14.10. Exotic options

14.10.1. *Introduction*

The derivative products of first generation concern the traditional calls and puts also called *plain vanilla options* and furthermore the anticipation of the investor leads to the construction of *strategies* for hedging or eventually for speculation.

However, these traditional options and the derived strategies generally have high costs and their exercise prices only depend of the value of the underlying assets at maturity. In particular, they do not work for some markets such as foreign currency and commodities markets.

That is why the market of derivative products has been enlarged with the *second generation options* or *exotic options*.

Their main characteristics are as follows:

a) a *prime reduction*, essentially for barrier, binary mean and compound options defined after;

b) introduction of a diversification with the use of options of the types *out performance, best of, worst of*;

c) use of options of the types lookback, option on the mean, etc.;

d) use of options linked to the exchange market like a quanto option.

All these options are in fact two types following the way on which the exercise price is defined:

(i) "*non-path dependent" options:* the exercise price is defined at the time of the conclusion of the option contract;

(ii) "*path dependent" options*: the exercise price is not known at the time of the conclusion of the option contract but the way to calculate it at the maturity time is given in the contract.

In practice, the market of such options is less liquid than the traditional market and also has a lack of organization due to the lack of standardized contracts.

However, the foreign currency options are available on organized markets as the PHLX (Philadelphia Options Exchange) created in 1983 with a clearing room, yet approximately 80% of the transactions are over the counter and in this last case, intermediaries are big banks supporting the counterpart for bid and ask for their customers.

14.10.2. *Garman-Kohlhagen formula*

For foreign currency options, we use the Black and Scholes model:

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \\ &S(0) = S_0. \end{aligned} \tag{14.104}$$

with the usual assumptions, but here $S(t)$ is the value of the *spot exchange rate* at time t. The domestic and foreign instantaneous interest rates, respectively noted r_d , r_f , are constant over the life of the considered option.

Under these assumptions, it is possible to calculate the value of a call with the following formula:

$$\begin{aligned} &C(S,t) = Se^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + \left(r_d - r_f + \frac{\sigma^2}{2}\right)(T-t)\right], \\ &d_2 = d_1 - \sigma\sqrt{T-t}, \\ &S = S(t). \end{aligned} \tag{14.105}$$

The calculation of the put value is done with the following *call put parity relation*:

$$\begin{aligned} &P_{fin}(S_0,0) = C_{fin}(S_0,0) - e^{-r_f T}S_0 + e^{-r_d T}K \\ &\text{or} \\ &P_{fin}(S\ ,t) = C_{fin}(S\ ,t) - e^{-r_f(T-t)}S\ + e^{-r_d(T-t)}K, \end{aligned} \tag{14.106}$$

so that:

$$\begin{aligned} &P(S,t) = Ke^{-r_d(T-t)}\Phi(-d_2) - Se^{-r_f(T-t)}\Phi(-d_1), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r_d - r_f + \frac{\sigma^2}{2})(T-t)\right], \\ &d_2 = d_1 - \sigma\sqrt{T-t}, \\ &S = S(t). \end{aligned} \tag{14.107}$$

Remark 14.6 Some empirical studies show that the G-K formula overestimates the observed market values.

14.10.3. *Greek parameters*

The values of the Greek parameters for calls and puts obtained by calculation as for the traditional Black and Scholes model are given below.

I. For the call:

1) delta$\left(=\frac{\partial C}{\partial S}\right)=e^{-r^{*}\tau}\Phi(d_1)>0$

2) gamma$\left(=\frac{\partial \Delta}{\partial S}\right)=e^{-r^{*}\tau}\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}>0$

3) vega$\left(=\frac{\partial C}{\partial \sigma}\right)=e^{-r\tau}K\sqrt{\tau}\Phi'(d_1)>0$

4) rho$\left(=\frac{\partial C}{\partial r}\right)=K\tau e^{-r\tau}\Phi(d_2)>0$

4') rho'$\left(=\frac{\partial C}{\partial r^{*}}\right)=-K\tau e^{-r^{*}\tau}\Phi(d_1)<0$

5) theta$\left(=\frac{\partial C}{\partial \tau}\right)=-r^{*}e^{-r^{*}\tau}S\Phi(d_1)+rKe^{-r\tau}\Phi(d_2)+\frac{\sigma K}{2\sqrt{\tau}}\Phi'(d_2)>0$

6) $\frac{\partial C}{\partial K}=-e^{-r\tau}\Phi(d_2)<0$

II. For the put:

1) delta$\left(=\frac{\partial P}{\partial S}\right)=e^{-r^{*}\tau}(\Phi(d_1)-1)=-e^{-r^{*}\tau}\Phi(-d_1)<0$

2) gamma$\left(=\frac{\partial \Delta}{\partial S}\right)=e^{-r^{*}\tau}\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}(=gamma_P)>0$

3) vega$\left(=\frac{\partial P}{\partial \sigma}\right)=Ke^{-r^{*}\tau}\sqrt{\tau}\Phi'(d_1)(=vega_C)>0$

4) rho$(=\frac{\partial P}{\partial r})=\tau Ke^{-r\tau}\Phi(-d_2)=\tau Ke^{-r\tau}\left[1-\Phi(d_2)\right]>0$

5) theta$\left(=\frac{\partial P}{\partial \tau}\right)=-r^{*}Se^{-r^{*}\tau}\Phi(-d_1)+\frac{\sigma Ke^{-r\tau}}{\sqrt{\tau}}\Phi'(d_2)+rKe^{-r\tau}\Phi(-d_2)$

14.10.4. *Theoretical models*

We know that there exist two ways for pricing derivative products:

(i) the resolution of a partial differential equation (PDE) with eventually a numerical solution of the risk neutral measure method;

(ii) the calculation of the present value of gain at maturity under the risk neutral measure.

Let us recall that the first gives a PDE for the call value using Itô's calculus and the assumption of absence of opportunity arbitrage (AOA). For non-plain vanilla options, the only way to work is to use numerical methods to obtain an approximate solution with, for example, *the finite differences* method.

It is also possible to use a discrete time model as the Cox-Rubinstein method, particularly useful for the American type.

The risk neutral measure method uses the Girsanov theorem to obtain a new probability measure Q instead of the historical probability measure P so that the call value is given with the present value of the "gain" at maturity.

Here, the new measure Q is obtained with a new trend in the SDE (14.104) given by $r_d - r_f$.

In this case, on [0, T], we obtain:

$$S(T) = S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{B}(T) - \tilde{B}(t))}, \tag{14.108}$$

$\tilde{B} = (\tilde{B}(s), 0 \le s \le T)$ being a new standard Brownian motion standard on the filtered probability space $(\Omega, \Im, (\Im_t), Q)$.

If h represents the "gain" at maturity for the considered derivative product, the value *V(t)* of this product at time t is given by:

$$V(t) = E_Q\left[e^{-r_d(T-t)} h \middle| \Im_t\right]. \tag{14.109}$$

For example, for a plain vanilla call of exercise price K, we obtain

$$h = (S(T) - K)_+, \tag{14.110}$$

and so:

$$V(t) = E\left[e^{-r_d(T-t)}(S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{B}(T) - \tilde{B}(t))} - K)_+ \middle| \Im_t\right]. \tag{14.111}$$

The process S being adapted to the basic filtration, we finally obtain:

$$\begin{aligned} V(t) &= C(S(t), t) \\ &= \frac{1}{\sqrt{2\pi}} \int_R e^{-r_d(T-t)}(S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t}} - K)_+ e^{-\frac{z^2}{2}})dz, \end{aligned} \tag{14.112}$$

An expression resulting in, after the change of variable $\frac{y}{\sqrt{T-t}} = x$, the Garman-Kohlhagen formula.

Similarly, for the put, we obtain:

$$V(t) = E\left[e^{-r_d(T-t)} \left(K - S(t) e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{B}(T) - \tilde{B}(t)\right)} \right)_+ \middle| \Im_t \right]. \qquad (14.113)$$

Remark 14.7 (options on shares and options on foreign currency options) Formally, the Garman-Kohlhagen formula is an simple extension of the Black and Scholes formula; indeed, setting $r_f = 0$ in the first formula, we obtain the second.

In particular, this means that all the results on exotic foreign currency options contain similar results for share options.

14.10.5. *Binary or digital options*

14.10.5.1. *Definition*

We will present the *"cash or nothing"* and *"asset or nothing"* options. In this case, the gain at maturity depends on the fact that, at maturity time, the underlying asset goes beyond a barrier called the *exercise price* and if so, the exercise of the option gives as gain a fixed amount mentioned in the contract signed at time of purchase of the considered option and independent of $S(T)$.

In other words, the purchaser of the option receives a coupon if the underlying asset is above the barrier and nothing in the other case.

Example 14.3: ***a standard cash or nothing call***

– *option type*: all or nothing call;

– *underlying asset*: CAC 40 index;

– *nominal*: 100,000;

– *device*: €;

– *index value at the issuing of the option*: 3,000 points;

– *exercise price*: 3,100 points;

– *coupon*: 10%;

– *issuing date*: 5/1/07;

– *maturity date*: 5/1/08;

– *premium option*: 2.9%.

So, if the CAC 40 index is larger than 3,100 points at maturity time, the counter part will pay an amount of €10,000.

The initial premium is €2,900, and the net return, excluding transaction costs, is 245%. On the other hand, if the CAC 40 index is smaller than 3,100 points at maturity time, the premium is lost.

14.10.5.2. *Pricing of a call cash or nothing*

Let N be the coupon of the option and K the exercise price. From the definition of the type of this option, we have, under the risk neutral measure Q:

$$\begin{aligned} C_{cn}(S(t),N,K,t) &= e^{-r_d(T-t)}E_Q\left[N.1_{\{S(T)\geq K\}}\right], \\ &= Ne^{-r_d(T-t)}E_Q\left[1_{S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}\geq K}\right], \\ &= Ne^{-r_d(T-t)}\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\ln\frac{S(t)}{K}+\left(\frac{r_d-r_f}{2}-\frac{\sigma}{2}\right)\sqrt{T-t}\right). \end{aligned} \tag{14.114}$$

and so:

$$C_{cn}(S(t),N,K,t) = Ne^{-r_d(T-t)}\Phi\left(d_2\right), \tag{14.115}$$

where, as for the Garman-Kohlhagen model:

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t}, \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S(t)}{K}+\left(r_d-r_f+\frac{\sigma^2}{2}\right)(T-t)\right]. \end{aligned} \tag{14.116}$$

14.10.5.3. *Case of the put cash or nothing*

For the put, we have:

$$N.1_{\{S(T)\leq K\}} = N - N.1_{\{K\leq S(T)\}}, \tag{14.117}$$

and so:

$$P_{cn}(S(t),N,K,t) = Ne^{-r_d(T-t)}\Phi(-d_2). \tag{14.118}$$

14.10.5.4. *Main Greek parameters for call and put cash or nothing*

We just consider the case of the delta, gamma and vega.

14.10.5.4.1. Case of the call

a) The delta

By definition, we have

$$\Delta_C = \frac{\partial C}{\partial S}, \tag{14.119}$$

so, by relation (14.115):

$$\Delta_C = Ne^{-r_d(T-t)}\Phi'(d_2)\frac{\partial d_2}{\partial S}, \tag{14.120}$$

and by relation (14.116):

$$\Delta_C = \frac{N}{\sigma\sqrt{T-t}.S}e^{-r_d(T-t)}\Phi'(d_2). \tag{14.121}$$

Delta being always positive, it follows that the call is an increasing function of S, the value of the underlying asset at time *t*. Furthermore, it is maximum for *S*=*K* and becomes infinite at maturity.

b) The gamma

We know that:

$$\gamma_C = \frac{\partial \Delta_c}{\partial S} = \frac{\partial^2 C}{\partial S^2}. \tag{14.122}$$

so, by relation (14.115):

$$\gamma_C = \frac{Ne^{-r_d(T-t)}}{\sigma\sqrt{T-t}}\left[\frac{-1}{S^2}\Phi'(d_2) + \frac{1}{S}\Phi''(d_2)\frac{\partial d_2}{\partial S}\right]. \tag{14.123}$$

and finally:

$$\gamma_C = \frac{Ne^{-r_d(T-t)}}{\sigma\sqrt{T-t}}\left[\frac{-1}{S^2}\Phi'(d_2)+\frac{1}{S}d_2\Phi'(d_2)\frac{1}{S\sigma\sqrt{T-t}}\right], \tag{14.124}$$

or

$$\gamma_C = \frac{Ne^{-r_d(T-t)}}{S^2\sigma\sqrt{T-t}}\Phi'(d_2)\left[-1+\frac{d_2}{\sigma\sqrt{T-t}}\right]. \tag{14.125}$$

This gamma is quasi-zero for large maturities, it changes its sign, from positive to negative values, at K, and at maturity, it becomes infinite.

14.10.5.4.2. Put case

From relation (14.117), we know that

$$P_{cn} = N - C_{cn}, \tag{14.126}$$

and so, we immediately obtain the following values:

$$\begin{aligned}\Delta_{P_{cn}} &= -\Delta_{C_{cn}},\\ \gamma_{P_{cn}} &= -\gamma_{C_{cn}},\\ \upsilon_{P_{cn}} &= -\upsilon_{C_{cn}}.\end{aligned} \tag{14.127}$$

14.10.6. *"Asset or nothing" options*

14.10.6.1. *Definition*

This type of option differs from the preceding one as it arrives at maturity at the money, the coupon paid is not a fixed amount *N* but a *multiple* of the underlying asset.

Example 14.4: *a standard asset or nothing*

– *option type*: call asset or nothing;

– *underlying asset*: share X;

– *nominal*: €800,000 (1,000 shares);

– *devise*: €;

– *share value at the issuing of the option*: €800;

– *exercise price*: €850;

– *percentage*: 10%;

– *payment*: in asset value at maturity;

– *issuing date*: 5/1/07;

– *maturity date*: 5/1/08;

– *option premium*: 4.25%.

So, if the asset value at maturity is above €850, for example €900, the counterpart has to pay, per share, an amount of 0.1 × 900 = 90, that is, a total amount of €90,000.

In this case, for an initial investment of €34,000, the net return, without transaction costs, is given by:

$$\frac{90,000-34,000}{340}=164.71\%.$$

Of course, if the asset value at maturity is less than €850, the holder of the call loses the premium of €34,000.

14.10.6.2. *Pricing a call asset or nothing*

Let M be the percentage of share to be paid in cash and K the exercise price.

Proceeding as before, under the risk neutral measure Q, we successively obtain:

$$\begin{aligned}
&C_{an}(S(t),M,K,t)=e^{-r_d(T-t)}E_Q\left[MS(T).1_{\{S(T)\geq K\}}\middle|\Im_t\right]\\
&=Me^{-r_d(T-t)}\\
&\cdot E_Q\left[S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}1_{S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}\geq K}\right] \qquad (14.128)\\
&=MS(t)e^{-r_f(T-t)}\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\ln\frac{S(t)}{K}+\left(\frac{r_d-r_f}{2}-\frac{\sigma}{2}\right)\sqrt{T-t}\right).
\end{aligned}$$

The final result is:

$$C_{an}(S(t),M,K,t)=MS(t)e^{-r_f(T-t)}\Phi\left(d_1\right). \qquad (14.129)$$

For a call asset or nothing on a share market, setting $r_f=0$, we obtain:

$$C_{an}(S(t),M,K,t)=MS(t)\Phi\left(d_1\right). \qquad (14.130)$$

14.10.6.3. *Premium of the put asset or nothing*

From the relation:

$$S(T).1_{\{S(T)\leq K\}}=S(T)-S(T).1_{\{K\leq S(T)\}}, \qquad (14.131)$$

we obtain:

$$P_{an}(S(t),M,K,t) = e^{-r_d(T-t)} E_Q\left[MS(T)\right] - P_{cn}(S(t),M,K,t). \tag{14.132}$$

As under Q, the drift of S is given by $r_d - r_f$, we can write that:

$$P_{an}(S(t),M,K,t) = e^{-r_d(T-t)} E_Q\left[MS(T)\right] - P_{cn}(S(t),M,K,t). \tag{14.133}$$

Thus,

$$P_{an}(S(t),M,K,t) = Me^{-r_f(T-t)} S(t)\Phi\left(-d_1\right). \tag{14.134}$$

On a share market, we obtain in this case $r_f = 0$:

$$P_{an}(S(t),N,K,t) = MS(t)\Phi\left(-d_1\right). \tag{14.135}$$

14.10.6.4. *Greek parameters for call and put asset or nothing*

Here too, we just consider the case of the delta, gamma and vega.

14.10.6.4.1. Case of the call

a) The delta

As

$$\Delta_C = \frac{\partial C}{\partial S}, \tag{14.136}$$

we obtain from relation (14.129):

$$\Delta_C = MS(t)e^{-r_f(T-t)}\Phi'\left(d_1\right)\frac{\partial d_1}{\partial S} + Me^{-r_f(T-t)}\Phi\left(d_1\right), \tag{14.137}$$

or

$$\Delta_C = \frac{M}{\sigma\sqrt{T-t}} e^{-r_d(T-t)}\Phi'\left(d_1\right) + Me^{-r_d(T-t)}\Phi\left(d_1\right). \tag{14.138}$$

The delta being always positive, it follows that the call is an increasing function of S, the value of the underlying asset at time *t*. Furthermore, it is maximum for S = K and, at maturity, it has the value M in the case of being in the money and 0 out of the money. At maturity and at the money, the delta becomes infinite.

b) The gamma

As

$$\gamma_C = \frac{\partial \Delta_c}{\partial S} = \frac{\partial^2 C}{\partial S^2}, \tag{14.139}$$

we obtain:

$$\gamma_C = Me^{-r_d(T-t)}\left[\frac{1}{\sigma\sqrt{T-t}}\Phi''\left(d_1\right)+\Phi'\left(d_1\right)\right]\frac{\partial d_1}{\partial S}, \tag{14.140}$$

and

$$\gamma_C = \frac{Me^{-r_d(T-t)}}{S(t)\sigma\sqrt{T-t}}\left[\frac{1}{\sigma\sqrt{T-t}}\Phi''\left(d_1\right)+\Phi'\left(d_1\right)\right], \tag{14.141}$$

or finally

$$\gamma_C = \frac{Me^{-r_d(T-t)}}{S(t)\sigma\sqrt{T-t}}\left[-\frac{d_1}{\sigma\sqrt{T-t}}+1\right]\Phi'\left(d_1\right). \tag{14.142}$$

This gamma changes sign, from positive to negative values, at K, and at maturity, it becomes infinite.

14.10.6.4.2. Case of the put

As

$$P_{an} = Me^{-r_f(T-t)}S - C_{an}, \tag{14.143}$$

we immediately have:

$$\begin{aligned} \Delta_{P_{an}} &= Me^{-r_f(T-t)} - \Delta_{C_{an}}, \\ \gamma_{P_{cn}} &= -\gamma_{C_{cn}}, \\ \upsilon_{P_{cn}} &= -\upsilon_{C_{cn}}. \end{aligned} \tag{14.144}$$

14.10.7. *The barrier options*

14.10.7.1. *Definitions*

Let us assume that a French enterprise has to pay one of its American furnishers in dollars and in three months.

If the exchange rate $/€ is 1.27, this enterprise *e* can be hedged against an increase of the exchange rate with a call in $ or a put in € in the money. However, if this enterprise anticipates that the rate will not be higher than 1.31, for example, it is possible to add a supplementary condition to the standard option contract as follows: if on [0, *T*]*, the rate goes beyond this value, then the option disappears and arrives at maturity without any value.*

This means that we introduce the concept of a *barrier*, here at a value of 1.31, and so this new type of option has a final value which depends on all the paths of the underlying asset and not only on its final value.

It is clear that this new type of options, called *barrier options*, will find a liquid enough market as their premiums are lower than the plain vanilla options. So, we have the following definition.

Definition 14.2 *A barrier option is a path-dependent option, the payoff of which depends on the payoff of a traditional option and whether a pre-specified barrier has been crossed.*

Most popular types are: *down-and-in options*, *down-and-out options*, *up-and-in options* and *up-and-down options.*

The definitions are as follows:

(i) *down-and-out options*: a lower barrier (i.e. smaller than $S(0)$) is specified. If the spot exchange rate falls below this barrier during the life of the option, that is on [0*, T*], the option ceases to exist and if not, the option remains traditional;

(ii) *down-and-in options*: the option becomes active only if the spot exchange rate goes below a given barrier; otherwise, the contract gives no right;

(iii) *up-and-out-options*: with a given specified upper barrier, if the spot exchange rate goes above the barrier on [*0, T*], the option ceases to exist; otherwise, it remains a traditional option;

(iv) *up-and-it-options*: with a given specified upper barrier, if the spot exchange rate does not go above the barrier on [*0, T*], the option is worthless; otherwise, it remains a traditional option.

14.10.7.2. *Examples of pricing*

Let us consider the case of a *down-and-in* call. It is clear that the value of the call is given by

$$C_{di}(S,t) = e^{-r_d(T-t)} E_Q\left[(S(T)-K)^+ 1_{\{T_H<T\}} \middle| \Im_t \right],$$

T_H being the hitting time of the barrier H for the process S:

$$T_H = \inf\{s>0 : S(s) \le H\}$$

Thus, we have:

$$K \le H:$$

$$C_{di}(S,t) = Se^{-r_f(T-t)}\left[\left(\frac{H}{S}\right)^{2\lambda}\Phi(y_1)+\Phi(x)-\Phi(x_1)\right] - Ke^{-r_d(T-t)}$$

$$\cdot\left[\left(\frac{H}{S}\right)^{2\lambda-2}\Phi(y_1-\sigma\sqrt{T-t})+\Phi(x-\sigma\sqrt{T-t})-\Phi(x_1-\sigma\sqrt{T-t})\right],$$

$$K > H:$$

$$C_{di}(S,t) = Se^{-r_f(T-t)}$$

$$\cdot\left[\left(\frac{H}{S}\right)^{2\lambda}\Phi(y)\right] - Ke^{-r_d(T-t)}\left[\left(\frac{H}{S}\right)^{2\lambda-2}\Phi(y-\sigma\sqrt{T-t})\right].$$

In these results, we have:

$$\begin{aligned} x &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{S}{K}+\lambda\sigma\sqrt{T-t}, y \\ &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{H^2}{SK}+\lambda\sigma\sqrt{T-t}, \\ x_1 &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{S}{H}+\lambda\sigma\sqrt{T-t}, y_1 \\ &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{H}{S}+\lambda\sigma\sqrt{T-t}, \\ \lambda &= \frac{1}{2}+\frac{r_d-r_f}{\sigma^2}. \end{aligned} \tag{14.145}$$

14.10.8. *Lookback options*

These are also called "no regrets options" and are path-dependent options favorable to the holder as they are generally expansive.

The two main types are:

– the "*standard lookback*" option: in the case of a call, the payoff is given by:

$$S(T) - \inf_{0 \le t \le T} S(t);$$

– the "*option on extrema*" with a given exercise price K has as payoff for a *call on maximum*:

$$\left(\sup_{[0,T]} S(s) - K \right)_+ . \tag{14.146}$$

They are only interesting if the underlying asset is highly increasing or decreasing on $[0,T]$ and with a high volatility.

14.10.9. *Asiatic (or average) options*

Such an option has as final payoff determined by the average price of the asset during a specified period, say $[a,b]$, included in $[0, T]$.

For a *fixed strike average option*, the payoff depends on the difference of the average and a fixed striking price; for a *floating strike average option*, the payoff at maturity depends on the difference of the spot price and the average.

Sometimes, the *geometric mean* is used instead of the arithmetic mean. The evaluation of such options is complicated and, in general, there is no explicit formula for the pricing except in the last case for which Vorst (1990) proved that it suffices to use the Garman-Kohlhagen formula with

$$\sigma' = \frac{\sigma}{\sqrt{3}}, r'_f = \frac{1}{2}\left(r_d + r_f + \frac{\sigma^2}{6} \right).$$

The "arithmetic mean" case was studied by Geman and Yor who gave the explicit form of the Laplace transform of the premium.

14.10.10. *Rainbow options*

They depend on at least two underlying assets in the same device and have generally lower prices; this is due to the correlation between the considered assets.

As example, let us present the *outperformance* or "*Margrabe option*" giving the right to the holder to receive the difference of returns between two assets if it is positive.

This means that the holder receives the outperformance of asset A on asset B at maturity time T, that is, $\left(S_2(T)-S_1(T)\right)_+$ where S_1 and S_2 are two foreign currency rates expressed in the same device.

The model to be considered is the following one:

$$\begin{aligned} &dS_i(t) = S_i(t)\left[\mu_i dt + \sigma_i dW_i(t)\right],\ i=1,2, \\ &E\left[dW_1(t)dW_2(t)\right] = \rho dt. \end{aligned} \tag{14.147}$$

so that

$$d(\ln\frac{S_2(t)}{S_1(t)}) = (\mu_2 - \mu_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2))dt + \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\, dW(t) \tag{14.148}$$

It is possible to prove the following result

$$\begin{aligned} &O_M(S_1(t), S_2(t), t) = S_2(t)e^{-r_2(T-t)}\Phi(d_1) - S_1(t)e^{-r_1(T-t)}\Phi(d_2), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}(\ln(\frac{S_2}{S_1}) + (r_1 - r_2 + \frac{\sigma^2}{2})(T-t) \\ &d_2 = d_1 - \sigma\sqrt{T-t} \\ &\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \end{aligned}$$

which is in fact the Garman-Kohlhagen formula with: $K = S_1, S = S_2$,$r_d=r_1$,$r_f=r_2$

$$\eta = (\mu_2 - \mu_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)), \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

14.11. The formula of Barone-Adesi and Whaley (1987): formula for American options

Using the PDE approach for pricing American options giving a continuous dividend at rate y and an approximation by solving an ordinary differential equation, Barone-Adesi and Whaley (1987) obtained the following good approximations for the American call and put:

1) For the call

$$C_{am}(S,T,K) = \begin{cases} S-K, & S \geq S^*, \\ C_{eur}(S,T,K) + A_2 \left(\dfrac{S}{S^*} \right)^{\gamma_2}, & S < S^*, \end{cases} \tag{14.149}$$

where A_2 and $d_1(S^*)$ are given by:

$$\begin{aligned} A_2 &= \left(\frac{S^*}{\gamma_2} \right) [1 - e^{-yT} \Phi(d_1(S^*)], \\ d_1(S^*) &= \frac{\ln \dfrac{S^*}{K} + \left(r - y + \dfrac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}, \end{aligned} \tag{14.150}$$

S* being the solution of the following algebraic equation to be solved by iteration:

$$\begin{aligned} S^* - K &= C_{eur}(S^*,T,K) + \left(\frac{S^*}{\gamma_2} \right) (1 - e^{-yT} \Phi(d_1(S^*)), \\ \gamma_2 &= \frac{1}{2}[-(\beta-1) + \sqrt{(\beta-1)^2 + 4\frac{\alpha}{1-e^{-rT}}} (<0), \\ \alpha &= \frac{2r}{\sigma^2}, \beta = \frac{2(r-y)}{\sigma^2}. \end{aligned} \tag{14.151}$$

2) For the put

$$P_{am}(S,T,K) = \begin{cases} K-S, & S^{**} \geq S, \\ P_{eur}(S,T,K) + A_1 \left(\dfrac{S}{S^{**}} \right)^{\gamma_1}, & S^{**} < S, \end{cases} \tag{14.152}$$

where

$$A_1 = -\left(\frac{S^{**}}{\gamma_1}\right)[1 - e^{-yT}\Phi(-d_1(S^{**})],$$
$$d_1(S^{**}) = \frac{\ln\frac{S^{**}}{K} + \left(r - y + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{14.153}$$

S** being the solution of the following algebraic equation to be solved by iteration:

$$K - S^{**} = P_{eur}(S^*, T, K) - \left(\frac{S^{**}}{\gamma_1}\right)(1 - e^{-yT}\Phi(d_1(S^{**})),$$
$$\gamma_1 = \frac{1}{2}\left[-(\beta - 1) - \sqrt{(\beta - 1)^2 + 4\frac{\alpha}{1 - e^{-rT}}}\right](> 0), \tag{14.154}$$
$$\alpha = \frac{2r}{\sigma^2}, \beta = \frac{2(r - y)}{\sigma^2}.$$

In these formulae, quantities S^* and S^{**} represent the thresholds to exercise respectively the call and put, i.e.:

$$S^* - K = C_{am}(S^*, T, K)(K - S^{**} = P_{am}(S^{**}, T, K). \tag{14.155}$$

These values are good for $T \to 0$ or $T \to \infty$ but not so good for mean maturity values.

Remark 14.8 *Interpolation method for American puts* (Johnson (1983), Broadie and Detemple (1996))

Johnson showed the following double inequality:

$$P_{eur}(S, T - t, K) \le P_{am}(S, T - t, K) \le P_{eur}(S, T - t, Ke^{-r(T-t)}). \tag{14.156}$$

Then, he gave the following result:

$$P_{am}(S, T - t, K) = \alpha P_{eur}(S, T - t, K) + (1 - \alpha)P_{eur}(S, T - t, Ke^{-r(T-t)}). \tag{14.157}$$

where the value of parameter α depends on the values of $S/K, r(T - t), \sigma^2(T - t)$.

Geske and Johnson model

Discretizing [0, *T*] with the subdivision $(t_1,...,t_n)$, it is possible to approach the put value with a type of Cox-Rubinstein model.

Parity relation

Without dividend repartition, the traditional *parity relation* is replaced by the following double inequality:

$$S - K \leq C_{am}(S,T,K) - P_{am}(S,T,K) \leq S - Ke^{-rT}. \qquad (14.158)$$

Furthermore, without dividend repartition, we can use the traditional parity relation for European options to obtain:

$$0 \leq P_{am}(S,T,K) - P_{eur}(S,T,K) \leq K(1-e^{-rT}). \qquad (14.159)$$

Relation of symmetry

Chesney and Gibson (1995) proved the following important result:

$$C_{am}(S,T,K,\sigma,r,y) = P_{am}(K,T,S,\sigma,y,r) \qquad (14.160)$$

so that, for the American options, every result on the call (respectively put) gives a result on the put (res. call) with the permutation of S and K and r and y.

Example 14.5 Let us suppose that we have to know the value of an American put with parameters:

$$S = 100, K = 95, T = 1, \sigma = 35\%, r = 2.75\%, y = 3\%,$$

we can solve the problem of an American call with parameters:

$$S = 95, K = 100, T = 1, \sigma = 35\%, r = 3\%, y = 2.75\%.$$

Example 14.6 Let us consider an asset with a value of €100 at t = 0 and suppose that the European call of maturity is three months and an exercise price of €102 has the value of €5.43. The European put with the same parameters has the value of €6.22.

Knowing that the asset gives no dividend on the considered period provides:

(i) the value of the American call with the same parameters;

(ii) a double inequality for the American put of same parameters;

(iii) the value of the risky instantaneous rate.

Answers

(i) Knowing that the asset gives no dividend on the considered period, we know that the American call has the same value as the European call: $C_{am} = €5.43$.

(ii) The American is always larger than the European put so that: $€\,6.22 \le P_{am}$.

From the double inequality (14.156), we obtain:

$$C_{am}(S,T,K) - S + Ke^{-rT} \le P_{am}(S,T,K) \le C_{am}(S,T,K) - S + K \ , \tag{14.161}$$

and from result (i) and the traditional parity relation, we obtain:

$$\begin{aligned} &C_{eur}(S,T,K) - S + Ke^{-rT} \le P_{am}(S,T,K) \le C_{am}(S,T,K) - S + K, \\ &\text{and} \\ &P_{eur}(S,T,K) \le P_{am}(S,T,K) \le C_{eur}(S,T,K) - S + K. \end{aligned} \tag{14.162}$$

From the second inequality, we obtain here:

$$P_{am}(S,T,K) \le C_{eur}(S,T,K) - S + K = 5.43 - 100 + 102 = €7.43.$$

The final reply is:

$$6.22 Euro \le P_{am} \le 7.43 Euro. \tag{14.163}$$

(i) From the traditional parity relation for European options, we have:

$$C_{eur}(S,T,K) - S + Ke^{-rT} = P_{eur}(S,T,K), \tag{14.164}$$

and so:

$$\begin{aligned} &e^{-rT} = \frac{P_{eur}(S,T,K) - C_{eur}(S,T,K) + S}{K} \\ &\text{and} \\ &r = -\frac{1}{T}\ln\frac{P_{eur}(S,T,K) - C_{eur}(S,T,K) + S}{K}. \end{aligned} \tag{14.165}$$

We finally obtain:

$$\begin{aligned} &r = -4\ln\frac{6.22 - 5,43 + 100}{102}, \\ &r = 0.04773. \end{aligned} \tag{14.166}$$

Chapter 15

Markov and Semi-Markov Option Models

15.1. The Janssen-Manca model

In this section, we present a new extension of the fundamental Black and Scholes (1973) formula in stochastic finance with the introduction of a random economic and financial environment using Markov processes, which we owe to Janssen and Manca (1999).

In preceding papers (Janssen, Manca and De Medici (1995), Janssen, Manca and Di Biase (1997), Janssen, Manca and Di Biase (1998), Janssen and Manca (2000)), these authors already show how it is useful to introduce Markov and semi-Markov theory to finance, with the assumption that the evolution of the asset follows a semi-Markov process, homogenous or non-homogenous, and how to price options in such new models. The main idea of this approach is to insert a strong dependence of the asset evolution as a function of the preceding value.

The construction of this new model starts from the traditional CRR model with one period to obtain a new continuous time model satisfying the absence of arbitrage assumption.

One of the main potential applications of our model concerns the possibility of obtaining a new way of using the Black and Scholes formula with information related to the economic and financial environment, particularly concerning the volatility of the underlying asset.

This new model also provides the possibility to take into account *anticipations* of investors in such a way as to incorporate them in their own option pricing.

In the same philosophy, the model can be used to construct scenarios, particularly in the case of stress in a VaR approach.

15.1.1. *The Markov extension of the one-period CRR model*

15.1.1.1. *The model*

Starting on a complete probability space $(\Omega, \Im, P)$, let us consider a one-period model for the evolution of one asset having the known value $S(0) = S_0$ at time 0 and random value $S(1)$ at time 1.

The economic and financial environment is defined with random variables J_0, J_1 representing the environment states respectively at time 0 and time 1. These random variables take their values in the state space $E = \{1,...,m\}$ and are defined on the probability space by:

$$
\begin{aligned}
&P(J_0 = i) = a_i, i = 1,...,m;\\
&P(J_1 | J_0 = i) = p_{ij}, i, j = 1,...,m,
\end{aligned}
\tag{15.1}
$$

where:

$$
\begin{aligned}
&a_i \geq 0, i = 1,...,m;\\
&\sum_{i=1}^{m} a_i = 1,\\
&p_{ij} \geq 0, i, j = 1,...,m,\\
&\sum_{j=1}^{m} p_{ij} = 1,\ i = 1,...,m.
\end{aligned}
\tag{15.2}
$$

Furthermore, let us introduce the following function of J_0, J_1: $u_{J_0 J_1}, d_{J_0 J_1}$, $q_{J_0 J_1}$ such that, a.s.:

$$
\begin{aligned}
&0 < d_{J_0 J_1} < r_{J_0 J_1} < u_{J_0 J_1},\\
&d_{J_0 J_1} < 1,\ 1 < r_{J_0 J_1},
\end{aligned}
\tag{15.3}
$$

$$
0 < q_{J_0 J_1} < 1. \tag{15.4}
$$

The *one-period model*, related to the process $\{S(0), S(1)\}$, is the following: given J_0, J_1 and that $S(0) = S_0$, the asset has the following evolution: it goes up from S_0 to $u_{J_0 J_1} S_0$ with the conditional probability $q_{J_0 J_1}$ or goes down from S_0 to

$d_{J_0J_1}S_0$ with the conditional probability $1-q_{J_0J_1}$; moreover, the non-risky interest rate of this period has the value $\nu_{J_0J_1}$ defined by:

$$\nu_{J_0J_1} = r_{J_0J_1} - 1. \tag{15.5}$$

Given J_0, J_1, we have:

$$\begin{aligned}
&P\left(S(1) = u_{J_0J_1}S_0 \,|J_0, J_1, S_0\right) = q_{J_0J_1},\\
&P\left(S(1) = d_{J_0J_1}S_0 \,\|J_0, J_1, S_0\right) = 1 - q_{J_0J_1},\\
&E\left(S(1)\|J_0, J_1, S_0\right) = q_{J_0J_1}u_{J_0J_1}S_0 + (1-q_{J_0J_1})d_{J_0J_1}S_0,\\
&E\left(S(1)\|J_0, S_0\right) = \sum_{j=1}^{m} p_{J_0j}(q_{J_0j}u_{J_0j}S_0 + (1-q_{J_0j})d_{J_0j}S_0),\\
&E\left(S(1)|S_0\right) = \sum_{i=1}^{m} P\left(J_0 = i\right)\sum_{j=1}^{m}\left[p_{ij}(q_{ij}u_{ij} + (1-q_{ij})d_{ij})\right]S_0.
\end{aligned} \tag{15.6}$$

One of the basic concepts of stochastic finance is the *absence of arbitrage possibility*. In fact, it is equivalent to state that the process $\{rS(0), S(1)\}$ is a martingale where $r = 1+\rho$ and ρ is an adequate non-risky interest rate for calculating the present value of $S(1)$ at time 0.

Here, we must take into account the possible information of the investor concerning the environment; at time 0, in addition to the knowledge of S_0, different information sets may be available. Three cases are possible:

1) *Knowledge of* $\left(J_0, J_1\right)$

In this case, the martingale condition:

$$E\left(S(1)\|J_0, J_1, S_0\right) = r_{J_0J_1}S_0 \tag{15.7}$$

becomes:

$$r_{J_0J_1}S_0 = q_{J_0J_1}u_{J_0J_1}S_0 + (1-q_{J_0J_1})d_{J_0J_1}S_0 \tag{15.8}$$

or

$$r_{J_0J_1} = q_{J_0J_1}u_{J_0J_1} + (1-q_{J_0J_1})d_{J_0J_1}. \tag{15.9}$$

This last condition is exactly the same as the CRR model; this means that the new conditional probability for which the martingale condition is satisfied is given by:

$$\tilde{q}_{J_0J_1} = \frac{r_{J_0J_1} - d_{J_0J_1}}{u_{J_0J_1} - d_{J_0J_1}}. \tag{15.10}$$

This value defines the *risk neutral conditional probability measure.*

As an example of its application in *option pricing*, let us consider that we want to study a European call option of maturity $T = 1$ and exercise price K bought at time 0.

It follows that at time 1 or at the end of the maturity, the value of the option will be given by the random variable:

$$C(S(1),0) = \max\left\{0, S(1) - K\right\}. \tag{15.11}$$

We calculate the price of the option at time 0 with a maturity period of value 1 as the conditional expectation under the risk neutral conditional probability measure, denoted $C_{J_0,J_1}(S_0,1)$, of the present value of the gain at time 1:

$$\begin{aligned} C_{J_0,J_1}(S_0,1) &= E\left(r_{J_0J_1}^{-1} \max\left\{0, S(1) - K\right\} \middle| J_0, J_1\right) \\ &= r_{J_0J_1}^{-1}\left[\tilde{q}_{J_0J_1} \max\left\{0, u_{J_0J_1} S_0 - K\right\} + (1 - \tilde{q}_{J_0J_1}) \max\left\{0, d_{J_0J_1} S_0 - K\right\}\right]. \end{aligned} \tag{15.12}$$

2) *Knowledge of* J_0

Let us begin to see what the martingale condition becomes.

We have:

$$E\left(S(1) \| J_0, S_0\right) = E\left(E\left(S(1) \| J_0, J_1, S_0\right) \middle| J_0, S_0\right). \tag{15.13}$$

As the assumption of AOA is now satisfied for the conditioning with J_0 and J_1, we can write that

$$E\left(S(1) \| J_0, S_0\right) = E\left(r_{J_0J_1} S_0 \middle| J_0, S_0\right), \tag{15.14}$$

and so:

$$E\left(S(1) \| J_{0,}, S_0\right) = S_0 E\left(r_{J_0J_1} \middle| J_0, S_0\right), \tag{15.15}$$

and finally:

$$E\left(S(1)\middle\|J_{0,},S_0\right)=\varsigma_{J_0}S_0 \tag{15.16}$$

where:

$$\varsigma_{J_0}=\sum_{j=1}^{m}p_{J_0j}r_{J_0j}\,. \tag{15.17}$$

These last two formulae show that, given, at time 0, the initial environment state, the AOA is still valid with risk neutral interest

$$\rho_{J_0}=1-\varsigma_{J_0}, \tag{15.18}$$

or

$$\rho_{J_0}=\sum_{j=1}^{m}p_{J_0j}\nu_{J_0j}, \tag{15.19}$$

with r_{J_0j} given by relation (15.5) which is perfectly coherent as relation (15.19) represents the conditional mean of the non-risky interest rate given J_0.

3) *No environment knowledge*

In this last case, the investor merely observes the initial value of the stock S_0 as in the CRR or the Black and Scholes models. As above, we can calculate the expectation of $S(1)$ as follows:

$$E\left(S(1)\middle|S_0\right)=E\left(E\left(S(1)\middle|J_0\right)\middle|S_0\right), \tag{15.20}$$

and from relation (15.16):

$$E\left(S(1)\middle|S_0\right)=S_0E\left(\varsigma_{J_0}\middle|S_0\right). \tag{15.21}$$

As, from relation (15.17), we obtain:

$$E\left(\varsigma_{J_0}\middle|S_0\right).=\sum_{i=1}^{m}a_i\sum_{j=1}^{m}p_{ij}r_{ij}\ , \tag{15.22}$$

it follows that the AOA is still true in this case with a non-risky interest rate ρ defined by:

$$\rho = 1 - \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij} r_{ij}. \tag{15.23}$$

From this last relation and relation (15.19), we obtain

$$\begin{aligned} \rho &= \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij}(1 - v_{ij}) \\ &= \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij} v_{ij} \\ &= \sum_{i=1}^{m} a_i v_i . \end{aligned} \tag{15.24}$$

Once more, these last two relations show the perfect coherence concerning the non-risky interest rates to be used with regard to the three environment information sets we can have.

15.1.1.2. *Calculational option pricing formula for the one-period model*

In the preceding section, relation (15.12) gives the value of a call option at time 0 given the initial and final environment states J_0 and J_1. We now calculate the price of the option, firstly with only the knowledge at time 0 of the initial environment state J_0, then with only the knowledge of the final state J_1 and finally with no knowledge of the initial and final states:

1) *with the knowledge of* J_0

This value, denoted by $C_{J_0}(S_0,1)$, is nothing other than the conditional expectation of $C_{J_0 J_1}(S_0,1)$ given J_0:

$$C_{J_0}(S_0,1) = E\left(C_{J_0 J_1}(S_0,1) \middle| J_0, S_0 \right), \tag{15.25}$$

or

$$C_{J_0}(S_0,1) = \sum_{j=1}^{m} p_{J_0 j} C_{J_0 j}(S_0,1). \tag{15.26}$$

2) *with the knowledge of* J_1

Let $C^j(S_0,1)$ represent the value of the call, in this case when $J_1 = j$; we have:

$$C^j(S_0,1) = \sum_{i=1}^{m} P\left(J_0 = i \middle| J_1 = j\right) C_{ij}(S_0,1). \tag{15.27}$$

From the Bayes formula, we obtain:

$$\begin{aligned} P\left(J_0 = i \middle| J_1 = j\right) &= \frac{P\left(J_0 = i, J_1 = j\right)}{P\left(J_1 = j\right)} \\ &= \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} \end{aligned} \tag{15.28}$$

and so, from relation (15.27):

$$C^j(S_0,1) = \sum_{i=1}^{m} \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} C_{ij}(S_0,1). \tag{15.29}$$

Let us note that this case is useful if the investor wants to anticipate the final value of the environment state at time 0.

3) *with no knowledge of* J_0 *and* J_1

In this case, with the help of relation (15.26), we can write that the call value represented by $C(S_0,1)$ is given by:

$$C(S_0,1) = \sum_{i=1}^{m} a_i C_i(S_0,1), \tag{15.30}$$

or with the help of relation (15.29) by:

$$C(S_0,1) = \sum_{j=1}^{m} \sum_{k=1}^{m} a_k p_{kj} C^j(S_0,0). \tag{15.31}$$

15.1.1.3. *Examples*

The application of our one-period model is already useful with only two or three states. Indeed, it is quite natural to consider one state, for example, state 0 to model the *normal* economic and financial environment; then we can add a supplementary state –1 to represent an *abnormal* situation like a crash or a doped situation.

With three states, we can separate the crash possibility represented by state –1 from the doped situation represented by state 1, state 0 always being the normal case.

Example 15.1 *A two-state model*

As stated just above, let the state set be:

$$I = \{0,1\} \tag{15.32}$$

with state 0 as the *normal* economic and financial situation environment and state 1 as the *exceptional* in the sense of, for example, a crash or doped situation.

Numerical data are the following:

$$\begin{aligned}
&\mathbf{a} = (0.95, 0.05),\\
&\mathbf{P} = \begin{bmatrix} 0.98 & 0.02 \\ 0.60 & 0.4 \end{bmatrix}, \underline{\mathbf{v}} = \begin{bmatrix} 1.03 & 1.05 \\ 1.05 & 1.03 \end{bmatrix},\\
&\underline{\mathbf{U}} = \begin{bmatrix} 1.3 & 1.1. \\ 1.06 & 1.2 \end{bmatrix}, \underline{\mathbf{D}} = \begin{bmatrix} 0.7 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}.
\end{aligned} \tag{15.33}$$

Example 15.2 *A three-state model*

Here, let the state set be:

$$I = \{-1,0,1\}. \tag{15.34}$$

State 0 represents the *normal* economic and financial situation environment, state –1 the *exceptionally bad* situation in the sense of, for example, a crash situation and state 1 as *exceptionally good* as a doped effect of the Stock Exchange, for example.

Numerical data are the following:

$$\begin{aligned}
&\mathbf{a} = (0.05, 0.90, 0.05),\\
&\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.02 & 0.96 & 0.02 \\ 0.6 & 0.35 & 0.05 \end{bmatrix}, \underline{\mathbf{v}} = \begin{bmatrix} 1.05 & 1.03 & 10.2 \\ 1.05 & 1.03 & 10.2 \\ 1.06 & 1.04 & 10.3 \end{bmatrix},\\
&\underline{\mathbf{U}} = \begin{bmatrix} 1.07 & 1.10 & 1.20 \\ 1.07 & 1.10 & 1.20 \\ 1.07 & 1.09 & 1.15 \end{bmatrix}, \underline{\mathbf{D}} = \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.8 \\ 0.65 & 0.7 & 0.8 \end{bmatrix}.
\end{aligned} \tag{15.35}$$

For both examples, we will consider a European call option with $S_0 = 100$ and $K = 80$ and 95.

Results are given in Table 15.1.

S	100										
K	95										
Example 1											
transition	A1	a2	a3	p(ij)	r(ij)	u(ij)	d(ij)	q(ij)	Cij(100,1)	Ci(100,1)	C(100,1)
0 to 0	0.95	0.05		0.98	1.03	1.3	0.7	0.55	2.6699	2.7038	
0 to 1				0.02	1.05	1.1	0.5	0.9167	4.3651		
1 to 0				0.6	1.05	1.06	0.4	0.9848	4.6898	4.2054	
1 to 1				0.4	1.03	1.2	0.6	0.7167	3.4790		
											2.7789
Example 2											
	0.05	0.9	0.05								
bad to bad				0.6	1.05	1.07	0.5	0.9649	4.5948	4.2280	
bad to normal				0.3	1.03	1.1	0.7	0.825	4.0049		
bad to good				0.1	1.02	1.2	0.8	0.55	2.6961		
normal to bad				0.02	1.05	1.07	0.6	0.9574	4.5594	4.3275	
normal to normal				0.96	1.03	1.07	0.7	0.8919	4.3296		
normal to good				0.02	1.02	1.07	0.8	0.8148	3.9942		
good to bad				0.6	1.02	1.2	0.65	0.6727	3.2977	3.2361	
good to normal				0.35	1.02	1.2	0.7	0.64	3.1373		
good to good				0.05	1.03	1.15	0.8	0.6571	3.1900		
											4.2679

Table 15.1. *European call option examples*

15.1.2. *The multi-period discrete Markov chain model*

Let us now consider a multi-period model over the time interval $[0,n]$, n being an integer larger than 1, always under the assumption of absence of arbitrage.

To obtain useful results, we will still follow the fundamental presentation of the CRR model (Cox, Rubinstein (1985)) but adapted for our Markov environment in such a way that tractable results may be found:

1) *result with knowledge of* $J_0,...,J_n$

Let us begin with a discrete time model with n periods and suppose that given $J_0,...,J_n,S(0)=S_0$ with $J_0=i,J_n=j$, the up and down parameters, the non-risky interest rate and the probabilities of an up movement for each period are the same for all periods and given respectively by u_{ij},d_{ij},r_{ij} and q_{ij}.

Then, the asset value $S(n)$ at time n is given by:

$$S(n)=V_{j_0j_1}\cdot\cdots\cdot V_{j_{n-1}j_n}S_0 \tag{15.36}$$

where the conditional distributions of the random variables V are defined as:

$$V_{J_{n-1}J_n}=\begin{cases} u_{ij} \text{ with probability } q_{ij}, \\ d_{ij} \text{ with probability } 1\text{-}q_{ij}, \end{cases} i,j\in I. \tag{15.37}$$

Moreover, we suppose that, for each n, the random variables $V_{J_0J_1},...,V_{J_{n-1}J_n}$ are conditionally independent given $J_0,...,J_n$.

