Elementary Differential Calculus on Discrete and Hybrid Structures

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Abstract. We set up differential calculi in the Cartesian-closed category CONV of *convergence spaces*. The central idea is to uniformly define the 3-place relation ___ **is a differential of** ___ **at** ___ for each pair of convergence spaces X, Y in the category, where the first and second arguments are elements of $\mathrm{Hom}(X, Y)$ and the third argument is an element of X, in such a way as to (1) obtain the chain rule, (2) have the relation be in agreement with standard definitions from real and complex analysis, and (3) depend only on the convergence structures native to the spaces X and Y. All topological spaces and all reflexive directed graphs (i.e. discrete structures) are included in CONV. Accordingly, ramified hybridizations of discrete and continuous spaces occur in CONV. Moreover, the convergence structure within each space local to each point, individually, can be discrete, continuous, or hybrid.

Keywords: Differential, convergence space, discrete structure, hybrid structure.

1 Introduction

With topology, continuity of functions generalizes from the contexts of classical analysis to a huge collection of structures, the topological spaces. The purpose here is to do the same for differentiability and also to allow for differentiation of such functions as, for example, functions between discrete structures (represented as reflexive directed graphs) as well as functions between discrete structures and topological spaces, particularly continua commonly occurring in elementary analysis.

Just as continuity itself neither presupposes any separation strength nor any notion of linearity, neither does differentiability. The familiar differential calculus on Euclidean spaces is of course intrinsically dependent on the vector space structure, but this is due to the choice of functions used to serve as differentials, and the consequent determination of the conditions under which functions are differentiable. What matters is the differentiability relation "differential g **is a differential of** f **at** x". Unless we demand of g that it satisfy some kind of linearity property, linearity does not intrinsically enter into the relation.

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A word about derivatives: The derivative of a function at a point is a differential. For example, the derivative of $\lambda x \,.\, x^2$ at 1 is $\lambda x \,.\, 2x$, the linear function with slope 2. The derivative of a function f on a subset of the function's domain is another function that maps each point x of the subset to the derivative of f at x. The point is that derivatives are differential-valued. In the case of $\mathbf{E}^1$, the real numbers with the standard Euclidean topology, the space of linear functions, i.e. the space of differentials, is taken with a topology making it homeomorphic to $\mathbf{E}^1$. For situations where no such homeomorphism is available, we expect the codomain of a derivative of f to be different from the codomain of f. This is evident already with 2-dimensional vector spaces over the reals.

We will set up differential calculi in the Cartesian-closed category CONV of *convergence spaces*. The central idea is to uniformly define the 3-place relation

__ is a differential of __ at __

for each pair of convergence spaces X, Y in the category, where the first and second arguments are elements of $\mathrm{Hom}(X, Y)$ and the third argument is an element of X, in such a way as to (1) obtain the chain rule, (2) have the relation be in agreement with standard definitions in real and complex analysis, and (3) depend only on the convergence structure native to the spaces X and Y.

Plan of the papers: In section 2 we define convergence spaces and the notion of continuity of functions at a point and discuss some of relevant properties of the resulting category CONV. The representation of reflexive digraphs and topological spaces as convergence structures is discussed in section 3. Section 4 presents the algebraic ideas that constitute the extraction of *linear structure* from the symmetries of a convergence space's convergence structure. Section 5 gives the definition of a differential calculus involving homogeneous spaces and the definition of the 3-place differentiability relation. Section 5 includes the statement and proof of the chain rule and identifies the differential calculi associated with CONV. Section 6 presents examples of differential calculi. Section 7 concludes the paper by extending the ideas to differential calculi that include nonhomogeneous spaces.

It is important to note that the spaces and functions of interest are naturally organized into categories and to note the nature of the containments and embeddings that are involved. In particular, any method for constructing a differential calculus for mappings between arbitrary convergence spaces gives such a method for all reflexive digraphs and all topological spaces. The results of this paper should thus be seen as constituing a tool-kit for setting up mathematical structures that import techniques from continuous mathematics into discrete contexts.

2 Convergence Spaces, CONV and Prior Work

There is a beautiful paper including a brief but powerful tutorial on convergence spaces due to R. Heckmann [2003]. We present a few of the fundamental ideas necessary for our work.

A *filter* on a set X is a collection of subsets of X closed under finite intersection and reverse inclusion. $\mathcal{F}$ is a *proper* filter if the empty set is not a member of $\mathcal{F}$. Let $\Phi(X)$ denote the set of all filters on X. For a subset A of X, $\{ B \mid A \subseteq B \subseteq X \}$ is a member of $\Phi(X)$. We denote this filter by $[A]$. In the special case where A is a singleton $\{x\}$ we denote $[A]$ by $[x]$ and call this the *point filter* at x.