If the random variable M_n represents the total number of up movements on $[0,n]$, the asset value at time n is given by:

$$S(n)=(u_{ij})^{M_n}(d_{ij})^{n-M_n}S_0 \tag{15.38}$$

and consequently:

$$\ln\frac{S(n)}{S_0}=M_n\ln u_{ij}+(n-M_n)\ln d_{ij}. \tag{15.39}$$

Given $J_0 = j_0,...,J_n = j_n, S(0) = S_0$, the conditional distribution of M_n is a binomial distribution with parameters (n, q_{ij}). It follows that:

$$E\left(\ln \frac{S(n)}{S_0} | J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = n(q_{ij} \ln u_{ij} + (1-q_{ij}) \ln d_{ij}). \quad (15.40)$$

Concerning the conditional variance, we obtain:

$$\operatorname{var}\left(\ln \frac{S(n)}{S_0} | J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = n\left[q_{ij}(1-q_{ij})\left(\ln \frac{u_{ij}}{d_{ij}}\right)^2\right]. \quad (15.41)$$

Choosing now for the up probability on the n periods, the risk neutral probability given by relation (15.10):

$$\tilde{q}_{ij_1} = \frac{r_{ij_1} - d_{ij_1}}{u_{ij_1} - d_{ij_1}}, \quad (15.42)$$

it is clear that, under our assumptions, for each n, given $J_0,...,J_n, S(0) = S_0$ with $J_0 = i,\ J_n = j,$ we have a CRR model, so that their results recalled in the beginning of this chapter concerning the European call are valid. Consequently, we obtain the value of the European call with exercise price and maturity n as the present value of the expectation of the "gain" at time n under the risk neutral measure, that is:

$$\begin{aligned} &C(S_0, 0 | J_0 = i, J_1,...,J_n = j) \\ &= \frac{1}{v_{ij}^n} \sum_{k=0}^{n} \binom{n}{k} \tilde{q}_{ij}^k (1-\tilde{q}_{ij})^{n-k} \max\left\{u_{ij}^k d_{ij}^{n-k} S_0 - K\right\}. \end{aligned} \quad (15.43)$$

After some calculation, we can obtain the following expression (see Cox and Rubinstein (1985)):

$$\begin{aligned} &C(S_0, n | J_0 = i, J_1,...,J_n = j) \\ &= \begin{cases} S_0 B(a_{ij}; n, \tilde{q}'_{ij}) - \dfrac{K}{v_{ij}^n} B(a_{ij}; n, \tilde{q}_{ij}), & \text{if } a_{ij} < n, \\ 0 & \text{if } a_{ij} > n, \end{cases} \end{aligned} \quad (15.44)$$

where $B(x;m,\alpha)$ is the value of the complementary binomial distribution function complementary with parameters m,α at point x and

$$a_{ij}=\left[\frac{\ln(K/d_{ij}^{n}S_0)}{\ln(u_{ij}/d_{ij})}+1\right],$$
$$\tilde{q}'_{ij}=\frac{u_{ij}}{r_{ij}}q_{ij}. \tag{15.45}$$

Result (15.44) can be seen as the *discrete time extension of the Black and Scholes formula* given the environment:

$$J_0=i,\dots,J_n=j,S(0)=S_0\,. \tag{15.46}$$

2) *result with knowledge of* $J_0=i$

If we only know the initial state of the environment $J_0=i$, it is clear that the value of the call is given by

$$C_i(S_0,n)=\sum_{j=1}^{m}p_{ij}^{(n)}C_{ij}(S_0,n) \tag{15.47}$$

where, of course:

$$\left[p_{ij}^{(n)}\right]=\mathbf{P}^n. \tag{15.48}$$

3) *result with knowledge of* $J_n=j$

Proceeding as in the previous section, the use of the Bayes formula provides the following result, now on *n* periods instead of one:

$$\begin{aligned}P\left(J_0=i\middle|J_n=j\right)&=\frac{P\left(J_0=i,J_n=j\right)}{P\left(J_n=j\right)}\\&=\frac{a_i p_{ij}^{(n)}}{\sum_{k=0}^{m}a_k p_{kj}^{(n)}}\end{aligned} \tag{15.49}$$

and so the value of the call given $J_n = j$, represented by $C^j(S_0, n)$, is given by:

$$C^j(S_0, n) = \sum_{i=1}^{m} \frac{a_i p_{ij}^{(n)}}{\sum_{k=0}^{m} a_k p_{kj}^{(n)}} C_{ij}(S_0, n). \tag{15.50}$$

4) *result with no environment knowledge*

Finally, if we have no knowledge on the initial environment state but know its probability distribution given by (15.1), the value of the call denoted $C(S_0, n)$ is given by

$$C(S_0, n) = \sum_{i=1}^{m} a_i C_i(S_0, n) \tag{15.51}$$

or by

$$C(S_0, n) = \sum_{j=1}^{m} \sum_{k=1}^{m} a_k p_{kj}^{(n)} C^j(S_0, n). \tag{15.52}$$

15.1.3. *The multi-period discrete Markov chain limit model*

To construct our continuous time model on the time interval [0,*t*], let us begin to consider a multi-period discrete Markov chain model with *n* periods, where each period has length *h* so that we have equidistant observations at time 0,*h*,2*h*,...,*nh* with $n = \lfloor t/h \rfloor$.

We also assume that in the approximated discrete time model, the environment process is a homogenous ergodic Markov chain defined by relations (15.1) and (15.2) and that (see Cox and Rubinstein (1985)), for each *n*, given $J_0, \ldots, J_n, S(0) = S_0$ with $J_0 = i, J_n = j$, we select, in each subinterval $[kh, (k+1)h]$, the following up and down parameters:

$$\begin{aligned} u_{j_k j_{k+1}} &= e^{\sigma_{ij}\sqrt{\frac{t}{n}}}, d_{j_k j_{k+1}} = e^{-\sigma_{ij}\sqrt{\frac{t}{n}}}, \\ q_{j_k j_{k+1}} &= \frac{1}{2} + \frac{1}{2}\frac{\mu_{ij}}{\sigma_{ij}}\sqrt{\frac{t}{n}}, \end{aligned} \tag{15.53}$$

thus depending on the two $m \times m$ non-negative matrices:

$$\left[\mu_{ij}\right],\left[\sigma_{ij}\right]. \tag{15.54}$$

From relations (15.40) and (15.41), it follows that, for all n:

$$E\left(\ln\frac{S(n)}{S_0}|J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = \mu_{ij}t, \tag{15.55}$$

$$\text{var}\left(\ln\frac{S(n)}{S_0}|J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = \sigma_{ij}^2 t. \tag{15.56}$$

As our conditioning implies that we can follow the reasoning of Cox and Rubinstein (1985), we know that, for $n \to +\infty$:

$$\ln\frac{S(t)}{S_0} \prec N(\mu_{ij}t, \sigma_{ij}^2 t), \tag{15.57}$$

where $j_0 = i$ as the initial environment state observed at $t = 0$ and j the environment state at time t.

Concerning the non-risky interest rates, we also suppose that, for all i and j, there exists $\nu_{ij} > 1$ such that the new return rate for all the periods $\left(kh,(k+1)h\right)$, denoted $\hat{r}_{ij}$, for $n \to +\infty$, satisfies the following condition:

$$(1+r_{ij})^n \to (1+\hat{r}_{ij})^t. \tag{15.58}$$

Now let $C_{ij}(S_0,n)$ represent the value at time 0 of a European call option with maturity n and exercise price K.

Using the proof of the Black and Scholes formula given by Cox and Rubinstein (1985)) but with our parameters depending on all on the environment states i and j, we obtain under conditions (15.53) and (15.58), for fixed t:

$$C_{ij}(S_0,n) \to C_{ij}(S_0,t) \tag{15.59}$$

where:

$$
\begin{aligned}
&C_{ij}\ (S_0,t)=S_0\Phi(d_{ij_t,1})-Kr_{ij}^{-t}\Phi(d_{ij_t,2}),\\
&d_{ij_t,1}=\frac{\ln\dfrac{S_0}{Kr_{ij}^{-1}}}{\sigma_{ij}\sqrt{t}}+\frac{1}{2}\sigma_{ij}\sqrt{t},\\
&d_{ij_t,2}=d_{ij,1}-\sigma_{ij}\sqrt{t}.
\end{aligned}
\tag{15.60}
$$

This result gives the value of the call at time 0 with *i* as the initial environment state and *j* as the environment state observed at time *t*, represented from now by J_t.

If we want to use the traditional notation in the Black and Scholes (1973) framework, we can define the instantaneous interest rate intensity ρ_{ij} such that:

$$
r_{ij}=e^{\rho_{ij}}
\tag{15.61}
$$

so that the preceding formula (15.60) now becomes:

$$
\begin{aligned}
&C_{ij_t}(S_0,t)=S_0\Phi(d_{ij_t,1})-Ke^{-\rho_{ij}t}\Phi(d_{ij_t,2}),\\
&d_{ij_t,1}=\frac{1}{\sigma_{ij}\sqrt{t}}\left(\ln\frac{S}{K}+\left(\rho_{ij}-\frac{\sigma_{ij}^2}{2}\right)t\right),\\
&d_{ij_t,2}=d_{ij,1}-\sigma_{ij}\sqrt{t}.
\end{aligned}
\tag{15.62}
$$

15.1.4. *The extension of the Black-Scholes pricing formula with Markov environment: the Janssen-Manca formula*

The last result (15.62) gives a first extension of the Black and Scholes formula in continuous time from the knowledge of the initial and final environment states, respectively J_0 and J_t where J_t represents, as stated above, the state of the environment at time *t*.

Now, always with the assumption that the Markov chain with matrix **P** is ergodic, we can extend results (15.44), (15.50) and (15.52) valid for our discrete multi-period model to our continuous time model, thus giving the following main result.

Proposition 15.1 (Janssen and Manca (1999))

Under the assumption that the Markov chain of matrix **P** *of the environment process is ergodic and given that the initial environment state* $i \in I$ *and the environment state at time t is* $j \in I$, *the non-risky rate is given by* ρ_{ij} *and the annual volatility by* σ_{ij}, *then we have the following results concerning the European call price at time* 0 *with exercise price K and maturity t:*

(1) *with knowledge of state* $J_0 = i, J_t = j$, *the call value is given by result* (15.62),

(2) *with knowledge of state* $J_0 = i$, *the call value represented by* $C_i(S_0,t)$ *is given by:*

$$C_i(S_0,t) = \sum_{j=1}^{m} \pi_j C_{ij}(S_0,t), \tag{15.63}$$

(3) *with knowledge of state* $J_t = j$, *the call value represented by* $C^j(S_0,t)$ *is given by:*

$$C^j(S_0,t) = \sum_{i=1}^{m} a_i C_{ij}(S_0,t), \tag{15.64}$$

(4) *without any environment knowledge, the call value represented by* $C(S_0,t)$ *is given by:*

$$C(S_0,t) = \sum_{i=1}^{m} a_i C_i(S_0,t) \tag{15.65}$$

or

$$C(S_0,t) = \sum_{j=1}^{m} \pi_j C^j(S_0,t). \tag{15.66}$$

Proof Result (1) is proved in the previous section.

Result (2) follows from relation (15.47), letting n go to $+\infty$ and then using result (1) and the assumption of ergodicity on the environment matrix chain **P**.

Result (3) can easily be deduced from result (2) and relation (15.50).

Finally, result (4) follows immediately from relations (15.51) or (15.52) and results (2) and (3). □

Example

Examples 15.1 and 15.2 of the preceding section are covered in Table 15.2 where “?” means “unknown”.

Example 1				
K	80		K	80
S	100		S	100
0 to 0			0 to 0	
	t	Cij(100,t)	t	Cij(100,t)
	0.25	22.18	0.25	11.84
	0.5	24.87	0.5	15.69
	0.75	27.24	0.75	18.7
	1	29.35	1	21.26
1 to 0	0.25	22.01	0.25	11.18
	0.5	24.54	0.5	14.86
	0.75	26.83	0.75	17.8
	1	28.91	1	20.32
? to 1	0.25	21.57	0.25	10.17
	0.5	23.64	0.5	13.42
	0.75	25.61	0.75	16.03
	1	27.43	1	18.29
? to ?	0.25	22.11	0.25	11.31
	0.5	24.35	0.5	14.54
	0.75	26.58	0.75	17.43
	1	28.62	1	19.93

Table 15.2. *Janssen Manca option model results*

In conclusion, the Janssen-Manca approach gives for the first time a new family of Black and Scholes formulae taking into account the economic and social environment showing that:

– a “good” extension of the traditional Cox Rubinstein model is possible;

– the model also extends the Black and Scholes model;

– numerical results are possible.

Moreover, as the Janssen-Manca formulae are linear combinations of the traditional Black-Scholes results, the Greek parameters can also be calculated and will be linear combinations of the Greek parameters given in section 14.6 and similarly for hedging coefficients.

We also add that, from our point of view, one of the main potential applications of our new model concerns the possibility of obtaining a new way of using the Black and Scholes formula with information related to the economic, financial and even political environment, provided it can be modeled by an ergodic homogenous Markov chain.

This model also provides the possibility of taking into account *anticipations* made by the investors in such a way as to incorporate them in their own option pricing and can also be used for models with financial crashes as well as to construct scenarios, and particularly in the case of stress in a VaR type approach.

15.2. The extension of the Black-Scholes pricing formula with a semi-Markov environment: the Janssen-Manca-Volpe formula (Janssen and Manca (2007))

15.2.1. *Introduction*

In this section, we present the semi-Markov (SM) extension of the Black and Scholes formula to the Janssen-Manca-Volpe model to eliminate one of the restrictions of the Black and Scholes model, that is, the assumption of constant volatility over time.

There have been many attempts to slacken this condition, as for example in the model of Hull and White (1985) where the concept of stochastic volatility is introduced, but to our knowledge, in practice, no generalized model really supplants the traditional Black and Scholes model.

Whilst comparing the Markovian Janssen-Manca model of the preceding section, we developed another type of model. More precisely, we present new semi-Markov models for the evolution of the volatility of the underlying asset.

In fact, the SM model presented here assumes a type of SM evolution for the volatility of an initial Black-Scholes model presented at the ETH Zurich (1995) by Janssen, and in a different approach by E. Çinlar at the First Euro-Japanese meeting on Insurance, Finance and Reliability held in Brussels in 1998 which led to a generalization of the traditional Black and Scholes formula for the pricing of European calls with easy numerical applications.

15.2.2. *The Janssen-Manca-Çinlar model*

The semi-Markov extension of the Black and Scholes model assumes a type of SM evolution for the volatility of an initial Black and Scholes model presented by Janssen (1995) and, more recently, in a different approach by Çinlar (1998).

Hereby, we present Janssen's initial model which is similar to the presentation of Çinlar, however he provides the formula for the pricing of a call option using the Markov renewal theory.

15.2.2.1. *The JMC (Janssen-Manca-Çinlar) semi-Markov model (1995, 1998)*

Let us consider a *two-dimensional* positive (*J-X*) process of kernel **Q** with state space:

$$I = \{1, ..., m\}. \tag{15.67}$$

This means that on the probability space $(\Omega, \Im, \mathrm{P})$, we define the *three-dimensional* process

$$\left((J_n, (X_n, \sigma_n)), n \geq 0 \right) \tag{15.68}$$

with:

$$J_n \in I, (X_n, \sigma_n) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{15.69}$$

such that:

$$\begin{aligned} &P\left(X_n \leq x, \sigma_n \leq \sigma, J_n = j \middle| (J_k, (X_k, \sigma_k)), k = 0.1.....n-1 \right) \\ &= Q_{J_{n-1}j}(x, \sigma), p.s. \end{aligned} \tag{15.70}$$

We know that the $Q_{ij}, i, j \in I$ can be written in the following form:

$$Q_{ij}(x, \sigma) = p_{ij} F_{ij}(x, \sigma) \tag{15.71}$$

where:

$$p_{ij} = P\left(J_n = j \middle| J_k, k \leq n-1, J_{n-1} = i \right), \tag{15.72}$$

$$F_{ij}(x, \sigma) = P\left(X_n \leq x, \sigma_n \leq \sigma \middle| (J_k, (X_k, \sigma_k)), k \leq n-1, J_{n-1} = i \right). \tag{15.73}$$

We also introduce the following r.v.:

$$\begin{aligned} &T_n = X_1 + \cdots + X_n, n \ge 0, \\ &N(t) = \sup\{n : T_n \le t\}, t \ge 0, \\ &Z(t) = J_{N(t)}, t \ge 0. \end{aligned} \tag{15.74}$$

As usual, the transition probability for the process $Z = \left(Z(t), T \ge 0\right)$ is designed by:

$$\phi_{ij}(t) = P\left(Z(t) = j \middle| Z(t) = i\right) \tag{15.75}$$

and the stochastic processes $(N(t), t \in \mathbb{R}^+), (Z(t), t \in \mathbb{R}^+)$ are respectively the Markov renewal counting and the semi-Markov processes.

To give the financial interpretation of our model, let us define on the probability space $\left(\Omega, \Im, P\right)$, the following filtration $\Im = (\Im_t, t \in \mathbb{R}^+)$,

$$\Im_t = \sigma((J_n, (X_n, \sigma_n)), n \le N(t)). \tag{15.76}$$

Given $\Im_t$, let us consider the random time interval $\left[T_{N(t)}, T_{N(t)+1}\right]$ on which we define the new stochastic process $(S(t), t \in \mathbb{R}^+)$, representing the value of the considered financial asset, as the solution of the stochastic differential equation:

$$\begin{aligned} &\frac{dS}{S(t')} = \mu_{J_{N(t)}J_{N(t)+1}} dt' + \sigma_{J_{N(t)}J_{N(t)+1}} dW_{J_{N(t)}J_{N(t)+1}}(t' - T_{N(t)}), t' \in \left[T_{N(t)}, T_{N(t)+1}\right], \\ &S(T_{N(t)}) = S(T_{N(t)}-), \end{aligned} \tag{15.77}$$

where process $(W_{J_{N(t)}J_{N(t)+1}}(t'), t' \ge 0)$ is a standard Brownian motion on $\left[T_{N(t)}, T_{N(t)+1}\right]$ defined on the basic probability space stochastically independent on $\left(J_{N(t)}, X_{N(t)}\right)$.

This model has the following financial interpretation: at $t = 0$, the asset starts from the known initial value S_0, with the known initial *j*-state J_0 representing the state of the initial economic and financial environment. On the time interval X_1, the asset has the random volatility σ_1 and has as stochastic dynamics the SDE (15.77) with $t = 0$; at time X_1, the *J* process has a transition to state J_1 and on the time interval $\left[T_1, T_2\right)$, the asset has the random volatility σ_2 and has as stochastic dynamics the SDE (15.77) with $N(t) = 1$, etc.

We always define $X_0 = 0,$ a.s.

So, it is now clear that we have in fact a disrupted Black and Scholes model due to this random change of volatility; note that this model is quite general as, in fact, we have a random volatility on each time interval $\left[T_{N(t)}, T_{N(t)+1}\right]$.

Of course, for *m* = 1, we recover the traditional Black-Scholes-Samuelson model for the description of an asset.

15.2.2.2. *The explicit expression of S(t)*

Given $J_{N(t)}, J_{N(t)+1}$, the Itô calculus gives the solution of the SDE (15.77):

$$S(t') = S_{N(t)} e^{\left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)t'} e^{\sigma_{J_{N(t)}J_{N(t)+1}} W(t'-T_{N(t)})}, \tag{15.78}$$
$$t' \in \left[T_{N(t)}, T_{N(t)+1}\right].$$

Starting from state S_0 at time 0 and given a scenario for the economic and financial environment $(J_0, J_1, ..., J_n, ...)$, this expression gives the explicit form of the trajectories of the process $(S(t), t \geq 0)$.

Now, given $(J_0, X_0, J_1, X_1, ..., J_{N(t)}, X_{N(t)}, J_{N(t)+1}, X_{N(t)+1})$, from relation (15.78), we obtain:

$$\ln\frac{S(t')}{S_{N(t)}} = \left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2})t' + \sigma_{J_{N(t)}J_{N(t)+1}} W\left(t' - T_{N(t)}\right)\right), \tag{15.79}$$
$$t' \in \left[T_{N(t)}, T_{N(t)+1}\right],$$

so that for $t' \in \left[T_{N(t)}, T_{N(t)+1}\right]$:

$$\ln\frac{S(t')}{S_{N(t)}} \prec N\left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)(t' - T_{N(t)}), \tag{15.80}$$
$$\sigma^2_{J_{N(t)}J_{N(t)+1}}(t' - T_{N(t)}).$$

$$E\left(\frac{S(t)}{S_{N(t)}} \middle| \Im_t, J_{N(t)+1}\right) = e^{\mu_{J_{N(t)},J_{N(t)+1}}(t'-T_{N(t)})}, \tag{15.81}$$

$$\operatorname{var}\left(\frac{S(t)}{S_{N(t)}}\Big|\Im_t, J_{N(t)+1}\right) = e^{2\mu_{J_{N(t)},J_{N(t)+1}}(t'-T_{N(t)})}\left(e^{\sigma^2_{J_{N(t)}J_{N(t)+1}}(t'-T_{N(t)})}-1\right). \quad (15.82)$$

Let us suppose that the random variables

$$S_0, J_0, X_1, J_1, \ldots, J_{N(t)}, X_{N(t)+1}, J_{N(t)+1}$$

are given; it follows that the conditional distribution function of $\frac{S(t)}{S_0}$ is a lognormal distribution, i.e.:

$$\ln\frac{S(t)}{S_0} \prec N\left(\mu_{J_0J_1}X_1 + \cdots + \mu_{J_{N(t)}J_{N(t)+1}}(t - T_{N(t)}), \sigma^2_{J_0J_1}X_1 + \cdots + \sigma^2_{J_{N(t)}J_{N(t)+1}}(t - T_{N(t)})\right). \quad (15.83)$$

15.2.3. *Call option pricing*

Now to obtain a useful model, let us proceed as in Janssen and Manca (1999); for a fixed *t*, we assume that all the parameters μ, σ only depend on $J_0, J_{N(t)}, J_{N(t)+1}$, and *t* is represented by

$$\mu_{J_0J_{N(t)}J_{N(t)+1}}, \sigma_{J_0J_{N(t)}J_{N(t)+1}} \quad (15.84)$$

so that from relation (15.83):

$$\ln\frac{S(t)}{S_0} \prec N\left(\left(\mu_{J_0J_{N(t)}J_{N(t)+1}} - \frac{1}{2}\sigma^2_{J_0J_{N(t)}J_{N(t)+1}}\right)t, \sigma^2_{J_0J_{N(t)}J_{N(t)+1}}t\right). \quad (15.85)$$

Of course, we can always simplify our basic assumption by suppressing the dependence with respect to $J_{N(t)+1}$ and even to $J_{N(t)}$.

Nevertheless, we think that the dependence from the future environment state $J_{N(t)+1}$ is quite important as it gives for the first time the possibility of modeling the stochastic asset evolution taking into account this anticipation of the next future state.

Let us now consider a European call option with *t* as the maturity time, and *K* as the exercise price that we must price at time 0.

If we want to assume that there is no arbitrage possibility, we must impose that

$$\mu_{J_0 J_{N(t)} J_{N(t)+1}} = \delta_{J_0 J_{N(t)} J_{N(t)+1}} \tag{15.86}$$

where $\delta_{J_0 J_{N(t)} J_{N(t)+1}}$ represents the equivalent instantaneous non-risky return on $[0,t]$ given $J_0, J_{N(t)}, J_{N(t)+1}$. Doing so, we will use the risk-neutral measure under which the forward value of the asset is a martingale, otherwise we work with the initial "physical" measure more appropriate for insurance than for finance.

Knowing $J_0, J_{N(t)}, J_{N(t)+1}$ and working with the risk neutral measure, we can calculate the value of the call at time 0 using the traditional Black and Scholes formula:

$$\begin{aligned}
&C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) = S_0 \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},1}) - K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-t} \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},2}),\\
&d_{J_0 J_{N(t)} J_{N(t)+1},1} = \frac{\ln \dfrac{S_0}{K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-1}}}{\sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t}} + \frac{1}{2} \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&d_{J_0 J_{N(t)} J_{N(t)+1},2} = d_{J_0 J_{N(t)} J_{N(t)+1},1} - \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&\nu_{J_0 J_{N(t)} J_{N(t)+1}} = e^{\delta_{J_0 J_{N(t)} J_{N(t)+1}}}.
\end{aligned} \tag{15.87}$$

To obtain the formula of the call only knowing S_0, J_0, we must use the following formula:

$$C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0 \right). \tag{15.88}$$

From the theory of semi-Markov processes, we obtain:

$$\begin{aligned}
&C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0 \right),\\
&C_{J_0}(t) = \sum_{j \in I} \sum_{k \in I} P_{J_0 j}(t) p_{jk} C_{J_0 jk}(S_0,t).
\end{aligned} \tag{15.89}$$

If we have no information about the initial state J_0, we of course obtain the following formula:

$$C(t) = E\left(C_{J_0}(t)\right) = E\left(E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0, t) \middle| J_0, S_0\right)\right),$$
$$C(t) = \sum_{i \in I} a_i C_i(t). \tag{15.90}$$

Remark 15.1 Numerical treatments are possible.

15.2.4. *Stationary option pricing formula*

In option pricing, it is nonsense to let t tend towards $+\infty$; nevertheless, we can use the limit reasoning proposed by Janssen by supposing that on the time horizon $[0, t]$, the semi-Markov environment has more and more transitions in this finite time period.

We can model this situation under the assumption that the conditional sojourn time means that $b_{ij}, i, j \in I$ satisfy the conditions

$$b_{ij} = \varepsilon \varsigma_{ij,} \ \varepsilon > 0,$$
$$b_{ij} = E\left(X_n \middle| J_{n-1} = i, J_n = j\right) \tag{15.91}$$

so that:

$$\eta_i = \sum_{j \in I} p_{ij} b_{ij} = \varepsilon \sum_{j \in I} p_{ij} \varsigma_{ij} = \varepsilon \theta_i, i \in I,$$
$$\theta_i = \sum_{j \in I} p_{ij} \varsigma_{ij}. \tag{15.92}$$

From the asymptotic theory of semi-Markov processes, we know that:

$$\lim_{\varepsilon \to 0} P\left(J_{N(t)} = j, J_{N(t)+1} = k\right) = \frac{\pi_i p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l}, i, j \in I, \tag{15.93}$$

where the vector $\left(\pi_1, ..., \pi_m\right)$ is the unique stationary distribution of the embedded Markov chain of matrix **P** assumed to be ergodic.

The new parameters ς_{jk}, $i, j, k \in I$ represent factors expressing the proportionality of the sojourn in each environment state.

Now result (15.89) becomes:

$$C_{J_0}(t) = \sum_{j \in I} \sum_{k \in I} \frac{\pi_j p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l} C_{J_0 jk}(S_0, t). \tag{15.94}$$

From (15.90), we obtain

$$C(t) = \sum_{i \in I} a_i \sum_{j \in I} \sum_{k \in I} \frac{\pi_j p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l} C_{ijk}(S_0, t). \tag{15.95}$$

This last formula replaces the Black and Scholes formula without any *a priori* information at time 0 except of course the initial value of the asset S_0.

In conclusion, the new model proposed here extends the traditional Black and Scholes formula in the case of the existence of an economic and financial environment modeled with a homogenous semi-Markov process taking into account this environment not only at the time of pricing but also before and after the maturity date.

This new family of Black and Scholes formulae seems to be more adapted to the reality, particularly when taking into account the anticipations of the investor or the consideration of stress scenario in the philosophy of the VaR approach.

15.3. Markov and semi-Markov option pricing models with arbitrage possibility

The aim of this last part is the presentation of new models for option pricing, discrete in time and within the framework of Markov and semi-Markov processes as an alternative to the traditional Cox-Rubinstein model and giving arbitrage possibilities. Both cases of European and American options are considered and possible extensions are given.

15.3.1. *Introduction*

Let us consider an asset observed on a discrete time scale

$$\{0, 1, ..., t, ...T\}, T < \infty \tag{15.96}$$

having $S(t)$ as market value at time t. To model the basic stochastic process

$$(S(t),\ t = 0,1,...,T), \tag{15.97}$$

we suppose that the asset has known minimal and maximal values so that the set of all possible values is the closed interval $[S_{\min}, S_{\max}]$ partitioned in a subset of m subclasses.

For example, if S_0 is the value of the asset at time 0, we can put:

$$\begin{aligned} S_0 &= \frac{S_{\max} - S_{\min}}{2}, \\ S_k &= S_0 + k\Delta, k = 1,...,\upsilon, \\ S_{-k} &= S_0 - k\Delta, k = 1,...,\upsilon, \\ \Delta &= \frac{S_{\max} - S_{\min}}{2\upsilon}, \end{aligned} \tag{15.98}$$

υ being arbitrarily chosen.

This implies that the total number of states is $2\upsilon + 1$. In the following, we will order these states in the natural increasing order and use the following notation for the state space:

$$I = \{-\upsilon, -(\upsilon - 1), ..., 0, 1, ..., \upsilon\}. \tag{15.99}$$

We can also introduce different step lengths following up or down movements and so consider respectively Δ, Δ'.

It is also possible to let

$$S_{\max} \to +\infty \tag{15.100}$$

and

$$T \to +\infty \tag{15.101}$$

particularly to obtain good approximation results.

Let us suppose we want to study a call option of maturity T and exercise price $K = k_0\Delta$ in both European and American cases bought at time 0.

So, in the European case, the intrinsic value of the option is given by:

$$C(T) = \max\{0, S(T) - K\}. \tag{15.102}$$

For the American case, the optimal time for exercising is given by the random time τ such that:

$$\max_{t=1,\ldots,T} \max\{0, S_t - K\} = \max\{0, S_\tau - K). \tag{15.103}$$

To obtain results, we must now introduce in the following section a stochastic model for the *S*-process.

15.3.2. *The homogenous Markov model for the underlying asset*

Let us suppose that we are working on the filtered probability space $(\Omega, \Im, (\Im_t)P)$.

In our first model, we will suppose that the underlying asset *S* is a homogenous Markov chain with matrix:

$$\mathbf{P} = \left[p_{ij} \right] \tag{15.104}$$

on the state space *I* given by relation (15.99).

It follows that, at time *t*, given the knowledge of the asset value $S(t) = S_t$, the market value of the option at time *t*, *C*(*t*), thus with a remaining maturity *T*-*t* and exercise price *K* given by $K = k_0\Delta,$ has as the probability distribution:

$$\begin{aligned} &P(C(T) = (j - k_0)\Delta) = p_{S,j}^{(T-t)}, j > k_0, \\ &P(C(T) = 0) = \sum_{l \le k_0} p_{S,j}^{(T-t)}. \end{aligned} \tag{15.105}$$

This result gives the possibility to calculate all interesting parameters concerning *C*. For example, the mean of *C*(*t*) has the value:

$$E(C(T) \big| S(t) = S_t) = \sum_{l > k_0} p_{S,j}^{(T-t)} (l - k_0)\Delta. \tag{15.106}$$

Of course, we have to calculate the present value at time *t* with the non-risky unit period interest rate *r* so that the value of the call at time *t* is given by:

$$\begin{aligned} &C(t) = v^{T-t} E(C(T) \big| S(t) = S_t) = v^{T-t} \sum_{l > k_0} p_{S,j}^{(T-t)} (l - k_0)\Delta, \\ &v = \frac{1}{1+r}. \end{aligned} \tag{15.107}$$

If matrix **P** is ergodic, then if *T-t* is large enough, results (15.105) and (15.106) can be well approximated by:

$$
\begin{aligned}
&P(C(T)=(j-k_0)\Delta)=\pi_j, j>k_0,\\
&P(C(T)=0)=\sum_{l\le k_0}\pi_l, j\le k_0,\\
&E(C(T)\,|\,S(t)=S_t)=\sum_{l>k_0}\pi_j(l-k_0)\Delta,\\
&C(t)=v^{T-t}\sum_{l>k_0}\pi_j(l-k_0)\Delta.
\end{aligned}
\tag{15.108}
$$

Of course, the vector

$$\boldsymbol{\pi}=(\pi_{-\nu},\ldots,\pi_0,\ldots,\pi_\nu) \tag{15.109}$$

is the steady-state vector related to the matrix **P**.

15.3.3. *Particular cases*

As we stated in our introduction, our homogenous Markov model contains as a very special case the famous CRR binomial model but with fixed minimal and maximal values. It suffices to select a Markov matrix **P** with the structure

$$
\begin{bmatrix}
* & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & * & 0 & * & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & * & 0 & * & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & *\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & *
\end{bmatrix}
\tag{15.110}
$$

and as the Cox-Rubinstein model has a multiplicative form, we can consider that:

$$
\Delta=\begin{cases}(u-1)S_0, u>1, S>S_0,\\ (1-d)S_0, d<1, S<S_0.\end{cases}
\tag{15.111}
$$

Remark 15.2 Under (15.100), matrix **P** has an infinite number of rows and columns.

We can also obtain the *trinomial model* if we put in (15.110) a non-zero main diagonal, etc.

15.3.4. *Numerical example for the Markov model*

To numerically illustrate our first model, let us suppose that we are interested in an asset whose possible values are restricted to the following ones:

– maximum value: state 3 = 1,650;

– intermediary values: state 2 = 1,600, state 1 = 1,550, state 0 = 1,500;

– state –1 = 1,450, state –2 = 1,400;

– minimum value: state –3 = 1,350.

With the used notation, this means that $S_0 = 1{,}500,\ \Delta = 50.$ Moreover, we also suppose that the transition matrix **P**, with the week as unit step, is given by

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \qquad (15.112)$$

It is easily seen that matrix **P** is ergodic with as unique stationary distribution:

(0.10002, 0.13336, 0.27228, 0.23737, 0.16927, 0.07539, 0.01231).

Then, starting at time 0 in state 1,500 with a maturity time of 16 weeks, the asymptotic value of the European call option expectation with 1,500 as exercise price is 41.95 and the call value at time 0 is 41.328.

Table 15.3 gives option expectations and option values with different exercise prices.

Exercise price	Option expectation	Option value
1,350	174.106	171.512
1,400	124.721	122.826
1,450	79.1059	77.927
1,500	41.9538	41.328
1,550	16.6704	16.422
1,600	5.00113	4.927
1,650	0	0

Table 15.3. *Markov option calculation*

Let us now consider the transient behavior, meaning that we will consider the maturity as a parameter expressed in *n* weeks. Table 15.4, gives option expectations, Table 15.5 option values with as exercise price 1,500 and for different maturity times from 1 to 16 weeks.

	STATE						
n	-3	-2	-1	0	1	2	3
1	75.00	75.00	57.14	25.00	14.29	7.14	0.00
2	60.71	53.57	46.93	38.39	30.10	20.41	16.96
3	50.02	48.40	43.39	40.60	37.08	31.61	31.39
4	45.70	44.92	42.79	41.11	39.61	37.39	37.44
5	43.70	43.30	42.35	41.57	40.84	39.87	39.81
6	42.76	42.58	42.13	41.78	41.45	40.98	40.96
7	42.33	42.24	42.04	41.87	41.72	41.50	41.50
8	42.13	42.09	41.99	41.92	41.84	41.75	41.74
9	42.03	42.02	41.97	41.94	41.90	41.86	41.86
10	41.99	41.98	41.96	41.95	41.93	41.91	41.91
11	41.97	41.97	41.96	41.95	41.94	41.93	41.93
12	41.96	41.96	41.96	41.95	41.95	41.94	41.94
13	41.96	41.96	41.95	41.95	41.95	41.95	41.95
14	41.96	41.96	41.95	41.95	41.95	41.95	41.95
15	41.95	41.95	41.95	41.95	41.95	41.95	41.95
16	41.95	41.95	41.95	41.95	41.95	41.95	41.95

Table 15.4. *Option expectation*

	STATE						
n	–3	–2	–1	0	1	2	3
1	70.93	74.93	57.09	24.98	14.27	7.14	0.00
2	60.60	53.47	46.85	38.32	30.05	20.37	16.93
3	49.88	48.27	43.26	40.48	36.98	31.53	31.31
4	45.53	44.75	42.63	40.26	39.45	37.25	37.30
5	43.50	43.10	42.15	41.38	40.65	39.68	39.63
6	42.22	42.34	41.90	41.54	41.21	40.75	40.73
7	42.05	41.97	41.76	41.60	41.45	41.23	41.22
8	41.81	41.77	41.68	41.60	41.53	41.43	41.43
9	41.68	41.66	41.62	41.58	41.55	41.51	41.50
10	41.60	41.59	41.57	41.55	41.54	41.52	41.52
11	41.54	41.54	41.53	41.52	41.51	41.50	41.50
12	41.49	41.49	41.49	41.48	41.48	41.47	41.47
13	41.45	41.45	41.45	41.44	41.44	41.44	41.44
14	41.41	41.41	41.41	41.41	41.41	41.41	41.40
15	41.37	41.37	41.37	41.37	41.37	41.37	41.37
16	41.33	41.33	41.33	41.33	41.33	41.33	41.33

Table 15.5. *Option value*

15.3.5. *The continuous time homogenous semi-Markov model for the underlying asset*

With the generalization of electronic trading systems, it seems more adaptive to construct a time continuous model for which the changes in the values of the underlying process may depend on the time it remained unchanged before a transition.

Also, let

$$((S_n, T_n)\ n = 0,1,...) \tag{15.113}$$

be the successive states and time changes of the considered asset.

The Janssen-Manca semi-Markov continuous model without AOA starts from the basic assumption that process (15.113) is a semi-Markov process of kernel **Q**.

It follows that, at time t in state $S(t) = S_t$, the market value of the considered European option with maturity $T - t$ has as probability distribution at maturity time

$$P(C(T)=(j-k_0))=\phi_{S_t j}(T-t), j>k_0,$$
$$P(C(T)=0)=\sum_{l\le K_0}\phi_{S_t j}(T-t), j\le k_0. \tag{15.114}$$

Of course, matrix $\mathbf{\Phi}(t)$ represents the transition probabilities for the considered semi-Markov process (see relation (12.101)).

This result gives the possibility to calculate all interesting parameters concerning C. For example, the mean of $C(T)$ has the value:

$$E(C(T)=\left|S(t)=S_t\right.)=\sum_{j>k_0}\phi_{S_t j}(T-t)(j-k_0)\Delta. \tag{15.115}$$

The pricing of the option at time t is here given by the conditional market value $C(t)$:

$$C(S_t,t)=v^{T-t}\sum_{j>k_0}\phi_{S_t j}(T-t)(j-k_0)\Delta \tag{15.116}$$

which is the Janssen-Manca-Di Biase formula for the considered semi-Markov model.

If the semi-Markov process is ergodic, then, if $(T-t)$ is large enough, results (15.114) can be well approximated by:

$$P(C(T)=(j-k_0))=\tilde{\pi}_j, j>k_0,$$
$$P(C(T)=0)=\sum_{l\le K_0}\tilde{\pi}_l, j\le k_0. \tag{15.117}$$

The *stationary* version of the Janssen-Manca-Di Biase formula is thus given by

$$C(S_t,t)=v^{T-t}\sum_{j>k_0}\tilde{\pi}_j\phi_{S_t j}(j-k_0)\Delta. \tag{15.118}$$

Of course the vector $(\tilde{\pi}_1,...,\tilde{\pi}_m)$ is the asymptotic distribution of the embedded semi-Markov process given by relation (12.15) .

Formally the evaluation of assets is continuous, but substantially is given in the discrete case; furthermore, the numerical solution of a continuous time semi-Markov process causes problems of numerical and stochastic convergence. For these reasons, it may be useful to deal with our problem with the discrete time homogenous semi-Markov process as introduced in Janssen and Manca (2007).

15.3.6. *Numerical example for the semi-Markov model*

We will only provide a numerical example for the semi-Markov model in the asymptotic case, i.e. values of the option expectation and of the options for large maturities.

We merely need as supplementary information, the conditional mean sojourn times given by relations (12.25). The used values are given by the following matrix $\mathbf{\Sigma}$:

$$\mathbf{\Sigma}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 2 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 & 2 & 1 & 1 \\ 2 & 1 & \frac{1}{2} & 1 & 2 & 2 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 2 & 1 \\ 1 & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}. \tag{15.119}$$

In this case, the asymptotic distribution for the semi-Markov process is:

(0.09487, 0.12650, 0.38238, 0.15352, 0.15013, 0.08358, 0.00902).

Then, starting at time 0 in state 1,500, the asymptotic value of the call option expectation with 1,500 as the exercise price is 46 and the call value is 45.315.

The following table gives option expectations and option values with different exercise prices.

Exercise price	Option expectation	Option value
1,350	178.78	176.119
1,400	129.234	127.308
1,450	83.8638	82.614
1,500	46.0002	45.315
1,550	15.8126	15.577
1,600	4.74378	4.673
1,650	0	0

Table 15.6. *Semi-Markov option calculation*

15.3.7. *Conclusion*

The JMD models presented here provide a semi-Markov approach for the pricing of option financial products working in discrete time and with a finite number of possible values for the imbedded asset, which is always the case from the numerical point of view.

The main interest of these models is that they work even when there are possibilities of arbitrage, that is to say, for the most common cases. Of course, one of the main difficulties in applying this model is the fitting of the needed data and this is only of interest in the case of asymmetric information so that the economic agent can believe in his own information, knowing that he will always be in a risky situation to expect gain but still worried about the possibility of losing as in the case of a real life situation!

It is also important to point out that the numerical examples are coherent; nevertheless, there are significant differences according to the model used, Markov or semi-Markov, so that it is very important to select the most concrete one.

Chapter 16

Interest Rate Stochastic Models – Application to the Bond Pricing Problem

This chapter first presents some basic definitions on bond investments and interest rates. The second part is devoted to the two basic interest rate stochastic models: the *Ornstein-Uhlenbeck-Vasicek* (OUV) and the *Cox-Ingersoll-Ross* (CIR).

In the third part, we use these two models to describe the stochastic dynamics of zero-bonds applied to the pricing problem of bonds.

16.1. The bond investments

16.1.1. *Introduction*

A bond of *nominal value P* with *coupons of value C* and of *maturity date s+S* gives the right for the investor buying this bond at time *s*, to receive the coupon value *C* at times $\{s+1, s+2, \ldots, s+S\}$ and the nominal value *P* at time *s+S*.

In the following, a will represent the cost of this investment at time *s*, in general fixed by the bond market of the Stock exchange.

It follows that the successive cash flows of this investment are given by:

– at times *s,s+1,…, st+S-1*: the coupon value *C*;

– at time *s+S*: the amount *P+C*.

If $A(t,T)$ $(s \le t \le s+S)$ represents at time t the value of the bond issuing at time $t+T$, the main problem is to evaluate its fair value in view of comparing it to the proposed market value at time t.

A *zero-bond*, an investment made for example at time s and of maturity date $s+S$, is a very simple investment for which it is paid the sum $P(s,S)$ at time s in view of receiving €1 at the maturity date $s+S$.

Thus, we can calculate the value of the above bond $A(t,T)$ with the following formula:

$$A(t,T) = P(t,1)C + P(t,2)C + ... + P(t,T-1)C + P(t,T)\left(C+P\right). \tag{16.1}$$

16.1.2. *Yield curve*

It is well known that the interest rate for a deposit at time t depends not only on this time t but also on its maturity T, so that this annual rate can be written as $i(t,T)$.

For a fixed time t, the graph of the function $T \mapsto i(t,T)$ represents the yield curve a time t and generally has the following form

Given this curve, we obtain the following value for a zero-bond:

$$P(t,T) = (1 + i(t,T))^{-T} \tag{16.2}$$

and using formula (16.1) for different bonds of different maturity times, we can calculate the values of the zero-bonds according to the market values of the observed bonds.

Example 16.1 Let us consider the case of T=2 and let us suppose we have two bonds, the first with a coupon of 5.2%, with 100 as nominal value and with 1 as maturity, the second with a coupon of 5.6%, with 100 as nominal value and with 2 as maturity. The market values of these two bonds at time T are respectively of 100 and 102.

Using formula (16.1) twice, we obtain:

– 100 = $P(t,1)(5.2 + 100)$;

– 102 = $P(t,1)5.6 + P(t,2)(5.6 + 100)$.

From the first equation, we obtain $P(t,1)$

$$P(t,1) = \frac{100}{105.2} (= 0.950570) \tag{16.3}$$

and then from the second, the value of $P(t,2)$:

$$P(t,2) = \frac{102 - 0.950570 \times 5.6}{105.6} (= 0.915228). \tag{16.4}$$

Consequently, the yield rates for one and two years are given by:

$$\begin{aligned} i(t,1) &= \left(0.950570\right)^{-1} - 1 = 5.2\%, \\ i(t,2) &= \left(0.915228\right)^{-1/2} - 1 = 4.53\%. \end{aligned} \tag{16.5}$$

Let us point out that, in this example, there is a phenomenon of *inversion of the yield curve* as the yield for a maturity of two years is smaller than the yield for a maturity of one year.

Of course, in practice, this method needs a bond market liquid enough to have all the data available for all maturities, and moreover a statistical treatment with the least squares method can be used to improve the method.

16.1.3. *Yield to maturity for a financial investment and for a bond*

Let us consider a financial investment of present value C generating the following financial flow:

$$F = \left\{ \left(F_j, t_j \right) j = 1, \ldots, n \right\}. \tag{16.6}$$

The *yield to maturity* is the constant discount rate or actuarial rate, the $i(F)$ solution of the polynomial equation:

$$C = \sum_{j=1}^{n} \left(1 + i(F)\right)^{-t_j} F_j. \tag{16.7}$$

Using the traditional Newton interpolation with the nominal or coupon rate as initial value, this solution is easily given.

For the particular case of a bond of *subscription price A* at time t and maturity time $t+T$, and with *coupon value C* and *nominal value P*, the corresponding financial flow is given by:

– at times $t+1,...,t+T-1$: payment of the coupon C;

– at time $t+T$: payment of the coupon C, and of the nominal value.

Equation (16.1) or (16.7) becomes:

$$A(t,T) = \left(1+i(F)\right)^{-1} C + \left(1+i(F)\right)^{-2} C + ... \\ + \left(1+i(F)\right)^{-(t+T-1)} C + \left(1+i(F)\right)^{-(t+T)} \left(C+P\right). \tag{16.8}$$

It is clear that the yield to maturity $i(F)$ is also a function of t and T.

16.2. Dynamic deterministic continuous time model for instantaneous interest rate

16.2.1. *Instantaneous interest rate*

In this section, we recall briefly some basic concepts fully described in section 3.7 using a discrete time model for the financial flows and the interest rates.

Now, we will use the traditional deterministic continuous time model (DCTM) for an investment on $[t,t+T]$ of amount $C(t)$ at time t producing a continuous yield of rate $r(s;t,T)$ at time s.

So, we see that this rate depends of t and T and on the "small" time interval $\left[s,s+\Delta s\right]\subset\left[t,t+T\right]$, one monetary unit at time t produces at the end of the interval a yield of value $r(s;t,T)\Delta s$. This rate is called the *continuous time instantaneous rate* or, in short, the *instantaneous rate* for an investment at time t and of maturity time $t+T$.

Let $C(s)$ be the capitalization value of $C(t)$ at time $s, s>t$.

From the definition of the instantaneous rate, it is clear that:

$$C(s+\Delta s) = C(s) + r(s;t,T)C(s)\Delta s \cdot \tag{16.9}$$

With traditional limit reasoning, we obtain the following relation:

$$\frac{C'(s)}{C(s)} = r(s;t,T) \tag{16.10}$$

and by integration:

$$C(s) = C(t)e^{\int_t^{t+s} r(u;t,T)du} \tag{16.11}$$

In particular, at maturity, we obtain:

$$C(t+T) = C(t)e^{\int_t^{t+T} r(u;t,T)du} \tag{16.12}$$

16.2.2. *Particular cases*

As r is a function of three variables, it is useful to distinguish the four following cases:

a) *stationarity in time*: r does not depend on t:
$r(s;t,T) = r(s;T)$,

b) *stationarity in maturity* : r does not depend on T:
$r(s;t,T)=r(s;t)$,

c) *stationarity in time and in maturity*: r does not depend both on t and T:
$r(s;t,T) = r(s)$,

d) *constant case*: r is independent of the three considered variables:
$r(s;t,T) = \delta$.

For the last case, we get back the well known result of section 3.7:

$$C(t) = C(0)e^{\delta T}.$$

16.2.3. *Yield curve associated with instantaneous interest rate*

For the preceding section, we know that for an investment of €1 at time t and of maturity time $t+T$, the capitalization value at maturity is given by

$$e^{\int_t^{t+T} r(u;t,T)du} \tag{16.13}$$

Using the *yield curve* $T \mapsto i(t,T), T \geq 0$, for a fixed *t*, corresponding to this investment for which *i*(*t,T*) represents the corresponding annual interest rate on [*t,t*+*T*] given the same capitalization values as (16.13), we obtain:

$$\left(1+i(t,T)\right)^T = e^{\int_t^{t+T} r(u;t,T)du} \tag{16.14}$$

and so:

$$i(t,T) = e^{\frac{1}{T}\int_t^{t+T} r(u;t,T)du} - 1 \tag{16.15}$$

The constant instantaneous rate $\delta(t,T)$ on [*t,t*+*T*] corresponding to this yield curve is defined as follows:

$$e^{\int_t^{t+T} \delta(t,T)du} = e^{\int_t^{t+T} r(u;t,T)du}$$

or

$$e^{\delta(t,T)T} = e^{\int_t^{t+T} r(u;t,T)du} \tag{16.16}$$

i.e.:

$$\delta(t,T) = \frac{1}{T}\int_t^{t+T} r(u;t,T)du$$

16.2.4. *Examples of theoretical models*

1) Constant case

For *r*(*s,t;T*)= δ , relation (16.16) gives for the yield curve of the traditional case of deterministic traditional finance:

$$i(t,T) = e^{\frac{1}{T}\int_t^{t+T} \delta du} - 1$$
or
$$i(t,T) = e^{\delta} - 1 \tag{16.17}$$

From this last relation, we obtain:

$$\delta = \ln(1+i) \tag{16.18}$$

2) Deterministic Ornstein-Uhlenbeck-Vasicek model (1973), (Janssen and Janssen (1996))

Starting from the following relation:

$$r(t+\Delta t) - r(t) \simeq a(b - r(t))\Delta t, \Delta t > 0, t \geq 0 \tag{16.19}$$

we obtain for $\Delta t \to 0$ the following differential equation:

$$dr(t) = a(b - r(t))dt \tag{16.20}$$

for which the general solution is given by:

$$r(t) = b - Ke^{-at} \tag{16.21}$$

With the initial condition:

$$r(0) = r_0$$

where r_0 is the observed instantaneous rate or *spot rate* observed at t=0, the constant K can be calculated to find the following unique solution:

$$r(t) = b + \left(r_0 - b\right)e^{-at} \tag{16.22}$$

or

$$r(t) = r_0 e^{-at} + b\left(1 - e^{-at}\right) \tag{16.23}$$

So, the function r is a linear convex combination of r_0 and parameter b.

To find the economic-financial significance of this last parameter, it suffices to let $t \to \infty$ to see that:

$$b = \lim_{t\to\infty} r(t) \tag{16.24}$$

which is the anticipated value of the long term spot rate.

To see what the other parameter represents, we obtain from relation (16.23):

$$r'(t) = -ae^{-at}\left(r_0 - b\right) \tag{16.25}$$

and so the sign of the derivative function of r is those of a if $r_0 < b$ or of $-a$ if $r_0 > b$, and moreover:

$$r'(0) = -a\left(r_0 - b\right) \tag{16.26}$$

In conclusion, if $r_0 < b$, function r is strictly increasing, starting from r_0 at t=0 and tending towards b for large t; on the other hand, if $r_0 > b$, function r is strictly decreasing, starting from r_0 at t=0 and tending towards b for large t.

In the two cases the absolute value of the slope at t=0 is an increasing function of a; this means that the convergence is faster for large values of a than for small values. This is why parameter a is often called the *convergence parameter*.

The frontier case $r_0 = b$ gives the very special case of a *flat yield curve*.

To obtain the yield curve corresponding to the instantaneous rate given by relation (16.23), it suffices to substitute the value of r in relation (16.15); this calculation (see Janssen and Janssen (1996)) gives the following result:

$$i(0,T) = e^{b + \frac{b - r_0}{bT}(e^{-aT} - 1)} - 1 \tag{16.27}$$

More generally, starting from t with r_0 as in initial rate, this last formula becomes

$$i(t,T) = e^{b + \frac{b - r_0}{bT}(e^{-aT} - 1)} - 1 \tag{16.28}$$

So, we see that we have a *stationary model*, as on $\left[t, t+T\right]$ there is no influence of time t.

16.3. Stochastic continuous time dynamic model for instantaneous interest rate

In finance, it is well known that the future values of the rates are uncertain as there is a large influence from many financial and economic parameters, also depending on political factors.

It follows that deterministic models are unsatisfactory and so the new discipline of mathematical finance called “stochastic finance” was born from the results of Samuelson (1965), Black, Merton and Scholes (1973).

In this section, we will present the three most important stochastic models used in practice: the *Ornstein-Uhlenbeck-Vasicek* (*OUV*) model, the *Cox, Ingersoll and Ross* (*CIR*) model and the *Heath, Jarrow and Morton* (*HJM*) model.

The first two models are related to the instantaneous rate or spot rate, and the last starts from the yield curve at time 0 to model this entire yield curve at time *t*.

Other models are possible; for example the *Brennan and Schwartz* model considers two rates: the spot and the long term rates both modeled with a system of two SDEs.

16.3.1. *The OUV stochastic model*

16.3.1.1. *The model*

As usual, we consider a complete filtered probability space $\left(\Omega,\Im,\left(\Im_t\right),P\right)$ on which all the defined stochastic processes will be adapted, in particular, the following standard Brownian motion.

The considered OUV model starts with the following stochastic dynamic for the spot rate process $r=\left(r(t),t\geq 0\right)$

$$\begin{aligned} &dr(t)=a(b-r(t))dt+\sigma dB(t)\\ &r(0)=r_0. \end{aligned} \tag{16.29}$$

This means that *r* is a special diffusion process extending the deterministic OUV case depending on four parameters: a,b,r_0,σ , assumed to be constant and known.

Using Itô’s calculus, it is possible to show (see Appendix to this chapter) that the unique solution of this SDE is given by:

$$r(t)=b+\left(r_0-b\right)e^{-at}+\sigma e^{-at}\int_0^t e^{as}dB(s) \tag{16.30}$$

Indeed, if we differentiate this function, we obtain:

$$dr(t) = -a\left(r_0 - b\right)e^{-at} - a\sigma e^{-at}\int_0^t e^{as}dB(s) + \sigma e^{-at}e^{at}dB(s)$$

and as

$$r(t) - b - \left(r_0 - b\right)e^{-at} = \sigma e^{-at}\int_0^t e^{as}dB(s)$$

$$dr(t) = -a\left(r_0 - b\right)e^{-at} - a\left(r(t) - b - \left(r_0 - b\right)e^{-at}\right) + \sigma dB(t)$$

and so:

$$dr(t) = a(b - r(t))dt + \sigma dB(t)$$

16.3.1.2. *Model and parameters interpretation*

From result (16.30) and the traditional rules of Itô differentiation seen in Chapter 4, we obtain the *mean of* $r(t)$:

$$m(t) = E\left[r(t)\right] = b + \left(r_0 - b\right)e^{-at} \tag{16.31}$$

So this mean is nothing other than the value of *r* in the deterministic OUV model and moreover $m(t)$ tends towards *b* for $t \to \infty$.

Consequently, the interpretation of the parameter *b* is the same as in the deterministic case that is the *anticipated spot rate for long term*.

Concerning the *variance* of *r(t),* we still use Itô differentiation:

$$\operatorname{var} r(t) = \operatorname{var}\left(\sigma e^{-at}\int_0^t e^{as}dB(s)\right) \tag{16.32}$$

So:

$$\operatorname{var} r(t) = \sigma^2 e^{-2at}\int_0^t e^{2as}dB(s) \tag{16.33}$$

and finally:

$$\operatorname{var} r(t) = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \tag{16.34}$$

Results (16.31) and (16.34) show that the financial and economic interpretations of parameters *a* and *b* are identical as in the deterministic OUV model but also that the *key parameter* here is

$$\lambda = \frac{\sigma^2}{2a} \tag{16.35}$$

as indeed, it represents the value of the asymptotic variance of $r(t)$ as t tends to $+\infty$, and moreover this asymptotic variance is a linear function of λ.

So, this variance is smaller for a weakly volatile market and larger for a market with large volatility, in conformity with *n* empirical studies.

Parameter *a* has an opposite effect: large (small) values of a give smaller (larger) values of the variance of r(t).

To conclude, we see that:

(i) the variance of $r(t)$ is increasing with time, confirming the fact that the uncertainty on the rate values increases with time;

(ii) the larger parameter σ, called *volatility*, is, the greater the impact of randomness;

(iii) the larger parameter *a*, called *convergence parameter*, is, greater the convergence of the spot rate towards *b*.

16.3.1.3. *Marginal distribution of r(t), fixed t*

To calculate the distribution of $r(t)$, for all fixed t, it suffices from relation via relation (16.30) to calculate one of the r.v. of $X(t)$ defined by:

$$X(t) = \int_0^t e^{as} dB(s) \cdot \tag{16.36}$$

Coming back to the definition of stochastic integral given in Chapter 13, let us consider a sequence of subdivisions of [*0,t*]:

$$\Pi_n = (t_0, \ldots, t_n), t_0 = 0, t_n = t, n \in \mathbb{N}_0.$$

Then we know that

$$\int_0^t e^{as} dB(s) = \lim_{\nu_n} \sum_{i=0}^{n} e^{at_i} \left[B(t_{i+1}) - B(t_i) \right], \tag{16.37}$$

ν_n being the norm of subdivision Π_n and using the uniform convergence in probability.

However, from the properties of the standard Brownian motion (see Chapter 10), we know that for each such subdivision, the distribution of the sum

$$\sum_{i=0}^{n} e^{at_i} \left[B(t_{i+1}) - B(t_i) \right] \tag{16.38}$$

is normal with zero mean and with variance

$$\sum_{i=0}^{n-1} e^{at_i} (t_{i+1} - t_i) . \tag{16.39}$$

As $\nu \to 0$, this variance converges to

$$\text{var}(X(t)) = \int_0^t e^{2as} ds (= \frac{e^{2at} - 1}{2}) . \tag{16.40}$$

As $X(t) \prec N(0, \frac{e^{2at} - 1}{2})$, we obtain from results (16.30), (16.31) and (16.33) that

$$r(t) \prec N\left(b + \left(r_0 - b \right) e^{-at}, \frac{\sigma^2}{2a} \left(1 - e^{-2at} \right) \right) . \tag{16.41}$$

16.3.1.4. *Confidence interval for r(t), fixed t.*

From result (16.41), we can easily give a confidence interval at level $1 - \alpha$, for example with $\alpha = 5\%$. Indeed, if λ_α is a quantile of a r.v. $X \prec N(0,1)$ such that:

$$P\left[|X| \leq \lambda_\alpha \right] = 1 - \alpha , \tag{16.42}$$

we obtain

$$P[\frac{r(t) - b - \left(r_0 - b \right) e^{-at}}{\sigma \sqrt{\frac{1 - e^{-2at}}{2a}}} \leq \lambda_\alpha] = 1 - \alpha . \tag{16.43}$$

Consequently, the confidence interval at level $1-\alpha$ is given by

$$\left(b+\left(r_0-b\right)e^{-at}\right)-\lambda_\alpha\sigma\sqrt{\frac{1}{2a}\left(1-e^{-2at}\right)},$$
$$\left(b+\left(r_0-b\right)e^{-at}\right)+\lambda_\alpha\sigma\sqrt{\frac{1}{2a}\left(1-e^{-2at}\right)} \tag{16.44}$$

From relation (16.15), we also find a confidence interval at level $1-\alpha$ for $i(t,T)$ given by:

$$e^{b+\frac{b-r_0}{aT}\left(e^{-\lambda T}-1\right)}e^{-\frac{\lambda_\alpha\sigma}{T}\int_0^T\sqrt{\frac{1-e^{-2as}}{2a}}ds}-1\le i(t,T)\le e^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)}e^{\frac{\lambda_\alpha\sigma}{T}\int_0^T\sqrt{\frac{1-e^{-2as}}{2a}}ds}-1. \tag{16.45}$$

As the length of the half interval for r(t) quickly tends $\frac{\lambda_\alpha\sigma}{\sqrt{2a}}$ for $T\to\infty$, we obtain approximatively:

$$e^{\frac{1}{T}\int_0^T\left(r_0e^{-as}+\left(1-e^{-as}\right)b-\lambda_\alpha\frac{\sigma}{\sqrt{2a}}\right)ds}-1\le i(t,T)\le e^{\frac{1}{T}\int_0^T\left(r_0e^{-as}+\left(1-e^{-as}\right)b+\lambda_\alpha\frac{\sigma}{\sqrt{2a}}\right)ds}-1, \tag{16.46}$$

and finally:

$$e^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)}e^{-\frac{\lambda_\alpha\sigma}{\sqrt{2a}}}-1\le i(t,T)\le e^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)}e^{\frac{\lambda_\alpha\sigma}{\sqrt{2a}}}-1. \tag{16.47}$$

In particular, if the basic coefficient $\frac{\lambda_\alpha\sigma}{\sqrt{2a}}$ is such that:

$$\frac{\lambda_\alpha\sigma}{\sqrt{2a}}<<<1, \tag{16.48}$$

we have with a probability near to $1-\alpha$

$$e^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)}\approx 1+i(t,T) \tag{16.49}$$

and so with condition (16.48), we see that the deterministic OUV model is a good approximation of the stochastic model.

More precisely, we have:

$$me^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)} \leq 1+i(t,T) \leq Me^{b+\frac{b-r_0}{aT}\left(e^{-aT}-1\right)},$$
$$m = e^{-\frac{\lambda_\alpha \sigma}{\sqrt{2a}}}, M = e^{\frac{\lambda_\alpha \sigma}{\sqrt{2a}}} (=1/m).$$

The following table gives some numerical values for *m* and *M*.

1-α	λ	σ	*a*	coefficient	λ coefficient	*m*	*M*
0.05	1.960	0.1	0.1	0.2236	0.01118	0.989	1.011
0.025	2.241	0.2	1.1	0.1348	0.00337	0.997	1.003
0.05	1.960	0.1	5	0.0316	0.00158	0.998	1.002
0.025	2.241	0.2	5	0.0632	0.00158	0.998	1.002

16.3.1.5. *Monte Carlo simulation method*

From the result (16.30) and the definition of the stochastic integral, we obtain the following approximation

$$r(t) \approx b + \left(r_0 - b\right)e^{-at} + \sigma e^{-at} \sum_{i=0}^{n} e^{at_i} \left[B(t_{i+1}) - B(t_i)\right], \tag{16.50}$$

corresponding to the following subdivision of the interval [0,*t*]

$$\Pi_n = \left(t_0, \ldots, t_n\right), \quad t_0 = 0, t_n = t. \tag{16.51}$$

To obtain a simulation of a sample path for the *r*-process on [*0,T*], it suffices to simulate a sample path ω of the standard Brownian process $\left(B(t), t \geq 0\right)$ giving the observed values

$$\left\{B(t_i,\omega), t_i \in \left\{t_0,\ldots,t_n\right\}, t_0 = 0, t_n = T\right\} \tag{16.52}$$

from which we deduce the observed values for the *r*-process given by:

$$\left\{\begin{array}{l} r(t_i,\omega) = b + (r_0 - b)e^{-at_i} + \sigma e^{-at_i} \sum_{i=1}^{n} e^{at_i} \left[B(t_{i+1},\omega) - B(t_i,\omega)\right], \\ t_i \in \left\{t_0,\ldots,t_n\right\}, t_0 = 0, t_n = T \end{array}\right\}. \tag{16.53}$$

16.3.2. *The CIR model (1985)*

16.3.2.1. *The model*

In 1985, Cox, Ingersoll and Ross presented a new model for the temporal structure of interest rates for which the inconvenience of having negative values, as is the case for the OUV model, were dropped.

To obtain the model, the authors introduced a factor $\sqrt{r(t)}$ in the coefficient of dB, which is why their model is also called the *square root model*.

The stochastic differential equation governing this model is the following one:

$$\begin{aligned} &dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dB(t), \\ &r(0) = r_0. \end{aligned} \tag{16.54}$$

with the same assumptions as in the OUV model.

16.3.2.2. *Model and parameters interpretation*

Now, we have a non-linear stochastic differential equation, but as the spirit of this model is the same for the OUV model as for $\sigma = 0$, we still obtain the deterministic OUV model.

16.3.2.3. *Marginal distribution of r(t), fixed t*

This non-linearity implies that there does not exist a simple "explicit" form of the solution of problem (3.26); nevertheless, the authors obtained the following explicit form of the conditional density function of $r(t)$, giving $r(s)$ ($s<t$):

$$\begin{aligned} &f(y,t\,|\,r(s) = x) = Ke^{-K(u+v)}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}), \\ &K = \frac{2a}{\sigma^2(1-e^{-a(t-s)})}, \\ &u = Kxe^{-a(t-s)}, v = Ky, \\ &q = \frac{2ab}{\sigma^2} - 1. \end{aligned} \tag{16.55}$$

I_q being the modified Bessel function of second kind of order q defined as the following convergent series:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(n+k+1)},$$
$$I_{-n}(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{-n+2k}}{k!\Gamma(-n+k+1)} \tag{16.56}$$

n being a positive natural number and Γ the usual Eulerian function of the second kind.

From result (16.55), it is possible to obtain the following expressions for the conditional mean and variance of $r(t)$:

$$E\left[r(t)\middle|r(s)=x\right] = xe^{-a(t-s)} + b(1-e^{-a(t-s)}),$$
$$\operatorname{var}\left[r(t)\middle|r(s)=x\right] = x\frac{\sigma^2}{a}(e^{-a(t-s)} - e^{-2a(t-s)}) + b\frac{\sigma^2}{2a}(1-e^{-a(t-s)})^2. \tag{16.57}$$

Remark 16.1 These last results immediately give the asymptotic forms of the mean and the variances as follows:

$$\lim_{t\to\infty} E\left[r(t)\middle|r(s)=x\right] = b$$
$$\operatorname{var}\left[r(t)\middle|r(s)=x\right] = b\frac{\sigma^2}{2a}. \tag{16.58}$$

These results show that the interpretation of the two parameters a and b is the same as the OUV models, and we see that the conditional variance is inversely proportional to a.

Let us also point out that for $a \to 0$, we obtain for fixed t:

$$E\left[r(t)\middle|r(s)=x\right] \to x,$$
$$\operatorname{var}\left[r(t)\middle|r(s)=x\right] \to x\sigma^2(t-s). \tag{16.59}$$

Remark 16.2 The authors show that:

(i) if $2ab \geq \sigma^2$, then the solution of the SDE (16.54) never becomes zero starting with a strictly positive value at time 0;

(ii) if $2ab < \sigma^2$, then it is possible that the spot rate takes on a 0 value but it will never take negative values as it could be for the OUV model.

16.3.2.4. *Confidence interval for r(t), fixed t*

As the conditional distribution of $r(t)$ is given by result (16.55), it is possible to construct a confidence interval for this spot rate at time t.

Moreover, asymptotically, the authors proved that:

$$\begin{aligned} &\lim_{t\to\infty} f(y,t\left|r(s)=x\right.)=\frac{\theta^{\nu}}{\Gamma(\nu)}y^{\nu-1}e^{-\theta y}, \\ &\theta=\frac{2a}{\sigma^2},\nu=\frac{2ab}{\sigma^2}. \end{aligned} \tag{16.60}$$

Of course, this last result gives results (16.58) for the asymptotic mean and variances.

From the practical point of view, it is seen that the bounds of the confidence interval for fixed t, quickly converge to the bounds of the asymptotic one.

16.3.2.5. *Monte Carlo simulation method*

As for the preceding model, the simulation of trajectories of the r-process is done with time discretization of the stochastic differential equation (16.54) with a fixed partition of $[0,t]$:

$$\begin{aligned} &\Pi_n=\left(t_0,\ldots,t_n\right), \\ &t_0=0,t_n=t, \end{aligned} \tag{16.61}$$

often with equal subintervals of length t/n.

This leads to the following non-linear system:

$$\left\{\begin{aligned} &r(t_{i+1},\omega)=r(t_i,\omega)+a(b-r(t_i,\omega)(t_{i+1}-t_i) \\ &\qquad\qquad +\sigma\sqrt{r(t_i,\omega}\left[B(t_{i+1},\omega)-B(t_i,\omega)\right], \\ &r(0)=r_0,t_i\in\left\{t_0,\ldots,t_n\right\},t_0=0,t_n=T. \end{aligned}\right\}. \tag{16.62}$$

This system being recursive is easily solved and so we obtain the following simulated trajectory:

$$(r_0, r(t_1,\omega),..., r(t_i,\omega),..., r(t_n,\omega)). \tag{16.63}$$

16.3.3. *The HJM model (1992)*

16.3.3.1. *Motivation*

This model starts from a quite different point of view than the two preceding models as the authors want to model the *entire yield curve* starting with the "actual" given yield curve, that is, at time 0.

Their general result is overall theoretical, and it provides two particular models known as the *Ho and Lee* and the *generalized Vasicek* model.

16.3.3.2. *The forward rates*

Let *f*(*t,s*) (*t*<*s*) be the instantaneous forward rate at time *t*, which will be attributed at time *s*.

This means that on the future interval $(s, s+\Delta s)$, the attributed yield will be approximatively $f(t,s)\Delta s$.

Under our AOA assumption, the investment of one monetary unit on [*0,s*] must produce the same yield as an investment of one monetary unit on [*0,t*] followed by the investment of the capitalized value at time *t* on the time interval [*t,s*]; so we must have the following relation:

$$e^{\int_0^t r(u)du} \times e^{\int_t^s f(t,u)du} = e^{\int_0^s r(u)du}, \tag{16.64}$$

or

$$e^{\int_t^s f(t,u)du} = e^{\int_t^s r(u)du} (=1/P(t,s)), \tag{16.65}$$

P(*t,s*) being the value of a zero coupon at time *t* of time maturity *s* years and *r* the instantaneous continuous rate function.

This last relation is equivalent to:

$$\int_t^s f(t,u)du = -\ln P(t,s), \tag{16.66}$$

or still by derivation:

$$f(t,s) = -\frac{\partial \ln P}{\partial s}(t,s). \tag{16.67}$$

Relation (16.67) also gives the *forward value of a zero-coupon*, as usual without default risk, calculated at t=0

$$P(t,T) = e^{-\int_t^T f(t,u)du}. \tag{16.68}$$

The instantaneous continuous rate $r(t)$ is given by

$$r(t) = \lim_{T \to t} f(t,T) = f(t,t) \tag{16.69}$$

assuming that function f is continuous.

Let us also recall the following links with the yield curve i, $i(t,s)$ being the equivalent annual interest rate for an investment of one monetary unit decided at time t up to time s.

$$(1+i(t,s))^{-(t-s)} = P(t,s). \tag{16.70}$$

From relation (16.67), we also obtain:

$$\begin{aligned} f(t,s) &= -\frac{\partial}{\partial s}\ln(1+i(t,s))^{-(t-s)}, \\ f(t,s) &= (t-s)\ln(1+i(t,s)). \end{aligned} \tag{16.71}$$

16.3.3.3. *The HJM methodology*

As already pointed out in section 3.3.1, the main idea of the authors is to build a stochastic model for the forward instantaneous rate curve at time t, $f(0,T), T \geq 0$ given at time 0, and the observable forward instantaneous rate curve $f(0,T), T \geq 0$.

From the preceding section, we now know that:

$$P(t,T) = e^{-\int_t^T f(t,u)du} \tag{16.72}$$

and

$$f(t,s) = -\frac{\partial \ln P}{\partial s}(t,s). \tag{16.73}$$

We assume that the stochastic dynamics of the zero-coupon is governed by the following SDE

$$dP(t,T) = \mu(t,T)P(t,T)dt + \sigma(t,T)P(t,T)dB(t), \tag{16.74}$$

B being a standard Brownian motion in the considered complete filtered probability space.

From Itô's formula, we obtain:

$$df(t,T) = \frac{\partial}{\partial T}(\frac{\sigma^2(t,T)}{2} - \mu(t,T))dt - \frac{\partial}{\partial T}(\sigma(t,T)dB(t), \tag{16.75}$$

a relation representing the SDE for the stochastic dynamics of the process $f(t,T), T>t$.

Remark 16.3 From relation (16.69), we know that

$$\begin{aligned} r(t) &= f(t,t), \\ &\text{or} \\ &= f(t^*,t) + \int_{t^*}^{t} df(s,t), (t^* < t). \end{aligned} \tag{16.76}$$

assuming that at time t^*, the values $f(t^*,t)$ are known for all $t \geq t^*$.

Using relation (16.75), we obtain:

$$r(t) = f(t^*,t) + \int_{t^*}^{t} [\sigma(s,t)\frac{\partial}{\partial t}\sigma(s,t) - \frac{\partial}{\partial t}\mu(t,s)]ds - \int_{t^*}^{t} \frac{\partial}{\partial T}(\sigma(t,s)dB(s), \tag{16.77}$$

The calculation of the Itô differential of r (see Wilmott (2000)) gives:

$$dr = \begin{pmatrix} \left[\dfrac{\partial f(t^*,t)}{\partial t} - \dfrac{\partial \mu(t,s)}{\partial s}\right]_{s=t} \\ + \displaystyle\int_{t^*}^{t} [\sigma(s,t)\frac{\partial^2}{\partial t^2}\sigma(s,t) + (\frac{\partial \sigma(s,t)}{\partial t})^2 \\ - \dfrac{\partial^2 \mu(s,t)}{\partial t^2})ds - \displaystyle\int_{t^*}^{t} [\frac{\partial^2 \sigma(s,t)}{\partial t^2} dB(s) \end{pmatrix} dt - \left.\frac{\partial \sigma(t,s)}{\partial s}\right|_{s=t} dB(t). \tag{16.78}$$

We give this result to observe that the last coefficient of dt depends on all the past processes from t^* up to time t, and so dynamics no longer defines a Markov process.

This is a serious complication for the HJM model as an infinite number of variables are needed to solve this equation. That is why we can only consider particular cases.

Under the risk-neutral measure Q, we know that the volatility still remains identical and the trend is the riskless instantaneous interest rate $m(t)$, so that:

$$\begin{aligned} &df(t,T) = m(t,T)dt + \nu(t,T)d\tilde{B}(t), \\ &\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T), \\ &f(0,T) = f^*(0,T) \end{aligned} \tag{16.79}$$

where f^* is the forward rate curve at time 0 and from the Girsanov theorem $\tilde{B}$ is a new standard Brownian motion.

To find the value of the trend under Q, let us start with the drift under the historical measure P given in relation (16.75):

$$\frac{\partial}{\partial T}(\frac{\sigma^2(t,T)}{2} - \mu(t,T)) \tag{16.80}$$

and as under the risk-neutral measure Q, the drift μ of the dynamics of the zero coupons is nothing other than $r(t)$, we obtain from the second equality of (16.79) and taking into account that $\sigma(t,t) = 0$, as $P(t,t)=1$:

$$\begin{aligned} \frac{\partial}{\partial T}(\frac{\sigma^2(t,T)}{2} - \mu(t,T)) &= \nu(t,T)\int_t^T \nu(t,s)ds - \frac{\partial}{\partial T}r(t), \\ &= \nu(t,T)\int_t^T \nu(t,s)ds. \end{aligned} \tag{16.81}$$

So, under the measure Q, the stochastic dynamics of the HJM model become:

$$\begin{aligned} &df(t,T) = \tilde{\mu}(t,T)dt + \nu(t,T)d\tilde{B}(t), \\ &\mu(t,T) = \nu(t,T)\int_t^T \nu(s,T)ds, \\ &f(0,T) = f^*(0,T). \end{aligned} \tag{16.82}$$

where, from the second equality of (16.79):

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T). \tag{16.83}$$

Thus, we can now provide the way to obtain the pricing of the zero coupons without default risk:

1) observe the forward instantaneous yield curve of the market at time t=0: $f^*(0,T)$;

2) determine the volatility $\nu(t,T)$;

3) calculate drift $\mu(t,T)$ from (16.82);

4) evaluate the forward instantaneous yield curve at time t under the risk-neutral measure:

$$f(t,T) = f^*(0,T) + \int_0^t \mu(s,T)ds + \int_0^t \nu(s,T)d\tilde{B}(s) \tag{16.84}$$

5) evaluate the instantaneous short term or spot rate at time t under the risk-neutral measure:

$$\begin{aligned} r(t) &= f(t,t), \\ &= f^*(0,t) + \int_0^t \mu(s,t)ds + \int_0^t \nu(s,t)d\tilde{B}(s) \end{aligned}; \tag{16.85}$$

6) evaluate zero coupons at time t of several maturities T under the risk-neutral measure:

$$\begin{aligned} P(t,T) &= e^{-\int_t^T f(t,s)ds} \\ &= e^{-\left[\int_t^T f^*(0,u)du + \int_t^T du \int_0^u \mu(s,u)ds + \int_t^T du \int_0^u \nu(s,u)d\tilde{B}(s)\right]} \end{aligned}. \tag{16.86}$$

16.3.3.4. *Particular cases of the HJM model: the Ho and Lee and generalized Vasicek models*

16.3.3.4.1. The Ho and Lee model

This is the simplest and the most useful model with the very particular assumption that the volatility is constant: $\nu(t,T) = \nu$.

With this assumption, the methodology given above provides many simplifications and leads to the following one:

1) observe the forward instantaneous yield curve of the market at time t=0: $f^*(0,T)$;

2) determine the volatility $\nu(t,T)=\nu$;

3) calculate drift $\mu(t,T)$ from (3.54): $\mu(t,T)=\nu^2(T-t)$;

4) evaluate the forward instantaneous yield curve at time t under the risk-neutral measure:

$$\begin{aligned} f(t,T) &= f^*(0,T)+\int_0^t \mu(s,T)ds+\int_0^t \nu(s,T)d\tilde{B}(s) \\ &= f^*(0,T)+\nu^2 t\left(T-\frac{t}{2}\right)+\nu\tilde{B}(t) \end{aligned} ; \quad (16.87)$$

5) evaluate the instantaneous short term or spot rate at time t under the risk-neutral measure:

$$\begin{aligned} r(t) &= f(t,t), \\ &= f^*(0,t)+\int_0^t \mu(s,t)ds+\int_0^t \nu(s,t)d\tilde{B}(s) \\ &= f^*(0,t)+\int_0^t \nu^2(t-s)ds+\int_0^t \nu d\tilde{B}(s) \\ &= f^*(0,t)+\frac{\nu^2 t^2}{2}+\nu\tilde{B}(t) \end{aligned} ; \quad (16.88)$$

6) evaluate zero coupons at time t of several maturities T under the risk-neutral measure:

$$\begin{aligned} P(t,T) &= e^{-\int_t^T f(t,s)ds} \\ &= e^{-\left[\int_t^T f^*(0,s)ds+\int_t^T \nu^2 t\left(s-\frac{t}{2}\right)ds+\int_t^T \nu\tilde{B}(t)ds\right]} \\ &= e^{-\int_t^T f^*(0,s)ds-\frac{\nu^2 Tt(T-t)}{2}-\nu\tilde{B}(t)(T-t)} \end{aligned} . \quad (16.89)$$

From (16.87), we immediately obtain:

$$\text{(i) } f(t,T) \prec N\left(f(0,T) + \frac{\nu^2}{2} t\left(T - \frac{t}{2}\right), \nu^2 t \right) \tag{16.90}$$

showing that, in the Vasicek model, negative values for *f* could be observed;

$$\text{(ii) } r \prec N\left(f(0,t) + \frac{\nu^2 t^2}{2}, \nu^2 t \right) \tag{16.91}$$

showing that, in the Vasicek model, negative values for *r* could be observed;

(iii) (distribution lognormality of the zero coupons)

Relation (16.89) leads to:

$$P(t,T) = e^{-\int_t^T f^*(o,s)ds - \frac{\nu^2 Tt(T-t)}{2} - \nu\tilde{B}(t)(T-t)}, \tag{16.92}$$

and so

$$\ln P(t,T) = -\int_t^T f^*(o,s)ds - \frac{\nu^2 Tt(T-t)}{2} - \nu\tilde{B}(t)(T-t). \tag{16.93}$$

As

$$\begin{aligned} &\ln P(t,T) = -\int_t^T f^*(0,s)ds - \frac{\nu^2 Tt(T-t)}{2} - \nu\tilde{B}(t)(T-t) \\ &E\left[-\int_t^T f^*(0,s)ds - \frac{\nu^2 Tt(T-t)}{2} - \nu\tilde{B}(t)(T-t) \right] = -\int_t^T f^*(0,s)ds - \frac{\nu^2 Tt(T-t)}{2} \\ &\text{var}\left[-\int_t^T f^*(0,s)ds - \frac{\nu^2 Tt(T-t)}{2} - \nu\tilde{B}(t)(T-t) \right] = \nu^2 (T-t)^2 t, \end{aligned} \tag{16.94}$$

it follows that:

$$P(t,T) \prec LN\left(-\int_t^T f^*(0,s)ds - \frac{\nu^2 Tt(T-t)}{2}, \nu^2 (T-t)^2 t \right); \tag{16.95}$$

(iv) as $P(0,u)=\int_0^u f^*(0,s)ds, \forall u \geq 0,$ we still have that:

$$e^{-\int_t^T f^*(0,s)ds} = \frac{P(0,T)}{P(0,t)} \tag{16.96}$$

and thus from relation (3.61), we can write:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\frac{v^2 Tt(T-t)}{2} - v\tilde{B}(t)(T-t)}. \tag{16.97}$$

This last result gives the possibility to calculate the forward values of zero coupons without the forward instantaneous yield curve of the market at time t=0: $f^*(0,T)$, and to easily simulate these values.

16.3.3.4.2. The generalized Vasicek model

For this model, the volatility is given by:

$$v(t,T) = ve^{-k(T-t)} \tag{16.98}$$

the volatility tending to 0 as $t \to T$.

The general HJM methodology now becomes:

$$(i)\ \mu(t,T) = \frac{v^2}{k}(e^{-k(T-t)} - e^{-2k(T-t)})$$

$$(ii)\ df(t,T) = [\frac{v^2}{k}(e^{-k(T-t)} - e^{-2k(T-t)})]dt + ve^{-k(T-t)}d\tilde{B}(t)$$

$$(iii)\ \text{f(t,T)=f*(0,T)-}\frac{v^2}{2k^2}(1-e^{-k(T-t)})^2 + \frac{v^2}{2k^2}(1-e^{-kT})^2 + v\int_0^t e^{-k(T-s)}d\tilde{B}(s)$$

$$(iv)\ \text{r(t)=f*(0,T)} + \frac{v^2}{2k^2}(1-e^{-kt})^2 + v\int_0^t e^{-k(t-s)}d\tilde{B}(s)$$

$$(v)\ P(t,T) = e^{-\int_t^T f(t,s)ds} = \frac{P(0,T)}{P(0,t)} e^{\left[-\frac{K^2(t,T)}{2}L(t)+K(t,T)(f(0,t)-r(t))\right]},$$

$$K(t,T) = \frac{1-e^{-k(T-t)}}{t}, L(t) = \nu^2 \int_0^t e^{-2k(T-s)} ds = \nu^2 e^{-2kT} \frac{e^{2kt}-1}{2k}$$

$$(vi)\ \text{f(t,T)} \prec \text{N(f*(0,T)-}\frac{\nu^2}{2k^2}(1-e^{-k(T-t)})^2 + \frac{\nu^2}{2k^2}(1-e^{-kT})), \nu^2 e^{-2kT} \frac{e^{2kt}-1}{2k}$$

$$(vii)\ \text{r(t)} \prec N(\text{f*(0,T)} + \frac{\nu^2}{2k^2}(1-e^{-kt})^2), \nu^2 \frac{1-e^{-2kt}}{2k}$$

$(viii)$ $P(t,T)$ has a lognormal distribution.

Exercise 16.2 For this model, calculate the parameters of the distribution of $P(t,T)$.

Remark 16.4 There are many other rate models. For example, the discrete time *Ho and Lee model* (Ho and Lee (1986) uses a binomial tree and the *Hull and White model* (Hull-White (1996)), uses the following stochastic dynamics for the spot rate:

$$\begin{aligned} &\Delta r = (\theta(t) - ar)\Delta t + \sigma \Delta B, \\ &\Delta r = r(t+\Delta t) - r(t), \\ &\theta(t) > 0, \forall t, \\ &a > 0, \\ &\sigma > 0 \text{ (volatility)}, \\ &\Delta B = B(t+\Delta t) - B(t),\ B \text{ MBS}. \end{aligned} \tag{16.99}$$

Let us also mention the *Black, Derman and Toy model* (Black, Derman and Toy (1990)), starting with the following discrete time model for the spot rate:

$$\Delta \ln r(t) = \mu(r,t)\Delta t + \sigma(t)\Delta B(t). \tag{16.100}$$

16.4. Zero-coupon pricing under the assumption of no arbitrage

In the HJM model, we have introduced the dynamics of the zero coupons; in this section, we will do the same in the general case and finally for our two basic models, the OUV and CIR models.

16.4.1. *Stochastic dynamics of zero-coupons*

As before, let $P(t,s)$($t<s$) represent at time t the value of a zero-coupon of time maturity s, thus, of maturity $T=t-s$ at time t. Let $T=s-t$ so that with our preceding notation:

$$P(t,s)=P(t,t+T) \tag{16.101}$$

and so

$$P(s,s)=1. \tag{16.102}$$

The general problem of the evaluation of zero coupons consists of studying the stochastic process P defined on the filtered probability space $\left(\Omega,\Im,\left(\Im_t\right),P\right)$ as follows:

$$P=\left\{P(t,s,\omega),t\in\left[0,s\right]\right\}. \tag{16.103}$$

To study this process is equivalent to the study of process R of the equivalent instantaneous yields:

$$R=\left\{R(t,s,\omega),t\in\left[0,s\right]\right\}, \tag{16.104}$$

where:

$$\begin{aligned} P(t,s,\omega) &= e^{-(s-t)R(t,s-t,\omega)}, \\ &= e^{-TR(t,T,\omega)}. \end{aligned} \tag{16.105}$$

Let us repeat that $R(t,T,\omega)$ is the equivalent instantaneous rate constant on the time interval $\left[t,t+T\right]$ given by present value, the value of the zero coupon at time t, $P(t,T,\omega)$.

From this last relation, we can provide the value of R:

$$R(t,T,\omega)=-\frac{1}{T}\log P(t,t+T,\omega). \tag{16.106}$$

For the *spot* rate r, we have:

$$\lim_{T\to 0} R(t,T,\omega)=r(t,\omega). \tag{16.107}$$

The equivalent annual rate is given by:

$$e^{R(t,T,\omega)} = 1 + i(t,T,\omega). \qquad (16.108)$$

Let us now assume a *general stochastic dynamics* for the spot rate process r defined by the following SDE:

$$dr(t) = f(r,t)dt + \rho(r,t)dB(t). \qquad (16.109)$$

Observing P also as a function of r, Itô's formula provides the value of the stochastic differential $dP(t,s,r)$:

$$dP(t,s,r) = \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} f(r,t) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\rho^2(r,t)\right]dt + \frac{\partial P}{\partial r}\rho(r,t)dB(t). \qquad (16.110)$$

With

$$\begin{aligned} \mu(t,s,r) &= \frac{1}{P(t,s,r)}\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2}{\partial r^2}\right)P(t,s,r), \\ \sigma(t,s,r) &= -\frac{1}{P(t,s,r)}\rho\frac{\partial}{\partial r}P(t,s,r), \end{aligned} \qquad (16.111)$$

we obtain

$$dP(t,s,r) = P(t,s,r)\mu(t,s,r)dt - P(t,s,r)\sigma(t,s,r)dB(t). \qquad (16.112)$$

16.4.2. *Application of the no arbitrage principle and risk premium*

Let us study an investor issuing at time *t, x* zero coupon bonds expiring at time s_1 and investing y in zero coupon bonds expiring at time s_2.

The value $W(t)$ of this portfolio at time t is given by

$$W(t) = -xP(t,s_1,r) + yP(t,s_2,r). \qquad (16.113)$$

From the linearity property of Itô's formula and from relation (16.112), we obtain

$$\begin{aligned} dW(t) &= \left[yP(t,s_2,r)\mu(t,s_2,r) - xP(t,s_1,r)\mu(t,s_1,r)\right]dt \\ &-\left[yP(t,s_2,r)\sigma(t,s_2,r) - xP(t,s_1,r)\sigma(t,s_1,r)\right]dB(t). \end{aligned} \qquad (16.114)$$

Now, the assumption of AOA has two consequences:

(i) firstly, we have to cancel the risk component and so the coefficient of *dB* must have a 0 value;

(ii) then, the instantaneous yield of this portfolio must be, in every time interval (*t,t+dt*), the same as a riskless investment at the spot rate *r*(*t*).

Thus, from relation (16.114) we obtain:

$$\begin{aligned} &yP(t,s_2,r)\sigma(t,s_2,r) - xP(t,s_1,r)\sigma(t,s_1,r) = 0, \\ &yP(t,s_2,r)\mu(t,s_2,r) - xP(t,s_1,r)\mu(t,s_1,r) = rW(t). \end{aligned} \tag{16.115}$$

Moreover, as the value of *W*(*t*) is given by relation (16.113), we obtain the following linear system for the two unknown values *x* and *y*:

$$\begin{aligned} &yP(t,s_2,r)\sigma(t,s_2,r) - xP(t,s_1,r)\sigma(t,s_1,r) = 0, \\ &yP(t,s_2,r)\left[\mu(t,s_2,r) - r\right] - xP(t,s_1,r)\left[\mu(t,s_1,r) - r\right] = 0. \end{aligned} \tag{16.116}$$

As this system is homogenous, from Rouchè's theorem, we know that there exists a non-trivial solution, i.e. a solution with at least one value different from 0, if and only if the determinant of the system is different from 0.

Thus, the condition to have a financial market is

$$\frac{\mu(t,s_1,r) - r}{\sigma(t,s_1,r)} = \frac{\mu(t,s_2,r) - r}{\sigma(t,s_2,r)}, \tag{16.117}$$

for all t, s_1 and s_2.

This condition means that the function $\dfrac{\mu(t,s,r) - r}{\sigma(t,s,r)}$ is independent of *s* or that the function λ defined by

$$\lambda(t,r) = \frac{\mu(t,s,r) - r}{\sigma(t,s,r)} \tag{16.118}$$

is independent of *s*. Function λ represents the *risk premium of the market* as the difference between the instantaneous yield of the bond and the riskless rate *r*, normed by the volatility value σ.

16.4.3. *Partial differential equation for the structure of zero coupons*

Substituting the values of μ, σ from relations (16.111) into relation (16.118), we obtain the following *structural partial differential equation* (PDE) of zero-coupon bonds:

$$
\begin{aligned}
& rP(t,s,r) = \frac{\partial}{\partial t} P(t,s,r) + (f(r,t) + \rho(r,t)\lambda(r,t)) \frac{\partial}{\partial r} P(t,s,r) \\
& + \frac{1}{2}\rho^2 \frac{\partial^2}{\partial r^2} P(t,s,r), \\
& P(s,s,r) = 1.
\end{aligned}
\tag{16.119}
$$

The next proposition gives its solution.

Proposition 16.1 Under the traditional regularity conditions on the coefficients, the solution of the structural PDE (16.119) is given, for all t≤s by:

$$
P(t,s,r) = E_{\Im_t}\left[e^{-\int_t^s r(\tau)d\tau - \frac{1}{2}\int_t^s \lambda^2(\tau, r(\tau)d\tau + \int_t^s \lambda(\tau, r(\tau)dB(\tau)} \right], \tag{16.120}
$$

where $\Im_t = \sigma(B(u), u \le t)$.

Proof If we introduce process V defined by

$$
V(u) = e^{-\int_t^u r(\tau)d\tau - \frac{1}{2}\int_t^u \lambda^2(\tau, r(\tau)d\tau + \int_t^u \lambda(\tau, r(\tau)dB(\tau)}, \tag{16.121}
$$

we can also write:

$$
V(u) = e^{g(u)}, \tag{16.122}
$$

where

$$
g(u) = -\int_t^u r(\tau)d\tau - \frac{1}{2}\int_t^u \lambda^2(\tau, r(\tau)d\tau + \int_t^u \lambda(\tau, r(\tau)dB(\tau), \tag{16.123}
$$

and so

$$dg(u) = -\left[r(u) + \frac{1}{2}\lambda^2(u, r(u)) \right] du + \lambda(u, r(u) dB(u). \tag{16.124}$$

Using Itô's calculus rules, we have

$$\begin{aligned} dV(u) = V(u) &\left[-\left(r(u) + \frac{1}{2}\lambda^2(u, r(u)) \right) + \frac{1}{2}\lambda^2(u, r(u) \right] du \\ &+ V(u)\lambda(u, r(u) dB(u). \end{aligned} \tag{16.125}$$

As

$$dP(t,s,r) = \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} f(r,t) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\rho^2(r,t) \right] dt + \frac{\partial P}{\partial r}\rho(r,t) dB(t), \tag{16.126}$$

we obtain:

$$d(P(u)V(u) = P(u)dV(u) + V(u)dP(u) + V(u)\lambda(u, r(u)\frac{\partial P}{\partial r}\rho(r,u)du. \tag{16.127}$$

Substituting into this last equality $dP(u)$ and $dV(u)$, given by relations (16.125) and (16.126), we obtain, after some calculations that:

$$\begin{aligned} d(PV) = V &\left(\frac{\partial P}{\partial t} + (f + \rho\lambda)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2 \frac{\partial^2 P}{\partial r^2} - rP \right) du \\ &+ V\frac{\partial P}{\partial r}\rho dB + PV\lambda dB. \end{aligned} \tag{16.128}$$

From PDE (16.119), this last equality becomes:

$$d(PV) = V\frac{\partial P}{\partial r}\rho dB + PV\lambda dB. \tag{16.129}$$

By integration from t to s ($t < s$), we obtain:

$$PV(s) - PV(t) = \int_t^s \left[\rho V \frac{\partial P}{\partial r} + PV\lambda \right] dB. \tag{16.130}$$

By conditional expectation with respect to $\Im_t$, we find:

$$\begin{aligned} E_{\Im_t}\left[P(s,s,r)V(s)-P(t,s,r)V(t)\right] &= E_{\Im_t}\left[\int_t^s\left[\rho V\frac{\partial P}{\partial r}+PV\lambda\right]dB\right],\\ &=\int_t^s E_{\Im_t}\left[\rho V\frac{\partial P}{\partial r}+PV\lambda\right]dE_{\Im_t}B,\\ &=0, \end{aligned} \tag{16.131}$$

as B is a SBM.

As we know that $P(s,s,r)=1$, relation (16.131) gives:

$$E_{\Im_t}\left[V(s)\right]=P(t,s,r)V(t) \tag{16.132}$$

and finally, as $V(t)=1$:

$$E_{\Im_t}\left[V(s)\right]=P(t,s,r), \tag{16.133}$$

that is, relation (16.120) is proved. □

16.4.4. *Values of zero coupons without arbitrage opportunity for particular cases*

We will now make use of relation (16.120) to evaluate the risk-neutral value of a zero coupon.

16.4.4.1. *The risk premium is* 0

From (16.118), we obtain:

$$\mu(t,s)=r(t),t\geq s, \tag{16.134}$$

and from the fundamental result (16.120), we obtain the traditional form of a zero-coupon value:

$$P(t,s,r(t))=E_{\Im_t}\left[e^{-\int_t^s r(u)du}\right]. \tag{16.135}$$

If, moreover, the spot rate is deterministic, we obtain the traditional formula:

$$P(t,s,r(t))=e^{-\int_t^s r(u)du} \tag{16.136}$$

16.4.4.2. *Constant premium rate*

Here result (16.120) becomes:

$$P(t,s,r) = E_{\Im_t}\left[e^{-\int_t^s r(\tau)d\tau - \frac{1}{2}\lambda^2(s-t)+\lambda\left[B(s)-B(t)\right]} \right]. \tag{16.137}$$

To provide an interpretation of this result with the introduction of the *risk-neutral measure*, let us now introduce *Girsanov's theorem*.

16.4.4.3. *Girsanov's theorem (Gikhman and Skorohod (1980))*

On $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),P\right)$, let us consider the adapted stochastic process $f=\left\{f(t),t\in\left[0,T\right]\right\}$ such that

$$\int_0^T f^2(s)ds < \infty \text{ a.s.} \tag{16.138}$$

Then, Girsanov's theorem introduces to the new stochastic process $\xi=\left\{\xi(t),t\in\left[0,T\right]\right\}$ defined by:

$$\xi(t)=\exp\left(\int_0^t f(s)dB(s)-\frac{1}{2}\int_0^t f^2(s)ds\right). \tag{16.139}$$

Moreover, process B is a SBM on $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),P\right)$.

Girsanov introduces a new probability measure dependent on f and noted $Q=Q(f)$ on $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right)\right)$ of density $\xi(T)$ with respect to the initial measure P, such that:

$$\frac{dQ(f)}{dP}=\xi(T). \tag{16.140}$$

This means that the expectation with respect to the new measure is related to the old one with the following relation:

$$\int X(\omega)dQ(f)=\int X(\omega)\xi(T,\omega)dP$$

or

$$\mathrm{E_Q}[X(\omega))]=\mathrm{E_P}[X(\omega)\xi(T,\omega)]. \tag{16.141}$$

Of course, we also have:

$$E_P\left[X(\omega)\right]=E_Q\left[X(\omega)\left(\xi(T,\omega)\right)^{-1}\right]. \tag{16.142}$$

The result of Girsanov's theorem is if the following condition is fulfilled:

$$E\left[\xi(T,\omega)\right]=1, \tag{16.143}$$

then the new process

$$\hat{B}=\left\{\hat{B}(t),t\in\left[0,T\right]\right\} \tag{16.144}$$

defined by

$$\hat{B}(t)=B(t)-\int_0^t f(s)ds \qquad (16.145))$$

is still a SBM but now on $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),Q\right)$.

Under *Novikov's condition*:

$$E\left[\exp\left(\frac{1}{2}\int_0^T f^2(s)ds\right)\right]<\infty \tag{16.146}$$

process ξ defined by (16.135) satisfies condition (16.143) and so Girsanov's theorem applies.

16.4.4.4. *Neutral risk measure*

Let us recall the stochastic dynamics of the zero coupons is defined by the SDE:

$$dP(t,s,r)=P(t,s,r)\mu(t,s,r)dt-P(t,s,r)\sigma(t,s,r)dB(t) \tag{16.147}$$

where

$$\mu(t,s,r)=r(t)+\lambda(t)\sigma(t,s,r). \tag{16.148}$$

Let us now introduce the new probability measure Q on $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right)\right)$ defined by:

$$\frac{dQ}{dP}=e^{\int_0^s\lambda(u)dB(u)-\frac{1}{2}\int_0^s\lambda^2(u)du}(=\xi(T)). \tag{16.149}$$

Through Girsanov's theorem, we know that process $\widehat{B}=\left\{\widehat{B}(u),u\in\left[0,T\right]\right\}$ defined by

$$\widehat{B}(u) = B(u) - \int_0^u \lambda(v)dv \tag{16.150}$$

is also a SBM but now on $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), Q\right)$ such that:

$$d\hat{B}(u) = dB(u) - \lambda(u)du\ . \tag{16.151}$$

Returning to relation (16.146), we now have:

$$dP(t,s,r) = P(t,s,r)\mu(t,s,r)dt - P(t,s,r)\sigma(t,s,r)\left[d\widehat{B}(t) + \lambda(t)dt\right] \tag{16.152}$$

or

$$\frac{dP(t,s,r)}{P(t,s,r)} = \left[\mu(t,s,r) - \sigma(t,s,r)\lambda(t)\right]dt - \sigma(t,s,r)d\widehat{B}(t)\ . \tag{16.153}$$

From relation (16.148), we also have:

$$\frac{dP(t,s,r)}{P(t,s,r)} = r(t)dt - \sigma(t,s,r)d\widehat{B}(t). \tag{16.154}$$

From result (16.120) stating that:

$$P(0,s,r) = E_{\Im_t}\left[e^{-\int_0^s r(\tau)d\tau - \frac{1}{2}\int_0^s \lambda^2(\tau,r(\tau)d\tau + \int_0^s \lambda(\tau,r(\tau)dB(\tau)} \right], \tag{16.155}$$

we obtain from relation (16.149):

$$\begin{aligned} P(0,s,r) &= E_Q\left[e^{-\int_0^s r(\tau)d\tau - \frac{1}{2}\int_0^s \lambda^2(\tau,r(\tau)d\tau + \int_0^s \lambda(\tau,r(\tau)dB(\tau)} . e^{\frac{1}{2}\int_0^s \lambda^2(\tau,r(\tau)d\tau - \int_0^s \lambda(\tau,r(\tau)dB(\tau)} \right], \\ &= E_Q\left[e^{-\int_0^s r(\tau)d\tau} \right]. \end{aligned} \tag{16.156}$$

Thus, under Q, the value of a zero coupon is formally given, as in the particular case of $\lambda = 0$, which is why the new measure Q is called the *risk-neutral measure*.

From the Markov property, this last result gives, starting at time t with $r(t)=r$, as the spot rate:

$$P(t,s,r|\Im_t)$$
$$= E_Q\left[e^{-\int_t^s r(\tau)d\tau - \frac{1}{2}\int_0^s \lambda^2(\tau,r(\tau)d\tau + \int_0^s \lambda(\tau,r(\tau)dB(\tau)}.e^{\frac{1}{2}\int_t^s \lambda^2(\tau,r(\tau)d\tau - \int_t^s \lambda(\tau,r(\tau)dB(\tau)} \middle| \Im_t \right], \quad (16.157)$$
$$= E_Q\left[e^{-\int_t^s r(\tau)d\tau} \middle| \Im_t \right].$$

16.4.4.5. *Examples*

Example 16.1 The OUV process as rate dynamics

We know that the OUV model is governed by the following SDE:

$$\begin{aligned} &dr(t) = a(b - r(t))dt + \sigma dB(t), |\Im_t \\ &r(0) = r_0. \end{aligned} \quad (16.158)$$

Assuming that the risk premium is constant with value λ on $[0,t]$, the risk-neutral measure given by relation (16.149):

$$\frac{dQ}{dP} = e^{\int_0^s \lambda(u)dB(u) - \frac{1}{2}\int_0^s \lambda^2(u)du} (= \xi(T)), \quad (16.159)$$

becomes:

$$dQ = e^{\lambda B(t) - \frac{1}{2}\lambda^2 t} dP \quad (16.160)$$

and by relation (16.150)

$$\hat{B}(u) = B(u) - \lambda(u). \quad (16.161)$$

Thus, under measure Q, the stochastic dynamics of process r are defined by the following SDE

$$\begin{aligned} &dr(t) = a(b - r(t))dt + \sigma\left[d\hat{B}(t) + \lambda dt \right], \\ &r(0) = r_0, \end{aligned} \quad (16.162)$$

or even:

$$\begin{aligned} &dr(t) = a(\theta - r(t))dt + \sigma\left[d\widehat{B}(t) + \lambda dt \right], \\ &r(0) = r_0, \\ &\theta = b + \frac{\lambda\sigma}{a}. \end{aligned} \tag{16.163}$$

On the time interval $[t, s]$, under Q, the basic results of section 16.3.1.1 take the form:

$$\begin{aligned} &r(s) = \theta + \left(r_t - \theta\right)e^{-at} + \sigma e^{-at}\int_t^s e^{au}\, d\widehat{B}(u), \\ &E_Q\left[r(s)\right] = \theta + \left(r_t - \theta\right)e^{-as}, \\ &\operatorname{var}_Q r(s) = \frac{\sigma^2}{2a}\left(1 - e^{-2as}\right). \end{aligned} \tag{16.164}$$

For the value of the zero coupon, we obtain from result (16.157):

$$P(t, s, r) = E_Q\left[e^{-\int_t^s r(\tau)d\tau} \middle| \Im_t \right]. \tag{16.165}$$

Let us now calculate the value of $P(0, s, r_0)$.

With

$$\beta(u) = \frac{1 - e^{-au}}{a}, \tag{16.166}$$

the first relation of relation (16.164) becomes:

$$r(u) = r_0 e^{-au}\theta + a\theta\beta(u) + \sigma\int_0^u e^{-a(u-v)}\, d\widehat{B}(v). \tag{16.167}$$

For

$$U(0, s) = \int_0^s r(u)du, \tag{16.168}$$

the Fubini theorem and result (16.167) lead to the following result:

$$U(0,s) = (r_0 - \theta)\beta(s) + \theta s + \sigma \int_0^u \beta(s-u) d\widehat{B}(u). \tag{16.169}$$

It follows that the distribution of $U(0,s)$ is normal with parameters given by:

$$E\left[U(0,s)\right] = (r_0 - \theta)\beta(s) + \theta s, \tag{16.170}$$

$$\begin{aligned} \operatorname{var}\left[U(0,s)\right] &= E\left[\left(\sigma \int_0^s \beta(s-u)) d\widehat{B}(u)\right)^2\right], \\ &= \sigma^2 \int_0^s \beta^2(s-u) du. \end{aligned} \tag{16.171}$$

As β is given by relation (16.166), we obtain:

$$\operatorname{var}\left[U(0,s)\right] = \frac{\sigma^2}{a^2} \int_0^s \left(1 - e^{-a(s-u})\right)^2 du. \tag{16.172}$$

The calculation of this traditional integral leads to:

$$\operatorname{var}\left[U(0,s)\right] = \frac{\sigma^2}{2a^3}\left[2as - e^{-2as} + 4e^{-as} - 3\right]. \tag{16.173}$$

Returning to result (16.165) with t=0, we find that:

$$\begin{aligned} P(0,s,r_0) &= E_Q\left[e^{-\int_0^s r(\tau)d\tau} \middle| \Im_0\right], \\ &= E_Q\left[e^{-\int_0^s r(\tau)d\tau}\right], \\ &= E_Q\left[e^{-U(0,s)}\right]. \end{aligned} \tag{16.174}$$

Consequently, the value of this zero coupon is given by the value of the generating function of $U(0,s)$ at s=-1, that is,

$$P(0,s,r_0) = e^{-E_Q[U(0,s)]+\frac{1}{2}\mathrm{var}_Q[U(0,s)]}. \tag{16.175}$$

Starting from a t different from 0, we obtain:

$$P(t,s,r_0) = e^{-E_Q[U(t,s)]+\frac{1}{2}\mathrm{var}_Q[U(t,s)]}. \tag{16.176}$$

Using results (16.163), (16.170) and (16.171), we obtain:

$$\begin{aligned} &P(t,s,r_t) = \exp\left[-(s-t)R_\infty - \beta(s-t)(r_t - R_\infty) - \frac{\sigma^2}{4a}\beta^2(s-t)\right], \\ &\text{where} \\ &R_\infty = b + \frac{\lambda\sigma}{a} - \frac{\sigma^2}{2a^2}, \\ &r_t = r(t). \end{aligned} \tag{16.177}$$

Using result (16.105), we obtain the instantaneous term structure:

$$\begin{aligned} &R(s,s-t) = R_\infty + \frac{\beta(s-t)}{s-t}(r_t - R_\infty) + \frac{\sigma^2}{4a}\frac{\beta^2(s-t)}{s-t}, \\ &\beta(u) = \frac{1-e^{-au}}{a}, \\ &R(t,\infty) = R_\infty, \\ &E = R_\infty - \frac{\sigma^2}{4a^2}, F = R_\infty + \frac{\sigma^2}{2a^2}. \end{aligned} \tag{16.178}$$

Example 16.2 The CIR process as rate dynamic

We can also use the CIR model defined and studied in section 16.3.2 by:

$$\begin{aligned} &dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dB(t), \\ &r(0) = r_0. \end{aligned} \tag{16.179}$$

We will express the premium risk in the form

$$\lambda(t,r) = -\frac{\pi}{\sigma}\sqrt{r(t)}. \tag{16.180}$$

It is clear that this premium risk is no longer constant as in the preceding example.

Here the PDE (16.119) for the zero coupon value takes the form:

$$\frac{\partial P}{\partial t}+\left[a(b-r-\pi r\right]\frac{\partial P}{\partial r}+\frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial^2 r}-rP=0, \qquad (16.181)$$
$$P(r,s,s)=1.$$

Cox, Ingersoll and Ross obtained the solution under the form:

$$P(r,t,s)=A(t,s)e^{-rD(t,s)},$$

where

$$A(t,s)=\left[\frac{2\gamma e^{\frac{(a+\pi+\gamma)(s-t)}{2}}}{(a+\pi+\gamma)(e^{\gamma(s-t)}-1)+2\gamma}\right]^{\frac{2ab}{\sigma^2}}, \qquad (16.182)$$

$$D(t,s)=\frac{2(e^{\gamma(s-t)}-1)}{(a+\pi+\gamma)(e^{\gamma(s-t)}-1)+2\gamma},$$
$$\gamma=\sqrt{(a+\pi)^2+2\sigma^2}.$$

So, as

$$P(r,t,s)=e^{-(s-t)R(r,t,s)}, \qquad (16.183)$$

we obtain from (16.182):

$$R(r,t,s)=\frac{rD(t,s)-\ln A(t,s)}{(s-t)}. \qquad (16.184)$$

The two limit cases give as results:

$$s\to t\Rightarrow R(r,t,s)\to r(=r(t)),$$
$$s\to\infty\Rightarrow R(r,t,s)\to\frac{2ab}{a+\pi+\gamma}. \qquad (16.185)$$

Remark 16.5 If we assume that the spot rate satisfies the very simple model:

$$\begin{aligned} &dr = \mu dt + \sigma dB(t), \\ &r(0) = r_0, \end{aligned} \tag{16.186}$$

which is not incredibly adequate since it has a linear trend, it can be shown that the value of the zero-coupon is given by:

$$P(t,s) = \exp\left[-r(s-t) - \frac{\mu(s-t)^2}{2} + \frac{\sigma^2(s-t)^3}{6}\right], \tag{16.187}$$

which is a very unsatisfactory solution since

$$\lim_{s \to \infty} P(t,s) = \infty! \tag{16.188}$$

This is why such a simple model must be rejected in the case of interest rate modeling.

16.4.5. *Value of a call on zero-coupon*

16.4.5.1. *General results*

Let us give at time 0, a zero coupon expiring at time s. On this asset, we consider a call of maturity T where s>T, with K as the exercise price and the value $C(0,T)$ at $t = 0$.

From relation (16.112) canceling the dependence with respect to r, the dynamics for the zero coupon is given by:

$$\frac{dP}{P} = \mu(t,T)dt - \sigma(t,T)dB(t)\,. \tag{16.189}$$

It is possible to show (Musiela and Rutkowski (1997)) that the value under AOA is given by

$$\begin{aligned} &C(0,T) = P(0,s)\Phi(d_1) - KP(0,T)\Phi(d_1 - H), \\ &H = \int_0^T \left[\sigma(u,s) - \sigma(u,T)\right]^2 du, \\ &d_1 = \frac{1}{H}\ln\frac{P(0,s)}{KP(0,T)} + \frac{H}{2}. \end{aligned} \tag{16.190}$$

More generally, for the evaluation at time t instead of 0, the preceding result becomes:

$$C(t,T) = P(t,s)\Phi(h_+) - KP(t,T)\Phi(h_-),$$
$$h_+ = \frac{1}{V(s,T)}\ln\frac{P(t,s)}{KP(t,T)} + \frac{V(t,T)}{2}, \tag{16.191}$$
$$h_- = \frac{1}{V(s,T)}\ln\frac{P(t,s)}{KP(t,T)} - \frac{V(t,T)}{2},$$

with

$$V(t,T)^2 = \int_t^T \left[\sigma(u,s) - \sigma(u,T)\right]^2 du.$$

For the puts, we use the call parity formula:

$$Call(t,T) - Put(t,T) = P(t,s) - KP(t,T), \tag{16.192}$$

and after calculation, the final result is given by

$$Put(t,T) = KP(t,T)N(-h_-) - P(t,s)N(-h_+). \tag{16.193}$$

16.4.5.2. *Particular case of the OUV model*

Here, we have

$$H^2 = \frac{\sigma^2}{a^2}\left(\frac{1-e^{2aT}}{2a}\right)(1-e^{-a(s-T)})^2. \tag{16.194}$$

16.4.6. *Option on bond with coupons (Jamshidian (1989))*

The exact value of an option on a bond with coupons was first given by Jamshidian as a linear combination of options on zero coupons.

Before giving his result, it is necessary to introduce some notions.

Let r^*b be the yield rate at time T, where T is the maturity of the option such that the price of the considered bond is equal to the exercise price of the call option.

Let s_j represent the jth date of coupon maturity with j=1,…, n, should be after time T: $(s_j > T, j = 1,...,n)$ and let $c_j, j = 1,\ldots,n$, be the value of the jth coupon.

Jamshidian also introduced the values $K_j, j = 1,...,n$ defined as:

$$K_j = P(r^*,T,s_j), j = 1,...,n \tag{16.195}$$

which is the value of a zero coupon at time *T* with *r*(*T*)=*r** and of maturity s_j.

As the price of a bond is a decreasing function of the spot rate *r*, the investor will exercise the call if and only if *r*<*r** and if a zero coupon of maturity s_j will be larger than $c_j K_j$, Jamshidian proved that the value *C*(*t,T*) of the European call on the bond is given by the following linear combination:

$$C(t,T) = \sum_{j=1}^{n} C(t,T,s_j,K_j), \tag{16.196}$$

$C(t,T,s_j,K_j)$ being the value of an European call of maturity time *T* with K_j as exercise price on a zero coupon expiring at time s_j.

It can also be proved that (El Karoui and Rochet (1989)):

$$\begin{aligned}
&C(0,T) = \sum_{j=1}^{n}\left[c_j P(0,s_j)\Phi(d_j) - KP(0,T)\Phi(d_0)\right],\\
&d_j = d_0 + H_j, j = 1,...,n,\\
&H_j^2 = \int_0^T \left[\sigma(u,s_j) - \sigma(u,T)\right]^2 du,\\
&d_0 : \sum_{i=1}^{n} c_i P(0,s_i) e^{-\frac{1}{2}H_i^2 + d_0 H_i} = KP(0,T).
\end{aligned} \tag{16.197}$$

For the put, the relation of parity leads to the following result:

$$Put(0,T) = KP(0,T)\Phi(-d_0) - \sum_{j=1}^{n} c_j P(0,s_j)\Phi(-d_j). \tag{16.198}$$

16.4.7. *A numerical example*

The next table provides the result of zero coupon values with CIR models with four scenarios given by the five parameters, selected as given by lines 2 and 3.

scen.	**I**	**II**	**III**	**IV**
par.	1-3.5%-3.5%	1-3.5%-3.5%	**5**-3.5%-3.5%	1-**6**%-3.5%
	3%-1%	25%-1%	3%-1%	3%-1%
mat.	**CIR**	**CIR**	**CIR**	**CIR**
0.25	0.99128664	0.99127952	0.99128701	0.98863114
0.5	0.98264681	0.98263459	0.98264889	0.97490507
1	0.96558803	0.96561683	0.96559715	0.94644076
3	0.90024357	0.90102057	0.90029773	0.8394155
5	0.83931463	0.84105891	0.83941429	0.74446771
7	0.78325086	0.78512914	0.78264807	0.66025964
10	0.70442635	0.7081375	0.70461487	0.55146515
20	0.49620034	0.5020042	0.49648109	0.30266638

Table 16.1. *Zero coupons values with CIR*

16.5. Appendix (solution of the OUV equation)

To solve the OUV equation:

$$\begin{aligned} &dr(t) = a(b - r(t))dt + \sigma dB(t), \\ &r(0) = r_0, \end{aligned} \tag{16.199}$$

let us start from the deterministic version

$$\begin{aligned} &dr(t) = a(b - r(t))dt \\ &r(0) = r_0, \end{aligned} \tag{16.200}$$

for which the general solution is given by

$$\begin{aligned} &r(t) = b + ce^{-at}, \\ &c \text{ constant.} \end{aligned} \tag{16.201}$$

Now let us suppose that c is also a function of t such that the function

$$r(t) = b + c(t)e^{-at}$$

is the solution of the SDE (16.199).

From Proposition 7.1 in Chapter 4, we know that:

$$dr = e^{-at} dc - ae^{-at} c(t) dt \tag{16.202}$$

and from relation (16.201), we obtain:

$$dr = e^{-at} dc + a(b - r(t))dt. \tag{16.203}$$

and comparing with relation (16.199), we obtain:

$$e^{-at} dc + a(b - r(t))dt = a(b - r(t))dt + \sigma dB(t), \tag{16.204}$$

and so

$$e^{-at} dc = \sigma dB(t).$$

It follows that:

$$dc(t) = \sigma e^{at} dB(t)$$

and by relation (A.3)

$$r(t) = b + e^{-at} (c_0 + \sigma \int_0^t e^{as} dB(s), \tag{16.205}$$

with

$$\mathrm{r}_0 = b + c_0$$

or

$$c_0 = r_0 - b.$$

Substituting this last value in the first equality of relation (16.205), we obtain the announced solution in section 16.3.1.1:

$$r(t) = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB(s).$$

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Chapter 17

Portfolio Theory

17.1. Quantitative portfolio management

Let us consider a financial market trading n risky assets called $a_1,...,a_n (n \geq 1)$. We assume that the absolute returns $\xi_1,...,\xi_n$ of these n assets on a fixed time period $[0,T]$ are random variables with means $\mu_1,...,\mu_n$ and variances $\sigma_1^2,...,\sigma_n^2$, and moreover as these returns are in general dependent, we have to introduce the following covariances:

$$\sigma_{ij} = E\left[\left(\xi_i - \mu_i\right)\left(\xi_j - \mu_j\right)\right], \quad i,j = 1,...,n. \tag{17.1}$$

The problem of the choice of a portfolio consists of selecting a vector

$$\mathbf{x} = (x_1,...,x_n)' \tag{17.2}$$

such that

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, i = 1,...,n \tag{17.3}$$

under a certain criteria depending on the attitude of the investor against risk. In general, the investors are risk adverse and thus manage their portfolio with a prudential attitude, but others may be risk lovers, attracted by the expectation of possible high returns.

Mathematically, risk adverse investors having a choice between two portfolios will select the one having a mean turn with the smallest variance provided that the performance of the portfolio is measured with the mean return.

17.2. Notion of efficiency

To find such a portfolio, Markowitz (1959) introduced the concept of efficiency or of efficient, portfolio.

Definition 17.1 A portfolio is efficient if for all the portfolios having the same expectation of return, it is of minimal variance.

Following this definition, there corresponds an efficient portfolio to each fixed expectation of return and so with such a return as variable we obtain a new function which graphs in a plane mean-variance and is called the efficient frontier.

The return of portfolio of vector **x**' at time T is given by:

$$R(\mathbf{x}) = \sum_{i=1}^{n} x_i \xi_i \tag{17.4}$$

Moreover, we have:

$$\begin{aligned} E(r(T)) &= \sum_{i=1}^{n} x_i \mu_i (= \mu) \\ \operatorname{var} r(T) &= E\left[\left(\sum_{i=1}^{n} x_i \xi_i - \sum_{i=1}^{n} x_i \mu_i\right)^2\right], \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j (= \sigma^2) \end{aligned} \tag{17.5}$$

where

$$\sigma_{jj} = \sigma_j^2, j = 1, \ldots, n.$$

The search for an efficient portfolio corresponding to a mean return of value *m* leads to the following mathematical optimization:

$$\min_{x_1,\dots,x_n} \sigma^2(x_1,\dots,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}x_ix_j, \tag{17.6}$$

under the constraints:

$$(i)\sum_{i=1}^{n} x_i\mu_i = \mu,$$
$$(ii)\sum_{i=1}^{n} x_i = 1,$$
$$(iii) x_i \geq 0, i = 1,\dots,n,$$

μ now being a mean return selected by the investor.

Remark 17.1

a) Condition (iii) excludes short sales.

b) The variables $x_i; i = 1,\dots,n$ represent the percentages of the *n* shares in the portfolio.

To solve this mathematical programming problem with constraints, we must introduce the Lagrangian function *L* of *n*+2 variables defined by

$$L(x_1,\dots,x_n,\lambda,\nu) =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}x_ix_j - 2\lambda\left(\sum_{i=1}^{n} x_i\mu_i - \mu\right) - 2\nu\left(\sum_{i=1}^{n} x_i - 1\right). \tag{17.7}$$

Taking the *n* partial differentials with respect to $x_i, i = 1,\dots,n$, we obtain the following linear system:

$$\sum_{j=1}^{n} \sigma_{kj}x_j - \lambda\mu_k - \nu = 0, \qquad k = 1,\dots,n$$

or

$$x = \mathbf{V}^{-1}(\lambda\boldsymbol{\mu} + \nu\mathbf{1}) \tag{17.8}$$

where the square matrix **V** represents the variance covariance matrix of the vector of returns $\xi = (\xi_1, ...\xi_n)'$:

$$\mathbf{V} = \left[\sigma_{ij}\right] \tag{17.9}$$

and with the following notations:

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)', \ \mathbf{1} = (1, \ldots, 1)'. \tag{17.10}$$

It is possible to show that the unique solution of this algebraic system (see, for example, Poncet, Portait and Hayat (1996)) is given by

$$\mathbf{x}^* = \mathbf{g} + \mu\mathbf{h} \tag{17.11}$$

where

$$\begin{aligned} \mathbf{g} &= \frac{1}{d}\left(b\mathbf{V}^{-1}\mathbf{1} - a\mathbf{V}^{-1}\boldsymbol{\mu}\right) \\ \mathbf{h} &= \frac{1}{d}\left(c\mathbf{V}^{-1}\boldsymbol{\mu} - a\mathbf{V}^{-1}\mathbf{1}\right) \end{aligned} \tag{17.12}$$

with

$$\begin{aligned} d &= bc - a^2 \\ a &= \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{\mu} \\ b &= \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \\ c &= \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \end{aligned} \tag{17.13}$$

The two Lagrange parameters are given by:

$$\begin{aligned} \lambda &= \frac{c\mu - a}{d}, \\ \nu &= \frac{b - a\mu}{d}. \end{aligned} \tag{17.14}$$

As it is also possible to prove that the second conditions order to obtain a maximum are satisfied, we now have the following proposition.

Proposition 17.1 In the plane (σ, μ), the efficient frontier of the considered portfolio is represented by a hyperbola of equation

$$\frac{\sigma^2}{A^2} - \frac{(\mu - C)^2}{B^2} = 1,$$

where:

$$\mathrm{A}^2 = \frac{1}{c}, B^2 = \frac{d}{c^2}, C = \frac{a}{c}. \tag{17.15}$$

coordinates of the vertex:

$$\left(\sqrt{\frac{1}{c}}, \frac{a}{c}\right);$$

coordinates of the center:

$$\left(0, \frac{a}{c}\right);$$

asymptotes:

$$\mu = \frac{a}{c} \pm \sqrt{\frac{d}{c}}\sigma\ .$$

Proposition 17.2 In the plane $\left(\sigma^2, \mu\right)$, the equation of the efficient frontier takes the form of a parabola of equation:

$$\sigma^2 = \frac{1}{d}(c\mu^2 - 2a\mu + b) \tag{17.16}$$

having as vertex $(1/c, a/c)$.

Proof If $\sigma^2_{\min}$ represents the minimum value under constraints of the function defined by relation (17.6), we can write:

$$\sigma^2_{\min} = \mathbf{x}^{*'}\mathbf{V}\mathbf{x}^*. \tag{17.17}$$

Replacing x from its value given by relation (17.8), we obtain:

$$\begin{aligned}\sigma_{\min}^2 &= \mathbf{x}^{*'}\mathbf{V}\mathbf{V}^{-1}(\lambda\boldsymbol{\mu}+\nu\mathbf{1})\\ &= \mathbf{x}^{*'}(\lambda\boldsymbol{\mu}+\nu\mathbf{1})\\ &= \lambda\mathbf{x}^{*'}\boldsymbol{\mu}+\nu\mathbf{x}^{*'}\mathbf{1}\\ &= \lambda\mu+\nu.\end{aligned}$$

$$\begin{aligned}\sigma_{\min}^2 &= x^{*'}VV^{-1}(\lambda\overline{\mu}+\nu\overline{1}),\\ &= x^{*'}(\lambda\overline{\mu}+\nu\overline{1}),\\ &=(\lambda x^{*'}\overline{\mu})+\nu x^{*'}\overline{1}),\\ &=\lambda\mu+\nu.\end{aligned} \tag{17.18}$$

From relations (17.14), we finally obtain:

$$\sigma_{\min}^2 = \frac{1}{d}\left(c\mu^2-2a\mu+b\right), \tag{17.19}$$

which is relation (17.16). □

Remark 17.2 It is clear that we must only use the upper branch of hyperbola (17.15).

Let us now introduce a non-risky asset of unitary return r on the considered time period $[0,T]$ so that the portfolio may also contain a proportion of this new asset.

If y represents the proportion of this asset in the portfolio, (1-x) will represent the part of the risky efficient portfolio. The mean and standard deviation of this new portfolio are:

$$\begin{aligned}\mu_P &= xr+(1-x)\mu,\\ \sigma_P &= (1-x)\sigma,\end{aligned} \tag{17.20}$$

By elimination of x between these two equations, we obtain:

$$\mu_P = r+\frac{\mu-r}{\sigma}\sigma_P \tag{17.21}$$

In the plane (σ_P, μ_P), this equation represents the tangent from the point (0,*r*) to the Markowitz hyperbola of equation

$$\frac{\sigma_P^2}{A^2} - \frac{(\mu_P - C)^2}{B^2} = 1. \tag{17.22}$$

Thus, the introduction of a non-risky asset modifies the structure of the curve of optimal portfolios called the efficacious frontier, composed with the tangent up to the efficient frontier and after of the part of the efficient frontier for portfolios without a risky asset.

When the tangent is above the efficient portfolio, the corresponding portfolio no longer satisfies result (17.6) condition (iii) as (1-x) is strictly greater than 1 or x<0. This means that the investor borrows from the bank at rate r to buy the risky asset part of his portfolio, increasing his mean return but also his risk!

Such an investor is clearly a risk lover attracted by high return expectation.

17.3. Exercises

1) Prove that every linear convex combination of two efficient portfolios is still an efficient portfolio.

2) Let us consider two efficient portfolios of mean returns $\mu_1, \mu_2 (\mu_1 \neq \mu_2)$; for a given mean return, show that the corresponding efficient portfolio can be given as a linear combination of the two given portfolios.

Answer

The following reasoning solves the two exercises.

Let us consider three efficient portfolios having different mean returns μ_1, μ_2, μ_3 respectively.

From result (17.11), we obtain for the constitution of these three portfolios:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{g} + \mu_1 \mathbf{h}, \\ \mathbf{x}^{(2)} &= \mathbf{g} + \mu_2 \mathbf{h}, \\ \mathbf{x}^{(3)} &= \mathbf{g} + \mu_3 \mathbf{h}. \end{aligned} \tag{17.23}$$

Let k be the real number such that:

$$\mu_3 = k\mu_1 + (1-k)\mu_2. \tag{17.24}$$

Let us now form a portfolio as follows:

$$\mathbf{x} = k\mathbf{x}^{(1)} + (1-k)\mathbf{x}^{(2)}. \tag{17.25}$$

By unicity of linear convex combinations, we have:

$$\mathbf{x} = \mathbf{x}^{(3)}. \tag{17.26}$$

17.4. Markowitz theory for two assets

Here, x will represent the proportion invested in asset A and of course (1-x) that invested in asset B always with $0 \le x \le 1$; moreover, without loss of generality, we assume that:

$$\begin{aligned} &\mu_1 < \mu_2, \\ &\sigma_1 < \sigma_2. \end{aligned} \tag{17.27}$$

In this case, the general results of section 17.2 become:

1) for the return on [0,T]:

$$R(x) = (1-x)\xi_1 + x\xi_2, \tag{17.28}$$

2) for the mean return on [0,T]:

$$\begin{aligned} E(R(x)) &= (1-x)\mu_1 + x\mu_2, \\ &= x(\mu_2 - \mu_1) + \mu_1, \end{aligned} \tag{17.29}$$

3) for the variance to be minimized:

$$\sigma^2 = (1-x)^2\sigma_1^2 + x^2\sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2 \tag{17.30}$$

where ρ is the correlation coefficient between the two assets.

We will now discuss the different possibilities with respect to the value of ρ.

Case 1: $\rho = 1$

Intuitively, this means that the two assets vary in the same sense and so we will not have a portfolio of risk less than that of asset *A*.

Indeed, from relation (17.30), we obtain:

$$\begin{aligned}\sigma^2 &= (1-x)^2\sigma_1^2 + x^2\sigma_2^2 + 2x(1-x)\sigma_1\sigma_2, \\ &= (x\sigma_2 + (1-x)\sigma_1)^2,\end{aligned} \tag{17.31}$$

so, from assumption (17.27):

$$\sigma = x(\sigma_2 - \sigma_1) + \sigma_1. \tag{17.32}$$

With relation (17.29), we obtain:

$$\begin{cases}\mu = x(\mu_2 - \mu_1) + \mu_1, \\ \sigma = x(\sigma_2 - \sigma_1) + \sigma_1\end{cases} \tag{17.33}$$

representing the parametric equations of a straight line in the Markowitz plane (σ, μ) having Cartesian equation:

$$\frac{\mu - \mu_1}{\mu_2 - \mu_1} = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \tag{17.34}$$

or:

$$\mu = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}(\mu_2 - \mu_1) + \mu_1. \tag{17.35}$$

The efficient frontier is given by the part of this straight line between the two points $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$.

It follows that the portfolio of minimum risk consists of investing all in asset *A* and that of maximum risk and also with maximum mean return all in asset *B*.

Case 2: $\rho = -1$

Here, we have:

$$\begin{aligned}\sigma^2 &= (1-x)^2\sigma_1^2 + x^2\sigma_2^2 - 2x(1-x)\sigma_1\sigma_2, \\ &= (x\sigma_2 - (1-x)\sigma_1)^2,\end{aligned} \tag{17.36}$$

or

$$\sigma = \left| x(\sigma_2 + \sigma_1) - \sigma_1 \right|. \tag{17.37}$$

In this case, it is possible to select a non-risky portfolio taking for x* the value such that σ equals 0:

$$x^* = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{17.38}$$

The parametric equations of the efficient frontier are:

$$\begin{cases} \mu = x(\mu_2 - \mu_1) + \mu_1, \\ \sigma = \begin{cases} x(\sigma_2 + \sigma_1) - \sigma_1, x > \dfrac{\sigma_1}{\sigma_2 + \sigma_1}, \\ -x(\sigma_2 - \sigma_1) + \sigma_1, x \le \dfrac{\sigma_1}{\sigma_2 + \sigma_1}. \end{cases} \end{cases} \tag{17.39}$$

Thus, it is formed by two straight line segments, the first one between the representative point of asset *A* (σ_1, μ_1) to the point representative of the portfolio without any risk, and the second from this last point and the point representative of asset *B* (σ_2, μ_2).

Figures 17.1 and 17.2 show the corresponding graphs.

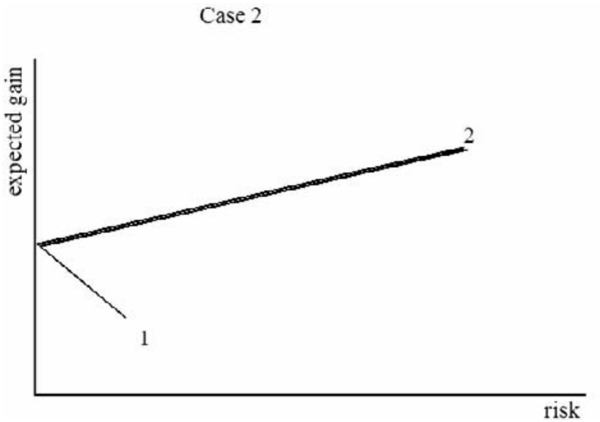

Figure 17.1. *Case 2*

Case 3: $-1 < \rho < 1$

In this case, the variance of the portfolio is given by:

$$\sigma^2 = (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2. \tag{17.40}$$

So, the parametric equations of the efficient frontier are:

$$\begin{cases} \mu = x\mu_1 + (1-x)\mu_{2,} \\ \sigma^2 = (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2. \end{cases} \tag{17.41}$$

To find the portfolio of minimum risk or minimum variance, we must find x* such that (17.40) is minimum. After some basic calculations, we obtain:

$$x^* = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \quad \sigma_{\min}^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{17.42}$$

It is easy to show that this variance is smaller than the minimum of the two variances of the assets, here σ_1^2.

This portfolio gives for a minimum risk a return larger than the minimum of the two asset returns, but it cannot exceed the value μ_2. To do that, we know that we must allow values of x>1.

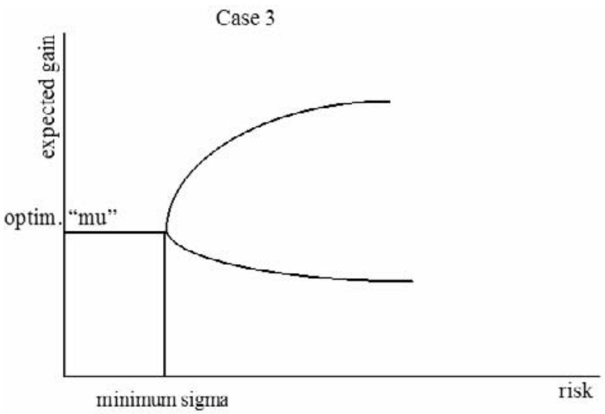

Figure 17.2. *Case 3*

17.5. Case of one risky asset and one non-risky asset

Let us consider a risky asset, which may be a portfolio having, on [0,*T*], a mean return and standard deviation μ_a and σ_a respectively. The non-risky asset will have a return *r* on the time period.

Here too, let *x* represent the proportion of the risky asset in the global portfolio; it follows that the mean return on [0,*T*] is given by

$$R(x) = xX + (1-x)r, \tag{17.43}$$

x being the random variable of the return of the risky asset on [0,*T*].

The mean and variances of the return of the global portfolio are given by:

$$\begin{aligned} \mu &= x\mu_a + (1-x)r, \\ \sigma &= x\sigma_a. \end{aligned} \tag{17.44}$$

By eliminating x, we obtain:

$$\mu = \frac{\mu_a - r}{\sigma_a}\sigma + \frac{\sigma_a}{\sigma}r. \tag{17.45}$$

This equation represents a straight line containing the point representative of the risky asset (σ_a, μ_a) with a slope of

$$\frac{\mu_a - r}{\sigma_a}. \tag{17.46}$$

This slope represents the *risk premium* for the investment in the risky asset and is in principle strictly positive except on very disturbed financial markets.

Remark 17.3

(i) If we introduce a second risky asset, we know the efficient frontier is a branch of a hyperbola and that we must consider the tangent with the maximum slope issuing from the point (0,r).

The optimal portfolios are now on this half tangent and after on the part of the efficient frontier for two risky assets (see Figure 17.3 on case 4).

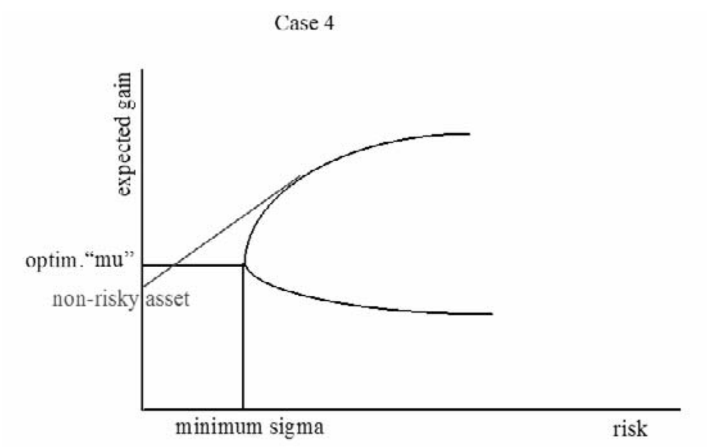

Figure 17.3. *Case 4*

Numerical example:

The following table gives data, exact results and simulation results for the efficient frontier in the case of two assets.

resolution data			simulation					
			proportion in the portfolio					
probability	active 1	active 2	active 1	active 2	average rend.	variance	standard deviation	
			1	0	0.125	0.026875	0.16393596	
0.25	0.1	0.8	0.9	0.1	0.1375	0.01741875	0.13198011	
0.5	0	0.2	0.8	0.2	0.15	0.0123	0.11090537	
0.25	0.4	-0.2	0.7	0.3	0.1625	0.01151875	0.10732544	opt. approched
			0.6	0.4	0.175	0.015075	0.12278029	
			0.5	0.5	0.1875	0.02296875	0.15155445	
mean	0.125	0.25	0.4	0.6	0.2	0.0352	0.18761663	
			0.3	0.7	0.2125	0.05176875	0.22752747	
	0.0425	0.19	0.2	0.8	0.225	0.072675	0.26958301	
			0.1	0.9	0.2375	0.09791875	0.31291972	
variance	0.026875	0.1275	0	1	0.25	0.1275	0.35707142	
standard deviation	0.16393596	0.35707142						
covariance	-0.03125							
correlation	-0.53385178							

xopt(ac2)	0.26801153
1-xopt(ac1)	0.73198847
sig2min	0.01129683
sigmin	0.10628655
value opt.	0.15850144

(ii) A numerical example with simulation is given as an exercise.

Exercise

An investor wants to invest a sum of €100,000 in a portfolio formed from two risky assets *A* and *B*.

The two-dimensional discrete distribution of the "returns" is given by the following table.

Probability	Possible values for A	Possible values for B
.1	−.05	0
.2	0	.05
.4	.1125	.0875
.2	.15	.10
.1	.20	.15

a) Show that the mean and variance covariance are given by:

$$\boldsymbol{\mu} = \begin{bmatrix} .09 \\ .08 \end{bmatrix}, \ \mathbf{V} = \begin{bmatrix} .00571 & .00267 \\ .00267 & .000141 \end{bmatrix}.$$

b) Give the graph of the efficient frontier.

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Chapter 18

Value at Risk (VaR) Methods and Simulation

18.1. VaR of one asset

18.1.1. *Introduction*

The VaR technique, due to J.P. Morgan and Company in 1994 in the follow up of Basel I prudential rules related to the quantification of credit and market risks, was distributed under the name of Riskmetrics as a way to measure the protection against the shortfall risk, that is, the critical risk of not having enough equity against facing a bad situation.

The aim of the VaR theory is to find, for a given risk, an amount of equity such that the probability of having a loss larger than this value is very small, for example 1%, and thus compatible with the attitude of the management against risk.

Of course, this determination always depends on the time horizon on which we are working: a day, a week, a month, etc.

This new tool achieved great success and its use is now reinforced not only in the recommendations of Basel II but also in Solvency II.

In fact, for actuaries, this approach is in the spirit of risk theory and ruin theory for insurance companies, but here defined more concretely in view of its real-life applications in finance and in insurance.

It is clear that the calculation of VaR values depends on the considered financial products: linear products (shares or bonds) or non-linear products (in fact, the optional products).

18.1.2. *Definition of VaR for one asset*

Let us consider an asset for which the stochastic time evolution on the time interval $[0,T], T>0$ is given by a stochastic process *S*,

$$S=\left(S(t), 0 \leq t \leq T\right) \tag{18.1}$$

defined on the complete filtered probability space $\left(\Omega, \Im, (\Im_t), P\right)$.

At t=0, we can observe the value of this asset on the market thus

$$S(0)=S_0, \tag{18.2}$$

S_0 being known.

On the time horizon *T*, for example 10 days, the eventual loss is given by the random variable:

$$S_0 - S(T) \tag{18.3}$$

where the value of the random variable S(T) is unknown at time *T* at which we have to calculate the VaR value.

It is also clear that there is a real loss if and only if the value of (18.3) is strictly positive.

As it is in general impossible to obtain a certain upper bound for the loss, except for the trivial one S_0, the only possibility is to construct a half confidence interval for (18.3) such that the probability of being outside of this interval is very small, let us say of value α.

Of course, the fixation of this value has a crucial value and is done by the supervisor.

The problem of calculating the VaR value at level α, noted VaR_α, is now formalized as follows:

$$P\left(S(0)-S(T) \leq VaR_\alpha\right)=\alpha. \tag{18.4}$$

Let us point out that VaR_α not only depends on α but also on the time interval [0, *T*] considered and of course on the distribution function of $S_0 - S(T)$. This is why we will now specify the choice of this distribution function.

18.1.3. *Case of the normal distribution*

18.1.3.1. *The VaR value*

Let us suppose that the d.f. of $S_0 - S(T)$ is normal with known parameters:

$$S_0 - S(T) \prec N(m_T, \sigma_T^2) \tag{18.5}$$

Thus, we have:

$$P\left(\frac{S_0 - S(T) - m_T}{\sigma_T} \leq z_\alpha\right) = \alpha$$

or

$$P\left(S_0 - S(T) \leq z_\alpha \sigma_T + m_T\right) = \alpha \tag{18.6}$$

and so that

$$VaR_\alpha = z_\alpha \sigma_T + m_T.$$

The following table gives some values of the *z*-quantile in function of the probability level α.

alpha	0.95	0.99	0.999	0.9999
z	1.6449	2.3263	3.1	3.7

Table 18.1. *z-quantile of the reduced normal distribution*

From this, we see the price of security: from level 0.95 to 0.99, the surplus with respect to the mean loss is multiplied by 1.41, by 1.89 to get to level 0.99, and finally by 2.24 to get to 0.9999! From 0.99 to 0.999, there is an increase of 33%.

18.1.3.2. *Numerical example I*

Let us suppose that a financial institution has 10,000 shares with individual value of €700.

On the basis of historical data, the global return on a time period of one year, for example, is estimated as having the following normal distribution:

$$S(1) - S_0 \prec N(60,1600).$$

It follows that the loss on the period has a normal distribution of mean –60 and standard deviation 40.

Using result (18.6) we obtain the VaR values given in Table 18.2 according to different security or probability levels.

α	0.95	0.99	0.999	0.9999
VaR	5.796	33.052	64	88

Table 18.2. *VaR values for one asset with the normal distribution*

The interpretation of these results uses the frequency interpretation of probability stating that the probability of an event can be seen as the ratio of the "favorable" cases, i.e. the realization of the considered event, over the total number of realizations, this last one assumed to be large so that this interpretation is confirmed by the law of large numbers.

So, with a level of 0.999, after one year there is one chance in 1,000 that the observed loss is over €64 per action.

If level 0.9999 is imposed, with one chance in 10,000, the loss per action is larger than €88, which is 40% larger than with the preceding level.

For the total of the investment, we obtain the following results.

α	0.95	0.99	0.999	0.9999
VaR	57,960	330,520	640,000	880,000

Table 18.3. *VaR values for the total investment with the normal distribution*

In percentage of the global investment, the part of the VaR is given by Table 18.4.

α	0.95	0.99	0.999	0.9999
VaR	0.00828	0.04722	0.09142	0.1257

Table 18.4. *VaR values in percentage of the global investment with the normal distribution*

So, to pass from the minimum level 0.95 up to the maximum level 0.9999, the amount of the VaR is multiplied by 15.18!

Conclusion for example I

This example shows both the interest of the concept of VaR and its difficulties to apply it due to the following:

– the selection a security level α : it is fixed by the supervisor;

– the estimation of the parameters from a good database on historical data of the considered asset and on the considered period;

– the use of normal distribution for the return is called the standard method in Basel I and II and thus there is no problem of authorization for the institution using except the justification of the parameter estimation;

– the risk of obtaining values too high for the VaR. In this case high amounts of equities could not be used for new investments.

18.1.4. *Example II: an internal model in case of the lognormal distribution*

One way for the financial institution to outline the last point is to build its own model, called an internal model, from which a VaR value can be calculated. If this internal model is approved by the supervisor, then the institution can use it instead of the standard method.

As an example, let us start with the assumption that the given asset has a stochastic dynamics governed by the Black and Scholes model seen in Chapter 14 in which we have seen that for such a model for stochastic process (18.1) with trend μ and volatility σ, the distribution of $S(t)/S_0$ is a lognormal distribution with parameters

$$\left(\mu-\frac{\sigma^2}{2}\right)t,\sigma^2 t$$

or

$$\ln\frac{S(t)}{S_0}\prec N\left(\left(\mu-\frac{\sigma^2}{2}\right)t,\sigma^2 t\right) \tag{18.7}$$

so that:

$$E[S(t)] = S_0 e^{\mu t},\\ \operatorname{var} S(t) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1). \tag{18.8}$$

To calculate VaR values at the time horizon T, we have to study the loss given by:

$$S_0 - S(T) = S_0\left(1 - \frac{S(T)}{S_0}\right), \tag{18.9}$$

and so, we successively obtain:

$$S_0 - S(T) = S_0\left(1 - \frac{S(T)}{S_0}\right), \tag{18.10}$$

$$P\left(S_0\left(1 - \frac{S(T)}{S_0}\right) \le VaR_\alpha\right) = \alpha,\\ P\left(\left(1 - \frac{S(T)}{S_0}\right) \le \frac{VaR_\alpha}{S_0}\right) = \alpha,\\ P\left(\left(1 - \frac{VaR_\alpha}{S_0}\right) \le \frac{S(T)}{S_0}\right) = \alpha,\\ P\left(\ln\left(1 - \frac{VaR_\alpha}{S_0}\right) \le \ln\frac{S(T)}{S_0}\right) = \alpha. \tag{18.11}$$

Using the reduced variable, we obtain:

$$P\left(\frac{\ln\left(1 - \frac{VaR_\alpha}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \le \frac{\ln\frac{S(T)}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \alpha. \tag{18.12}$$

And so:

$$1 - \Phi\left(\frac{\ln\left(1 - \frac{VaR_\alpha}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \alpha$$

or

$$\Phi\left(\frac{\ln\left(1-\frac{VaR_\alpha}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)=1-\alpha, \tag{18.13}$$

from which

$$\frac{\ln\left(1-\frac{VaR_\alpha}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=-z_\alpha.$$

To obtain the VaR value, we have to solve the following equation:

$$\frac{\ln\left(1-\frac{VaR_\alpha}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=-z_\alpha$$

or (18.14)

$$\ln\left(1-\frac{VaR_\alpha}{S_0}\right)=-z_\alpha\sigma\sqrt{T}+\left(\mu-\frac{\sigma^2}{2}\right)T.$$

This last result gives the explicit form of the VaR for the lognormal case:

$$1-\frac{VaR_\alpha}{S_0}=e^{-\sigma\sqrt{T}z_\alpha+\left(\mu-\frac{\sigma^2}{2}\right)T}$$

or (18.15)

$$VaR_\alpha=S_0\left(1-e^{-\sigma\sqrt{T}z_\alpha+\left(\mu-\frac{\sigma^2}{2}\right)T}\right).$$

Here, we see that the crucial problem in determining this VaR value is to calculate and then to estimate the two basic parameters: the trend and the volatility.

To do this, let us recall the following results (see Chapter 10):

$$E\left[S(T)\right]=S_0e^{\mu T},$$
$$\operatorname{var}S(T)=S_0^2e^{2\mu T}(e^{\sigma^2T}-1). \tag{18.16}$$

By inversion, we obtain the values of the two parameters μ,σ as a function of the mean and variance of $S(T)$:

$$\mu=\frac{1}{T}\ln\frac{E\left[S(T)\right]}{S_0},$$
$$\sigma^2=\frac{1}{T}\ln(1+\frac{\operatorname{var}S(T)}{S_0^2e^{2\mu T}}). \tag{18.17}$$

Let us consider the preceding example for the financial institution having at time 0 10,000 shares, each of value €700 and knowing on the time period T, that the mean return is €60 and the standard deviation 40.

Formula (18.17) gives as a result:

$$\mu=0.0822,$$
$$\sigma^2=0.0027665, \tag{18.18}$$
$$\sigma=0.052597.$$

The second result of (18.15) gives Table 18.5.

alpha	0.95	0.99	0.999	0.9998
VaR	3.95	28.45359	55.232	74.39682

Table 18.5. *VaR values for the lognormal distribution*

The two next tables compare the results of the two models: standard (normal) and internal (lognormal), first for one asset (Table 18.6) and secondly (Table 18.7) for all the investment.

alpha	0.95	0.99	0.999	0.9999
VaR I	5.796	33.052	64	88
VaR II	3.95	28.45359	55.232	74.39682

Table 18.6. *Comparisons of VaR values for one asset between models I and II*

For the portfolio, we obtain the following.

alpha	0.95	0.99	0.999	0.9999
VaR I	57,960	330,520	640,000	880,000
VaR II	39,500	284,536	552,320	743,968

Table 18.7. *Comparisons of VaR values for the global investment between models I and II*

Finally, the VaR values in percentage of the global investment are given by Table 18.8.

alpha	0.95	0.99	0.999	0.9999
VaR I	0.00828	0.04722	0.09142	0.1257
VaR II	0.0056	0.04064	0.07890	0.1062

Table 18.8. *Comparisons of VaR values in percentage for the global investment between models I and II*

The last table, Table 18.9, gives the VaR I as a percentage of the VaR II.

alpha	0.95	0.99	0.999	0.9999
VaR I	1.479	1.1619	1.1587	1.1836

Table 18.9. *VaR I as a percentage of the VaR II*

We see that this percentage reduces when the security level α increases.

Conclusion for example II

– Here, the internal model gives lower values than the standard model, and so the institution is lucky to use such a model! It will probably be accepted without any problem as the Black and Scholes model is well used for modeling share evolution.

– These two examples well illustrate the modelization risk as indeed with two acceptable models, we obtain different values of the VaR indicators. This is why the control authorities often use supplementary guarantees; for example, in Basel I, they used the level 0.99 and took a final VaR value three times the value at this level!

– Of course, if the internal model is favorable to the institution this model will be used to calculate VaR values provided this internal model will be validated by the control authorities.

– In any cases, this new VaR value cannot go too low with respect to the standard one, for example a reduction no more than 80%.

18.1.5. *Trajectory simulation*

When it is not possible to obtain explicit results of the VaR values or even when it is possible it may be useful to simulate *N* sample paths of the considered asset and observe for each of them the value of the loss at time *T*, that is, the values

$$S_0 - \widehat{S}_i(T), i = 1,...,N \ . \tag{18.19}$$

Then from the histogram of these values, it is possible to deduce an estimation of the VaR at different security levels: VaR_α , α varying.

Here we must mention that the control authorities always ask the financial institutions to periodically recalculate their VaR values VaR_α .

So, for example, in case of a VaR_α calculated daily for a time period of *M* days and with the day as time unit, the first simulation called *S*(1) will give the estimation $VaR_\alpha(M;1)$ valuable for the first day. For the second day, we must perform another simulation staring this time with $S(1) = \widehat{S}_1(1)$ to obtain the estimation of estimation $VaR_\alpha(M;2)$, etc. This means that for the j^{th} day, we will start from $S(j-1) = \widehat{S}_i(j-1), j = 2,...,M$.

Each sample path (*i*=1,…,*N*), will give the observed value $(\widehat{S}_i(j-1) - \widehat{S}_i(j), j = 1,2,...,M-1)$ from which we deduce the observed VaR values.

18.2. Coherence and VaR extensions

18.2.1. *Risk measures*

The notion of VaR represents well a risk measure or risk indicator for this investor. Generally, let us consider a given risk represented by the r.v. *X*, for example the loss at the end of a time period as before, and a risk measure defined as a functional θ associated with the given risk a positive $\theta(X)$, which provides the level of danger in the economic and financial environment of this investor for the given risk with subjective choice, as before the fixation of the security level α . In practice, $\theta(X)$ will always be an amount of money representing the capital needed to hedge the given risk, and of course we pose:

$$\theta(0) = 0 \tag{18.20}$$

Artzner, Delbaen, Eber and Heath (1999) introduced the concept of coherent risk measure imposing the following conditions:

(i) invariance by translation: $\theta(X+c)=\theta(X)+c, \forall c$;

(ii) sub-additivity: $\theta(X+Y)\leq\theta(X)+\theta(Y)$, for all risks X,Y; (18.21)

(iii) homogenity: $\theta(cX)=c\theta(X), \forall c>0$;

(iv) monotonicity: $P(X\leq Y)=1\Rightarrow\theta(X)\leq\theta(Y)$.

With Denuit and Charpentier (2004), the following condition, only useful for insurance, is added:

$$\text{(v) } \theta(X)\geq E(X) \tag{18.22}$$

stating that the amount of hedging is always higher than or equal to the mean loss.

From property (iii), it follows that if loss X is equal to a constant, then $\theta(c)=c$, which is a very intuitive condition.

Property (ii) of sub-additivity implies that every diversification leads to a risk reduction or at least does not increase the risk, in conformity with Markowitz's theory developed in Chapter 17.

Finally, in insurance management, property (v) explains that the ruin event is certain without introducing a load factor in the pure premium of value $E(X)$.

18.2.2. *General form of the VaR*

In the preceding section we have seen that for a risk measured by the r.v. X having a normal distribution

$$X \prec N(m_X,\sigma_X^2) , \tag{18.23}$$

the VaR value at level α is given by the quantile of order $(1-\alpha)$ (α small!) of the d.f. of X, that is:

$$VaR_\alpha = z_\alpha\sigma_X + m_X$$

because (18.24)

$$P\left(X\leq z_\alpha\sigma_X+m_X\right)=\alpha.$$

Now, for a risk X having a general d.f. F_X, the VaR level α satisfies the following equality:

$$F_X(VaR_\alpha) = \alpha \tag{18.25}$$

and if function F_X is strictly increasing, we obtain:

$$F_X^{-1}(\alpha) = VaR_\alpha. \tag{18.26}$$

This relation gives the way to calculate the VaR value at a given level provided we know the d.f. of X. The knowledge of this function is in general not easy, and so we can proceed with a parametric model as in the Black and Scholes model in section 1 or use simulation methods.

For such a definition of the VaR, it is possible to show that it defines a risk measure that is invariant under translation, homogenous and monotone but not always sub-additive for every d.f. F_X.

Denuit and Charpentier (2005) give the following counter-example: let X and Y be two independent risks having a Pareto distribution, that is,

$$F_X(x) = F_Y(x) = \begin{cases} 1-\left(\dfrac{\theta}{x+\theta}\right)^{\beta}, x \geq 0, \\ 0, x < 0. \end{cases} \tag{18.27}$$

but with $\theta = \beta = 1$ so that:

$$P(X \leq x) = P(Y \leq x) = \frac{x}{1+x}, x > 0. \tag{18.28}$$

From relation (18.25), we obtain at level α :

$$P(X \leq VaR_\alpha(X)) = \frac{VaR_\alpha(X)}{1+VaR_\alpha(X)} = \alpha \tag{18.29}$$

and so

$$VaR_\alpha(X) = \frac{\alpha}{1-\alpha}. \tag{18.30}$$

From the fact that X and Y have the same distribution, we also obtain:

$$VaR_\alpha(Y) = \frac{\alpha}{1-\alpha}. \tag{18.31}$$

The d.f. of the sum X+Y is given by the following manipulation:

$$F_{X+Y}(z) = \int_0^z F_X(z-u)dF_Y(z) = F_X^{(2)}(z). \tag{18.32}$$

A little calculation gives the final result:

$$F_{X+Y}(z) = 1 - \frac{2}{2+z} + 2\frac{\ln(1+z)}{(2+z)^2}, z > 0. \tag{18.33}$$

If we replace

$$z = VaR_\alpha(X) + VaR_\alpha(Y), \tag{18.34}$$

we obtain:

$$z = VaR_\alpha(X) + VaR_\alpha(Y) = 2VaR_\alpha(X) = 2\frac{\alpha}{1-\alpha}, \tag{18.35}$$

and so:

$$F_{X+Y}(2VaR_\alpha(X)) = \alpha - \frac{(1-\alpha)^2}{2}\ln\left(\frac{1+\alpha}{1-\alpha}\right) < \alpha. \tag{18.36}$$

Now, if the property of sub-additivity was satisfied for this Pareto distribution, we must:

$$VaR_\alpha(X+Y) \le VaR_\alpha(X) + VaR_\alpha(Y) \tag{18.37}$$

and by relation (18.35):

$$F_{X+Y}(VaR_\alpha(X+Y)) \le F(2VaR_\alpha(X)) \tag{18.38}$$

or by relations (18.25) and (18.36):

$$\alpha = \alpha - \frac{(1-\alpha)^2}{2}\ln\left(\frac{1+\alpha}{1-\alpha}\right) \tag{18.39}$$

However, this is impossible as the second member of (18.39) is strictly less than α.

This contradiction proves well that, in general, VaR is not always a coherent risk measure.

Remark 18.1 Of course, this contradiction does not imply that VaR is never a coherent risk measure whatever the d.f. F_X is, and particularly for the standard case of Basel rules, it is so.

To prove this result, let us consider two risks *X* and *Y* such that:

$$\begin{aligned} &X \prec N(m_X, \sigma_X^2), Y \prec N(m_Y, \sigma_Y^2), \\ &\rho(X,Y) = \rho. \end{aligned} \tag{18.40}$$

From relation (18.6), we know that:

$$\begin{aligned} &VaR_\alpha(X) = z_\alpha \sigma_X + m_X, \\ &VaR_\alpha(Y) = z_\alpha \sigma_Y + m_Y, \\ &VaR_\alpha(X+Y) = z_\alpha \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} + m_X + m_Y. \end{aligned} \tag{18.41}$$

As $|\rho| \leq 1$, we obtain:

$$\begin{aligned} &VaR_\alpha(X+Y) \\ &= z_\alpha \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} + m_X + m_Y \leq z_\alpha(\sigma_X + \sigma_Y) + m_X + m_Y \end{aligned} \tag{18.42}$$

and so from the first two relations of (18.41), we obtain:

$$VaR_\alpha(X+Y) \leq VaR_\alpha(X) + VaR_\alpha(Y), \tag{18.43}$$

thus proving that for the standard case, the VaR is well sub-additive. As an additional result, this result also shows that using the optimal diversification principle of Markowitz, we also reduce the VaR of the optimal portfolio with respect to the sum of all individual VaR of each component.

18.2.3. *VaR extensions: TVaR and conditional VaR*

The search for new indicators having, if possible, better properties than the VaR begins with the consideration that the fixation of the security level is of course subjective and so, the idea is that we effectively fix this level in a reasonable way at value α but to take into account all the values larger than α, we take the mean of all the corresponding VaR values to obtain a new indicator called Tail-VaR, denoted $TVaR_\alpha(X)$ and defined as:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\xi(X)d\xi. \tag{18.44}$$

With the change of variable $\xi \mapsto x$, where $\xi = F_X(x)$, we obtain:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_{VaR_\alpha(X)}^\infty xdF_X(x). \tag{18.45}$$

It follows that:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\left[\int_{VaR_\alpha(X)}^\infty xdF_X(x)\right] \tag{18.46}$$

or:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\left[E(X) - \int_0^{VaR_\alpha(X)} xdF_X(x)\right] \tag{18.47}$$

and with the same change of variable as above, we obtain:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\left[E(X) - \int_0^\alpha VaR_\alpha(X)d\xi\right]. \tag{18.48}$$

As the VaR is a function of α, it is also possible to show that the function TVaR of variable α is also decreasing and so, in particular:

$$TVaR_\alpha(X) \geq TVaR_0(X) = E(X). \tag{18.49}$$

Of course, from relation (18.44), we also have:

$$TVaR_\alpha(X) \geq VaR_\alpha(X). \tag{18.50}$$

To continue, let us now consider the loss if this loss is effectively greater than the VaR, that is, what we call a scenario catastrophe. To measure this new risk of catastrophic loss, we introduce three new risk indicators:

(i) the conditional tail expectation or CTE level $\alpha : CTE_\alpha$;

(ii) the conditional VaR or CVaR at level $\alpha : CVaR_\alpha$;

(iii) the expected shortfall or ES at level $\alpha : ES_\alpha$.

Their definitions are as follows:

$$\begin{aligned}&\text{(i) } CTE_\alpha(X) = E[X \mid X > VaR_\alpha(X))],\\&\text{(ii) } CVaR_\alpha(X) = E\left[X - VaR_\alpha(X) \mid X > VaR_\alpha(X)\right],\\&\text{(iii) } ES_\alpha(X) = E\left[\max\left\{X - VaR_\alpha(X), 0\right\}\right].\end{aligned} \tag{18.51}$$

Clearly, we have:

$$CTE_\alpha(X) = CVaR_\alpha(X) + VaR_\alpha(X). \tag{18.52}$$

Thus, $CTE_\alpha(X)$ represents the expectation value of the total loss given that this loss is larger than $VaR_\alpha(X)$ and $CVaR_\alpha(X)$, the expectation value of the excess of less beyond the $VaR_\alpha(X)$.

$ES_\alpha(X)$ represents the mean loss leveled at $VaR_\alpha(X)$.

It is possible to show the following results (Denuit and Charpentier (2004)):

$$\begin{aligned}TVaR_\alpha(X) &= VaR_\alpha(X) + \frac{1}{1-\alpha} ES_\alpha(X),\\CTE_\alpha(X) &= VaR_\alpha(X) + \frac{1}{1 - F_X(VaR_\alpha(X))} ES_\alpha(X).\end{aligned} \tag{18.53}$$

Moreover, if the d.f. F_X is continuous, we know that in this case $F_X(VaR_\alpha(X)) = \alpha$ and so the two right members in (18.53) are equal giving the next result

$$CTE_\alpha(X) = TVaR_\alpha(X). \tag{18.54}$$

Finally, it is possible to show that the TVaR indicator is coherent.

Example 18.1 *The conditional tail expectation $CTE_\alpha(X)$ in the standard case.*

For X having a normal distribution of parameters $m_X = m, \sigma_X^2 = \sigma^2$, let us calculate

$$CTE_\alpha(X) = E(X \mid X > VaR_\alpha(X)). \tag{18.55}$$

As:

$$P(X > y \mid X > x) = \frac{P(X > y}{P(X > x)}, y > x, \tag{18.56}$$

we have:

$$CTE_\alpha(X) = E(X \mid X > VaR_\alpha(X)) = \frac{E(X1_{X>VaR_\alpha(X)})}{P(X > VaR_\alpha(X))} = \frac{E(X1_{X>VaR_\alpha(X)})}{1-\alpha}. \tag{18.57}$$

The value of the denominator is given by

$$v(1-\alpha) + \int_v^\infty x f_X\, dx,$$

where (18.58)

$$v = VaR_\alpha(X),$$
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

With the following change of variable: $y = \dfrac{x-m}{\sigma}$, we obtain:

$$\begin{aligned}\int_v^\infty x f_X\, dx &= \frac{1}{\sqrt{2\pi}} \int_{\frac{v-m}{\sigma}}^\infty (\sigma y + m) e^{-\frac{y^2}{2}} dy,\\ &= \frac{1}{\sqrt{2\pi}} \sigma \int_{\frac{v-m}{\sigma}}^\infty y e^{-\frac{y^2}{2}} dy + \frac{1}{\sqrt{2\pi}} m \int_{\frac{v-m}{\sigma}}^\infty e^{-\frac{y^2}{2}} dy,\\ &= \frac{1}{\sqrt{2\pi}} \sigma I_1 + m\left(1 - \Phi\left(\frac{v-m}{\sigma}\right)\right),\end{aligned} \tag{18.59}$$

$$I_1 = \int_{\frac{v-m}{\sigma}}^\infty y e^{-\frac{y^2}{2}} dy.$$

For the calculation of I_1, we successively have:

$$\int_z^{\infty} y e^{-\frac{y^2}{2}} dy = \int_z^{\infty} e^{-\frac{y^2}{2}} d\frac{y^2}{2},$$

$$= \int_{\frac{z^2}{2}}^{\infty} e^{-u} du, \tag{18.60}$$

$$= e^{-\frac{z^2}{2}}.$$

Thus, by the last relation of (18.59):

$$I_1 = e^{-\frac{(v-m)^2}{2\sigma^2}}. \tag{18.61}$$

Now from the last equality of (18.59), we can write:

$$\int_v^{\infty} x f_X dx = \frac{1}{\sqrt{2\pi}} \sigma e^{-\frac{(v-m)^2}{2\sigma^2}} + m\left(1 - \Phi\left(\frac{v-m}{\sigma}\right)\right),$$

$$= \sigma\varphi\left(\frac{v-m}{\sigma}\right) + m\left(1 - \Phi\left(\frac{v-m}{\sigma}\right)\right),$$

where (18.62)

$$\varphi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Returning to relations (18.55), and (18.57), we finally obtain:

$$CTE_\alpha(X) = E(X \mid X > VaR_\alpha(X)),$$

$$= \frac{(1-\alpha)VaR_\alpha + \int_{VaR_\alpha}^{\infty} x f_X(x) dx}{1-\alpha}, \tag{18.63}$$

that is,

$$CTE_{\alpha}(X) = VaR_{\alpha}(X) + \frac{1}{1-\alpha}[\sigma_X \varphi(z_{\alpha}) + m_X(1-\Phi(z_{\alpha}))],$$
$$= VaR_{\alpha}(X) + \frac{\sigma_X \varphi(z_{\alpha})}{1-\alpha} + m_X. \tag{18.64}$$

As the normal distribution is of continuous type, we also have:

$$CTE_{\alpha}(X) = TVaR_{\alpha}(X). \tag{18.65}$$

Finally, from relations (18.45) and (18.44), we obtain:

$$ES_{\alpha}(X) = (1-\alpha)[TVaR_{\alpha}(X) - VaR_{\alpha}(X)],$$
$$CVaR_{\alpha}(X) = TVaR_{\alpha}(X) - VaR_{\alpha}(X). \tag{18.66}$$

Remark 18.2

1) With the lognormal assumption, $X \prec LN(\mu, \sigma^2)$, Besson and Partrat (2005) show that:

$$CVaR_{\alpha}(X) = e^{\mu+\frac{\sigma^2}{2}} \frac{\bar{\Phi}\left(\frac{\ln VaR_{\alpha}(X)-\mu}{\sigma} - \sigma\right)}{\bar{\Phi}\left(\frac{\ln VaR_{\alpha}(X)-\mu}{\sigma}\right)} - VaR_{\alpha}(X), \tag{18.67}$$

$$\bar{\Phi} = 1 - \Phi.$$

2) Here, we are directly interested with the loss assumed to be positive without introducing a stochastic dynamic model of the considered asset as in example II.

18.3. VaR of an asset portfolio

As we mentioned in Chapter 17 related to the Markowitz theory, for a portfolio composed of several assets, the main difficulty for applying this theory is the estimation of the variance-covariance matrix of the vector of assets constituting this portfolio.

Of course, this problem also exists when we have to calculate the VaR of such a portfolio of several assets.

Three basic methods can be used to calculate the VaR:

– the method of variance-covariance matrix;

– the simulation method;

– the historic method.

In the next sections, we will briefly describe them.

18.3.1. VaR methodology

Theoretically, it is not difficult to extend the VaR method for one asset to a portfolio composed of n assets.

Let

$$\left(S_1(t),...,S_n(t)\right), t \in [0,T] \tag{18.68}$$

be the stochastic process of the vector of the n considered assets; on $[0,T]$, the relative returns are given by

$$\xi_i = \frac{S_i(T) - S_i(0)}{S_i(0)}, i = 1,...,n \tag{18.69}$$

so that:

$$S_i(T) - S_i(0) = \xi_i S_i(0), i = 1,...,n. \tag{18.70}$$

If

$$x = \left(x_1,...,x_n\right)'$$

with (18.71)

$$x_i \geq 0, i = 1,...,n,$$
$$\sum_{i=1}^{n} x_i = 1,$$

represents the vector of the percentages of repartition of the considered assets in the global portfolio, we have:

$$S(t)=\sum_{i=1}^{n}x_i S(t),t\in[0,T] \tag{18.72}$$

and the return of the given global portfolio

$$\begin{aligned}S(T)-S(0)&=\sum_{i=1}^{n}x_i S_i(T)-\sum_{i=1}^{n}x_i S_i(0),\\&=\sum_{i=1}^{n}x_i\left[S_i(T)-S_i(0)\right],\\&=\sum_{i=1}^{n}x_i\xi_i S_i(0).\end{aligned} \tag{18.73}$$

To continue, we need to introduce the mean vector and the variance covariance matrix of the vector $\xi=(\xi_1,...,\xi_n)'$:

$$\begin{aligned}&E[\xi]=(m_1,...,m_n)',\\&\Sigma_\xi=(\sigma_{ij})\end{aligned} \tag{18.74}$$

so that for the global portfolio, we obtain from the last equality of (18.73):

$$\begin{aligned}&m=E\left[S(T)-S(0)\right]=\sum_{i=1}^{m}S_i(0)x_i m_i,\\&\sigma^2=\text{var}\left[S(T)-S(0)\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}S_i(0)S_j(0)x_i x_j.\end{aligned} \tag{18.75}$$

From these results, it follows that if the vector of returns ξ has a multi-normal distribution, then the loss of the global portfolio also has a normal distribution of parameters $N\left(-m,\sigma^2\right)$.

Thus, we reach the conclusion that in the standard case, the VaR calculation of the global portfolio of the *n* assets as similar to the case of the VaR for one asset developed in section 18.1.3, relation (18.6).

In the next section, we will show how to implement this method for real applications.

18.3.2. *General methods for VaR calculation*

18.3.2.1. *The variance-covariance matrix method*

This method is also known as the Riskmetrics method developed by J.P. Morgan (1996) under the assumption of the multidimensional normality of the vector of returns.

The three steps of the methods are as follows:

(i) calculate the present value of the portfolio, on the time horizon T;

(ii) estimate the mean return vector and of the variance-covariance matrix that must be actualized every day in principle;

(iii) calculate the VaR at the fixed level α.

18.3.2.2. *The simulation method*

As always in this case, this method is based on a simulation model for the evolution of the considered assets on the time horizon T, the model depending of course of a number of parameters that must be estimated, and to be useful it needs many simulations.

The steps are as follows:

(i) choose a distribution for the vector of returns on the time horizon T;

(ii) simulate a large number of sample paths on $[0,T]$;

(iii) estimate the VaR at the fixed level α.

18.3.2.3. *The historic method (Chase Manhattan Bank 1996)*

The basic principle of this method is to assume that the distribution of the asset returns in the future is identical to the one in the past.

Of course, this assumption may only be valid on a relatively short time interval and is very sensitive to the quality of the data.

Its main interest is that no assumption is made on the distribution of the asset returns as we start from the observed data in the past to estimate this distribution.

The main steps are as follows:

(i) calculate the present value of the portfolio, on the time horizon T;

(ii) estimate historical returns on the basis of the retained risk factors (asset values, bond values, exchange rates, options values, etc.);

(iii) calculate the historical values of gains and losses of the considered portfolio;

(iv) estimate the VaR at the fixed level α.

Let us also mention that complementing these three methods, the method of scenarios is often used meaning that we can select stressing scenarios corresponding to catastrophic events, which of course are rare and of very small probability, to see how the VaR and TVaR indicators resist in the extreme situations.

18.3.3. *VaR implementation*

The use of VaR methods depends on the model retained for the time stochastic evolution of the considered n assets of the portfolio.

From section 18.3.1, using the standard model means that vector $\xi = (\xi_1,...,\xi_n)'$ has a multi-normal distribution with mean and variance covariance matrix given by (18.74):

$$\begin{aligned} &E[\xi] = (m_1,...,m_n)', \\ &\Sigma_\xi = (\sigma_{ij}) \end{aligned} \tag{18.76}$$

so that for the global portfolio, from (18.73), we know that:

$$\begin{aligned} &m = E[S(T) - S(0)] = \sum_{i=1}^{m} S_i(0) x_i m_i, \\ &\sigma^2 = \operatorname{var}[S(T) - S(0)] = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} S_i(0) S_j(0) x_i x_j. \end{aligned} \tag{18.77}$$

As

$$P\left(\frac{S_0 - S(T) + m}{\sigma} \le z_\alpha\right) = \alpha$$

or (18.78)

$$P(S_0 - S(T) \le z_\alpha \sigma - m) = \alpha,$$

from (18.73), we have for the VaR value at level α :

$$VaR_\alpha = z_\alpha \sigma - m. \tag{18.79}$$

The number of parameters to be estimated for the application of this last result is in general high as indeed, we have the n values of the means, the n values of the variance and the $n(n\text{-}1)/2$ covariance values (for two distinct assets), so that we have $n(n+2)/2$ parameters.

For example, if n=50, which is not a large number of assets for a big bank, we have 1,300 parameters and for n=100, 5,100 parameters!

In fact, as mentioned in Chapter 17, the situation is the same as for the calculation of the efficient frontier and so the crucial problem is to see how to reduce this high number of parameters to estimate, which is a curse in dimension reducibility.

Two possibilities exist: the first possibility, as in Riskmetrics, is based on the evolution of the financial cash flows producing the returns of the portfolio, and the second on the use of econometric models for the asset market.

18.3.3.1. *The Riskmetrics method*

For the considered portfolio, let

$$\left(F_k, t_k, k = 1,...,m\right) \tag{18.80}$$

be the future produced cash flows: at time t_k, the asset will produce a return of value $F_k, k = 1,...,m$.

For the given yield curve, the present value of the portfolio is:

$$\sum_{j=1}^{m}(1+i_{t_k})^{-t_k} F_k, \tag{18.81}$$

where i_{t_k} represents the equivalent annual rate for maturity $t_k, k = 1,...,m$.

It is clear that each cash flow value is random and so must be treated carefully and a complete treatment requires the calculation of the correlations between all the components of the cash flow which, as we have seen above, is a problem with combinatory explosion!

Moreover, there will surely be a lack of enough data.

The proposed solution is called the mapping method in which we redistribute on a restricted number of standard maturities, for example: 1, 3, 6 months, 1, 2, 3, 5, 7, 10, 15, 20 and 30 years, all the maturities of cash flows necessary to analyze the different returns.

This needs to solve the following problem: how to split a cash amount of present value M and maturity t between the two nearest standard maturities retained $t_k, t_{k+1} (t_k < t_{k+1})$ and with respective present values M_1, M_2.

This problem is solved with the introduction of two conditions, the first one imposing the equality of the present values and the second the invariance of the duration (see Chapter 9):

$$\begin{aligned} M &= M_1 + M_2, \\ tM &= t_k M_1 + t_{k+1} M_2. \end{aligned} \tag{18.82}$$

This linear system in M_1, M_2 has the following unique solution:

$$\begin{aligned} M_1 &= \frac{t_{k+1} - t}{t_{k+1} - t_k} M, \\ M_2 &= \frac{t - t_k}{t_{k+1} - t_k} M. \end{aligned} \tag{18.83}$$

For variances $\sigma_k^2, \sigma_{k+1}^2, \sigma^2$ of the corresponding returns ξ_k, ξ_{k+1}, ξ , we obtain:

$$\sigma^2 M = \sigma_k^2 M_1^2 + \sigma_{k+1}^2 M_2^2 + 2\rho_{12} \sigma_k \sigma_{k+1} M_1 M_2 \tag{18.84}$$

where, of course, ρ_{12} is the correlation coefficient between the two cash amounts.

Without any information on it, we use the following two inequalities

$$\left(\sigma_k M_1 - \sigma_{k+1} M_2\right)^2 \leq \sigma^2 M^2 \leq \left(\sigma_k M_1 + \sigma_{k+1} M_2\right)^2 \tag{18.85}$$

If we want retain an assumption of maximum volatility, we have to use the second inequality and so:

$$\hat{\sigma} = \frac{\sigma_k M_1 + \sigma_{k+1} M_2}{M}. \tag{18.86}$$

This will lead to an overestimation of the VaR and of course the use of the first inequality will lead to a sub-estimation.

These two problems of over and sub-estimation can be avoided using a linear interpolation:

$$\begin{aligned} &\sigma = \alpha\sigma_1 + (1-\alpha)\sigma_2, \\ &\alpha = \frac{M_1}{M}, 1-\alpha = \frac{M_2}{M} \end{aligned} \tag{18.87}$$

and the two following conditions:

$$\begin{aligned} &M = M_1 + M_2, \\ &\sigma^2 M = \sigma_k^2 M_1^2 + \sigma_{k+1}^2 M_2^2 + 2\rho_{12}\sigma_k\sigma_{k+1}M_1M_2. \end{aligned} \tag{18.88}$$

Using the last equality of (18.87) for the substitution of M_1, M_2, we obtain the following equation for α:

$$\alpha^2(\sigma_k^2 + \sigma_{k+1}^2 - 2\sigma_k\sigma_{k+1}\rho_{12}) + 2\alpha(\sigma_k\sigma_{k+1}\rho_{12} - \sigma_{k+1}^2) + (\sigma_{k+1}^2 - \sigma^2) = 0. \tag{18.89}$$

18.3.3.2. *VaR for an asset portfolio with Sharpe model*

Another way for reducing the number of parameters of the covariance matrix is the use of some economic market models; we will see how this method works with two such models: the Sharpe and the MEDAF models.

Let us consider a market with *n* assets. If $r_j, j = 1,...,n$ represents the return function of asset *j*, the Sharpe model assumes that the variations of these returns satisfy the following relations:

$$\Delta r_j = \alpha_j + \beta_j \Delta I + \varepsilon_j, j = 1,...,n \tag{18.90}$$

where ΔI represents the variation of a reference market on the considered time horizon.

The "slack" variables are assumed to be independent and $N(0, \sigma_{\varepsilon_j}^2)$. ΔI is assumed normal and moreover independent of the slack variables.

As the global variation of the portfolio on the considered period is given by

$$\Delta r = \sum_{j=1}^{n} n_j \Delta r_j, \tag{18.91}$$

with

$$\begin{aligned} n_j &= x_j S_j(0), \\ \Delta r_j &= \frac{S_j(1) - S_j(0)}{S_j(0)}, \end{aligned} \tag{18.92}$$

we obtain:

$$\begin{aligned} E[\Delta r] &= \sum_{j=1}^{n} n_j \left[\alpha_j + \beta_j m_I \right], \\ &= \sum_{j=1}^{n} n_j \alpha_j + \left(\sum_{j=1}^{n} n_j \beta_j \right) m_I. \end{aligned} \tag{18.93}$$

Furthermore, we also have:

$$\operatorname{var}[\Delta r] = \left(\sum_{j=1}^{n} n_j \beta_j \right)^2 \sigma_I^2 + \sum_{j=1}^{n} n_j^2 \sigma_{\varepsilon_j}^2. \tag{18.94}$$

Remark 18.3 Fortunately, the independence between the error variables does not imply the independence of the returns of the assets as indeed:

$$\operatorname{cov}(\Delta r_i, \Delta r_j) = \beta_i \beta_j \sigma_I^2, i \neq j. \tag{18.95}$$

From this result, it follows that for a portfolio of n assets, the calculation of the $n(n\text{-}1)/2$ covariances reduces to the knowledge of the n^2 beta parameters and the volatility of the market index.

Such a calculation seems more realistic and so using the traditional approach of the VaR with the normal distribution, we obtain the following result:

$$VaR\Delta r = z_{\alpha}\sqrt{\operatorname{var}\Delta r} - E(\Delta r). \tag{18.96}$$

Exercise

Let us consider a portfolio with three assets with $n_1 = 3, n_2 = 6, n_3 = 1$ satisfying the following Sharpe model:

$$\begin{aligned} \Delta r_1 &= 0.014 + 0.60\Delta I + \varepsilon_1, \varepsilon_1 \sim N(0, 0.006), \\ \Delta r_2 &= 0.014 + 0.60\Delta I + \varepsilon_1, \varepsilon_1 \sim N(0, 0.006), \\ \Delta r_3 &= -0.200 + 1.32\Delta I + \varepsilon_3, \varepsilon_3 \sim N(0, 0.012) \end{aligned} \tag{18.97}$$

Moreover, the reference market index has a normal distribution

$N(0.0031,0.0468)$

and:

$$X_1(0) = 120, X_2(0) = 15, X_3(0) = 640.$$

Calculate the VaR value at confidence level 99%.

Answer: 21.4421.

18.3.3.3. *VaR for an asset portfolio with the MEDAF model*

Let us consider a portfolio of value $S(t)$ at time t constituted at time 0 with n assets such that in t, $x_i, i = 1,...,n$ and $S_i(t)$ represent successively the number and value of assets of type $i = 1,...,n$.

Thus, we have:

$$S(t) = \sum_{i=1}^{n} x_i S_i(t) . \tag{18.98}$$

On time horizon T, we have T:

$$\Delta S_i(T) = S_i(T) - S_i(0),$$
$$r_i(T) = \frac{S_i(T) - S_i(0)}{S_i(0)}. \tag{18.99}$$

These relations lead to:

$$S(T) = \sum_{i=1}^{n} x_i S_i(0)(1 + r_i(T)). \tag{18.100}$$

The MEDAF model assumes that:

$$r_i(T) = r_0 + \beta_i (r_m(T) - r_0) + \varepsilon_i(T)$$
$$i = 1, \ldots, n \tag{18.101}$$

where

– $r_m(T)$ is the return of the market portfolio S_m on $[0, T]$;

– r_0 is the non-risky return on the same period;

– r.vs. $\varepsilon_i(T), i = 1, \ldots n$ are independent, with normal distribution and uncorrelated with the return of the market portfolio so that:

$$E(\varepsilon_i) = 0, E(\varepsilon_i r_m) = 0, i = 1, \ldots, n; \tag{18.102}$$

– β_i represents the coefficient β of i, i=1,…,n.

Using this model, we obtain:

$$\Delta S(T) = S(0)\left[r_0 + \overline{\beta}(r_m(T) - r_0) + \varepsilon(T) \right],$$
$$\overline{\beta} = \frac{\sum_{i=1}^{n} \beta_i x_i S_i(0)}{S(0)}, \varepsilon(T) = \frac{\sum_{i=1}^{n} \varepsilon_i(T) x_i S_i(0)}{S(0)}. \tag{18.103}$$

For the mean and variance of the portfolio return, we obtain:

$$E\left[\Delta S(T)\right] = S(0)\left[r_0 + \overline{\beta}(E[r_m(T)] - r_0)\right],$$
$$\operatorname{var}\left[\Delta S(T)\right] = (S(0))^2\left[(\sigma_m\overline{\beta})^2 + \sum_{i=1}^{n}[\frac{S_i(0)}{S(0)}]^2\sigma_i^2\right]. \tag{18.104}$$

Consequently, the VaR value at confidence level α is given here by:

$$VaR_{\alpha}\Delta r = z_{\alpha}\sqrt{\operatorname{var}\Delta S(T)} - E(\Delta S(T)). \tag{18.105}$$

18.3.4. *VaR for a bond portfolio*

Let us consider a portfolio of value $S(t)$ at time t constituted at time 0 with n bonds such that in t, $x_i, i = 1,...,n$ and $v_i(t)$ represent successively the number and value of bonds of type $i = 1,...,n$.

So, we have:

$$p(t) = \sum_{i=1}^{n} x_i v_i(t)\ . \tag{18.106}$$

Let $X_1,...,X_k$ be the k risk factors such that:

$$v_j(t) = a_{j1}X_1(t) + ... + a_{jk}X_k(t), j = 1,...,n. \tag{18.107}$$

The portfolio value at time T is given by:

$$p(t) = \sum_{j=1}^{n} x_j v_j(t),$$
$$= \sum_{\nu=1}^{k}(\sum_{j=1}^{n} x_j a_{j\nu})X_{\nu}(t). \tag{18.108}$$

To simplify, let us work on a time horizon of length 1 on which:

$$\begin{aligned}
\Delta p &= p(1) - p(0),\\
&= \sum_{\nu=1}^{k}\left(\sum_{j=1}^{n} x_j a_{j\nu}\right)[X_\nu(1) - X_\nu(0)],\\
&= \sum_{\nu=1}^{k}\left(\sum_{j=1}^{n} x_j a_{j\nu}\right)X_\nu(0)\left[\frac{X_\nu(1) - X_\nu(0)}{X_\nu(0)}\right],\\
&= \sum_{\nu=1}^{k}\left(\sum_{j=1}^{n} x_j a_{j\nu}\right)X_\nu(0) r_\nu(0),\\
r_\nu(0) &= \frac{X_\nu(1) - X_\nu(0)}{X_\nu(0)}, \nu = 1,\ldots k.
\end{aligned} \tag{18.109}$$

To calculate VaR values, we must study the k random returns $r_\nu(0)$.

Using for example the historic method, we can obtain the following estimations:

$$\begin{aligned}
&E\left[r_\nu(0)\right] = \hat{m}_\nu,\\
&\operatorname{cov}\left[r_\nu(0), r_{\nu'}(0)\right] = \sigma_{\nu\nu'}.
\end{aligned} \tag{18.110}$$

Now, from relations (18.109), we obtain:

$$\begin{aligned}
&\Delta p = \sum_{\nu=1}^{k}\left(\sum_{j=1}^{n} x_j a_{j\nu}\right)X_\nu(0) r_\nu(0),\\
&\Delta p = \sum_{j=1}^{k} y_\nu(0) r_\nu(0),\\
&y_\nu(0) = \sum_{j=1}^{n} x_j a_{j\nu} X_\nu(0), \nu = 1,\ldots k.
\end{aligned} \tag{18.111}$$

For the mean and variance, we obtain:

$$\begin{aligned}
&E[\Delta p] = \sum_{j=1}^{k} y_\nu(0)\hat{m}_\nu,\\
&\operatorname{var}[\Delta p] = \sum_{\nu=1}^{k}\sum_{\nu'=1}^{k} y_\nu y_{\nu'}(0)\hat{\sigma}_{\nu\nu'},, \nu = 1,\ldots k.
\end{aligned} \tag{18.112}$$

Once more, with the assumption of the normal distribution of the return, we obtain:

$$VaR_{\alpha}\Delta r = z_{\alpha}\sqrt{\operatorname{var}\Delta p} - E(\Delta p). \tag{18.113}$$

18.4. VaR for one plain vanilla option

We know that the value of a plain vanilla option, for example, a call, depends on the following:

– S: value of the underlying asset at the time t of evaluation;

– T-t: maturity;

– K: exercise price;

– σ : volatility of the underlying asset;

– r: the instantaneous non-risky interest rate.

For small variations of these parameters in the short time interval $(t, t+\Delta t)$ characterized by the vector $(\Delta S, \Delta K, \Delta\tau, \Delta\sigma, \Delta r)$, we know from Chapter 14 that the corresponding variation of the call is linearly approximated with the aid of the Greek parameters:

$$\Delta C = \Delta\Delta S + \frac{1}{2}\Gamma(\Delta S)^2 + \frac{\partial C}{\partial K}\Delta K + \theta\Delta\tau + \upsilon\Delta\sigma + \rho\Delta r. \tag{18.114}$$

With:

$$\Delta u = \frac{\partial C}{\partial K}\Delta K + \theta\Delta\tau + \upsilon\Delta\sigma + \rho\Delta r, \tag{18.115}$$

we obtain:

$$\Delta C = \Delta\Delta S + \frac{1}{2}\Gamma(\Delta S)^2 + \Delta u, \tag{18.116}$$

so that:

$$P(\Delta C \le c) = P(\Delta\Delta S + \frac{1}{2}\Gamma(\Delta S)^2 \le c - \Delta u). \tag{18.117}$$

Now, let $F_{\Delta c}$ be the d.f. of $\Delta S + \frac{1}{2}\Gamma(\Delta S)^2$.

So, neglecting the small variation Δu, the determination of the VaR for the call is identical to the calculation of this distribution function as:

$$F_{\Delta C}(VaR_\alpha) = \alpha. \tag{18.118}$$

We will solve this problem for the two following cases:

(i) linear approximation in ΔS and normal case;

(ii) linear approximation in ΔS and lognormal case.

This means that the small variation of the call value is well approximated by:

$$\Delta C = \Delta \Delta S + \Delta u. \tag{18.119}$$

(i) *Normal case*

We are now working with the assumption that $S(t) \prec N(m,\sigma^2)$ and so:

$$P(\Delta C \le c) = P(\Delta S \le \frac{c-\Delta u}{\Delta}) \approx P(\Delta S \le \frac{c}{\Delta}). \tag{18.120}$$

From the normality assumption, we obtain:

$$P(\Delta C \le c) = \Phi\left(\frac{1}{\sigma}\left[\frac{c-\Delta u}{\Delta} - m\right]\right) \approx \Phi\left(\frac{1}{\sigma}\left[\frac{c}{\Delta} - m\right]\right). \tag{18.121}$$

(ii) *Lognormal case*

Using the Black and Scholes model of Chapter 14, the distribution of the $S(t)/S_0$ t is a lognormal distribution with parameters:

$$\ln\frac{S(t)}{S_0} \prec N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right). \tag{18.122}$$

It follows that:

$$\begin{aligned} E\left[S(t)\right] &= S_0 e^{\mu t}, \\ \operatorname{var}\left[S(t)\right] &= S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1) \end{aligned} \tag{18.123}$$

and

$$\mu = \frac{1}{T}\ln\frac{E[S(T)]}{S_0},$$
$$\sigma^2 = \frac{1}{T}\ln\left(1+\frac{\text{var}[S(T)]}{S_0^2 e^{2\mu T}}\right). \tag{18.124}$$

From result (18.120):

$$P(\Delta C \le c) = P\left(S(T) \le S_0 + \frac{c-\Delta u}{\Delta}\right) \approx P\left(S(T) \le S_0 + \frac{c}{\Delta}\right), \tag{18.125}$$

we now obtain:

$$P(\Delta C \le c) = P\left(\ln S(T) \le \ln\left[S_0 + \frac{c-\Delta u}{\Delta}\right]\right) \approx P\left(\ln S(T) \le \ln\left[S_0 + \frac{c}{\Delta}\right]\right), \tag{18.126}$$

or

$$P(\Delta C \le c) = P\left(\ln\frac{S(T)}{S_0} \le \ln\left[1+\frac{c-\Delta u}{S_0\Delta}\right]\right) \approx P\left(\ln\frac{S(T)}{S_0} \le \ln\left[1+\frac{c}{S_0\Delta}\right]\right). \tag{18.127}$$

As

$$\ln\frac{S(t)}{S_0} \prec N\left(\left(\mu-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

this last result becomes:

$$P(\Delta C \le c) = \Phi\left(\frac{\ln\left[1+\frac{c-\Delta u}{S_0\Delta}\right]-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \approx \Phi\left(\frac{\ln\left[1+\frac{c}{S_0\Delta}\right]-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \tag{18.128}$$

18.5. VaR and Monte Carlo simulation methods

18.5.1. *Introduction*

If we have no good reason to assume that the distribution of the asset return is normal or lognormal, the only way of proceeding is to use an historical database to approximate this distribution by the simulation of a histogram of these observed values.

If we do not know the distribution but if we know its type depending of a number of parameter, the situation is easier as it suffices to estimate these unknown parameters by traditional statistical approaches.

18.5.2. *Case of one risk factor*

Let X be the risk factor, for example, the value itself of a share or the interest rate at a fixed maturity.

At time 0, the relative return:

$$\Delta = \frac{X(1) - X(0)}{X(0)} \tag{18.129}$$

follows a probability law that could be estimated with the historical method based on a good internal database with data up to time $-T$:

$$\Delta = \frac{X(t) - X(t-1)}{X(t-1)}, t =, -T+1, \ldots, -1, 0. \tag{18.130}$$

However, this method must be used carefully as it is very sensitive to eventual irregularities of the market and to extreme values; so the choice of the time horizon T is important.

This is why we prefer, when it is possible, to use a parametric method based on some models given in this chapter: normal or lognormal distributions or even another one like, for example, Weibull or gamma distributions, all depending on a small number of parameters. Sometimes, it is also good to use a leptokurtic distribution, that is, with a positive coefficient γ_2 .

The algorithmic method is based on the following steps:

1) generate a random sample of N values of X: $x_1,...,x_N$;

2) repeat step 1 M times to obtain the following NM values: $x_{1j},...,x_{Nj}, j=1,...,M$;

3) on the simulated values, estimate $x_{1j},...,x_{Nj}, j=1,...,M$, and estimate the unknown parameters;

4) use the selected theoretical model to calculate the VaR on the given horizon at time T, value at confidence level α;

5) observe the likelihood of the results;

6) validate the obtained results.

For example, for the mean estimation, for each j, we know that the estimator

$$\overline{x}_{Nj} = \frac{x_{1j} + ... + x_{Nj}}{N} \tag{18.131}$$

is unbiased and so the histogram of the values $\overline{x}_1,...,\overline{x}_M$ will give an approximation of the distribution of an estimator of the mean m with variance $\frac{\sigma^2}{N}$.

18.5.3. *Case of several risk factors*

Let us consider the case of two correlated factors X_1, X_2.

If p is the given asset function of the two risk factors, as for one factor, we can consider the following relative returns:

$$\begin{aligned}
&p(1) = f(X_1, X_2),\\
&\Delta_i = \frac{X_i(1) - X_i(0)}{X_i(0)}, i = 1,2,\\
&E(\Delta_1) = \mu_1, E(\Delta_2) = \mu_2,\\
&\operatorname{var}\Delta_1 = \sigma_1^2, \operatorname{var}\Delta_2 = \sigma_2^2, \operatorname{cov}(\Delta_1, \Delta_2) = \sigma_{12}.
\end{aligned} \tag{18.132}$$

With historical data, we construct the following random sample:

$$\begin{bmatrix} \Delta_{11} \\ \Delta_{21} \end{bmatrix}, ..., \begin{bmatrix} \Delta_{1N} \\ \Delta_{2N} \end{bmatrix}. \tag{18.133}$$

If σ_{12} represents the covariance between the two factors, the correlation coefficient is given by:

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}. \tag{18.134}$$

In fact, it is possible to express the random vector $\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$ with another vector $\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$, but with uncorrelated components, using the following transformation:

$$\begin{aligned} \Delta_1 &= \sigma_1 \delta_1 + \mu_1, \\ \Delta_2 &= \sigma_2 \rho \delta_1 + \sigma_2 \sqrt{1-\rho^2}\, \delta_2 + \mu_2, \end{aligned} \tag{18.135}$$

as indeed, it is possible to show that:

$$\begin{aligned} &E(\delta_i) = 0, i = 1,2, \\ &\text{var}(\delta_i) = 1, i = 1,2, \text{cov}(\delta_1, \delta_2) = 0. \end{aligned} \tag{18.136}$$

Introducing the Cholewsky lower-triangular matrix L:

$$\mathbf{L} = \begin{bmatrix} \sigma_1 & 0 \\ \rho \sigma_2 & \sqrt{1-\rho^2}\, \sigma_2 \end{bmatrix}, \tag{18.137}$$

transformation (18.135) can be written as the following matrix notation:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \mathbf{L} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{18.138}$$

Moreover, we have:

$$\begin{aligned} &\mathbf{LL}' = \mathbf{\Sigma}, \\ &\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \text{ variance covariance matrix of } \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}. \end{aligned} \tag{18.139}$$

This decomposition of the covariance matrix is called the Cholesky decomposition or Cholesky factorization and remains true for n dimensions (n>2).

From the numerical point of view, matrix L is found by identification.

Exercise

For the covariance matrix, with

$$\mathbf{\Sigma} = \begin{bmatrix} 5 & -3 & -4 \\ -3 & 10 & 2 \\ -4 & 2 & 8 \end{bmatrix}, \tag{18.140}$$

show that the Cholesky matrix is given by:

$$\mathbf{L} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ -\frac{3\sqrt{5}}{5} & \frac{\sqrt{205}}{5} & 0 \\ \frac{-4\sqrt{5}}{5} & \frac{-2\sqrt{205}}{205} & \frac{14\sqrt{41}}{41} \end{bmatrix}. \tag{18.141}$$

With the Cholesky decomposition, it is now easy to generate sample (18.133) using the generated sample

$$\begin{bmatrix} \delta_{11} \\ \delta_{21} \end{bmatrix}, \ldots, \begin{bmatrix} \delta_{1N} \\ \delta_{2N} \end{bmatrix}. \tag{18.142}$$

Let us point out that with the normality assumption, the two components δ_1, δ_2 are furthermore independent.

Using relation (18.129), we can reconstruct the risk factors:

$$X_{1j}(1) = X_{1j}(0)(1 + \delta_{1j}), j = 1, \ldots, N \tag{18.143}$$

to finally obtain a sample of N values of the considered asset. For example at t=1:

$$p_j(1) = f(X_{1j}(1), X_{2j}(1)), j = 1, \ldots, N. \tag{18.144}$$

As we know the asset value at time 0:

$$p(0) = f(X_1(0), X_2(0)), \tag{18.145}$$

the simulated variations of the asset on [0,1] are:

$$\left(p_1(1)-p_1(0),...,p_N(1)-p_N(0)\right). \tag{18.146}$$

The VaR value is then estimated from the histogram of the corresponding estimated distribution. The observed distribution is reliable provided we can generate a large number of simulations.

18.5.4. *Monte Carlo simulation scheme for the VaR calculation of an asset portfolio*

As we know that the VaR is not necessarily sub-additive, we proceed carefully if we do not assume the normality assumption of the vector of returns.

In this case, we cannot add the individual VaR but we have to calculate the VaR of the entire portfolio using the following algorithm:

1) choice of a parametric model (that must be valid!) including different risk factors;

2) with available databases, estimate the distribution of the risk;

3) simulation of a big number N of vector risk factors:

$$\Delta_{ij}, i=1,...,n; j=1,...,N$$

so that at time 1, we have

$$X_{ij}(1)=X_{ij}(0)(1+\Delta_{ij}), i=1,...,n; j=1,...,N;$$

4) from the known relations given the M asset values is a function of the risk factors, we obtain the simulated values of the future prices at t=1:

$$p_{jk}(1)=f_j(X_{1k},...,X_{nk}), j=1,...,M, k=1,...,N$$

5) for each of the k simulations or scenarios, k=1,…,N, calculate the M values of the global portfolio at t=1: $P_k(1)=\sum_{j=1}^{M} n_j p_{jk}(1)$, which is a function of the number of shares of each asset of type j: $n_j, j=1,...,M$.

6) with the known value of the portfolio at time 0 given by

$$\sum_{j=1}^{M} n_j p_j(0), p_j(0) = f_j(X_1(0),...,X_n(0)), j = 1,...,M ,$$

obtain the N variations of the global portfolio on [0,1]:

$$\Delta P_k = P_k(1) - P(0), k = 1,...,N$$

7) based on the obtained ranked results, construct the histogram and the VaR value estimation given the confidence level α .

Remark 18.4

(i) With the normality assumption, we can directly write:

$$VaR_\alpha = -E\left[\Delta S(T)\right] + \lambda_\alpha \operatorname{var}\left[\Delta S(T)\right]$$

and so simply estimate the mean and variance of the global portfolio using, if necessary, reduction of the number of covariances and the Cholesky decomposition.

(ii) For the periodicity of the VaR calculation, the regulator asks for daily VaR values based on an account of 100 days minimum.

Chapter 19

Credit Risk or Default Risk

19.1. Introduction

As mentioned by Basel I and Basel II Committees, the credit risk problem is one of the most important contemporary problems for banks and insurance companies. Financial studies have been developed both from theoretical and practical points of views. They consist of calculating the default probability of a firm.

There is a very wide range of research on credit risk models (see, for example, Bluhm *et al.* (2002), Crouhy *et al.* (2000), Lando (2004), etc.).

In the 1990s, Markov models were introduced to study credit risk problems. Many important papers on these kinds of models were published (see Jarrow and Turnbull (1995), Jarrow *et al.* (1997), Nickell *et al.* (2000), Israel *et al.* (2001), and Hu *et al.* (2002)), mainly for solving the problem of the evaluation of the transition matrices. In Lando and Skodeberg (2002) some problems regarding the duration of the transition are expressed, but never, as far as the authors know, a model in which the randomness of time in the states transitions has been constructed.

Semi-Markov models were introduced by Janssen, Manca and D'Amico (2005a) and Janssen and Manca (2007) firstly in the homogenous case. The non-homogenous case was developed in Janssen, Manca and D'Amico (2004a) and Janssen and Manca (2007). With these new models, it is possible to generalize the Markov models introducing the randomness of time for transitions between the states.

19.2. The Merton model

19.2.1. *Evaluation model of a risky debt*

The *Merton* (1974) model or the *firm model* considers the case of a firm that borrows an amount M of money at time 0, for example in the form of a zero coupon bond with facial value F (interests included) representing the amount to reimburse at time T.

As the borrower has the risk that the firm will be in default at time T, the debt is called a *risky debt* of value $D(0)$ at time 0. This value of the risky debt must use a stochastic model, called here the Merton model.

After the loan, we have:

$$V(0)=A+M, \tag{19.1}$$

$V(0)$ representing the value of the firm at time 0.

At the maturity of debt T, two situations are possible following this value $V(0)$ with respect to F. They are given by the next table.

At time T	$V(T)<F$	$V(T)>F$
Borrowers	$V(T)$	F
Shareholders	0	$V(T)-F$

Table 19.1. *Situation at maturity time*

Using the concept of plain vanilla options, it is clear that the values of $A(T)$ and $D(T)$ representing respectively the *equities of the shareholders* and the *value of the risky debt* are given by:

$$\begin{aligned} A(T) &= \max\{0, V(T)-F\}, \\ D(T) &= \min\{V(T), F\}\,(= F-\max\{0, F-V(T)\}). \end{aligned} \tag{19.2}$$

Thus, at t=0, with the Black and Scholes approach for the evaluation of options, under the risk neutral measure Q and with F as exercise price, we obtain

$$\begin{aligned} A(0) &= e^{-rT} E_Q[\max\{0, V(T)-F\}] \text{ (value of the call)},\\ D(0) &= Fe^{-rT} - e^{-rT} E_Q[\max\{0, F-V(T)\}] \; (= e^{-rT}F - \text{ put}), \end{aligned} \tag{19.3}$$

r being, as usual, the instantaneous non-risky interest rate.

From this last relation, we obtain:

$$Fe^{-rT} - D(0) = e^{-rT} E_Q[\max\{0, F-V(T)\}], \tag{19.4}$$

which shows that the difference between the non-risky debt and the risky debt is simply the value of the put in the hands of the shareholders taking account of the possibility of default.

Let us recall that Merton uses the traditional Black and Scholes model given in Chapter 14.

So, on the complete filtered space $(\Omega, \Im, (\Im_t), Q)$, the process *value of the firm* $V = (V(t), t \in [0,T])$ satisfies:

$$\begin{aligned} dV &= V(t)rdt + V(t)\sigma dW(t),\\ V(0) &= V_0, \end{aligned} \tag{19.5}$$

and we know that:

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1),\\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\right],\\ d_2 &= d_1 - \sigma\sqrt{T-t},\\ S &= S(t). \end{aligned} \tag{19.6}$$

We have:

$$K = F, S = V(0), t = 0,$$

and thus:

$$P(V(0),T) = [Fe^{-rT}\Phi(-d_2) - V(0)\Phi(-d_1)],$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r + \frac{\sigma^2}{2})T\right], \tag{19.7}$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

From relation (19.4), the value of the risky debt is given by:

$$D(0) = Fe^{-rT} - [Fe^{-rT}\Phi(-d_2) - V(0)\Phi(-d_1)],$$

where (19.8)

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r + \frac{\sigma^2}{2})T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

19.2.2. *Interpretation of Merton's result*

From relation (19.8), we can write $D(0)$ in the following form:

$$D(0) = Fe^{-rT} - \Phi(-d_2)[Fe^{-rT} - V(0)\frac{\Phi(-d_1)}{\Phi(-d_2)}],$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r + \frac{\sigma^2}{2})T\right], \tag{19.9}$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

The first term is nothing other than the present value at time 0 of the non-risky debt of amount F; the second term is the product of the default probability at time $T, P(V(T) < F)$ and the present value of the expected loss amount

$$\left[Fe^{-rT} - V(0)\frac{\Phi(-d_1)}{\Phi(-d_2)}\right].$$

Let us show for example that $\Phi(-d_2)$ is the *default probability* $(P(V(T) < F)$.

Indeed, from the lognormality property of $V(T)/V(0)$, we successively obtain:

$$\begin{aligned}
P(V(T)<F) &= P\left(\frac{V(T)}{V_0}<\frac{F}{V_0}\right)\\
&= P\left(\ln\frac{V(T)}{V_0}<\ln\frac{F}{V_0}\right),\\
&= P\left(\frac{\ln\frac{V(T)}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}<\frac{\ln\frac{F}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}\right),\\
&= \Phi\left(\frac{\ln\frac{F}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}\right).
\end{aligned} \tag{19.10}$$

From the Black and Scholes result, we have:

$$\begin{aligned}
d_1 &= \frac{\ln\left(\frac{V_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\\
d_2 &= \frac{\ln\left(\frac{V_0}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}(=d_1-\sigma\sqrt{T}),\\
K &= F.
\end{aligned} \tag{19.11}$$

So, we obtain the desired result.

19.2.3. *Spreads*

The value of the risky debt $D(0)$ may be seen as the present value of F using a rate r' defined by:

$$D(0)=e^{-r'T}F, \tag{19.12}$$

so that

$$\begin{aligned} &D(0) = e^{-r'T} F, \\ &r' = -\frac{1}{T} \ln \frac{F}{D(0)}. \end{aligned} \tag{19.13}$$

The corresponding spread is thus given by:

$$spread = r' - r. \tag{19.14}$$

To compute the interest rate corresponding to the corresponding non-risky debt, we define the rate $r"$ such that:

$$M = e^{-r"T} F \tag{19.15}$$

and so:

$$r" = -\frac{1}{T} \ln \frac{F}{M}. \tag{19.16}$$

This gives another spread as the difference of risky and non-risky rates called actuarial spread:

$$\text{actuarial spread} = r" - r. \tag{19.17}$$

Example 19.1 (Farber *et al.*, (2004)) A firm has an initial capital of €2,500,000 and for future investments it is necessary to receive a loan of €2,000,000 to be reimbursed in two years.

The firm finds a bank agreeing this loan in the form of a zero coupon bond with facial value €3,000,000, interests included and of course of maturity 2 years.

This gives a rate $r"$ of 22.5%!

The next table gives the result related to the value of the risky debt.

Data of the firm		
Initial capital A(0)		2,500,000
Facial value F(T)		3,000,000
Volatility		0.6931
Maturity T		2
Amount M		2,000,000
Firm value at t=0:V(0)		4,500,000
Non-risky rate		
Annual		0.02
Instantaneous		0.01980263
Results		
Present value of F		2,883,506.34
d(1)	0.94416045	
d(2)	–0.03603097	
phi(-d(1))	0.172543815	
phi(-d(2))	0.514371227	
Default probability		0.51437123
Current value of recovering		1373998.96
Value of the risky debt: D(0)		2176760.82
Conclusions		
Instantaneous rate of the loan		0.20273255
Annual rate of the loan r''		0.22474487
Instantaneous rate of risky debt		0.16038719
Annual rate of risky debt r'		0.17396533
Spread		0.06435768
Spread with the non-risky rate		
With r''		0.20474487
With r'		0.15396533
Actuarial spread		0.06435768

Table 19.2. *Merton model*

19.3. The Longstaff and Schwartz model (1995)

To improve the Merton model, Longstaff and Schwartz (1995) have introduced a threshold K such that the firm is in default if its value is below K.

To compute the default risk $PDF(T)$ before time T, from the Merton model:

$$\begin{aligned} dV &= \mu V dt + \sigma V dW(t), \\ V(0) &= V_0, \end{aligned} \tag{19.14}$$

we know that

$$\ln \frac{V(t)}{V_0} = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t). \tag{19.15}$$

It follows that:

$$PDF(T) = P(V(T) < K), \tag{19.16}$$

and so:

$$PDF(T) = P\left(\ln \frac{V(t)}{V_0} < \ln \frac{K}{V_0} \right). \tag{19.17}$$

As from relation (19.15), we obtain:

$$\ln \frac{V(t)}{V_0} \prec N\left(\left(\mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right) \tag{19.18}$$

And we obtain from relation (19.17):

$$PDF(T) = \Phi \left(\frac{\ln \frac{K}{V_0} - \left(\mu - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right). \tag{19.19}$$

This is the result of Longstaff and Schwartz (1995) in their model called the KMV Credit Monitor.

It must be clear that this model gives the possibility to be in default at time *t* and no more in default at time *s*, *s*>*t*.

If we introduce, as in Janssen (1993), the concept of lifetime of the firm as the stopping time τ defined as:

$$\tau = \inf\{t: V(t) < K\} \tag{19.20}$$

or as:

$$\tau = \inf\left\{t: \ln\frac{V(t)}{V_0} < \ln\frac{K}{V_0}\right\}. \tag{19.21}$$

With result (19.15), we have:

$$\tau = \inf\left\{t: (\mu - \frac{\sigma^2}{2})t + \sigma W(t) < \ln\frac{K}{V_0}\right\}. \tag{19.22}$$

It follows that:

$$\tau = \inf\left\{t: \ln\frac{V_0}{K} + (\mu - \frac{\sigma^2}{2})t + \sigma W(t) < 0\right\}. \tag{19.23}$$

Finally, with:

$$u = \ln\frac{V_0}{K}, \mu' = \left(\mu - \frac{\sigma^2}{2}\right)t, \tag{19.24}$$

we can write:

$$P(\tau < t) = \Psi(u,t), \tag{19.25}$$

Using the fundamental results of Cox and Miller (1965) on diffusion processes, we finally obtain:

$$\Psi(u,t) = 1 - \Phi\left(\frac{u + \mu' t}{\sigma\sqrt{t}}\right) + e^{-\frac{2\mu' u}{\sigma^2}}\Phi\left(\frac{-u + \mu' t}{\sigma\sqrt{t}}\right). \tag{19.26}$$

This probability is called the ruin probability before t in the actuarial *risk theory*, and so the *non-ruin probability before t* is given by:

$$\phi(u,t) = 1 - \Psi(u,t). \tag{19.27}$$

For $t \to \infty$, we obtain:

$$\Psi(u) = \lim_{t\to\infty} \Psi(u,t) = \begin{cases} 1, \mu' \leq 0, \\ e^{-2\frac{\mu'}{\sigma^2}u}, \mu' > 0 \end{cases} \tag{19.28}$$

and so:

$$\phi(u) = \lim_{t\to\infty} \phi(u,t) = \begin{cases} 0, \mu' \leq 0, \\ 1 - e^{-2\frac{\mu'}{\sigma^2}u}, \mu' > 0. \end{cases} \tag{19.29}$$

Remark 19.1 It is clear that the default probability of Longstaff and Schwartz is always smaller than the ruin probability computed by the Janssen model.

19.4. Construction of a rating with Merton's model for the firm

19.4.1. *Rating construction*

In this section, we will develop an elaboration of a rating model using the traditional Merton model for the firm (1974), which is used in Creditmetrics initialized by J.P. Morgan as a sequel of the Riskmetrics computer program dedicated to the VaR methods (see Janssen and Manca (2007)).

In the Merton model (1974), value V of the firm is modeled with a Black and Scholes stochastic differential equation with trend μ and instantaneous volatility σ (see Chapter 14)

$$\begin{aligned} dV &= V(t)\mu dt + V(t)\sigma dW(t), \\ V(0) &= V_0, \end{aligned} \tag{19.30}$$

so that its value time at t is given by

$$V(t) = V_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)} \tag{19.31}$$

V_0 being the value of the firm at time 0 and $W = \left(W(t), t \in [0,T]\right)$ a standard Brownian motion defined on the filtered probability space $\left(\Omega, \Im, (\Im_t), P\right)$.

If V_{def} is the threshold beyond which the firm defaults, called the threshold default, the probability P_{def} that the company defaults before time *t* is given by:

$$\begin{aligned} P_{def}\left(V_{def}, t\right) &= P\left(V(t) < V_{def}\right) \\ &= P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t) < \ln\frac{V_{def}}{V_0}\right) \\ &= P\left(W(t) < \frac{1}{\sigma}\left(\ln\frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \end{aligned} \tag{19.32}$$

As, for all positive *t*, $W(t)/\sqrt{t}$ has a normal distribution, we obtain:

$$P_{def}\left(V_{def}, t\right) = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \tag{19.33}$$

So, if we fix value V_{def}, we can compute the corresponding value of P_{def} using the quartiles of the normal distribution.

Of course, the inverse is possible: first fix P_{def} and then compute the corresponding level V_{def}.

In the following, let us suppose that we fix the default probability V_{def} so that we compute the corresponding quantile Z_{CCC} given by

$$\begin{aligned} &P_{def}\left(V_{def}, t\right) = \Phi\left(Z_{CCC}\right), \\ &Z_{CCC} = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right). \end{aligned} \tag{19.34}$$

This means that if *Z* is below or equal to Z_{CCC}, with *Z* defined by:

$$Z = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right), \tag{19.35}$$

the considered firm is supposed to be in default and theoretically has to stop all activities.

On the contrary, if the value of Z is larger than Z_{CCC}, corresponding to the threshold value V_{CCC}, but before the quartile Z_B, corresponding to the threshold value V_B, the rating given to the firm is noted *CCC*, etc. So, with a fixed scale of firm threshold values:

$$V_{def} = V_{CCC} < V_B < V_{BB} < V_{BBB} < V_A < V_{AA} < V_{AAA} \tag{19.36}$$

we obtain a scale of increasing thresholds quartiles represented by:

$$Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA}, \tag{19.37}$$

assigning a credit rating or grade to firms as an estimate of their creditworthiness.

If Z represents the observed value of Z for the considered firm, the scale used here is the rating used by the famous credit rating agencies Standard and Poor's, and Moody's given below.

Zobs value	**notation**
Zobs<ZCCC	default
ZCCC<Zobs<ZB	CCC
ZB<Zobs<ZBB	B
ZBB<Zobs<ZBBB	BB
ZBBB<Zobs<ZA	BBB
ZA<Zobs<ZAA	A
ZAA<Zobs<ZAAA	AA
ZAAA<Zobs	AAA

Table 19.3. *Rating agencies*

It is clear that the credit ratings depend on time t and also on the selection of the probabilities

$$P_{def}\ P(Z_{CCC}),\ P(Z_B),\ P(Z_{BB}),\ P(Z_{BBB}),\ P(Z_A),\ P(Z_{AA}),\ P(Z_{AAA}) \tag{19.38}$$

or on the threshold scale of firm values

$$Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA} \tag{19.39}$$

chosen by the credit rating agency.

We can also compute the following relations:

$$\begin{aligned}
P_{déf} &= P(Z_{obs} < Z_{CCC}),\\
P_{CCC} &= P(Z_{CCC} < Z_{obs} < Z_B),\\
P_B &= P(Z_B < Z_{obs} < Z_{BB}),\\
P_{BB} &= P(Z_{BB} < Z_{obs} < Z_{BBB}),\\
P_{BBB} &= P(Z_{BBB} < Z_{obs} < Z_A),\\
P_A &= P(Z_A < Z_{obs} < Z_{AA}),\\
P_{AA} &= P(Z_{AA} < Z_{obs} < Z_{AAA}),\\
P_{AAA} &= P(Z_{AAA} < Z_{obs}),
\end{aligned} \tag{19.40}$$

and so:

$$\begin{aligned}
&P_B = P_{def} + P_{CCC,}\\
&P_{def} + P_{CCC} + P_B + \cdots + P_{AA} + P_{AAA} = 1.
\end{aligned} \tag{19.41}$$

Using relation (19.35), we obtain:

$$\begin{aligned}
&P_{def} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{def} + P_{CCC} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_B}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{def} + P_{CCC} + P_B = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&\ldots\\
&P_{def} + P_{CCC} + P_B + P_{BB} + \cdots + P_{AA} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right);
\end{aligned} \tag{19.42}$$

and moreover:

$$
\begin{aligned}
P_{déf} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_{CCC} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln V\frac{Z_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_B &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
&\dots\\
P_{AA} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_{AAA} &= 1-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right).
\end{aligned}
\tag{19.43}
$$

All these relations show how the grades are time dependent, which is why we will now study the dynamics of ratings.

19.4.2. *Time dynamic evolution of a rating*

19.4.2.1. *Continuous time model*

In continuous time, the rating process is nothing other than the stochastic process defined by relation (19.33),

$$Z = \{Z_t, 0 \le t \le T\} \tag{19.44}$$

where r.v. Z_t represents the credit rating at time t given by:

$$P_{def}(V_t, t) = \Phi(Z_t),$$

or (19.45)

$$Z_t = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_t}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right).$$

Here, grade Z_t represents exactly the value inside one of the classes defined above and no longer only the class.

Substituting the value of V_t from relation (19.30) in (19.45), we obtain:

$$Z_t = \frac{W(t)}{\sqrt{t}}, t > 0, \tag{19.46}$$

so that

$$P\left(Z_{t+\Delta t} \le j \middle| Z_t\right) = P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \le j \middle| \frac{W(t)}{\sqrt{t}} = i\right), \Delta t > 0, i, j > Z_{CCC}. \tag{19.47}$$

As the standard Brownian process has stationary and independent increments (see Definition 10.27), we also obtain:

$$\begin{aligned} &P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \le j \middle| \frac{W(t)}{\sqrt{t}} = i\right) \\ &= P\left(W(t+\Delta t) - W(t) \le j\sqrt{t+\Delta t} - W(t) \middle| \frac{W(t)}{\sqrt{t}} = i\right), \end{aligned} \tag{19.48}$$

or using relation (19.47):

$$\begin{aligned} P(Z_{t+\Delta t} \le j | Z_t) &= P\left(\frac{W(t+\Delta t) - W(t)}{\sqrt{\Delta t}} \le \frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}} \middle| Z_t = i\right) \\ &= \Phi\left(\frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}}\right), \end{aligned} \tag{19.49}$$

the last equality coming from the normality of the increments of a standard Brownian motion.

We can also write this last result in the form:

$$P(Z_s \le j | Z_t = i) = \Phi\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{19.50}$$

The corresponding density function is given by:

$$\frac{d}{dj}\left(P(Z_s \le j | Z_t = i)\right) = \frac{\sqrt{s}}{\sqrt{s-t}} \Phi'\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{19.51}$$

This last result is correct only for $i \geq Z_{CCC}$. On the other hand, for $i < Z_{CCC}$, the default state being considered as an absorbing state, we have necessarily for $j \geq i$:

$$P(Z_s \leq j | Z_t = i) = 1. \tag{19.52}$$

In conclusion, as the transition probability given by (4.21) depends on both *s* and *t* and not only on *t* – *s*, we proved that the *Z* process is *a non-homogenous Markov process*, introduced in Chapter 3.

19.4.2.2. *Discrete time model*

Let us define $\{1,...,m\}$ as the set of the *m* credit ratings ranked in increasing order with Moody's scale: $1 = D_{def}$ (*default*), $2=Z_{CCC},..., m=Z_{AAA}$.

Except for the extreme classes, the rating classes defined below will now be represented by their centers as follows:

$$\begin{array}{ll} (-\infty,1] : & 1 \\ (1,2] : & \frac{3}{2} \\ .... & \\ (i-1,i] : & \frac{2i\text{-}1}{2} \\ ... & \\ (m-1,m] : & \frac{2m\text{-}1}{2} \\ (m,\infty) : & m \end{array} \tag{19.53}$$

Let $Z_t = i$, *i* being a class center different from 1; from result (19.50), we have:

$$P(j-1 < Z_s \leq j | Z_t = i) = \Phi\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right) - \Phi\left(\frac{(j-1)\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right), \quad s > t. \tag{19.54}$$

To obtain a discrete-time, let us suppose that we give notations at times $0, u, 2u, ..., ku$ representing for example one year or a semester. Now transition probabilities become:

$$P(j-1 < Z_{ku+1} \leq j | Z_{ku} = i) = \Phi\left(\frac{j\sqrt{ku+1} - i\sqrt{ku}}{\sqrt{u}}\right) - \Phi\left(\frac{(j-1)\sqrt{ku+1} - i\sqrt{ku}}{\sqrt{u}}\right), k = 0,1,..... \tag{19.55}$$

Of course, if Z_{ku} equals Z_{Def} , we know from relation (19.52) that

$$P(j-1<Z_{ku+1}\leq j\left|Z_{ku}=Z_{D}\right.)=\begin{cases}0, j>1,\\1, j\leq 1.\end{cases} \quad (19.56)$$

Relations (19.54) and (19.55) define a sequence of probability transition matrices **P**(k), k=0,1,... with:

$$\mathbf{P}(k)=\left[p_{ij}(k)\right] \quad (19.57)$$

and

$$p_{ij}(k)=P(j-1<Z_{ku+1}\leq j\left|Z_{ku}=i\right.),i,j=1,...,m,k=0,1,... \quad (19.58)$$

It follows that the credit rating process Z in discrete-time $Z=(Z_{ku},k=0,1,...)$ is what we call a non-homogenous Markov chain defined in Chapter 12.

Of course, in the very particular and unrealistic case where the probability transition matrices **P**(k), k=0,1,... are independent of t, the process in discrete-time $Z=(Z_{ku},k=0,1,...,)$ is then a homogenous Markov chain as defined in Chapter 11.

19.4.2.3. *Example*

In real-life economics, credit rating agencies play a crucial role; they compile data on individual companies or countries to estimate their probability of default, represented by their scale of credit ratings at a given time and also by the probability of transitions for successive credit ratings.

A change in the rating is called a *migration*. Migration to a higher rating will of course increase the value of a company's bond and decrease its yield, giving what we call a negative *spread*, as it has a lower probability of default, and the inverse is true with a migration towards a lower grade with consequently a positive spread.

Here we have an example of a possible transition matrix for migration from one year to the next one.

	AAA	AA	A	BBB	BB	B	CCC	D	Total
AAA	0.90829	0.08272	0.00736	0.00065	0.00066	0.00014	0.00006	0.00012	1
AA	0.00665	0.9089	0.07692	0.00583	0.00064	0.00066	0.00029	0.00011	1
A	0.00092	0.0242	0.91305	0.05228	0.00678	0.00227	0.00009	0.00041	1
BBB	0.00042	0.0032	0.05878	0.87459	0.04964	0.01078	0.0011	0.00149	1
BB	0.00039	0.00126	0.00644	0.0771	0.81159	0.08397	0.0097	0.00955	1
B	0.00044	0.00211	0.00361	0.00718	0.07961	0.80767	0.04992	0.04946	1
CCC	0.00127	0.00122	0.00423	0.01195	0.0269	0.11711	0.64479	0.19253	1
D	0	0	0	0	0	0	0	1	1

Table 19.4. *Example of transition matrix of credit ratings*

We clearly see that the probabilities of no migration, given by the elements of the principal diagonal, are the highest elements of the matrix but that they decrease with the poor quality of the rating.

Here, we see for example that a company with rank AA has more or less nine chances out of 10 to keep its rating next year but it will move to rank AAA with only six chances in 1,000.

On the other hand, a company with a CCC as a rating will be in default next year with 20 chances out of 100.

As a more concrete example, the next table gives the transition probability matrix of *Standard and Poor's* credit ratings for 1998 (see ratings performance, Standard and Poor's) for a sample of 4,014 companies.

Let us point out the presence of a "new" state called NR (*rating withdrawn*) meaning that for a company in such a state, the rating has been withdrawn and that this event does not necessary lead to default the following year, thus explaining the last row of the above matrix.

Effec.		**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**	**NR**	**Total**
165	**AAA**	90	6	0	0.61	0	0	0	0	3.03	100
560	**AA**	0.18	89.8	5.61	0.18	0	0	0	0	4.23	100
1,095	**A**	0.09	1.5	87.18	5.11	0.18	0	0	0	5.94	100
896	**BBB**	0	0	2.79	84.93	4.46	0.67	0.22	0.34	6.59	100
619	**BB**	0.32	0.2	0.16	5.33	75.4	5.98	2.75	0.65	9.21	100
649	**B**	0	0	0.15	0.62	6.16	76.27	5.09	4.47	7.24	100
30	**CCC**	0	0	3.33	0	0	20	33.31	36.69	6.67	100
	NR	0	0	0	0	0	0	0	0	100	100
4,014											

Table 19.5. *Example with rating withdrawn*

Here, we see for example that companies in state AA will not be in default the next year but that 5.61% of them will degrade to A and 0.18% to a BBB and 0.18% will upgrade to an AAA.

Under the assumption of a homogenous Markov chain, we obtain the following results:

(i) *the probability that an AA company defaults after two years*

$$P^{(2)}(D/AA)=0.0018\cdot 0.0034=0.0006\%,$$

which is still very low;

(ii) *the probability that a BBB company defaults in one of the next two years*

This probability is given by:

$$\begin{aligned}
&P(D/BBB;2)=P(D/BBB)+P(BBB/BBB)P(D/BBB)\\
&+P(BB/BBB)P(D/BB)+P(B/BBB)P(D/B)+P(CCC/BBB)P(D/CCC)\\
&\qquad=0.34\%+(84.93\%\cdot 0.34\%)+(4.46\%\cdot 0.65\%)+(0.67\%\cdot 4.47\%)+(0.22\%\cdot 36.67\%)\\
&\qquad=0.77\%;
\end{aligned}$$

(iii) *the probability for a company BBB to default between year 1 and year 2*

Using the standard definition of conditional probability (see Chapter 1) we obtain

$$\begin{aligned}
P(D\ at\ 2/non\text{-}def.\ at\ 1) &= P(D\ at\ 2\ and\ non\text{-}def.\ at\ 1)/P(non\text{-}def.\ at\ 1)\\
&=(0.77\%-0.34\%)/(1-0.34\%)\\
&=0.43\%.
\end{aligned}$$

Let us point out that these illustrative results are true under the homogenous Markov chain model and moreover give similar results for all the companies of the panel in the same credit rating.

In fact, in real life applications, credit rating agencies also study each company on its own account so that specific information is also determined for giving the final grade.

19.4.2.4. *Ratings and spreads on zero bonds*

Let us first recall that a zero coupon bond is a contract paying a known fixed amount called the *principal*, at some given future date, called the *maturity date*.

So, if the principal is one monetary unit and T the maturity date, the value of this zero coupon at time 0 is given by:

$$B(0,T) = e^{-\delta T} \tag{19.59}$$

if δ is the considered constant instantaneous intensity of interest rate.

Of course, the investor in zero coupons must take into account the risk of default of the issuer. To do so, we consider that, in a risk neutral framework, the investor has no preference between the following two investments:

(i) to receive almost surely at time 1 the amount e^{δ} as counterpart of the investment at time 0 of one monetary unit;

(ii) to receive at time 1 the amount $e^{(\delta+s)}(s>0)$ with probability $(1-p)$ or 0 with probability p, as counterpart of the investment at time 0 of one monetary unit, p being the default probability of the issuer.

The positive quantity s is called the *spread* with respect to the non-risky instantaneous interest rate δ as counterpart of this risky investment in zero coupon bonds.

From the indifference given above, we obtain the following relation:

$$e^{\delta} = (1-p)e^{(\delta+s)} \tag{19.60}$$

or

$$1 = (1-p)e^{s}, \tag{19.61}$$

$$s = -\ln(1-p). \tag{19.62}$$

$$s \approx p,$$

$$s \cong p + \frac{1}{2}p^2. \tag{19.63}$$

Let us now consider a more positive and realistic situation in which the investor can obtain an amount $\alpha, (0<\alpha<1)$ if the issuer defaults at maturity or before.

In this case, the expectation equivalence principle relation (19.60) becomes:

$$e^{\delta} = (1-p)e^{\delta+s} + p\alpha e^{\delta}, \tag{19.64}$$

or

$$1 = (1-p)e^{s} + p\alpha. \tag{19.65}$$

It follows that in this case the value of the spread satisfies the equation

$$e^s = \frac{1-p\alpha}{1-p} \tag{19.66}$$

and so the spread value is

$$s = \ln\frac{1-p\alpha}{1-p}. \tag{19.67}$$

As above, using the MacLaurin formula respectively of order 1 and 2, we obtain the two following approximations for the spread:

$$\begin{aligned} s &\approx \frac{p}{1-p}(1-\alpha), \\ s &\approx \frac{p}{1-p}(1-\alpha) - \frac{1}{2}\left(\frac{p}{1-p}(1-\alpha)\right)^2. \end{aligned} \tag{19.68}$$

19.5. Discrete time semi-Markov processes

19.5.1. *Purpose*

In this section, we will present both discrete-time homogenous (DTHSMP) and non-homogenous (DTNHSMP) semi-Markov processes and how to apply semi-Markov models to the credit risk environment.

Although, in general, time in real-life problems is continuous, the real observation of the considered system is almost always made up of discrete-time even if the used time unit may in some cases be very small.

The choice of this time unit depends on what we observe and what we wish to study.

For example, if we are studying the random evolution of the earthquake activity in a tectonic fracture zone, then it could be observed with a unitary time scale of ten years. If we are studying the behavior of a disablement resulting from a job related illness, the unitary time could be one year, etc.

Thus, it results that while the phenomenon of time evolution is continuous, usually, the observations are discrete in time.

Consequently, if we construct a model to be fitted with real data, in our opinion, it would be better to begin with discrete-time models.

The rating changes can be followed by a Markov chain model.

In some papers, the problem of the unfitting of Markov process in the credit risk environment was outlined (see Altman (1998), Nickell *et al.* (2000), Kavvathas (2001), Lando and Skodeberg (2002)).

The principal problems of non-Markovianity that are highlighted are as follows:

(i) the duration inside a state. The probability of changing rating depends on the time that a firm remains at the same rating;

(ii) the dependence on time of the rating evaluation (ageing phenomenon). This means that, in general, the rating evaluation depends on the time at which it is done and, more importantly, on the business cycle. The rating evaluation done at time t is generally different from the one done at time s, if $s \neq t$;

(iii) the dependence of the new rating on the previous ones, not only the last rating, but also the one before last.

As the first approach, the first problem can be solved by means of semi-Markov processes (SMP). In fact, in SMP the transition probabilities are a function of the waiting time spent in a state of the system. Furthermore, in a semi-Markov backward recurrence time conditioning the problem is resolved successfully.

As a general approach, the second problem can be faced by means of a non-homogenous environment and, using a more particular approach, by means of different scenarios in the model.

The third effect exists in the case of downward moving ratings but not in the case of upward moving ratings; see Kavvathas (2001). More precisely, if a firm obtains a lower rating, then there is a higher probability that the next rating will be lower than the preceding one. In the case of an upward movement, this phenomenon does not hold.

The credit risk semi-Markov approach was developed in D'Amico *et al.* (2005a), D'Amico *et al.* (2004a), D'Amico *et al.* (2004b) and D'Amico *et al.* (2005b). In the last sections of this chapter we will present the models and their theoretical background.

It should be mentioned that Koopman *et al.* (2005) and Vasileiou and Vassiliou (2006), in other environments, show how semi-Markov processes are more suitable than the Markov ones in the credit risk transition models.

19.5.2. *DTSMP definition*

Though DTHSMP and DTNHSMP definitions are similar to the continuous ones given in Chapter 3, we will give these definitions for discrete-time using directly the terminology used for continuous time models.

Let $I=\{1, 2, \ldots, m\}$ be the state space and let $\{\Omega, \Im, P\}$ be a probability space. Let us also define the following r.vs.:

$$J_n : \Omega \to I, \qquad T_n : \Omega \to \mathbb{N}. \tag{19.69}$$

Definition 19.1 *The process* (J_n, T_n) *is a discrete-time homogenous Markov renewal process or a discrete-time non-homogenous Markov renewal process if the kernels* **Q** *associated with the process are defined respectively in the following way*:

$$\mathbf{Q} = [Q_{ij}(t)] = [P(J_{n+1} = j, T_{n+1} - T_n \le t \mid J_n = i)] \quad i, j \in I, t \in \mathbb{N}, \tag{19.70}$$

$$\mathbf{Q} = [Q_{ij}(s,t)] = [P(J_{n+1} = j, T_{n+1} \le t \mid J_n = i, T_n = s)] \; i, j \in I, s,t \in \mathbb{N} \tag{19.71}$$

As in the continuous time case, it results that for the homogenous case, we define:

$$\mathbf{P} = [p_{ij}] = \left[\lim_{t\to\infty} Q_{ij}(t)\right]; i, j \in I, t \in \mathbb{N}. \tag{19.72}$$

For the non-homogenous case, we obtain:

$$\mathbf{P} = [p_{ij}(s)] = \left[\lim_{t\to\infty} Q_{ij}(s,t)\right]; i, j \in I, s,t \in \mathbb{N}, \tag{19.73}$$

P being the transition matrix of the *embedded Markov chain* of the process.

Furthermore it is necessary to introduce the probability that the process will leave state i before or at time t:

$$\mathbf{H} = [H_i(t)] = [P(T_{n+1} - T_n \le t \mid J_n = i)], \tag{19.74}$$

$$\mathbf{H} = [H_i(s,t)] = [P(T_{n+1} \le t \mid J_n = i, T_n = s)]. \tag{19.75}$$

From the results of Chapter 12, we know that:

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t) \text{ and } H_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t). \tag{19.76}$$

Probability (19.77) only has sense in the discrete-time case and to be concise, we present first the definition for the homogenous case and then for the non-homogenous case.

Definition 19.2 *Matrix* **B** *is defined as follows:*

$$\mathbf{B} = \left[b_{ij}(t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} - T_n = t \mid J_n = i \right) \right], \tag{19.77}$$

$$\mathbf{B} = \left[b_{ij}(s,t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} = t \mid J_n = i, T_n = s \right) \right]. \tag{19.78}$$

From Definition 19.1 it results that:

$$b_{ij}(t) = \begin{cases} Q_{ij}(0) = 0 & \text{if} \quad t = 0, \\ Q_{ij}(t) - Q_{ij}(t-1) & \text{if} \quad t = 1,2,..., \end{cases} \tag{19.79}$$

$$b_{ij}(s,t) = \begin{cases} Q_{ij}(s,s) = 0 & \text{if} \quad t \le s, \\ Q_{ij}(s,t) - Q_{ij}(s,t-1) & \text{if} \quad t > s. \end{cases} \tag{19.80}$$

Definition 19.3 *The discrete-time conditional distribution functions of the waiting times given the present and the next states, are given by:*

$$\mathbf{F} = \left[F_{ij}(t) \right] = \left[P\left(T_{n+1} - T_n \le t \mid J_n = i, J_{n+1} = j \right) \right], \tag{19.81}$$

$$\mathbf{F} = \left[F_{ij}(s,t) \right] = \left[P\left(T_{n+1} \le t \mid J_n = i, J_{n+1} = j, T_n = s \right) \right]. \tag{19.82}$$

Obviously, the related probabilities can be obtained by means of the following formulae:

$$F_{ij}(t) = \begin{cases} Q_{ij}(t) / p_{ij} & \text{if} \quad p_{ij} \ne 0, \\ U_1(t) & \text{if} \quad p_{ij} = 0, \end{cases} \tag{19.83}$$

$$F_{ij}(s,t) = \begin{cases} Q_{ij}(s,t) / p_{ij}(s) & \text{if} \quad p_{ij}(s) \ne 0, \\ U_1(s,t) & \text{if} \quad p_{ij}(s) = 0, \end{cases} \tag{19.84}$$

where $U_1(t) = U_1(s,t) = 1\ \forall s,t$.

Now, we can introduce the *discrete-time semi-Markov process* $Z = \left(Z(t), t \in \mathbb{N} \right)$ where $Z(t) = J_{N(t)}$, $N(t) = \max\left\{ n : T_n \le t \right\}$ represents the state occupied by the process at time *t*.

For $i,j=1,\ldots,m$, the transition probabilities are defined in the following way:

$$\phi_{ij}(t) = \mathrm{P}\left(Z_t = j \mid Z_0 = i\right) \tag{19.85}$$

for the homogenous case; for the non-homogenous case, we have:

$$\phi_{ij}(s,t) = \mathrm{P}\left(Z_t = j \mid Z_s = i, N(s^+) = N(s^-) + 1\right). \tag{19.86}$$

They are obtained by solving the following evolution equations:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta), \tag{19.87}$$

$$\phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t), \tag{19.88}$$

where, as usual, δ_{ij} represents the Kronecker symbol.

The first part of relations (19.87) and (19.88)

$$\delta_{ij}(1 - H_i(t)) \tag{19.89}$$

$$\delta_{ij}(1 - H_i(s,t)) \tag{19.90}$$

give the probability that the system does not have transitions up to time t given that it was in state i at time 0 in the homogenous case and at time s in the non-homogenous case. Relations (19.89) and (19.90) in the rating migration case represent the probability that the rating organization does not give any new rating evaluation in a time t in homogenous case and from the time s up to the time t in non-homogenous case. This part has sense if and only if $i=j$ and this is the reason of Kronecker δ.

In the second parts

$$\sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta)$$

$$\sum_{\beta=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t) \tag{19.91}$$

$b_{i\beta}(\vartheta)$ and $b_{i\beta}(s,\vartheta)$ represent the probability that the system was at time s in the state I, remained in this state up to time ϑ and that it went to the state β just at time ϑ. After the transition, the system will go to state j following one of the possible trajectories that go from state β at time ϑ to state j within time t. In the credit risk environment, this means that in a time t, in the homogenous case, and from time s up to time ϑ, in the non-homogenous, the rating company does not offer any other evaluation of the firm; at time ϑ the rating company gave the new rating β for the evaluation firm. After this, the rating will arrive at state j within the time t following one of the possible rating trajectories.

19.6. Semi-Markov credit risk models

The rating process, generated by the rating agency, gives a reliability rating to a firm's bond.

For example, in Standard and Poor's case, there are the eight different classes of rating which means having the following set of states:

$$I = \{\text{AAA, AA, A, BBB, BB, B, CCC, D}\}.$$

The first seven states are good states and the last one is the only bad state that is also the only absorbing state. The two subsets are the following:

$$U = \{\text{AAA, AA, A, BBB, BB, B, CCC}\},\ \ D = \{\text{ D}\}.$$

Solving systems (19.88) and (19.89) we will obtain the following results:

1) $\phi_{ij}(t)$ and $\phi_{ij}(s,t)$ represent the probabilities of being in state j starting in state i after time t in the homogenous case, or starting at time s in state i in the non-homogenous one. Both the results take into account the different probabilities of changing state during the permanence of the system in the same state (duration problem). In the non-homogenous case, the problem of the different probabilities of changing state as a function of the different time of evaluation (aging problem) is also solved.

2) $A_i(t) = \sum_{j \in U} \phi_{ij}(t)$ and $A_i(s,t) = \sum_{j \in U} \phi_{ij}(s,t)$ represent the probability that the system never goes in the default state in time t in homogenous case and from time s up to the time t in the non-homogenous one.

3) $1 - H_i(t)$ and $1 - H_i(s,t)$ represent the probability that in time t or from time s up to the time t, no new rating evaluation was done for the firm.

Before giving another result that can be obtained in an SMP environment, we have to introduce the concept of the first transition after time t. More precisely, we suppose that the system at time 0 or at time s was in state i, and we know that with probability $\left(1-H_i(t)\right)$ or $\left(1-H_i(s,t)\right)$ the system does not move from state *i*. According to these hypotheses we would know the probability that the next transition will be to state *j*. This probability will be denoted by $\varphi_{ij}(t)$ in the homogenous case and by $\varphi_{ij}(s,t)$ in the non-homogenous case. These probabilities have the following meaning:

$$\varphi_{ij}(t) = P\left[\, X_{n+1} = j \mid X_n = i,\, T_{n+1} - T_n > t \,\right] \tag{19.92}$$

$$\varphi_{ij}(s,t) = P\left[\, X_{n+1} = j \mid X_n = i,\, T_{n+1} > t, T_n = s \,\right]. \tag{19.93}$$

These probabilities can be obtained by means of the following relations:

$$\varphi_{ij}(t) = \frac{p_{ij} - Q_{ij}(t)}{1 - H_i(t)} \tag{19.94}$$

$$\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1 - H_i(s,t)}. \tag{19.95}$$

After definitions (19.92) and (19.93) by means of SMP, it is possible to obtain the following results:

4) $\varphi_{ij}(t)$ and $\varphi_{ij}(s,t)$ represent, respectively, the probabilities of obtaining rank *j* at the next rating if the previous state was *i* and no rating evaluation was done in time *t* in the homogenous case, or from time *s* up to time *t* in the non-homogenous one. In this way, for example, if the transition to the default state is possible and if the system does not move from time *s* up to time *t* from state *i*, we know the probability that in the next transition the system will go to the default state.

The downward problem can be solved introducing six other states. The set of the states becomes the following:

$$I = \{\text{AAA, AA,AA-, A,A-, BBB,BBB-, BB,BB-, B,B-, CCC,CCC-, D}\}$$

For example, state BBB is divided into BBB and BBB-. The system will be in state BBB if it arrived from a lower rating. On the other hand, it will be in state BBB- if it arrived in the state from a better rating (a downward transition).

It is also possible to suppose that if there is a virtual transition, then if the system is in the BBB- state it will go to the BBB state, but in our models this assumption will not be made.

The first 13 states are good states and the last one is the only bad state. According to this hypothesis, the two subsets become the following:

$$U = \{\text{AAA, AA,AA-, A,A-, BBB,BBB-, BB,BB-, B,B-, CCC,CCC-}\},$$
$$D = \{\text{ D}\}.$$

The homogenous and non-homogenous models do not change. The simple introduction of the states makes it possible to solve the downward problem.

19.7. NHSMP with backward conditioning time

Now we introduce non-homogenous backward semi-Markov process, that is, a generalization of the SMP. We state only the non-homogenous case. To explain the backward introduction in Figure 7.1 a trajectory of a SMP with backward recurrence time is shown.

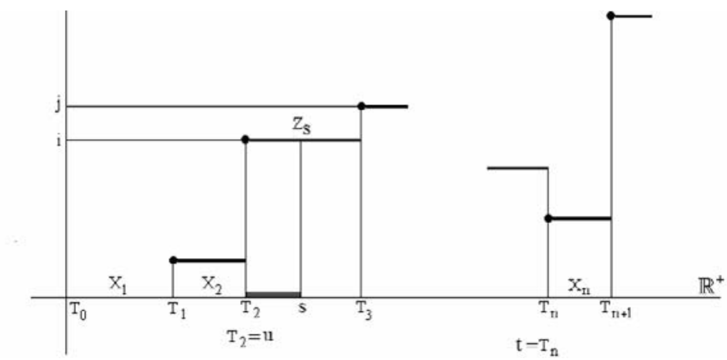

Figure 19.1. *Backward time conditioning*

With non-homogenous semi-Markov processes we know that at time s the system entered into state I, then the probability of being in state j at time t is given by $\phi_{ij}(s,t)$. Taking into account backward time, we consider that we entered into the state i at time u, and we remained in state i up to time s (backward recurrence time s-u). The transition probabilities are conditioned to the entrance time into state i and to the fact that the system does not have transitions up to time s. So, we introduce the new conditional probabilities

$$H_i(u,s,t) = \mathrm{P}\left[T_{N(s)+1} \le t \mid J_{N(s)} = i, T_{N(s)} = u, T_{N(s)+1} > s\right], t > s,$$
$$Q_{ij}(u,s,t) = \mathrm{P}\left[T_{N(s)+1} \le t, J_{N(s)+1} = j \mid J_{N(s)} = i, T_{N(s)} = u, T_{N(s)+1} > s\right], t > s$$

It is clear that $H_i(s,s,t), \dot{Q}_i(s,s,t)$ and $\phi_{ij}(s,s,t)$ are equal respectively to $H_i(s,t), \dot{Q}_i(s,t)$ and $\phi_{ij}(s,t)$ of the non-homogenous semi-Markov process.

According to this hypothesis, relations (19.71), (19.91), (19.88) and (19.90) are rewritten in the following way:

$$Q_{ij}(u,s,t) = \frac{Q_{ij}(u,u,t)}{1-H_i(u,u,s)} \tag{19.96}$$

$$b_{ij}(u,s,t) = \frac{b_{ij}(u,u,t)}{1-H_i(u,u,s)} \tag{19.97}$$

$$\phi_{ij}(u,s,t)=D_{ij}(u,s,t)+\sum_{j\in I}\sum_{\vartheta=s+1}^{t}\phi_{\beta j}(\vartheta,\vartheta,t)b_{i\beta}(u,s,\vartheta) \tag{19.98}$$

where

$$D_{ij}(u,s,t) = \begin{cases} \dfrac{1-H_i(u,u,t)}{1-H_i(u,u,s)} & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases} \tag{19.99}$$

and $\phi_{ij}(u,s,t)$ is the probability of being in state j at time t given that at time s the system was in state i and that it entered into this state at time u and has not moved from state i up to time s.

With this generalization of the model it is possible to consider the complete time of duration into a state in the rating migration model.

The results given in the previous section with backward conditioning recurrence time become the following:

1) $\phi_{ij}(u,s,t)$ represents the probability of being in state j at time t being in state i at time s and moreover given that the system arrived at state i at time u and that from u to s $(u<s)$ there was no transition. These results take into account the different probabilities of changing state during the permanence of the system in the same state (duration problem) considering the arrival time in the state and, in a complete way, the duration inside a state. Furthermore, it also considers the different probabilities of changing state as a function of the different time of evaluation (aging problem). The different probability values given for the two states that are obtained because of the downward problem solve the third Markovian model problem.

2) $A_i(u,s,t) = \sum_{j \in U} \phi_{ij}(u,s,t)$ represents the probability that the system never goes in the default state from time *s* up to time *t*.

3) $D_{ii}(u,s,t)$ represents the probability that from time *s* up to time *t* no one new rating evaluation was done for the firm, taking into account that there were no transitions from *u* to *s* either.

In this case, $\varphi_{ij}(s,t)$ does not make sense because the backward gives no more information as regards the case without recurrence times.

19.8. Examples

In this section, we present examples for the homogenous case and for downward and backward non-homogenous models; for a simple non-homogenous case see Janssen and Manca (2007). The data were extracted from Standard and Poor's Credit Review (1993), and Standard and Poor's (2001).

19.8.1. *Homogenous SMP application*

The first example is given using the transition matrix given in Jarrow *et al.* (1997), who presented one the first applications of Markov processes to the problem of credit risk.

Real data were not available and this matrix is used only in order to show how the model can work and the results that can be obtained by means of a homogenous semi-Markov process model.

The matrix was constructed starting from the 1 year transition matrix given in Standard and Poor's Credit Review (1993). The matrix is given in Table 19.6 for the sake of completeness.

The d.f. of waiting times are not known and they were constructed by means of random number generators.

The results at 5 years and at 10 years of the matrix are reported $\phi_{ij}(t)$ respectively in Tables 19.7 and 19.8.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.891	0.0963	0.0078	0.0019	0.003	0	0	0
AA	0.0086	0.901	0.0747	0.0099	0.0029	0.0029	0	0
A	0.0009	0.0291	0.8896	0.0649	0.0101	0.0045	0	0.0009
BBB	0.0006	0.0043	0.0656	0.8428	0.0644	0.016	0.0018	0.0045
BB	0.0004	0.0022	0.0079	0.0719	0.7765	0.1043	0.0127	0.0241
B	0	0.0019	0.0031	0.0066	0.0517	0.8247	0.0435	0.0685
CCC	0	0	0.0116	0.0116	0.0203	0.0754	0.6492	0.2319
D	0	0	0	0	0	0	0	1

Table 19.6. *1 year transition matrix*

For example, element 0.03046 in row **A** and in column **BBB** represents the probability that a firm that at time 0 has a rating **A** will have rating **BBB** at time 5.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.94730	0.04462	0.00474	0.00142	0.00185	0.00007	0.00000	0.00001
AA	0.00437	0.93638	0.04961	0.00616	0.00166	0.00176	0.00002	0.00005
A	0.00049	0.01130	0.94901	0.03046	0.00516	0.00289	0.00005	0.00065
BBB	0.00036	0.00232	0.03778	0.91369	0.03290	0.00886	0.00140	0.00268
BB	0.00027	0.00123	0.00366	0.04166	0.89871	0.03611	0.00472	0.01363
B	0.00000	0.00102	0.00219	0.00577	0.02916	0.90182	0.01727	0.04277
CCC	0.00000	0.00004	0.00570	0.00497	0.00718	0.02673	0.86863	0.08675
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 19.7. *Probabilities* $\phi_{ij}(5)$

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.83816	0.13758	0.01566	0.00376	0.00437	0.00038	0.00003	0.00007
AA	0.01099	0.85822	0.10525	0.01633	0.00460	0.00422	0.00012	0.00029
A	0.00133	0.03697	0.85606	0.08135	0.01528	0.00688	0.00026	0.00185
BBB	0.00087	0.00628	0.08211	0.79992	0.07740	0.02256	0.00292	0.00794
BB	0.00051	0.00300	0.01241	0.08615	0.73333	0.11574	0.01495	0.03391
B	0.00003	0.00282	0.00533	0.01253	0.06824	0.75073	0.05572	0.10460
CCC	0.00001	0.00027	0.01325	0.01395	0.02199	0.08238	0.61142	0.25673
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 19.8. *Probabilities* $\phi_{ij}(10)$

In Table 19.9, the $A_i(t)$ values are reported, the probabilities of not having a default in a time *t* (row index) starting in state *i* (column) at time 0.

	AAA	AA	A	BBB	BB	B	CCC	D
1	1.00000	1.00000	0.99995	0.99946	0.99826	0.98987	0.98989	0.0
2	1.00000	1.00000	0.99982	0.99902	0.99379	0.98636	0.98264	0.0
3	1.00000	0.99999	0.99969	0.99848	0.99296	0.98072	0.95688	0.0
4	0.99999	0.99997	0.99954	0.99825	0.99081	0.97239	0.93484	0.0
5	0.99999	0.99995	0.99935	0.99732	0.98637	0.95723	0.91325	0.0
6	0.99998	0.99993	0.99924	0.99633	0.98303	0.95100	0.86029	0.0
7	0.99998	0.99989	0.99900	0.99560	0.97817	0.93408	0.82584	0.0
8	0.99997	0.99984	0.99882	0.99444	0.97353	0.91576	0.77271	0.0
9	0.99995	0.99978	0.99850	0.99327	0.96946	0.90660	0.76244	0.0
10	0.99993	0.99971	0.99815	0.99206	0.96609	0.89540	0.74327	0.0

Table 19.9. *Probabilities of not having a default*

	AAA	AA	A	BBB	BB	B	CCC	D
1	0.85082	0.92221	0.90662	0.90109	0.92033	0.85765	0.93533	1.0
2	0.72879	0.85294	0.78671	0.81736	0.82019	0.65152	0.90909	1.0
3	0.69140	0.77216	0.67244	0.78712	0.79283	0.61430	0.85917	1.0
4	0.63930	0.65477	0.62791	0.62841	0.72874	0.58226	0.73706	1.0
5	0.47396	0.50142	0.58289	0.56618	0.68413	0.54727	0.61716	1.0
6	0.32902	0.37689	0.41751	0.51725	0.60283	0.32242	0.54618	1.0
7	0.28210	0.32079	0.39316	0.40741	0.47414	0.27700	0.45527	1.0
8	0.12558	0.24453	0.36959	0.25555	0.33608	0.21594	0.32597	1.0
9	0.11273	0.15467	0.19339	0.15823	0.16723	0.20158	0.16877	1.0
10	0.08805	0.02465	0.00905	0.04343	0.02941	0.03959	0.04901	1.0

Table 19.10. *Probability of remaining in the starting state*

As explained before, these results can assume great relevance in the computation of interest rates.

In Table 19.10, the probabilities of remaining in the starting state without transitions are reported.

In Tables 19.11 and 19.12, the probability $\varphi_{ij}(t)$ at 5 years and 10 years are reported. For example, 0.06644 represents the probability that the next transition of a firm that was at time 0 in the state **A** and that remained in this state up to time 5 will go to state **BBB** in the next transition.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.85843	0.12677	0.00993	0.00182	0.00305	0.00000	0.00000	0.00000
AA	0.00949	0.91387	0.06126	0.00951	0.00303	0.00283	0.00000	0.00000
A	0.00084	0.03299	0.88487	0.06644	0.01045	0.00369	0.00000	0.00073
BBB	0.00051	0.00417	0.05568	0.85789	0.06149	0.01481	0.00113	0.00432
BB	0.00024	0.00166	0.00735	0.05012	0.80635	0.10368	0.01260	0.01801
B	0.00000	0.00194	0.00278	0.00436	0.05011	0.82294	0.05565	0.06223
CCC	0.00000	0.00000	0.01027	0.01145	0.02199	0.08057	0.63240	0.24333
D	0.85843	0.12677	0.00993	0.00182	0.00305	0.00000	0.00000	0.00000

Table 19.11. *Probability* $\varphi_{ij}(5)$

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.98206	0.00684	0.00766	0.00164	0.00179	0.00000	0.00000	0.00000
AA	0.03037	0.86667	0.07300	0.01369	0.00639	0.00988	0.00000	0.00000
A	0.00342	0.14461	0.15548	0.64345	0.01673	0.03479	0.00000	0.00153
BBB	0.00025	0.00631	0.10388	0.76402	0.09709	0.02065	0.00147	0.00634
BB	0.00107	0.00242	0.00308	0.06613	0.63752	0.17163	0.04287	0.07528
B	0.00000	0.00016	0.00765	0.00036	0.09770	0.69738	0.04657	0.15017
CCC	0.00000	0.00000	0.01742	0.01122	0.03786	0.02115	0.50411	0.40823
D	0.98206	0.00684	0.00766	0.00164	0.00179	0.00000	0.00000	0.00000

Table 19.12. *Probability* $\varphi_{ij}(10)$

As was mentioned before, by means of this matrix it is possible, for example, to know the probability of going into the default state at the next transition.

Finally, Tables 19.13 and 19.14 present the discrete-time distribution functions of the first time of default in a time horizon of 10 years.

	1	2	3	4	5
AAA	0.00000	0.000000	0.000002	0.000005	0.000010
AA	0.00000	0.000004	0.000014	0.000029	0.000047
A	0.00005	0.000181	0.000311	0.000462	0.000648
BBB	0.00054	0.000980	0.001521	0.001749	0.002678
BB	0.00174	0.006214	0.007042	0.009187	0.013634
B	0.01013	0.013635	0.019281	0.027613	0.042766
CCC	0.01011	0.017358	0.043122	0.065157	0.086749
D	1.00000	1.00000	1.00000	1.00000	0.000010

Table 19.13. *Distribution function from 1 to 5*

	6	7	8	9	10
AAA	0.000015	0.000024	0.000033	0.000046	0.000066
AA	0.000074	0.000112	0.000158	0.000218	0.000290
A	0.000756	0.001005	0.001180	0.001503	0.001854
BBB	0.003667	0.004402	0.005563	0.006727	0.007937
BB	0.016971	0.021832	0.026472	0.030544	0.033914
B	0.049002	0.065918	0.084239	0.093395	0.104603
CCC	0.139715	0.174160	0.227294	0.237560	0.256734
D	1.00000	1.00000	1.00000	1.00000	1.00000

Table 19.14. *Distribution function from 6 to 10*

19.8.2. *Non-homogenous downward example*

To solve the downward problem we constructed the non-homogenous embedded Markov chain using the transition matrices given in Standard and Poor's (2001) Table 15 as a basis. In these matrices, the state No Rating was present. Each element $p_{ij}(s)$ of the embedded non-homogenous Markov chain should be constructed directly from the data. Constructing the MC, all the possible transitions from state i to state j starting from year s should be taken into account. Since we do not have the raw data, we used the one year transition matrices given in Standard and Poor's publication.

The publication reports a 20-year history (one year transition matrices from 1981 to 2000). The example works from year 0, corresponding to 1981 to year 19 that corresponds to year 2000. The $\mathbf{P}(s)$ in the semi-Markov environment should give the transition probabilities that there are, theoretically, from time s up to ∞. This fact means that if there is a transition from i to j at time $t, s < t$ then $p_{ij}(k) > 0,\ s \le k \le t$. Standard and Poor's transition matrix was rearranged taking into account this property. Furthermore, we rearranged the obtained matrix giving the transition probabilities of the downward states starting from the probability transitions constructed without the added states.

In the new states, the transition probabilities of remaining in the state or of obtaining a better rating are lower than those of the corresponding non-downward state.

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.906284	0	0.074012	0	0.016665	0	0.003039
AA	0.019456	0.890148	0	0	0.068095	0	0.004009
AA-	0.016895	0	0.851902	0	0.085766	0	0.016418
A	0.006028	0.04704	0	0.87366	0	0	0.06546
A-	0.00435	0.040023	0	0	0.818887	0	0.102706
BBB	0.00324	0.006481	0	0.04782	0	0.886292	0
BBB-	0.002331	0.005244	0	0.03467	0	0	0.847431
BB	0	0.005712	0	0.008785	0	0.044019	0
BB-	0	0.005106	0	0.0077	0	0.035284	0
B	0	0.001342	0	0.011884	0	0.006518	0
B-	0	0.001242	0	0.009932	0	0.004489	0
CCC	0.012308	0	0	0.010443	0	0.011375	0
CCC-	0.00027	0	0	0.007712	0	0.007495	0
D	0	0	0	0	0	0	0

Table 19.15. *Embedded MC at time 0 - **I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0	0	0	0	0	0	0
AA	0	0.008809	0	0.007235	0	0.002249	0
AA-	0	0.009641	0	0.013364	0	0.006014	0
A	0	0.00208	0	0.00168	0	0	0.004052
A-	0	0.005128	0	0.003674	0	0	0.025232
BBB	0	0.04782	0	0.003044	0	0.002258	0.003044
BBB-	0	0.089055	0	0.004953	0	0.007478	0.008838
BB	0.598413	0	0	0.294862	0	0.004392	0.043817
BB-	0	0.566994	0	0.316054	0	0.019595	0.049267
B	0.047345	0	0.875595	0	0	0.023673	0.033643
B-	0.042402	0	0	0.846713	0	0.056063	0.03916
CCC	0.011096	0	0.084755	0	0.847646	0	0.022378
CCC-	0.010123	0	0.064831	0	0	0.707889	0.20168
D	0	0	0	0	0	0	1

Table 19.16. *Embedded MC at time 0 - **II***

The probabilities of obtaining a lower rating are higher compared to that of the original state.

In Tables 19.15, 19.16, 19.17 and 19.18, two years of the non-homogenous embedded MC are reported.

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.899545	0	0.095701	0	0.004754	0	0
AA	0	0.918598	0	0	0.081402	0	0
AA-	0	0	0.87046	0	0.12954	0	0
A	0.00172	0.005059	0	0.919263	0	0	0.070619
A-	0.001502	0.004106	0	0	0.872126	0	0.100619
BBB	0	0.008356	0	0.052747	0	0.858366	0
BBB-	0	0.007356	0	0.043275	0	0	0.815366
BB	0	0	0	0	0	0.080374	0
BB-	0	0	0	0	0	0.072374	0
B	0	0.003848	0	0	0	0.003848	0
B-	0	0.003102	0	0	0	0.003483	0
CCC	0	0	0	0	0	0.018525	0
CCC-	0	0	0	0	0	0.014452	0
D	0	0	0	0	0	0	0

Table 19.17. *Embedded M.C. at time 10 - **I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0	0	0	0	0	0	0
AA	0	0	0	0	0	0	0
AA-	0	0	0	0	0	0	0
A	0	0.003339	0	0	0	0	0
A-	0	0.021647	0	0	0	0	0
BBB	0	0.061103	0	0.008356	0	0.005536	0.005536
BBB-	0	0.101103	0	0.010356	0	0.006536	0.016008
BB	0.799118	0	0	0.075855	0	0.017861	0.026791
BB-	0	0.754912	0	0.104586	0	0.027861	0.040267
B	0.061352	0	0.747004	0	0	0.034524	0.149423
B-	0.052135	0	0	0.7002	0	0.124524	0.116555
CCC	0.03705	0	0.074099	0	0.518468	0	0.351858
CCC-	0.034205	0	0.06741	0	0	0.483847	0.400086
D	0	0	0	0	0	0	1

Table 19.18. *Embedded M.C. at time 10 - **II***

To apply the model, it is also necessary to construct the d.f. of the waiting time in each state *i*, given that the state successively occupied is known. We do not have data and we constructed them by means of random number generators.

In Tables 19.19 and 19.20, the probabilities $1-H_i(s,t)$ of remaining in the state from *s* to *t* without any transition are given.

Probabilities of no transition from year s to year t								
years		**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	1	0.872181	0.803709	0.86605	0.884755	0.859781	0.950899	0.925139
0	2	0.77428	0.740025	0.84935	0.81325	0.807756	0.92007	0.774197
0	3	0.699214	0.719164	0.68683	0.662296	0.792684	0.840488	0.721149
0	4	0.579474	0.607234	0.604108	0.562696	0.748248	0.677993	0.712246
0	5	0.496353	0.431912	0.44989	0.503661	0.676747	0.644425	0.652798
0	6	0.375035	0.370249	0.33209	0.466088	0.624805	0.594064	0.477243
0	7	0.302027	0.324532	0.266083	0.338748	0.505539	0.411709	0.314356
0	8	0.206926	0.270204	0.179743	0.223977	0.354685	0.229692	0.292188
0	9	0.12002	0.204777	0.139748	0.116117	0.17136	0.115246	0.121753
0	10	0.048179	0.031125	0.056915	0.081423	0.036674	0.083032	0.07514
6	7	0.814415	0.783328	0.899957	0.795045	0.764473	0.731597	0.814112
6	8	0.514639	0.456258	0.78702	0.638951	0.570292	0.4426	0.573669
6	9	0.312577	0.307947	0.187436	0.245597	0.266901	0.299585	0.281691
6	10	0.042352	0.026528	0.085198	0.058273	0.037473	0.011313	0.04723

Table 19.19. *Probabilities* $1 - H_i(s,t)$

Probabilities of no transition from year s to year t							
years		**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	1	0.985533	0.894593	0.83837	0.914242	0.883844	0.955493
0	2	0.947469	0.848314	0.693836	0.743081	0.725237	0.840868
0	3	0.858208	0.729037	0.618079	0.713725	0.682691	0.684999
0	4	0.701267	0.636971	0.575733	0.681344	0.578327	0.56305
0	5	0.577156	0.585617	0.537296	0.660163	0.502152	0.521922
0	6	0.463875	0.471031	0.401942	0.492227	0.480255	0.439965
0	7	0.359584	0.351512	0.302279	0.454226	0.344989	0.288244
0	8	0.242471	0.212772	0.241288	0.292127	0.193908	0.219999
0	9	0.136879	0.152736	0.185382	0.114899	0.14218	0.108098
0	10	0.064375	0.043551	0.011903	0.084761	0.043851	0.088229
6	7	0.644256	0.61234	0.682245	0.778307	0.692634	0.662672
6	8	0.539284	0.505809	0.639596	0.460034	0.38039	0.473847
6	9	0.265781	0.178203	0.260959	0.188567	0.158466	0.315536
6	10	0.055535	0.078714	0.04885	0.072713	0.083376	0.021339

Table 19.20. *Probabilities* $1 - H_i(s,t)$

In Tables 19.21, 19.22, 19.23 and 19.24 the probabilities $\varphi_{ij}(s,t)$ are reported. These values give the probability that the next transition from the state *i* will be to the state *j* given that there was no transition from the time *s* to the time *t*.

For example, element 0.014583 gives the probability that next transition from rating **AA-** will be to rating **AAA** given that from time 0 up to time 4 there will be no *real* or *virtual* transitions.

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.899421	0	0.080348	0	0.017233	0	0.002998
AA	0.019706	0.879427	0	0	0.078952	0	0.003367
AA-	0.014583	0	0.837132	0	0.099825	0	0.01853
A	0.008036	0.060021	0	0.833406	0	0	0.088538
A-	0.0039	0.033986	0	0	0.839978	0	0.093764
BBB	0.002525	0.005556	0	0.045356	0	0.886689	0
BBB-	0.002212	0.004901	0	0.030534	0	0	0.865008
BB	0	0.00567	0	0.010089	0	0.028391	0
BB-	0	0.003461	0	0.007602	0	0.026574	0
B	0	0.000756	0	0.011137	0	0.007193	0
B-	0	0.001141	0	0.01029	0	0.003632	0
CCC	0.010151	0	0	0.00688	0	0.01011	0
CCC-	0.000246	0	0	0.00715	0	0.006269	0
D	0	0	0	0	0	0	0

Table 19.21. *Probabilities of remaining in state i from years 0 to 4 and after to go in j-**I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	0	0	0	0	0	0	0
AA	0	0.00794	0	0.008128	0	0.00248	0
AA-	0	0.010174	0	0.015311	0	0.004445	0
A	0	0.002865	0	0.001739	0	0	0.005395
A-	0	0.004857	0	0.002694	0	0	0.020821
BBB	0	0.051625	0	0.003346	0	0.001876	0.003027
BBB-	0	0.07824	0	0.004769	0	0.006562	0.007773
BB	0.677291	0	0	0.232917	0	0.004475	0.041167
BB-	0	0.587472	0	0.309078	0	0.019533	0.04628
B	0.047536	0	0.87162	0	0	0.025383	0.036375
B-	0.028513	0	0	0.863333	0	0.049553	0.043538
CCC	0.009922	0	0.079238	0	0.86236	0	0.021339
CCC-	0.009463	0	0.083273	0	0	0.671109	0.22249
D	0	0	0	0	0	0	1

Table 19.22. *Probabilities of remaining in state i from years 0 to 4 and after to go in j-**II***

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.904587	0	0.086438	0	0.005801	0	0.003173
AA	0.003638	0.935816	0	0	0.026268	0	0.008966
AA-	0.001835	0	0.875386	0	0.059341	0	0.030207
A	0.006464	0.044629	0	0.909804	0	0	0.030872
A-	0.004883	0.053664	0	0	0.852282	0	0.074755
BBB	0.002969	0.007921	0	0.066346	0	0.855137	0
BBB-	0.002141	0.005289	0	0.019131	0	0	0.814275
BB	0	0.005274	0	0.007533	0	0.029585	0
BB-	0	0.003526	0	0.010214	0	0.018422	0
B	0	0.001895	0	0.007703	0	0.005509	0
B-	0	0.000877	0	0.005706	0	0.00593	0
CCC	0.023288	0	0	0.021232	0	0.018959	0
CCC-	0.0005	0	0	0.007729	0	0.007913	0
D	0	0	0	0	0	0	0

Table 19.23. *Probabilities of remaining in state i from years 2 to 7 and after to go in j-**I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0	0	0	0	0	0	0
AA	0	0.015499	0	0.008795	0	0.001018	0
AA-	0	0.007653	0	0.016357	0	0.009222	0
A	0	0.004798	0	0.001643	0	0	0.00179
A-	0	0.009357	0	0.003793	0	0	0.001266
BBB	0	0.055705	0	0.007023	0	0.001336	0.003563
BBB-	0	0.093151	0	0.049235	0	0.011567	0.005211
BB	0.804949	0	0	0.123423	0	0.012818	0.016418
BB-	0	0.789015	0	0.145932	0	0.011718	0.021173
B	0.029954	0	0.892309	0	0	0.007011	0.05562
B-	0.028292	0	0	0.872399	0	0.048835	0.03796
CCC	0.013332	0	0.173902	0	0.56689	0	0.182397
CCC-	0.016184	0	0.169612	0	0	0.741203	0.056858
D	0	0	0	0	0	0	1

Table 19.24. *Probabilities of remaining in state i from years 2 to 7 and after to go in j-**II***

Tables 19.25, 19.26, 19.27 and 19.28 report $\phi_{ij}(s,t)$ (the element of the evolution equation matrix).

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.973331	3.23E-05	0.019182	1.87E-06	0.006311	2.30E-08	0.001037
AA	0.005264	0.964317	7.93E-05	8.58E-06	0.02358	6.34E-06	0.001527
AA-	0.005071	0.000101	0.962186	2.13E-05	0.01918	9.75E-06	0.004675
A	0.000823	0.014209	1.02E-05	0.976499	6.96E-05	7.74E-07	0.006184
A-	0.00111	0.009958	1.95E-05	5.49E-05	0.95434	2.97E-06	0.023357
BBB	0.000997	0.002635	1.08E-05	0.014953	3.16E-05	0.968121	6.51E-05
BBB-	0.000774	0.001457	1.13E-05	0.009248	1.43E-05	4.19E-05	0.946116
BB	6.72E-06	0.001589	1.39E-08	0.001036	1.66E-05	0.018562	6.41E-06
BB-	7.57E-06	0.002329	3.96E-08	0.002624	1.84E-05	0.012877	1.75E-05
B	4.30E-06	0.000761	4.08E-08	0.003514	4.35E-06	0.002204	2.02E-05
B-	4.39E-06	0.000435	3.27E-08	0.002285	5.53E-06	0.001897	1.72E-05
CCC	0.004433	3.51E-05	4.9E-05	0.004916	3.86E-06	0.005539	2.86E-05
CCC-	0.000115	2.04E-05	1.24E-06	0.00339	2.59E-07	0.002993	1.85E-05
D	0	0	0	0	0	0	0

Table 19.25. *Probabilities of being in j at time 3 given that at time 0 was in i-**I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	8.87E-08	3.78E-05	1.10E-08	4.01E-05	0	7.71E-06	1.91E-05
AA	9.85E-06	0.003022	4.00E-06	0.001399	0	0.00074	4.21E-05
AA-	1.1E-05	0.003049	2.19E-05	0.002839	0	0.002671	0.000164
A	3.58E-06	0.000493	4.60E-08	0.000766	0	1.33E-05	0.000929
A-	6.41E-06	0.001648	1.10E-07	0.00124	0	2.64E-05	0.008237
BBB	2.36E-06	0.010917	6.47E-06	0.00062	0	0.000719	0.00092
BBB-	8.89E-06	0.034942	1.9E-05	0.002013	0	0.002519	0.002836
BB	0.888772	0.000174	3.50E-06	0.07717	0	0.000701	0.011961
BB-	0.000454	0.853982	2.88E-05	0.106066	0	0.007009	0.014587
B	0.015196	1.5E-05	0.961312	7.42E-05	0	0.006078	0.010816
B-	0.021377	1.74E-05	0.000144	0.951864	0	0.015383	0.00657
CCC	0.003701	4.63E-05	0.039577	2.31E-05	0.924516	0.000114	0.017018
CCC-	0.00324	2.86E-05	0.016846	1.08E-05	0	0.902156	0.071179
D	0	0	0	0	0	0	1

Table 19.26. *Probabilities of being in j at time 3 given that at time 0 was in i-**II***

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.720751	0.000567	0.219784	0.000791	0.041623	0.00018	0.010661
AA	0.024902	0.817118	0.001019	0.00115	0.114982	0.00059	0.017173
AA-	0.021196	0.003109	0.724453	0.001831	0.168534	0.001272	0.030429
A	0.00389	0.040986	0.00013	0.820117	0.003373	0.001402	0.091656
A-	0.004086	0.039947	0.000146	0.008838	0.719879	0.002352	0.147647
BBB	0.004809	0.024665	0.000241	0.171344	0.00158	0.606366	0.012126
BBB-	0.003751	0.011209	0.000154	0.148196	0.000727	0.011129	0.540028
BB	0.000229	0.003772	7.32E-06	0.029637	0.000174	0.158403	0.001726
BB-	0.000156	0.003413	4.02E-06	0.021329	0.000162	0.107219	0.001524
B	0.000107	0.008409	2.46E-06	0.019755	0.000388	0.022386	0.001531
B-	0.000156	0.004512	3.57E-06	0.017595	0.000365	0.020248	0.001568
CCC	0.013874	0.002387	0.000357	0.023905	0.000151	0.028082	0.001715
CCC-	0.000702	0.002271	2.23E-05	0.019848	9.75E-05	0.022471	0.001308
D	0	0	0	0	0	0	0

Table 19.27. *Probabilities of being in j at time 10 given that at time 3 was in i-**I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0.000752	0.002525	1.92E-05	0.001931	0	0.000526	0.000567
AA	0.000605	0.006092	0.000229	0.010242	0	0.002915	0.002983
AA-	0.001153	0.01247	0.00063	0.021024	0	0.007958	0.005942
A	0.00039	0.022088	4.57E-05	0.010469	0	0.002099	0.003356
A-	0.000883	0.037623	9.19E-05	0.023517	0	0.00391	0.011079
BBB	0.001973	0.091162	0.000496	0.0492	0	0.013299	0.02274
BBB-	0.005375	0.114259	0.001431	0.092938	0	0.027454	0.043349
BB	0.566792	0.008926	0.002051	0.136216	0	0.038457	0.05361
BB-	0.008583	0.535043	0.002757	0.188448	0	0.054521	0.076843
B	0.079673	0.001372	0.681411	0.009225	0	0.051393	0.124346
B-	0.064503	0.001347	0.005869	0.635897	0	0.092434	0.155501
CCC	0.035086	0.001532	0.21134	0.003613	0.341717	0.012572	0.323669
CCC-	0.034429	0.001224	0.183819	0.00365	0	0.386965	0.343194
D	0	0	0	0	0	0	1

Table 19.28. *Probabilities of being in j at time 10 given that at time 3 was in i-**II***

For example, 0.000493 represents the probability of being in state **BB-** at time 3 given that the rating evaluation was **A** at time 0.

Finally, in Tables 19.29 and 19.30 the $A_i(s,t)$ probabilities are reported. These elements give the probability that a firm, that is, at a given rating at time *s*, will not have a default up to the time *t*.

$A_i(s,t)$								
years		**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	1	1	1	1	0.999968	0.996925	0.999764	0.998961
0	2	0.999991	0.999987	0.999936	0.999423	0.993572	0.999435	0.998299
0	3	0.999981	0.999958	0.999836	0.999071	0.991763	0.99908	0.997164
0	4	0.999968	0.999911	0.999709	0.998793	0.988904	0.998726	0.995916
0	5	0.999956	0.999812	0.999476	0.998475	0.985322	0.998057	0.99488
0	6	0.999933	0.999633	0.999143	0.997611	0.982541	0.997147	0.992688
0	7	0.999894	0.999401	0.998674	0.996742	0.979183	0.99623	0.990739
0	8	0.999835	0.998948	0.997753	0.995935	0.976017	0.994732	0.986505
0	9	0.999749	0.998206	0.996333	0.994831	0.973398	0.992288	0.982006
0	10	0.999314	0.995626	0.991587	0.990844	0.964015	0.981222	0.957506
6	7	1	1	1	1	1	0.998896	0.995888
6	8	0.999993	0.999954	0.999886	0.99988	0.999831	0.997163	0.994382
6	9	0.999983	0.999888	0.999719	0.999635	0.999343	0.993383	0.991843
6	10	0.999846	0.999329	0.998343	0.997602	0.99556	0.981921	0.9661

Table 19.29. *Probability of not going into default from years s to t*

$A_i(s,t)$								
years		**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
0	1	0.996257	0.994695	0.996146	0.999959	0.996284	0.972282	0
0	2	0.992331	0.991607	0.993892	0.996661	0.991224	0.953744	0
0	3	0.988039	0.985413	0.989184	0.99343	0.982982	0.928821	0
0	4	0.984554	0.977635	0.983544	0.986963	0.976653	0.917664	0
0	5	0.976896	0.969593	0.976542	0.976842	0.96483	0.878435	0
0	6	0.966982	0.959714	0.969033	0.968583	0.950608	0.84797	0
0	7	0.959384	0.955638	0.95667	0.958528	0.937731	0.819395	0
0	8	0.950314	0.939394	0.947369	0.947336	0.918583	0.775739	0
0	9	0.938124	0.92374	0.929473	0.917325	0.819802	0.724016	0
0	10	0.897052	0.873723	0.876134	0.844328	0.700216	0.581562	0
6	7	0.99906	0.996682	0.987736	0.998923	0.960108	0.972323	0
6	8	0.996519	0.987823	0.977231	0.990127	0.917491	0.86902	0
6	9	0.989132	0.970269	0.960058	0.93397	0.798357	0.819678	0
6	10	0.961048	0.917886	0.892609	0.861352	0.640223	0.670431	0

Table 19.30. *Probability not going into default from* years *s to t*

19.8.3. *Non-homogenous downward backward example*

In this example, we use the same inputs as in the previous section and thus we will only report the results connected with the backward case.

In Tables 19.31 and 19.32, the probabilities $D_{ii}(u,s,t)$ of remaining in the state from s to t without any transition given that the system arrived at time u in state i and remained in this state from u to s are reported (backward recurrence time s-u).

Probabilities of no transition									
u	s	t	**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	0	1	0.872181	0.803709	0.86605	0.884755	0.859781	0.950899	0.925139
0	0	2	0.77428	0.740025	0.84935	0.81325	0.807756	0.92007	0.774197
0	0	3	0.699214	0.719164	0.68683	0.662296	0.792684	0.840488	0.721149
0	0	4	0.579474	0.607234	0.604108	0.562696	0.748248	0.677993	0.712246
0	0	5	0.496353	0.431912	0.44989	0.503661	0.676747	0.644425	0.652798
0	0	6	0.375035	0.370249	0.33209	0.466088	0.624805	0.594064	0.477243
0	0	7	0.302027	0.324532	0.266083	0.338748	0.505539	0.411709	0.314356
0	0	8	0.206926	0.270204	0.179743	0.223977	0.354685	0.229692	0.292188
0	0	9	0.12002	0.204777	0.139748	0.116117	0.17136	0.115246	0.121753
0	0	10	0.048179	0.031125	0.056915	0.081423	0.036674	0.083032	0.07514
2	6	7	0.89922	0.813412	0.754061	0.806292	0.768666	0.805227	0.665298
2	6	8	0.746307	0.542539	0.450393	0.56255	0.434755	0.682026	0.560027
2	6	9	0.216419	0.44434	0.315451	0.281127	0.208424	0.585145	0.328494
2	6	10	0.179681	0.022597	0.047193	0.049291	0.08731	0.02137	0.077891
4	6	7	0.917817	0.770902	0.939876	0.677056	0.604429	0.729913	0.823655
4	6	8	0.619728	0.552357	0.603088	0.386159	0.388559	0.620946	0.55475
4	6	9	0.285654	0.343981	0.476964	0.26715	0.251663	0.557772	0.283716
4	6	10	0.063009	0.042074	0.042642	0.129126	0.140237	0.041711	0.196981

Table 19.31. *Probabilities* $D_{ii}(u,s,t)$ - ***I***

Probabilities of no transition								
u	s	t	**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	0	1	0.985533	0.894593	0.83837	0.914242	0.883844	0.955493
0	0	2	0.947469	0.848314	0.693836	0.743081	0.725237	0.840868
0	0	3	0.858208	0.729037	0.618079	0.713725	0.682691	0.684999
0	0	4	0.701267	0.636971	0.575733	0.681344	0.578327	0.56305
0	0	5	0.577156	0.585617	0.537296	0.660163	0.502152	0.521922
0	0	6	0.463875	0.471031	0.401942	0.492227	0.480255	0.439965
0	0	7	0.359584	0.351512	0.302279	0.454226	0.344989	0.288244
0	0	8	0.242471	0.212772	0.241288	0.292127	0.193908	0.219999
0	0	9	0.136879	0.152736	0.185382	0.114899	0.14218	0.108098
0	0	10	0.064375	0.043551	0.011903	0.084761	0.043851	0.088229
2	6	7	0.729997	0.925012	0.937741	0.920945	0.925065	0.716396
2	6	8	0.396545	0.695417	0.558071	0.832468	0.672505	0.553408
2	6	9	0.221682	0.459819	0.262421	0.396267	0.342161	0.283633
2	6	10	0.129046	0.123599	0.110453	0.036925	0.02891	0.051503
4	6	7	0.831486	0.872701	0.785129	0.91658	0.664807	0.865949
4	6	8	0.68755	0.552467	0.501403	0.567741	0.428559	0.583168
4	6	9	0.469288	0.303929	0.426686	0.248322	0.295415	0.321166
4	6	10	0.040333	0.029382	0.049693	0.201695	0.163986	0.150543

Table 19.32. *Probabilities* $D_{ii}(u,s,t)$ - ***II***

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.87914	7.43E-05	0.109719	7.32E-05	0.006752	5.27E-06	0.003312
AA	0.006296	0.934405	5E-05	0.000182	0.02957	0.000291	0.008368
AA-	0.004272	0.000385	0.909481	0.00054	0.045075	0.000118	0.023819
A	0.002963	0.041231	6.75E-05	0.912063	0.000969	0.000102	0.030523
A-	0.004478	0.035515	7.56E-05	0.00142	0.871933	0.000163	0.063676
BBB	0.002678	0.00514	5.46E-05	0.058214	0.000107	0.85594	0.001179
BBB-	0.001799	0.003705	3.45E-05	0.024706	0.0001	0.001665	0.826728
BB	6.51E-05	0.003005	3.21E-07	0.011852	8E-05	0.041455	0.000366
BB-	4.31E-05	0.003697	2.71E-07	0.004487	8.75E-05	0.024516	0.000136
B	2.56E-05	0.002082	1.07E-07	0.00639	3.82E-05	0.006298	0.000109
B-	3.35E-05	0.002076	1.67E-07	0.005122	5.98E-05	0.004665	9.97E-05
CCC	0.012057	0.000273	0.000163	0.014286	2.06E-05	0.010276	0.000285
CCC-	0.000297	0.000207	3.61E-06	0.010776	3.04E-06	0.01226	0.000238
D	0	0	0	0	0	0	0

Table 19.33. *Probabilities* $\phi_{ij}(2,4,8)$ - ***I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	5.57E-07	0.000554	2.00E-07	0.000324	0	1.67E-05	2.95E-05
AA	6.04E-05	0.01208	7.3E-05	0.00659	0	0.00156	0.000475
AA-	4.05E-05	0.005448	0.000129	0.005271	0	0.004953	0.000469
A	1.8E-05	0.006937	4.60E-06	0.003268	0	0.000261	0.001594
A-	4.17E-05	0.010847	1.07E-05	0.008864	0	0.000479	0.002497
BBB	9.45E-05	0.05323	0.000153	0.01481	0	0.004568	0.003831
BBB-	0.000542	0.06678	0.000358	0.055484	0	0.010978	0.00712
BB	0.807165	0.000809	0.000606	0.102833	0	0.020182	0.011582
BB-	0.000537	0.816197	0.000448	0.111314	0	0.017023	0.021514
B	0.029385	0.000196	0.891416	0.000712	0	0.017088	0.046261
B-	0.013616	0.000117	0.00294	0.83323	0	0.065388	0.072652
CCC	0.013842	0.000186	0.10049	0.000363	0.723444	0.003141	0.121172
CCC-	0.013936	0.000276	0.09612	0.000494	0	0.75511	0.110281
D	0	0	0	0	0	0	1

Table 19.34. *Probabilities* $\phi_{ij}(2,4,8)$ - ***II***

In Tables 19.33, 19.34, 19.35 and 19.36, some of the evolution equation sub-matrices of the $\phi_{ij}(u,s,t)$ are reported.

For example, 0.006937 represents the probability of being in state **BB-** at time 8 given that the rating evaluation was **A** at time 4 and the system entered into this state at time 2 (backward recurrence time 4-2).

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.893323	0.000122	0.08903	9.24E-05	0.013534	4.33E-06	0.003405
AA	0.006692	0.886695	0.000113	0.000456	0.074917	0.00097	0.008796
AA-	0.004456	0.001082	0.821541	0.001036	0.105953	0.000486	0.031362
A	0.007464	0.043065	0.000116	0.858171	0.003494	0.000777	0.06479
A-	0.004565	0.051651	0.000171	0.002344	0.771791	0.001026	0.132614
BBB	0.002391	0.013464	9.49E-05	0.057016	0.00083	0.854738	0.003984
BBB-	0.002063	0.009745	5.01E-05	0.048088	0.000341	0.004701	0.70638
BB	3.04E-05	0.005834	1.25E-07	0.008943	0.000311	0.049101	0.000341
BB-	3.76E-05	0.003823	2.20E-07	0.014951	0.000228	0.059871	0.000431
B	2.56E-05	0.004185	1.90E-07	0.007884	0.00018	0.010822	0.000403
B-	2.74E-05	0.00255	1.73E-07	0.006393	5.64E-05	0.010861	0.000339
CCC	0.026797	0.00159	0.000761	0.026286	0.000126	0.023591	0.001191
CCC-	0.000717	0.001173	1.76E-05	0.012282	2.26E-05	0.013133	0.000242
D	0	0	0	0	0	0	0

Table 19.35. *Probabilities* $\phi_{ij}(5,7,10)$ - ***I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	2.15E-07	0.00033	5.33E-08	8.77E-05	0	1.81E-05	5.31E-05
AA	0.000278	0.008518	1.39E-05	0.009602	0	0.001855	0.001094
AA-	0.000531	0.009151	0.000274	0.013886	0	0.005551	0.004692
A	4.03E-05	0.01449	3.74E-06	0.004256	0	0.000759	0.002575
A-	0.000128	0.021767	7.39E-06	0.009632	0	0.001292	0.003012
BBB	0.000347	0.046795	6.12E-05	0.010801	0	0.003264	0.006214
BBB-	0.000414	0.123791	0.000338	0.063061	0	0.014662	0.026367
BB	0.74951	0.001672	0.000397	0.126385	0	0.021406	0.03607
BB-	0.006158	0.636701	0.000714	0.183819	0	0.041211	0.052055
B	0.048595	0.000295	0.808251	0.003852	0	0.029791	0.085716
B-	0.055889	0.000264	0.001062	0.734109	0	0.09917	0.089279
CCC	0.019177	0.000997	0.164981	0.001588	0.43202	0.005078	0.295819
CCC-	0.029484	0.00043	0.161747	0.001591	0	0.536183	0.242977
D	0	0	0	0	0	0	1

Table 19.36. *Probabilities* $\phi_{ij}(5,7,10)$ - ***II***

$A_i(s,t)$									
u	s	t	**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	0	1	1	1	1	0.999968	0.996925	0.999764	0.998961
0	0	2	0.999991	0.999987	0.999936	0.999423	0.993572	0.999435	0.998299
0	0	3	0.999981	0.999958	0.999836	0.999071	0.991763	0.99908	0.997164
0	0	4	0.999968	0.999911	0.999709	0.998793	0.988904	0.998726	0.995916
0	0	5	0.999956	0.999812	0.999476	0.998475	0.985322	0.998057	0.99488
0	0	6	0.999933	0.999633	0.999143	0.997611	0.982541	0.997147	0.992688
0	0	7	0.999894	0.999401	0.998674	0.996742	0.979183	0.99623	0.990739
0	0	8	0.999835	0.998948	0.997753	0.995935	0.976017	0.994732	0.986505
0	0	9	0.999749	0.998206	0.996333	0.994831	0.973398	0.992288	0.982006
0	0	10	0.999314	0.995626	0.991587	0.990844	0.964015	0.981222	0.957506
2	6	7	1	1	1	0.999717	0.99652	0.998982	0.999799
2	6	8	0.999993	0.999909	0.999681	0.998935	0.991975	0.997865	0.995446
2	6	9	0.999986	0.999532	0.998959	0.998718	0.989737	0.995587	0.990746
2	6	10	0.999891	0.997734	0.995524	0.996719	0.984762	0.985371	0.969673
4	6	7	1	1	1	0.999957	0.999766	0.999656	0.998659
4	6	8	0.999996	0.999844	0.999806	0.99937	0.999451	0.998292	0.995586
4	6	9	0.99999	0.999614	0.998885	0.998971	0.999054	0.996909	0.991069
4	6	10	0.999889	0.998192	0.994393	0.997174	0.996021	0.99036	0.966736

Table 19.37. *Probability of not defaulting from s to t with backward recurrence time s-u-**I***

Finally, in Tables 19.37 and 19.38 the probabilities of never going into default are reported. These elements give the probability that a firm, that is, at a given rating at time s, will not have a default up to time t, given that it had the rating at time u (backward recurrence time s-u).

$A_i(s,t)$								
u	s	t	**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	0	1	0.996257	0.994695	0.996146	0.999959	0.996284	0.972282
0	0	2	0.992331	0.991607	0.993892	0.996661	0.991224	0.953744
0	0	3	0.988039	0.985413	0.989184	0.99343	0.982982	0.928821
0	0	4	0.984554	0.977635	0.983544	0.986963	0.976653	0.917664
0	0	5	0.976896	0.969593	0.976542	0.976842	0.96483	0.878435
0	0	6	0.966982	0.959714	0.969033	0.968583	0.950608	0.84797
0	0	7	0.959384	0.955638	0.95667	0.958528	0.937731	0.819395
0	0	8	0.950314	0.939394	0.947369	0.947336	0.918583	0.775739
0	0	9	0.938124	0.92374	0.929473	0.917325	0.819802	0.724016
0	0	10	0.897052	0.873723	0.876134	0.844328	0.700216	0.581562
2	6	7	0.998572	0.978718	0.969794	0.977368	0.987032	0.979671
2	6	8	0.996016	0.970403	0.954563	0.96299	0.97334	0.95212
2	6	9	0.99033	0.962454	0.940606	0.946127	0.880292	0.891065
2	6	10	0.956603	0.928904	0.89047	0.86583	0.756136	0.732838
4	6	7	0.999171	0.998713	0.992329	0.984648	0.99576	0.963201
4	6	8	0.992429	0.986815	0.980369	0.971697	0.955405	0.942493
4	6	9	0.986339	0.980006	0.956142	0.947635	0.820178	0.88657
4	6	10	0.958009	0.942448	0.905999	0.888396	0.684876	0.725764

Table 19.38. *Probability of not defaulting from s to t with backward recurrence time s-u-**II***

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Chapter 20

Markov and Semi-Markov Reward Processes and Stochastic Annuities

20.1. Reward processes

The association of a sum of money a state of the system and a state transition assumes great relevance in the study of financial events. This can be done by linking a reward structure to a stochastic process. This structure can be thought of as a random variable associated with the state occupancies and transitions (see Howard (1971)).

The rewards can be of different kinds, but in the financial environment only amounts of money will be considered as rewards. These amounts can be positive if they are seen as a benefit for the system, and negative if they are considered as a cost.

In this chapter, reward structures for discrete-time Markov and semi-Markov processes and how they can be considered a generalization of deterministic annuities will be described. Only the case of discrete-time reward structures and their relations to the discrete-time annuities will be presented.

A simple classification scheme of the different kinds of Discrete-time Markov ReWard Processes (DTMRWP) and Semi-Markov ReWard Processes (DTSMRWP) given in Janssen and Manca (2006, 2007) will be reported.

Process classification

Homogenous		
Non-homogenous		
Continuous time		
Discrete-time		
Not discounted		
Discounted	Fixed interest rate	
	Variable interest rate	Homogenous interest law
		Non-homogenous interest law

Reward classification

Time fixed rewards	
Time variable rewards	Homogenous rewards
	Non-homogenous rewards
Transition (impulse) rewards	
Permanence (rate) rewards	Immediate
	Due
	Independent on next transition
	Dependent on next transition

Some clarifications as regards the homogenity concept are necessary.

It is assumed that a phenomenon depends on time. We follow the phenomenon in the interval times $[s_1, t_1]$ and $[s_2, t_2]$ where $t_1 - s_1 = t_2 - s_2$. If the phenomenon behaves in the same way in the two time intervals and in each interval for the same time period, we say that it is homogenous. On the other hand, in the case in which the phenomenon changes not only for time duration but also because of the initial time, then the phenomenon is non-homogenous.

In general, this distinction is made in the stochastic processes environment, but also, as described in previous chapters, an interest rate law can be homogenous or non-homogenous. It is homogenous if the discount factor is a function of only the length of the financial operation, and is non-homogenous if the discount factor also takes into account not only the duration but also the initial time of the operation.

For the same reason, rewards can also be fixed in time, can depend only on the duration or can be non-homogenous in time.

It should be stated that in finance and insurance problems reward processes without discount do not normally make sense, but in some reliability problems they could have some meaning. Furthermore, the absence of interest rates simplifies the model. In this chapter, we will develop only discounted processes, the non-discounted DTMRWP and DTSMRWP description can be found in Janssen and Manca (2007). A very short description of reward processes with the study of some properties can be found in Rolski *et al.* (1999).

In a discrete-time process and as a first approach, the rewards that depend on permanence in the state could be considered as a generalization of discrete-time annuity. As for the annuities, there are *immediate permanence* rewards that are paid at the end of a period and *due permanence* rewards that are paid at beginning of a period.

All the hypotheses imply different formulae of the system evolution equation. The general relations in both homogenous and non-homogenous environments will be given.

Discounting factors

As regards the financial notations, it is assumed that we are working in a general environment with variable interest rates. In the homogenous case, the following

$$r(1), r(2), \ldots, r(t), \ldots$$

will denote the interest rates and

$$\nu(t) = \begin{cases} 1 & \text{if} \quad t = 0, \\ \prod_{h=1}^{t} \left(1 + r(h)\right)^{-1} & \text{if} \quad t > 0, \end{cases} \tag{20.1}$$

the t-period discount factor, if it begins at time 0. In this case, we can also obtain:

$$\nu(s,t) = \begin{cases} 1 & \text{if} \quad t = s, \\ \prod_{h=s+1}^{t} \left(1 + r(h)\right)^{-1} & \text{if} \quad t > s. \end{cases} \tag{20.2}$$

In the non-homogenous interest rate case, the following notations will be used:

$$r(s, s+1), r(s, s+2), \ldots, r(s, s+t), \ldots,$$

for the discrete-time non-homogenous interest rates and:

$$\dot{v}(s,t)=\begin{cases} 1 & \text{if} \quad t=s, \\ \prod_{h=s+1}^{t}\left(1+r(s,h)\right)^{-1} & \text{if} \quad t>s, \end{cases} \tag{20.3}$$

for the non-homogenous discount factors.

Reward notation

(i) ψ_i, $\psi_i(t)$, $\psi_i(s,t)$ denote the reward that is given for permanence in the *i*th state; it is also called rate reward (see Qureshi and Sanders (1994)); the first is paid in cases in which the period amount in state *i* is constant in time, the second when the payment is a function of the state and of the duration inside the state (homogenous payment) and the third when there is a non-homogenous period amount (the payment is a function of the state, the time of entrance into the state and the time of payment). $\boldsymbol{\psi}$ represents the vector of these rewards.

(ii) $\psi_{ij}, \psi_{ij}(t), \psi_{ij}(s,t)$ have the same meaning as given previously, the difference being that, in this case, the rewards depend on the future transition. $\boldsymbol{\Psi}$ represents the related matrix. It should be said that these kinds of permanence rewards are usually presented in the other works (see Papadopoulou and Tsaklidis (2006)) and can be seen as a generalization of case (i). In a financial environment, this kind of generalization will not make sense, so we will not present them; the interested reader can refer to Janssen and Manca (2006) and (2007).

(iii) γ_{ij}, $\gamma_{ij}(t)$, $\gamma_{ij}(s,t)$ denote the reward that is given for the transition from the *i*th state to the *j*th one (impulse reward); the difference between the three symbols is the same as in the previous cases. $\boldsymbol{\Gamma}$ is the matrix of the transition rewards.

The different kinds of ψ rewards represent an annuity that is paid due to remaining in a state. In the *immediate case*, the reward will be paid at the end of the period before the transition; in the *due case* the reward will be paid at the beginning of the period. On the other hand, γ represents lump sums that, theoretically, are paid at the instant of transition.

As far as the impulse reward γ is concerned, in the case of discounting it is only necessary to calculate the present value of the lump sum paid at the moment of the related transition and that does not present any difficulties.

Reward structure can be considered a very general structure linked to the problem being studied. The reward random variable evolves together with the evolution of the Markov or semi-Markov process with which it is linked. When the considered system, which evolves dynamically in a random way, is in a state, then a reward of type ψ can be paid; once there is a transition, an impulse reward of γ type can be paid.

This behavior is particularly efficient at constructing models which are useful for following, for example, the dynamic evolution of insurance problems.

Usually, in fact, permanence in a state involves the periodic payment of a premium or the periodic receipt of a claim. Furthermore, the transition from one state to another can often give rise to some other cost or benefit.

In the last part of this section, some matrix operation notation useful for describing the evolution equation of the reward processes will be given.

Matrix operations

Given the two matrices **A**, **B** with the notations

$$\mathbf{A} * \mathbf{B} \text{ and } \mathbf{A} \cdot \mathbf{B}$$

respectively the usual row column and the element by element matrix multiplication are denoted. It is clear that in the first case the number of columns in **A** should be equal to the number of the rows in **B** and that in the second operation the two matrices should have the same order of rows and columns.

Definition 20.1 *Given two matrices* $\mathbf{A},\mathbf{B}$ *that have row order equal to m and column order equal to n, the following operation is defined:*

$$\mathbf{c} = \mathbf{A} \circ \mathbf{B} \tag{20.4}$$

where **c** *is the m elements vector in which the ith component is obtained in the following way:*

$$c(i) = \sum_{j=1}^{n} a_{ij} b_{ij} = \mathbf{a}_{i*} * \mathbf{b}_{i*}. \tag{20.5}$$

20.2. Homogenous and non-homogenous DTMRWP

In our opinion, Markov reward processes should be considered a class of stochastic processes, each having different evolution equations. The differences from the analytic point of view can be considered irrelevant but from the algorithmic point of view the differences are very significant and in the construction of the algorithm the differences must be taken into account.

V_i and $\ddot{V}_i$ represent the *mean present value* of the *rewards* (RMPV) paid in the investigated horizon time in the homogenous immediate and due cases respectively.

For the sake of classification, first we present the simplest evolution equation case in immediate and due hypotheses and only in the homogenous case; subsequently, only the general relations in the discrete-time environment will be given.

The immediate homogenous Markov evolution equation in the case of fixed permanence and without transition rewards is the first relation presented. The DTMRWP present value after one payment is:

$$V_i(1) = (1+r)^{-1}\psi_i = (1+r)^{-1}\psi_i, \tag{20.6}$$

after two payments,

$$V_i(2) = (1+r)^{-1}\psi_i + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k = V_i(1) + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k , \tag{20.7}$$

and in general, taking into account the recursive nature of relations, at the n^{th} period is:

$$V_i(n) = V_i(n-1) + \nu^n \sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k , \tag{20.8}$$

that in matrix form becomes:

$$\mathbf{V}(n) = \nu\boldsymbol{\psi} + \cdots + \left(\nu^n \mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi} \tag{20.9}$$

Now the related due case is given:

$$\ddot{V}_i(1) = \psi_i ,$$

$$\ddot{V}_i(2) = \psi_i + (1+r)^{-1}\sum_{k=1}^{m} p_{ik}\psi_k = \ddot{V}_i(1) + (1+r)^{-1}\sum_{k=1}^{m} p_{ik}^{(1)}\psi_k , \tag{20.10}$$

$$\ddot{V}_i(n) = \ddot{V}_i(n-1) + (1+r)^{-n+1}\sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k , \tag{20.11}$$

that in matrix form is:

$$\ddot{\mathbf{V}}(n) = \mathbf{I} * \boldsymbol{\psi} + \left(\nu\mathbf{P}\right) * \boldsymbol{\psi} + \cdots + \left(\nu^{n-1}\mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi} \tag{20.12}$$

Now the general case with variable permanence, transition rewards and interest rates is presented. The present value after one period is:

$$V_i(1) = \nu(1)\left(\psi_i(1) + \sum_{j=1}^{m} p_{ij}\gamma_{ij}(1) \right), \tag{20.13}$$

after two payments,

$$\begin{aligned} V_i(2) &= \nu(1)\left(\psi_i(1) + \sum_{j=1}^{m} p_{ij}\gamma_{ij}(1) \right) \\ &\quad + \nu(2)\sum_{k=1}^{m} p_{ik}\left(\psi_k(2) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \right) \\ &= V_i(1) + \nu(2)\sum_{k=1}^{m} p_{ik}\left(\psi_k(2) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \right), \end{aligned} \tag{20.14}$$

and in general, taking into account the recursive nature of relations, at the n^{th} period is:

$$V_i(n) = V_i(n-1) + \nu(n)\sum_{k=1}^{m} p_{ik}^{(n-1)}\left(\psi_k(n) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(n) \right). \tag{20.15}$$

This relation can be written in matrix notation in the following way:

$$\begin{aligned} &\mathbf{V}(n) = \nu(1)\boldsymbol{\psi}(1) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)} \right) * \boldsymbol{\psi}(n) \\ &+\nu(1)\left(\mathbf{P} \circ \boldsymbol{\Gamma}(1) \right) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)} \right) * \left(\mathbf{P} \circ \boldsymbol{\Gamma}(n) \right). \end{aligned} \tag{20.16}$$

In the case of one period payment due, i.e. the permanence reward is paid at the beginning of the period and the transition reward at the end, we have:

$$\ddot{V}_i(1) = \psi_i(1) + \nu(1)\sum_{j=1}^{m} p_{ij}\gamma_{ij}(1), \tag{20.17}$$

with two payments we obtain:

$$\begin{aligned} \ddot{V}_i(2) &= \psi_i(1) + \nu(1)\sum_{k=1}^{m} p_{ik}\gamma_{ik}(1) \\ &\quad + \nu(1)\sum_{j=1}^{m} p_{ij}\psi_j(2) + \nu(2)\sum_{k=1}^{m} p_{ik}\sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \\ &= \ddot{V}_i(1) + \nu(1)\sum_{k=1}^{m} p_{ik}\psi_k(1) + \nu(2)\sum_{k=1}^{m} p_{ik}\sum_{j=1}^{m} p_{kj}\gamma_{kj}(2). \end{aligned} \tag{20.18}$$

At last, the general relation in the due homogenous Markov case is:

$$\ddot{V}_i(n) = \ddot{V}_i(n-1) + \nu(n)\sum_{k=1}^{m} p_{ik}^{(n-1)} \sum_{j=1}^{m} p_{kj}\gamma_{kj}(n) + \nu(n-1)\sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k(n), \quad (20.19)$$

which in matrix notation is:

$$\begin{aligned}\ddot{\mathbf{V}}(n) &= \mathbf{I} * \boldsymbol{\psi}(1) + \left(\nu(1)\mathbf{P}\right) * \boldsymbol{\psi}(2) + \cdots + \left(\nu(n-1)\mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi}(n) \\ &+ \left(\nu(1)\mathbf{P}\right) \circ \boldsymbol{\Gamma}(1) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)}\right) * \left(\mathbf{P} \circ \boldsymbol{\Gamma}(n)\right).\end{aligned} \quad (20.20)$$

Now the non-homogenous formulae with non-homogenous interest rates and rewards are reported. The first gives the immediate case, that is:

$$V_i(s,t) = V_i(s,t-1) + \dot{\nu}(s,t)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\left(\psi_k(s,t) + \sum_{j=1}^{m} p_{kj}(t)\gamma_{kj}(s,t)\right), \quad (20.21)$$

where $t = s + n$.

In matrix form, (20.21) becomes:

$$\begin{aligned}&\mathbf{V}(s,t) = \dot{\nu}(s,s+1)\boldsymbol{\psi}(s,s+1) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s+1)\right) \circ \boldsymbol{\Gamma}(s,s+1) \\ &+ \cdots + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \boldsymbol{\psi}(s,t) + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \left(\mathbf{P}(t) \circ \boldsymbol{\Gamma}(s,t)\right),\end{aligned} \quad (20.22)$$

where $\mathbf{P}^{(n)}(s) = \mathbf{P}(s+1) * \mathbf{P}(s+2) * \cdots * \mathbf{P}(t)$ and $\mathbf{P}(s)$ is the non-homogenous transition matrix at time *s*.

The related due case has the following notation:

$$\begin{aligned}&\ddot{V}_i(s,t) = \ddot{V}_i(s,t-1) + \dot{\nu}(s,t-1)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\psi_k(s,t) \\ &+ \dot{\nu}(s,t)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\sum_{j=1}^{m} p_{kj}(t)\gamma_{kj}(s,t),\end{aligned} \quad (20.23)$$

which in matrix formula becomes:

$$\begin{aligned}&\ddot{\mathbf{V}}(s,t) = \boldsymbol{\psi}(s,s+1) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s)\right) * \boldsymbol{\psi}(s,s+2) + \cdots \\ &+ \left(\dot{\nu}(s,t-1)\mathbf{P}^{(n-1)}(s)\right) * \boldsymbol{\psi}(s,t) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s+1)\right) \circ \boldsymbol{\Gamma}(s,s+1) \\ &+ \cdots + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \left(\mathbf{P}(t) \circ \boldsymbol{\Gamma}(s,t)\right).\end{aligned} \quad (20.24)$$

Remark 20.1 In this section, general formulae were presented. In the construction of the algorithms the differences between the possible cases should be taken into account and it is possible to construct a generalization. For example, in the non-discounting case $\nu(k)=1,\ k=1,\dots,n$ can be stated.

20.3. Homogenous and non-homogenous DTSMRWP

20.3.1. *The immediate cases*

20.3.1.1. *First model*

We assume that all the rewards are discounted at time 0 in the homogenous case and at time *s* in the non-homogenous case. Let us point out that these models, as we will see below, are very important for insurance applications. In the first formulation of this case we suppose that:

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate *r* is fixed.

In this case, $V_i(t)$ represents the mean present value of all the rewards (RMPV) paid or received in a time *t*, given that at time 0 the system is in state *i*.

Under these hypotheses, a similar reasoning as before leads to the following result for the evolution equation, firstly for the homogenous case. Trivially it results in:

$$\begin{aligned} V_i(0) &= 0, \\ V_i(1) &= \left(1-H_i(1)\right)\psi_i\nu^1+\sum_{k=1}^{m}b_{ik}(1)\psi_i\nu^1+\sum_{k=1}^{m}\sum_{\vartheta=1}^{1}b_{ik}(\vartheta)V_k(1-\vartheta)\nu^1 \\ &= \psi_i\nu^1, \end{aligned} \tag{20.25}$$

and in general:

$$V_i(t) = (1-H_i(t))\psi_i a_{\overline{t}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\psi_i a_{\overline{\vartheta}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}. \tag{20.26}$$

For the non-homogenous case, this last result becomes:

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\psi_i a_{\overline{t-s}|r} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r} \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)v^{\vartheta-s}. \end{aligned} \tag{20.27}$$

To explain these results, we divide the evolution equation into three parts. The meaning is the same as given in the previous cases but we use annuity formulae.

Let us just give the following comments:

– The term $(1-H_i(s,t))\psi_i a_{\overline{t-s}|r}$ represents the present value of the rewards obtained without state changes. More precisely, $(1-H_i(s,t))$ is the probability to remain in state i and $\psi_i a_{\overline{t-s}|r}$ is the present value of a constant annuity of t-s installments ψ_i.

– The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r}$ gives the present value of the rewards obtained before the change of state.

– The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)v^{\vartheta-s}$ gives the present value of the rewards paid or earned after the transitions and as the change of state happens at time ϑ, it is necessary to discount the reward values at time s.

As for DTMRWP we will give the matrix equation of each given relation.

To present the matrix form of the previous relations we have to define the following matrices:

$$D_{ij}(t) = \begin{cases} 1-H_i(t) & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases}, \quad D_{ij}(s,t) = \begin{cases} 1-H_i(s,t) & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

Relations (20.26) and (20.27) respectively become in matrix form:

$$\begin{aligned} \mathbf{V}(t) &= (\mathbf{D}(t) * \mathbf{1}) \cdot (\boldsymbol{\psi} a_{\overline{t}|r}) + \sum_{\vartheta=1}^{t} (\mathbf{B}(\vartheta) * \mathbf{1}) \cdot (\boldsymbol{\psi} a_{\overline{\vartheta}|r}) \\ &+ \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * (\mathbf{V}(t-\vartheta)v^{\vartheta}), \end{aligned} \tag{20.28}$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} a_{\overline{t-s}|r} \right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} a_{\overline{\vartheta-s}|r} \right)$$
$$+ \sum_{\vartheta=1}^{t} \mathbf{B}(s,\vartheta) * \left(\mathbf{V}(\vartheta,t) \nu^{\vartheta-s} \right),$$

where **1**, as specified in previous chapters, represents the sum vector whose elements are all equal to 1.

20.3.1.2. *Second model*

Now we introduce the case of variable interest rates with as assumptions:

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate r is variable.

Under these hypotheses, it can be shown that we obtain the following formulae:

$$V_i(t) = (1 - H_i(t))\psi_i \sum_{h=1}^{t} \nu(h) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\psi_i \sum_{h=1}^{\vartheta} \nu(h)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) V_k(t-\vartheta)\nu(\vartheta), \qquad (20.29)$$

$$V_i(s,t) = (1 - H_i(s,t))\psi_i \sum_{h=s+1}^{t} \nu(s,h) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i \sum_{h=s+1}^{\vartheta} \nu(s,h)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) V_k(\vartheta,t)\nu(s,\vartheta). \qquad (20.30)$$

The matrix forms related to (20.29) and (20.30) are:

$$\mathbf{V}(t) = \left(\mathbf{D}(t) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \vec{a}(t) \right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \vec{a}(\vartheta) \right)$$
$$+ \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\mathbf{V}(t-\vartheta)\nu(\vartheta) \right), \qquad (20.31)$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \vec{a}(s,t) \right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \vec{a}(s,\vartheta) \right)$$
$$+ \sum_{\vartheta=s+1}^{t} \mathbf{B}(\vartheta) * \left(\mathbf{V}(\vartheta,t)\nu(s,\vartheta) \right), \qquad (20.32)$$

where respectively it holds:

$$\vec{a}(t)=\sum_{h=1}^{t}\nu(h),\quad \vec{a}(s,t)=\sum_{h=s+1}^{t}\nu(s,h).$$

20.3.1.3. *Third model*

The next step is the introduction of the variability of rewards so we assume that:

a) rewards are variable in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate r is fixed.

In this case the following results hold:

$$\begin{aligned} V_i(t) &= (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu^h+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu^h \\ &\quad +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}, \end{aligned} \tag{20.33}$$

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu^{h-s}+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu^{h-s} \\ &\quad +\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \tag{20.34}$$

(20.33) and (20.34) in matrix form are:

$$\begin{aligned} \mathbf{V}(t) &= (\mathbf{D}(t)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(t)*\underline{\nu}^{(t)}\right)+\sum_{\vartheta=1}^{t}(\mathbf{B}(\vartheta)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta)*\underline{\nu}^{(\vartheta)}\right) \\ &\quad +\sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)*\left(\mathbf{V}(t-\vartheta)\nu^{\vartheta}\right), \end{aligned} \tag{20.35}$$

$$\begin{aligned} \mathbf{V}(s,t) &= (\mathbf{D}(s,t)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(s,t)*\underline{\nu}^{(t-s)}\right) \\ &\quad +\sum_{\vartheta=s+1}^{t}(\mathbf{B}(s,\vartheta)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(s,\vartheta)*\underline{\nu}^{(\vartheta-s)}\right) \\ &\quad +\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\left(\mathbf{V}(\vartheta,t)\nu^{\vartheta-s}\right), \end{aligned} \tag{20.36}$$

where

$$\underline{\boldsymbol{\psi}}(t)=\begin{bmatrix}\psi_1(1) & \psi_1(2) & \cdots & \psi_1(t)\\ \psi_2(1) & \psi_2(2) & \cdots & \psi_2(t)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_m(1) & \psi_m(2) & \cdots & \psi_m(t)\end{bmatrix}, \underline{\nu}^{(h)}=\begin{bmatrix}\nu^1\\ \nu^2\\ \vdots\\ \nu^h\end{bmatrix}$$

and

$$\underline{\boldsymbol{\psi}}(s,t)=\begin{bmatrix}\psi_1(s+1) & \psi_1(s+2) & \cdots & \psi_1(t)\\ \psi_2(s+1) & \psi_2(s+2) & \cdots & \psi_2(t)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_m(s+1) & \psi_m(s+2) & \cdots & \psi_m(t)\end{bmatrix}.$$

20.3.1.4. *Fourth model*

For the case of variable interest rates with variable rewards, we assume that:

a) rewards are variable in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rates are variable in time.

Here, the evolution equation takes the form:

$$\begin{aligned}V_i(t) &= (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu(h)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu(h)\\ &\quad+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta),\end{aligned} \tag{20.37}$$

$$\begin{aligned}V_i(s,t) &= (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu(s,h)+\\ &\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu(s,h)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta).\end{aligned} \tag{20.38}$$

Matrix forms of (20.37) and (20.38) respectively are:

$$\begin{aligned}\mathbf{V}(t) &= (\mathbf{D}(t)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(t)*\underline{\nu}(t)\right)+\sum_{\vartheta=1}^{t}(\mathbf{B}(\vartheta)*\mathbf{1})\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta)*\underline{\nu}(\vartheta)\right)\\ &\quad+\sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)*(\mathbf{V}(t-\vartheta)\nu(\vartheta)),\end{aligned} \tag{20.39}$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1}\right) \cdot \left(\underline{\boldsymbol{\psi}}(s,t) * \underline{\nu}(s,t)\right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right) \cdot \left(\underline{\boldsymbol{\psi}}(s,\vartheta) * \underline{\nu}(s,\vartheta)\right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\mathbf{V}(\vartheta,t)\nu(s,\vartheta)\right) \tag{20.40}$$

where

$$\underline{\nu}(t) = \begin{bmatrix} \nu(1) \\ \nu(2) \\ \vdots \\ \nu(t) \end{bmatrix}, \ \underline{\nu}(s,t) = \begin{bmatrix} \nu(s+1) \\ \nu(s+2) \\ \vdots \\ \nu(t) \end{bmatrix}.$$

20.3.1.5. *Fifth model*

The next step will introduce the γ rewards in the case of a fixed interest rate.

We have the following assumptions:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the end of the period;

d) interest rate r is fixed.

Under these hypotheses, the homogenous general formula is the following:

$$V_i(t) = (1 - H_i(t)) \sum_{h=1}^{t} \psi_i(h)\nu^h + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \sum_{h=1}^{\vartheta} \psi_i(h)\nu^h + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}. \tag{20.41}$$

Here too, the meaning of relation (20.41) can be easily understood with a subdivision into four parts.

Due to of the presence of lump sums in the RMPV, given or taken at change of state times, let us say that the sum of the last two terms

$$\sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta} \tag{20.42}$$

concerning the rewards $\gamma_{ik}(\vartheta)$ are paid or received at the transition moment and so must be discounted for a time of ϑ periods as $V_k(t-\vartheta)$.

The corresponding non-homogenous formula is the following:

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu^{h-s} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu^{h-s} \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta-s} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \tag{20.43}$$

Matrix forms of (20.41) and (20.43) are:

$$\begin{aligned} \mathbf{V}(t) &= \left(\mathbf{D}(t)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(t)*\underline{\nu}^{(t)}\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta)*\underline{\nu}^{(\vartheta)}\right) \\ &+ \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta}\right)\right)*\mathbf{1} + \sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)*\left(\mathbf{V}(t-\vartheta)\nu^{\vartheta}\right), \end{aligned} \tag{20.44}$$

$$\begin{aligned} \mathbf{V}(s,t) &= \left(\mathbf{D}(s,t)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,t)*\underline{\nu}^{(t-s)}\right) + \sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\left(\mathbf{V}(\vartheta,t)\nu^{\vartheta-s}\right) \\ &+ \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta-s}\right)\right)*\mathbf{1} + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,\vartheta)*\underline{\nu}^{(\vartheta-s)}\right). \end{aligned} \tag{20.45}$$

20.3.1.6. *Sixth model*

The next model extends the preceding model with the variability of interest rates that is under the following assumptions:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the end of the period;

d) interest rates are variable in time.

All these hypotheses lead us to the following relations:

$$\begin{aligned} V_i(t) &= (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu(h) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu(h) \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta). \end{aligned} \tag{20.46}$$

$$V_i(s,t) = (1 - H_i(s,t)) \sum_{h=s+1}^{t} \psi_i(h)\nu(s,h) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{h=s+1}^{\vartheta} \psi_i(h)\nu(s,h)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)\nu(s,\vartheta) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta). \quad (20.47)$$

(20.46) and (20.47) matrix forms are:

$$\mathbf{V}(t) = (\mathbf{D}(t) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(t) * \underline{\nu}(t)\right) + \sum_{\vartheta=1}^{t} (\mathbf{B}(\vartheta) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(\vartheta) * \underline{\nu}(\vartheta)\right)$$
$$+ \sum_{\vartheta=1}^{t} (\mathbf{B}(\vartheta) \cdot (\boldsymbol{\Gamma}(\vartheta)\nu(\vartheta))) * \mathbf{1} + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * (\mathbf{V}(t-\vartheta)\nu(\vartheta)), \quad (20.48)$$

$$\mathbf{V}(s,t) = (\mathbf{D}(s,t) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(s,t) * \underline{\nu}(s,t)\right)$$
$$+ \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * (\mathbf{V}(\vartheta,t)\nu(s,\vartheta)) + \sum_{\vartheta=s+1}^{t} (\mathbf{B}(s,\vartheta) \cdot (\boldsymbol{\Gamma}(\vartheta)\nu(s,\vartheta))) * \mathbf{1} \quad (20.49)$$
$$+ \sum_{\vartheta=s+1}^{t} (\mathbf{B}(s,\vartheta) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(s,\vartheta) * \underline{\nu}(s,\vartheta)\right).$$

20.3.1.7. *Seventh model*

For our last case, we consider non-homogenous rewards and interest rate. Therefore, the basic assumptions are:

a) rewards are non-homogenous;

b) rewards are also given at the transitions;

c) rewards are paid at the end of the period;

d) interest rate is non-homogenous.

It can easily be verified that the evolution equation takes the form:

$$V_i(s,t) = (1 - H_i(s,t)) \sum_{\tau=s+1}^{t} \psi_i(s,\tau)\dot{\nu}(s,\tau)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s+1}^{\vartheta} \psi_i(s,\tau)\dot{\nu}(s,\tau) \quad (20.50)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\dot{\nu}(s,\vartheta)\gamma_{ik}(s,\vartheta) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\dot{\nu}(s,\vartheta)V_k(\vartheta,t).$$

(20.50) in matrix form becomes

$$\begin{aligned}\mathbf{V}(s,t) &= \mathbf{D}(s,t)\sum_{\tau=s+1}^{t}\boldsymbol{\psi}(s,\tau)\dot{\nu}(s,\tau)\\ &+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\sum_{\tau=s+1}^{\vartheta}\boldsymbol{\psi}(s,\tau)\dot{\nu}(s,\tau)\\ &+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(s,\vartheta)\dot{\nu}(s,\vartheta)\right)\right)*\mathbf{1}+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\left(\mathbf{V}(\vartheta,t)\dot{\nu}(s,\vartheta)\right)\end{aligned} \tag{20.51}$$

To conclude this section, we will present the most significant due cases. The reasoning is quite similar to the models for the immediate case but nevertheless, it is useful to classify the most interesting models.

As above, we systematically treat the homogenous and non-homogenous cases.

20.3.2. *The due cases*

20.3.2.1. *First model*

For the due case, our first model has the following assumptions

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the beginning of the period;

d) interest rate r is fixed.

Here, $\ddot{V}_i(t)$ ($\ddot{V}_i(s,t)$) represents the RMPV given that at time 0, (s) the system in state i and the rewards being paid at the beginning of the period.

Under our hypotheses, the evolution equations take the form:

$$\ddot{V}_i(t) \;=\; (1-H_i(t))\psi_i\ddot{a}_{\overline{t}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\psi_i\ddot{a}_{\overline{\vartheta}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta)\nu^{\vartheta}, \tag{20.52}$$

$$\begin{aligned}\ddot{V}_i(s,t) \;&=\; (1-H_i(s,t))\psi_i\ddot{a}_{\overline{t-s}|r}+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\psi_i\ddot{a}_{\overline{\vartheta-s}|r}\\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t)\nu^{\vartheta-s}.\end{aligned} \tag{20.53}$$

The matrix forms of (20.52) and (20.53) respectively are

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1}\right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{t}|r}\right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1}\right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{\vartheta}|r}\right) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta) v^{\vartheta}\right) \tag{20.54}$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1}\right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{t-s}|r}\right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{\vartheta-s}|r}\right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t) v^{\vartheta-s}\right) \tag{20.55}$$

20.3.2.2. *Second model*

We now consider variable rewards and variable interest rates to obtain the following assumptions:

a) rewards are variable in time,

b) rewards are given only for the permanence in the state,

c) rewards are paid at the beginning of the period,

d) interest rates are time dependent.

The related evolution equations are:

$$\ddot{V}_i(t) = (1 - H_i(t)) \sum_{\tau=0}^{t-1} \psi_i(\tau+1) v(\tau) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \sum_{\tau=0}^{\vartheta-1} \psi_i(\tau+1) v(\tau) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \ddot{V}_k(t-\vartheta) v(\vartheta), \tag{20.56}$$

$$\ddot{V}_i(s,t) = (1 - H_i(s,t)) \sum_{\tau=s}^{t-1} \psi_i(\tau+1) v(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s}^{\vartheta-1} \psi_i(\tau+1) v(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \ddot{V}_k(\vartheta,t) v(s,\vartheta). \tag{20.57}$$

The matrix forms of (20.56) and (20.57) are

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1}\right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(t) * \underline{\ddot{v}}(t)\right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1}\right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(\vartheta) * \underline{\ddot{v}}(\vartheta)\right) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta) v(\vartheta)\right) \tag{20.58}$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(s,t) * \underline{\ddot{\nu}}(s,t) \right)$$
$$+ \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(s,\vartheta) * \underline{\ddot{\nu}}(s,\vartheta) \right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)\nu(s,\vartheta) \right) \tag{20.59}$$

where

$$\underline{\ddot{\boldsymbol{\psi}}}(t) = \begin{bmatrix} \psi_1(1) & \psi_1(2) & \cdots & \psi_1(t) \\ \psi_2(1) & \psi_2(2) & \cdots & \psi_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m(1) & \psi_m(2) & \cdots & \psi_m(t) \end{bmatrix}, \quad \underline{\ddot{\nu}}(t) = \begin{bmatrix} 1 \\ \nu(1) \\ \vdots \\ \nu(t-1) \end{bmatrix}$$

and

$$\underline{\ddot{\boldsymbol{\psi}}}(s,t) = \begin{bmatrix} \psi_1(s+1) & \psi_1(s+2) & \cdots & \psi_1(t) \\ \psi_2(s+1) & \psi_2(s+2) & \cdots & \psi_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m(s+1) & \psi_m(s+2) & \cdots & \psi_m(t) \end{bmatrix}, \quad \underline{\ddot{\nu}}(s,t) = \begin{bmatrix} 1 \\ \nu(s,s+1) \\ \vdots \\ \nu(s,t-1) \end{bmatrix}.$$

20.3.2.3. *Third model*

With the introduction of γ rewards and with a fixed interest rate, the assumptions of our third model are:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the beginning of the period;

d) interest rate r is fixed.

Under these hypotheses the equations are:

$$\ddot{V}_i(t) = (1 - H_i(t))\sum_{\tau=0}^{t-1}\psi_i(\tau+1)\nu^{\tau} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(\tau+1)\nu^{\tau}$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu^{\vartheta} b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu^{\vartheta} b_{ik}(\vartheta)\gamma_{ik}(\vartheta), \tag{20.60}$$

$$\ddot{V}_i(s,t) = \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu^{\vartheta-s} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=s}^{\vartheta-1}\psi_i(\tau+1)\nu^{\tau-s}$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu^{\vartheta-s} b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t) + (1-H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(\tau+1)\nu^{\tau-s}. \tag{20.61}$$

(20.60) and (20.61) matrix forms are:

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1}\right)\cdot\left(\ddot{\underline{\boldsymbol{\psi}}}(t) * \ddot{\underline{\nu}}^{(t)}\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta) * \mathbf{1}\right)\cdot\left(\ddot{\underline{\boldsymbol{\psi}}}(\vartheta) * \ddot{\underline{\nu}}^{(\vartheta)}\right)$$
$$+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta}\right)\right) * \mathbf{1} + \sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta)\nu^{\vartheta}\right) \tag{20.62}$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1}\right)\cdot\left(\ddot{\underline{\boldsymbol{\psi}}}(s,t) * \ddot{\underline{\nu}}^{(t-s)}\right) + \sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)\nu^{\vartheta-s}\right)$$
$$+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta-s}\right)\right) * \mathbf{1} + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right)\cdot\left(\ddot{\underline{\boldsymbol{\psi}}}(s,\vartheta) * \ddot{\underline{\nu}}^{(\vartheta-s)}\right) \tag{20.63}$$

where

$$\ddot{\underline{\nu}}^{(t)} = \begin{bmatrix} \nu^0 \\ \nu^1 \\ \vdots \\ \nu^{t-1} \end{bmatrix} = \begin{bmatrix} 1 \\ (1+r)^{-1} \\ \vdots \\ (1+r)^{-t+1} \end{bmatrix}.$$

20.3.2.4. *Fourth model*

Our last model introduces non-homogenous rewards and interest rates with the following assumptions:

a) rewards are non-homogenous in time;

b) rewards are also given at the transitions;

c) rewards are paid at the beginning of the period;

d) the interest rate is non-homogenous.

For this, the evolution equation has the following form:

$$\begin{aligned}\ddot{V}_i(s,t) \;&=\; (1 - H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(s,\tau+1)\dot{v}(s,\tau) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(s,\tau+1)\dot{v}(s,\tau) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\dot{v}(s,\vartheta)b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\dot{v}(s,\vartheta)b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta).\end{aligned} \tag{20.64}$$

The matrix form of (20.64) is given by

$$\begin{aligned}\ddot{\mathbf{V}}(s,t) &= \left(\mathbf{D}(s,t)\cdot\sum_{\tau=s}^{t-1}\ddot{\boldsymbol{\psi}}(s,\tau)\dot{v}(s,\tau)\right) * \mathbf{1} \\ &+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)v(s,\vartheta)\right) + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(s,\vartheta)\dot{v}(s,\vartheta)\right)\right) * \mathbf{1} \\ &+\sum_{\vartheta=s+1}^{t}\left(\left(\mathbf{B}(s,\vartheta)\cdot\sum_{\tau=s}^{\vartheta-1}\ddot{\boldsymbol{\psi}}(s,\tau)\dot{v}(s,\tau)\right) * \mathbf{1}\right)\end{aligned} \tag{20.65}$$

20.4. MRWP and stochastic annuities

20.4.1. *Stochastic annuities*

The annuity concept is very simple and can easily be understood by means of the following figure.

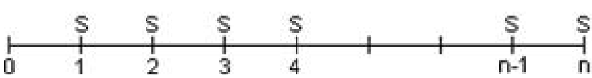

Figure 20.1. *Constant payment annuity-immediate*

where S represents the constant annuity payment.

Figure 20.1 shows the simplest immediate case.

The due case can be shown by the following figure.

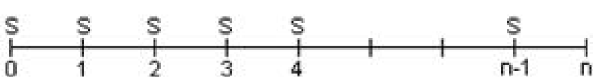

Figure 20.2. *Constant payment annuity-due*

Clearly, the payment can be variable. The simple problems to be dealt with are how to calculate the value at time 0 (present value) or at time *n* (capitalization value) of the annuity (see the first half of this book)

S can be considered not as a simple variable but rather as a random variable. This r.v. can assume, in the case of payments that vary only because of state, the following values:

$$S = \{ S_1, S_2, \ldots, S_m \}, \tag{20.66}$$

where S_i can be considered as the payment related to state *i*.

Furthermore, if it is set that the value at time *k* will depend only on the value at time *k*-1, we are in Markov process hypotheses. A sum is associated with each state which means that we are in a Markov reward environment. The problem of calculating the present value of this first simple case corresponds to the simplest case of DTHMRWP presented.

In this light, it now is possible to give the following definition.

Definition 20.2 *Discrete-time homogenous (non-homogenous) constant stochastic annuity*

Let:

$$I = \{1, 2, \ldots, m\}$$

be the states of a system and *A, B* two persons.

Furthermore, let

$$\{S_1, S_2, \ldots, S_m\},\ S_i \in \mathbb{R} \tag{20.67}$$

be sums.

The sum S_i will be paid or received from *A* to *B* if the system is in state *i*. These "payments" will be made from time $s+1$ [respectively *s*] up to time $s+n=T$ [respectively $s+n-1=T-1$].

We say that this financial operation is an *immediate* [respectively *due*] *homogenous* (*non-homogenous*) *discrete-time constant stochastic annuity* if:

i) the transitions among the states are governed by a homogenous (non-homogenous) discrete-time Markov Chain **P** $\left(\mathbf{P}(t) = \left[p_{ij}(t) \right]\right)$;

ii) when there is a transition from *i* to *j*, it is possible that a sum γ_{ij} is paid or received.

Each stochastic annuity can be seen as a discrete-time Markov reward process. The randomness is given by the fact that the periodic payment annuity is a r.v. Also, transition payments are allowed.

In the case of a simple immediate annuity, Figure 20.1 becomes Figure 20.3, and the annuity value can assume one of the values of r.v. (20.66).

Figure 20.4 gives the corresponding due case.

We are concerned with outlining the fact that, by means of the figures, it is possible to see quite easily that Markov reward processes can be considered a natural generalization of the annuity concept.

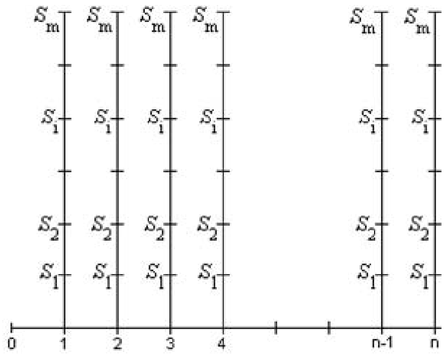

Figure 20.3. *Constant stochastic annuity-immediate*

It is clear that the reward structure could have a more complex structure that, in any case, can be seen as a generalization of the example shown in the two figures.

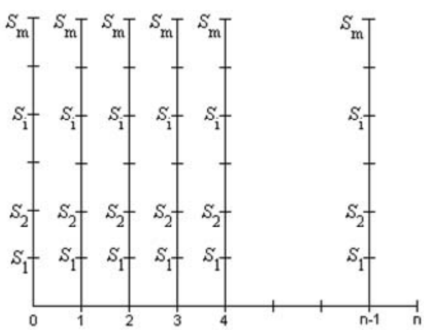

Figure 20.4. *Constant stochastic annuity-due*

It should be stated that this approach is not new in the actuarial environment; see, for example, Wolthuis (2003) and Daniel (2004). By means of our approach it is carried out in a more systematic way using the Markov reward process as the *natural stochastic generalization of the annuity* concept.

It is our opinion that the connection between Markov reward processes and annuities is natural and that an annuity can be seen as the Markov reward process with only one state and only permanence rewards.

In this light, within the field of finance it is possible to define *Markov reward processes* as *stochastic annuities.*

This first step also allows the generalization of the payments of the annuities in case of permanence rewards and transition rewards. Furthermore, the permanence rewards can be dependent or independent on the transition. All these rewards can be fixed or can vary due to time.

In the case of simple annuity, the payment can only vary due to time yet, in the case of *stochastic annuity,* clearly it can vary in the same way as the rewards, since rewards represent the payment generalization.

20.4.2. *Motorcar insurance application*

Stochastic annuities have many applications in the fields of finance and insurance.

In a general sense, actuarial mathematics can be seen as a branch of financial mathematics. In any actuarial mathematics application, we have to tackle a stochastic event within a financial environment. As it is well known, actuarial mathematics uses mathematical tools for insurance problems. In this light, DTMRWP could be seen as a useful tool to directly solve insurance problems.

In this section, DTMRWP will be applied to motor car *bonus malus* insurance rules that apply in Italy.

For a general reference on *bonus malus* systems and their properties, see Lemaire (1995) and Sundt (1993).

This example will use a transition matrix related to the motor car *bonus malus* insurance rules that apply in Italy. In this case, the Markov model fits quite well because:

1) the position of each insured person is given at the beginning of each year;

2) there are precise rules that give the change of states as a function of the behavior of the insured person during the year;

3) the future state depends only on the present one.

The number of states is 18.

Table 20.1 gives the evolution rules that hold in Italy for *bonus malus* insurance contract.

	Arriving state according to claims				
Starting state	0 claim	1 claim	2 claims	3 claims	4 or more
1	1	3	6	9	12
2	1	4	7	10	13
3	2	5	8	11	14
4	3	6	9	12	15
5	4	7	10	13	16
6	5	8	11	14	17
7	6	9	12	15	18
8	7	10	13	16	18
9	8	11	14	17	18
10	9	12	15	18	18
11	10	13	16	18	18
12	11	14	17	18	18
13	12	15	18	18	18
14	13	16	18	18	18
15	14	17	18	18	18
16	15	18	18	18	18
17	16	18	18	18	18
18	17	18	18	18	18

Table 20.1. *Italian bonus malus evolution rules*

We are in possession of the history of 105,627 insured persons over a period of three years (1998, 1999, 2000). This means that it was possible consider 316,881 real or virtual transitions. The Markov transition matrix that was obtained from the available data and taking into account the bonus malus Italian rules is given in Tables 20.2, 20.3 and 20.4.

States	1	2	3	4	5	6
1	0.941655	0	0.056264	0	0	0.001973
2	0.935097	0	0	0.062379	0	0
3	0	0.941646	0	0	0.056611	0
4	0	0	0.948892	0	0	0.049364
5	0	0	0	0.945231	0	0
6	0	0	0	0	0.949204	0
7	0	0	0	0	0	0.934685
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.2. *Transition matrix I*

States	7	8	9	10	11	12
1	0	0	0.000081	0	0	0.000027
2	0.002427	0	0	0.000097	0	0
3	0	0.001574	0	0	0.000169	0
4	0	0	0.001744	0	0	0
5	0.052354	0	0	0.002314	0	0
6	0	0.04908	0	0	0.00157	0
7	0	0	0.061856	0	0	0.00339
8	0.92227	0	0	0.073137	0	0
9	0	0.914103	0	0	0.082621	0
10	0	0	0.923854	0	0	0.071989
11	0	0	0	0.92933	0	0
12	0	0	0	0	0.930156	0
13	0	0	0	0	0	0.937854
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.3. *Transition matrix II*

States	13	14	15	16	17	18
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0.000067	0	0	0.000034	0	0
6	0	0.000146	0	0	0	0
7	0	0	0.000069	0	0	0
8	0.004246	0	0	0.00026	0	0.000087
9	0	0.003185	0	0	0	0.000091
10	0	0	0.003827	0	0	0.00033
11	0.066723	0	0	0.003696	0	0.000251
12	0	0.066697	0	0	0.002994	0.000153
13	0	0	0.059651	0	0	0.002495
14	0.920681	0	0	0.074704	0	0.004615
15	0	0.885204	0	0	0.107143	0.007653
16	0	0	0.777568	0	0	0.222432
17	0	0	0	0.876733	0	0.123267
18	0	0	0	0	0.888614	0.111386

Table 20.4. *Transition matrix III*

The payment of a claim by the insurance company can be seen as a lump sum (impulse or transition reward) paid by the insurer to the insured person. The model can be used to follow the financial evolution of a motor car insurance contract.

In Table 20.5, the premiums (which can be seen as permanence rewards) that are paid in Naples for a car of 2,300 c.c. are reported.

The example is constructed from the point of view of the insurance company and premiums are income for the company. It should be noted that these values correspond to the real premiums paid by an insured person in 2001 and officially given on the website of Assicurazioni Generali for that year.

States	Permanence rewards
1	1,037.5
2	1,099.75
3	1,162
4	1,224.25
5	1,286.5
6	1,369.5
7	1,452.5
8	1,535.5
9	1,618.5
10	1,701.5
11	1,826
12	1,950.5
13	2,075
14	2,386.25
15	2,697.5
16	3,112.5
17	3,631.25
18	4,150

Table 20.5. *Naples premiums*

In this case, permanence and impulse rewards should increase roughly in line with the inflation rate. In this light and with the aim of simplification, we suppose that the rewards are fixed in time. It is clear that the model can manage time variable premiums and benefits.

We suppose that we have a yearly fixed discount factor of 1/1.03. In the model, a stochastic interest rate could be easily introduced (see Janssen and Manca (2002)), but we do not think that this aspect is central in the presentation of our model.

Tables 20.6, 20.7 and 20.8 give the mean values of the expenses that the insurance company should pay for the claims made by the insured person.

More clearly stated, the element –7,772.51 represents the expenses that, on average, the company has to pay for the two accidents that an insured person who was in state 1 (lowest *bonus malus* class) had and which then took him to state 6.

These tables were constructed starting from the observed data in our possession.

From the point of view of the model, the elements of these three tables are transition rewards. More precisely, and as already mentioned, they can be seen as lump sums (impulse rewards) paid by the company at the time of the accident. In this case, being expenses for the company, they are negative.

States	1	2	3	4	5	6
1	0	0	–2,185.57	0	0	–7,772.51
2	0	0	0	–1,956.4	0	0
3	0	0	0	0	–2,188.25	0
4	0	0	0	0	0	–2,853.19
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.6. *Mean insurance payments I*

States	7	8	9	10	11	12
1	0	0	–3,240.77	0	0	–7,728.78
2	–3,196.16	0	0	–9,004.43	0	0
3	0	–2,846.52	0	0	–4,498.34	0
4	0	0	–2,920.39	0	0	0
5	–2,245.02	0	0	–3,945.44	0	0
6	0	–2,676.12	0	0	–3,076.05	0
7	0	0	–2,086.66	0	0	–3,391.18
8	0	0	0	–2,198.02	0	0
9	0	0	0	0	–2,017.77	0
10	0	0	0	0	0	–2,103.01
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.7. *Mean insurance payments II*

States	13	14	15	16	17	18
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	–3,240.77	0	0	–6,274.95	0	0
6	0	–6,703.61	0	0	0	0
7	0	0	–1,572.09	0	0	0
8	–4,027.26	0	0	–3,286.39	0	–3,629.14
9	0	–6,397.63	0	0	0	–3,687.5
10	0	0	–4,931.93	0	0	–5,165.44
11	–3,110.63	0	0	–4,710.94	0	–5,993.19
12	0	–3,048.69	0	0	–3,893.94	–1,1602.3
13	0	0	–2,613.27	0	0	–8,271.51
14	0	0	0	–3,564.01	0	–4,145.45
15	0	0	0	0	–2,468.23	–7,356.78
16	0	0	0	0	0	–2,883.68
17	0	0	0	0	0	–3,764.32
18	0	0	0	0	0	–2,578.55

Table 20.8. *Mean insurance payments III*

Tables 20.9, 20.10 and 20.11 report the present values of the mean total rewards that the company earns in 1 year, in 2 years and so on up to 20 years. Each column represents the starting state at time 0.

The permanence reward (insurance premium) increases as a function of the state and therefore the money earned by the company increases as a function of the starting state as well.

The results are interesting and show that, in this case, the company will earn a lot of money from this kind of insurance contract. The illustrated case is very particular. In Naples, the premiums are higher than in the other parts of Italy, the car is big and for this reason too the premiums are very high.

	Starting state					
Years	1	2	3	4	5	6
1	902.77	972.89	1,036.64	1,082.56	1,163.11	1,236.34
2	1,791.82	1,866.47	1,935.73	2,048.02	2,191.17	2,312.57
3	2,656.91	2,732.23	2,809.19	2,930.74	3,083.67	3,270.51
4	3,497.68	3,573.83	3,652.22	3,775.21	3,941.71	4,143.27
5	4,314.34	4,390.67	4,469.36	4,594.66	4,764.41	4,969.33
6	5,107.32	5,183.70	5,262.80	5,388.66	5,559.25	5,768.62
7	5,877.27	5,953.71	6,032.93	6,158.92	6,330.56	6,541.29
8	6,624.83	6,701.29	6,780.54	6,906.73	7,078.71	7,289.89
9	7,350.64	7,427.09	7,506.39	7,632.64	7,804.75	8,016.42
10	8,055.31	8,131.77	8,211.08	8,337.36	8,509.59	8,721.46
11	8,739.46	8,815.92	8,895.24	9,021.54	9,193.83	9,405.77
12	9,403.68	9,480.15	9,559.48	9,685.79	9,858.10	10,070.11
13	10,048.57	10,125.03	10,204.36	10,330.68	10,503.01	10,715.05
14	10,674.67	10,751.13	10,830.46	10,956.78	11,129.12	11,341.18
15	11,282.53	11,359.00	11,438.33	11,564.65	11,736.99	11,949.07
16	11,872.69	11,949.16	12,028.49	12,154.81	12,327.16	12,539.24
17	12,445.66	12,522.13	12,601.46	12,727.78	12,900.13	13,112.22
18	13,001.94	13,078.41	13,157.74	13,284.06	13,456.42	13,668.51
19	13,542.02	13,618.49	13,697.82	13,824.14	13,996.50	14,208.59
20	14,066.37	14,142.84	14,222.17	14,348.49	14,520.85	14,732.94

Table 20.9. *Present values of Naples mean total rewards I*

	Starting state					
Years	7	8	9	10	11	12
1	1,315.92	1,361.69	1,436.54	1,534.53	1,606.13	1,740.04
2	2,469.70	2,593.59	2,753.00	2,900.66	3,052.08	3,286.69
3	3,492.55	3,674.17	3,909.79	4,134.20	4,367.87	4,657.60
4	4,382.20	4,634.06	4,938.46	5,222.52	5,525.47	5,891.88
5	5,228.55	5,505.30	5,835.84	6,187.59	6,556.49	6,983.41
6	6,034.35	6,319.67	6,678.49	7,060.54	7,457.96	7,950.03
7	6,809.20	7,103.35	7,473.42	7,867.06	8,296.11	8,822.13
8	7,560.03	7,857.72	8,232.06	8,638.05	9,081.02	9,621.21
9	8,287.45	8,586.54	8,965.12	9,376.39	9,824.94	10,380.44
10	8,992.84	9,293.23	9,673.65	10,087.09	10,541.51	11,104.04
11	9,677.48	9,978.47	10,359.67	10,775.25	11,232.34	11,797.92
12	10,341.97	10,643.23	11,025.15	11,441.72	11,899.99	12,468.63
13	10,986.98	11,288.47	11,670.74	12,087.76	12,547.16	13,117.28
14	11,613.17	11,914.78	12,297.21	12,714.64	13,174.59	13,745.41
15	12,221.08	12,522.75	12,905.32	13,322.96	13,783.17	14,354.63
16	12,811.27	13,112.98	13,495.62	13,913.36	14,373.81	14,945.60
17	13,384.26	13,686.00	14,068.67	14,486.50	14,947.07	15,519.02
18	13,940.55	14,242.30	14,625.01	15,042.88	15,503.51	16,075.60
19	14,480.63	14,782.40	15,165.12	15,583.01	16,043.70	16,615.86
20	15,004.99	15,306.76	15,689.48	16,107.40	16,568.11	17,140.31

Table 20.10. *Present values of Naples mean total rewards II*

	Starting state					
Years	13	14	15	16	17	18
1	1,903.62	2,109.18	2,386.09	2,489.76	3,180.75	3,871.15
2	3,538.69	3,915.36	4,368.44	4,882.38	5,882.82	6,518.34
3	4,997.82	5,490.12	6,098.49	6,905.42	8,018.71	8,901.15
4	6,314.29	6,882.64	7,623.49	8,611.49	9,873.82	10,918.70
5	7,472.94	8,130.11	8,978.07	10,108.28	11,489.33	12,626.28
6	8,505.23	9,232.41	10,168.21	11,433.54	12,903.74	14,115.84
7	9,409.12	10,207.05	11,226.54	12,600.42	14,143.75	15,430.51
8	10,243.01	11,083.99	12,154.42	13,629.21	15,242.53	16,587.94
9	11,019.16	11,881.88	13,001.76	14,549.49	16,208.08	17,608.71
10	11,750.15	12,634.42	13,783.26	15,376.53	17,078.46	18,518.44
11	12,451.75	13,347.40	14,511.96	16,143.98	17,872.64	19,335.34
12	13,126.18	14,027.61	15,206.00	16,861.44	18,604.95	20,088.86
13	13,776.58	14,683.14	15,869.25	17,538.23	19,294.90	20,791.12
14	14,406.36	15,315.68	16,505.98	18,185.46	19,949.58	21,452.49
15	15,016.42	15,927.19	17,120.85	18,806.47	20,574.67	22,083.25
16	15,607.81	16,519.76	17,715.31	19,404.45	21,175.93	22,687.72
17	16,181.59	17,094.19	18,290.79	19,982.49	21,755.83	23,269.37
18	16,738.36	17,651.32	18,848.70	20,541.89	22,316.27	23,831.17
19	17,278.73	18,191.95	19,389.78	21,083.84	22,858.99	24,374.67
20	17,803.26	18,716.63	19,914.72	21,609.38	23,384.97	24,901.09

Table 20.11. *Present values of Naples mean total rewards III*

20.5. DTSMRWP and generalized stochastic annuities (GSA)

20.5.1. *Generalized stochastic annuities (GSA)*

The semi-Markov reward process is a generalization of the Markov reward process.

In discrete-time, the generalization of the SMRWP has the property that the waiting time before a transition is a r.v.

In the discrete-time Markov case, the transitions occur at each time step (the d.f that rules the transition is geometric). They can be *real* transitions, in the case where the system that goes all over the given system changes the state, or *virtual* in the case in which after the transition it remains in the same state. However, at each period there is a transition.

In the discrete-time Markov chain case, the time evolution of a trajectory can be described by means of Figure 20.5.

The time evolution of a trajectory of a discrete-time semi-Markov process is described in Figure 20.6.

The behavior of the two trajectories is not the same. The transition time in semi-Markov processes is random. A virtual transition case is shown in Figure 20.6.

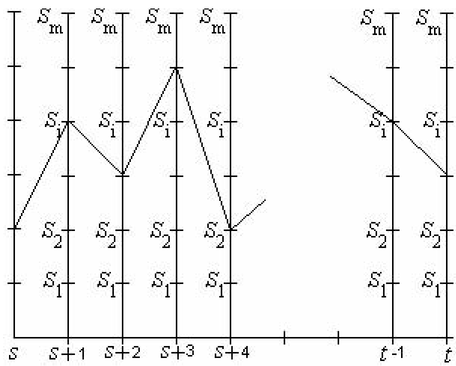

Figure 20.5. *A trajectory of discrete-time Markov reward process*

We attach a reward structure to the related process. In this light we can give the following definition.

Definition 20.3 A *generalized homogenous (non-homogenous) discrete-time stochastic annuity is a homogenous (non-homogenous) discrete-time stochastic annuity in which the following property holds:*

i) the transitions among the states follow a homogenous (non-homogenous) discrete-time semi-Markov process.

This financial concept naturally corresponds to the homogenous (non-homogenous) discrete-time semi-Markov reward process as defined in section 20.3.

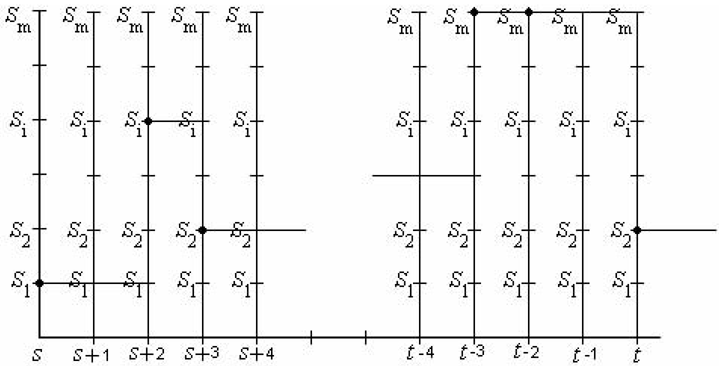

Figure 20.6. *A trajectory of DTNH semi-Markov reward process*

20.5.2. *GSA examples*

In Haberman and Pitacco (1999), Figure 20.7 is given to illustrate a trajectory of the stochastic process that describes a general insurance contract.

The depicted model has four states. It is evident that the transition time is naturally random.

A general insurance contract can be considered naturally to evolve in a semi-Markov environment. In the figures there are the states of the systems on the *y* axis, the time on the *x* axis; besides in Figure 20.7 the premiums and benefits of the insurance contract are also considered.

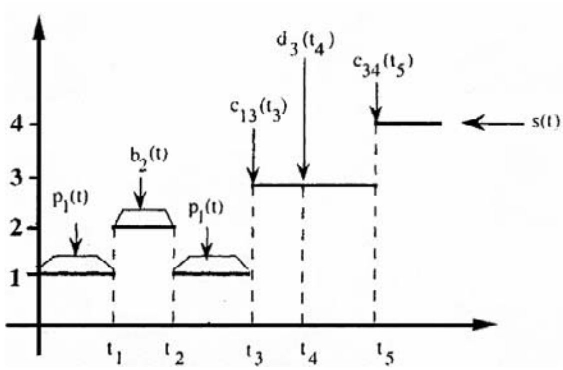

Figure 20.7. *Trajectory evolution of a general insurance contract*

Clearly, premiums and benefits can be considered as rewards. More precisely, in an HSMRWP environment we have the following:

• $p_1(t) = \psi_1(t)$ represents the premium paid by the insured. It is a permanence reward that can be constant or variable in time depending on the insurance contract;

• $b_2(t) = \psi_2(t)$ gives a benefit flow paid by the insurance company. Also in this case it is a constant or variable permanence reward;

• $d_3(t_4) = \psi_3(t)$ represents a discontinuous variable benefit flow where $\psi_3(t) = \begin{cases} k(t) & \text{if} \quad t \neq t_4 - t_3 \\ b & \text{if} \quad t = t_4 - t_3 \end{cases}$ clearly could also be $k(t) = 0 \ \ \forall t \neq t_4 - t_3$;

• $c_{13}(t_3) = \gamma_{13}(t)$ and $c_{34}(t_5) = \gamma_{34}(t)$ are transition rewards.

In this light we can say that any insurance contract can be modeled by means of SMRWP (MRWP can be seen as a particular case of SMRWP)!

In some cases, the homogenous environment is enough to model the insurance phenomenon. In other cases, the non-homogenity has to be introduced. Furthermore, in more composite cases the non-homogenous environment must be generalized to model the phenomenon (see Manca and Janssen (2007)).

In this first approach, we will consider the first examples that are reported in Haberman and Pitacco (1999). The related rewards evolution equations will be written.

The values that represent premiums and benefits have opposite algebraic signs. In these examples we will apply the discounted DTHSMRWP. Furthermore, we will suppose that the interest rate intensity δ is constant.

20.5.2.1. *Two states examples*

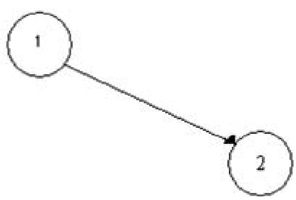

Figure 20.8. *1 = alive state; 2 = dead state*

Figure 20.8 can be used to depict three different cases of insurance:

(i) temporary assurance;

(ii) endowment assurance;

(iii) deferred annuity.

(i) In the case of death a constant sum c is assured and a premium at a constant rate p is paid at beginning of period. The policy ends at time T. So we have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad t\{0,1\ldots,T-1\} \\ 0 & \text{if} \quad t \geq T \end{cases}, \qquad \gamma_{12} = c,\ 0 < t \leq T\,.$$

In all three cases, state 2 is an absorbing state and after time T the insurance contract is extinguished. The premiums are always anticipated. Furthermore, in this case $\psi_2 = 0$. $\psi_1(t)$ can be considered as a constant permanence reward and the evolution equations will follow the system for a time T.

The evolution equation is the equation with a fixed interest rate, fixed time due permanence rewards and fixed time transition reward.

Under all these hypotheses we can write the following evolution equations:

$$\begin{aligned} \ddot{V}_1(t) &= (1 - H_1(t))\,\ddot{a}_{\overline{t}|\delta}\psi_1 + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\ddot{a}_{\overline{\vartheta}|\delta}\psi_1 \\ &\quad + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\nu^{\vartheta}\gamma_{12} \end{aligned} \qquad 1 \leq t \leq T \tag{20.68}$$

$$\ddot{V}_2(t) = 0, \quad \forall t. \tag{20.69}$$

$\ddot{V}_1(t)$ represents the present value of the temporary assurance at time 0 for a time period t (backward reserve).

(ii) In the endowment assurance, a sum c is insured in both the cases of death and of survival to maturity T. We can have the following positions:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad t = \{1,\ldots,T\} \\ 0 & \text{if} \quad t > T \end{cases}, \quad \gamma_{12} = c,\ 0 < t \leq T$$

Also, in this case $\psi_2 = 0$. $\psi_1(t)$ can be considered as a variable permanence reward and the evolution equations will follow the system for a time T.

The evolution equation is the equation with a fixed interest rate, variable time permanence rewards and fixed time transition reward.

We can write the following evolution equation:

$$\ddot{V}_1(t) = \left(1 - H_1(t)\right)\sum_{\theta=0}^{t-1}\psi_1(\theta+1)v^\theta + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\sum_{\theta=0}^{\vartheta-1}\psi_1(\theta+1)v^\theta + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)v^\vartheta\gamma_{12} \qquad 0 < t \le T$$

Also, in this case $\ddot{V}_1(t)$ represents the backward reserve at time 0 for a period t and (20.69) holds.

(iii) In the third case, the deferred annuity premiums are paid over the time period $\{1,\ldots,T_1\}$ when the insured person is in state 1. Also, the benefits are paid continuously from time T_1 until the death of the insured, and as usual the premiums are anticipated and the claim amounts unknown, we recognize the well known *phenomenon of inversion of the production cycle* in insurance.

In this case we have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T_1, \\ b & \text{if} \quad T_1 \le t < \omega - x, \end{cases} \tag{20.70}$$

where ω represents the maximum age reachable by a person and x the insured age at the formation the contract.

In this case, $\psi_2 = 0$. $\psi_1(t)$ can be considered as a variable permanence reward and the evolution equations will follow the system for a time $\omega - x$.

The evolution equation is the equation with a fixed interest rate, variable time permanence rewards and no transition rewards.

We do not present this case but we can easily write the following evolution equation:

$$\ddot{V}_1(t) = \left(1 - H_1(t)\right)\sum_{\theta=0}^{t-1}\psi_1(\theta+1)v^\theta + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\sum_{\theta=0}^{\vartheta-1}\psi_1(\theta+1)v^\theta \tag{20.71}$$

In this case, $V_1(t)$ represents the backward reserve at time t and (20.69) holds.

In these three cases, the dead state does not give any permanence reward. It allows for the end of the contract and, in the first two cases, before the natural maturity.

Another two states example, given in Haberman and Pitacco, is as follows.

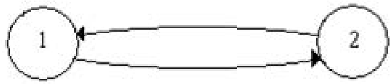

Figure 20.9. *1 = employed; 2 = unemployed*

In this case, the model can be used to study the annuity benefit in the case of unemployment. The dead state, in this two states model, is not considered because, as specified in Haberman and Pitacco (1999), the age range covered by such insurance contracts is characterized by low probabilities of death relative to the probabilities of moving from state 1 to state 2 or from state 2 to state 1, and because the financial effects of death may be small in relation to that of unemployment.

We will suppose that the premiums and benefits are fixed in time, but it is also possible to consider them variable without any difficulty. Under these hypotheses we obtain:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \quad \text{and} \quad \psi_2(t) = \begin{cases} b & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \tag{20.72}$$

where p is the premium paid by the insured, b is the benefit that he receives in the unemployment case and $T = W - x$, W is the maximum working age and x the insured age at the contractual formation. $\psi_1(t)$ and $\psi_2(t)$ could be considered as constant permanence rewards and the evolution equations will follow the system for a time T. In this case, because of the different period of payments we have a due case for premiums and an immediate case for claims.

The evolution equations will be the following:

$$\begin{aligned} \ddot{V}_1(t) = {} & \left(1 - H_1(t)\right) \ddot{a}_{\overline{t}|\delta} \psi_1 + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) \ddot{a}_{\overline{\vartheta}|\delta} \psi_1 \\ & + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) v^{\vartheta} V_2(t-\vartheta) \end{aligned} \tag{20.73}$$

$$\begin{aligned} V_2(t) = {} & \left(1 - H_2(t)\right) a_{\overline{t}|\delta} \psi_2 + \sum_{\vartheta=1}^{t} b_{21}(\vartheta) a_{\overline{\vartheta}|\delta} \psi_2 \\ & + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) v^{\vartheta} \ddot{V}_1(t-\vartheta) \end{aligned} \tag{20.74}$$

(20.73) represents the mean present value that an insured has at time t if it starts at time 0 in state 1. (20.74) has the same meaning but this time starting in state 2.

In all the cases that we have considered there are no possibility of virtual transitions, which means that $p_{ii} = 0$ and so also $Q_{ii} = 0$ and in the evolution equation only b_{12} or b_{21} is considered.

20.5.2.2. *Three states examples*

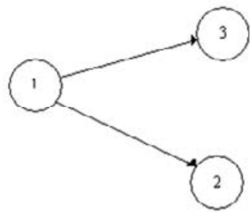

Figure 20.10. *Three state graph for the two three state examples*

Figure 20.10 can be used for the description of two cases:

(i) a temporary assurance with a rider benefit in the case of accidental death;

(ii) a lump sum benefit in the case of permanent and total disability.

In the first case, the three states will have the following meaning:

1 = alive;

2 = dead (other causes);

3 = dead (accident).

Two different causes of death are considered and the lump sums are a function of the death cause. We have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \qquad \gamma_{12} = c, \;\; \gamma_{13} = c', \;\; 0 < t \le T$$

The evolution equation is similar to (20.68).

$$\begin{aligned} \ddot{V}_1(t) = {} & \left(1 - H_1(t)\right)\ddot{a}_{\overline{t}|\delta}\psi_1 + \sum_{k=2}^{3}\sum_{\vartheta=1}^{t} b_{1k}(\vartheta)\ddot{a}_{\overline{\vartheta}|\delta}\psi_1 \\ & + \sum_{k=2}^{3}\sum_{\vartheta=1}^{t} b_{1k}(\vartheta)v^{\vartheta}\gamma_{1k} \end{aligned} \qquad 0 < t \le T$$

$$V_2(t)=0,\quad V_3(t)=0,\quad \forall t. \tag{20.75}$$

In the other example related to Figure 20.10 a lump sum will be paid in the case of a permanent and total disability. The states are:

1 = active;

2 = disabled (permanent disability);

3 = dead.

The considered rewards are:

$$\psi_1(t)=\begin{cases} p & \text{if} \quad 0<t\le T \\ 0 & \text{if} \quad t>T \end{cases} \qquad \gamma_{12}=c,\ 0<t\le T$$

The evolution equation is the following:

$$\begin{aligned}\ddot{V}_1(t) &= \left(1-H_1(t)\right)\ddot{a}_{\overline{t}|\delta}\psi_1+\sum_{k=2}^{3}\sum_{\vartheta=1}^{t} b_{1k}(\vartheta)\ddot{a}_{\overline{\vartheta}|\delta}\psi_1 \\ &\quad +\sum_{\vartheta=1}^{t} b_{12}(\vartheta)\nu^{\vartheta}\gamma_{12}\end{aligned} \tag{20.76}$$

(20.75) holds also in this case.

Remark 20.2 The time continuous version of the theory and other examples are given in Janssen and Manca (2006, 2007).

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