Definition 1. *[1964, 2003] A* convergence structure *on X is a relation $\downarrow$ (read as "converges to") between members of $\Phi(X)$ and members of X such that for each $x \in X$: (1) $[x]$ converges to x, and (2) the set of filters converging to x is closed under reverse inclusion. A pair $(X, \downarrow)$ consisting of a set X and a convergence structure $\downarrow$ on X is called a* convergence space*.*

A function $f : X \longrightarrow Y$ where X and Y are sets, induces functions $\hat{f} : 2^X \longrightarrow 2^Y$ and $\hat{\hat{f}} : \Phi(X) \longrightarrow \Phi(Y)$. $\hat{f}$ is defined by $\hat{f}(A) = \{f(a) \mid a \in A\}$, which we call the f-*image* of A. For $\mathcal{F} \in \Phi(X)$ note that the collection of all supersets of f-images of members of $\mathcal{F}$ forms a filter which we call $\hat{\hat{f}}(\mathcal{F})$. Hereafter we overload notation and drop the $\hat{}$ and $\hat{\hat{}}$ annotations.

When convenient, we will refer to a convergence space $(X, \downarrow)$ by its carrier, X.

Definition 2. *[1964, 2003] Let $f : X \longrightarrow Y$ where X and Y are convergence spaces, and let $x_0 \in X$. f is* continuous at x_0 *iff for each $\mathcal{F} \in \Phi(X)$, if $\mathcal{F} \downarrow x_0$ in X, then $f(\mathcal{F}) \downarrow f(x_0)$ in Y. f is* continuous *iff f is continuous at every point of X.*

Continuity can be characterized in terms of filter members, which play a role analogous to the role played by neighborhoods, as supersets of open sets, in topological spaces.

Proposition 1. *Let $f : X \longrightarrow Y$ where X and Y are convergence spaces, and let x_0 be a point of X. f is continuous at x_0 iff for every filter $\mathcal{F}$ converging to x_0 in X, there is a filter $\mathcal{G}$ converging to $f(x_0)$ in Y such that $(\forall V \in \mathcal{G})(\exists U \in \mathcal{F})[f(U) \subseteq V]$.*

Definition 3. *[1964] A* homeomorphism *between two convergence spaces is a continuous bijection whose inverse is continuous.*

The objects of the category of convergence spaces CONV are the convergence spaces. For convergence spaces X and Y, $\mathrm{HOM}(X, Y)$ is the set of continuous functions from X to Y.

The category CONV includes all topological spaces but enjoys several substantial advantages over the category TOP of topological spaces. Importantly for computation, CONV contains multiple representations of all reflexive directed graphs (finite and infinite). Among digraphs, continuity is the property of being *edge-preserving*, i.e. a digraph homomorphism. But, powerfully, and unlike TOP, CONV is a Cartesian-closed category. Several immediate consequences of Cartesian-closure and the relationship between TOP and CONV are: (1) convergence spaces preserve the notion of continuity on topological spaces;

(2) convergence spaces allow fine control over continuity, and in various circumstances allow for strengthening the conditions for continuity; (3) at one's option, there is a uniform way of regarding all spaces of continuous functions as convergence spaces, but other topological structures, (for example, a structure derived from a norm) are available, and (4) function composition and application are continuous.

Over time, a number of researchers have sought to generalize differentiability to spaces where the generalization is non-obvious. Some of the more serious and sophisticated results in this direction have employed one or another restriction of the notion of convergence space, often near to pre-topological spaces, or else stayed within TOP [1946, 1968, 1966, 1966, 1945, 1966, 1974, 1983, 1963, 1940]. These explorations assumed the existence of additional structure characterizing linearity. [1983] recognized the importance of Cartesian-closure for obtaining a robust chain-rule.

3 Reflexive Digraphs and Topological Spaces as Convergence Spaces

Definition 4. *Let x be a point of a convergence space X, and let U be a subset of X. U is said to be a* neighborhood *of x iff U belongs to every filter converging to x.*

Definition 5. *[1947] A convergence space $(X, \downarrow)$ is said to be a* pretopological space *if and only if $\downarrow$ is a* pretopology*, i.e. for each $x \in X$, the collection of all neighborhoods of x converges to x.*

Proposition 2. *Let $f : X \longrightarrow Y$ where X and Y are pretopological spaces, and let $x_0 \in X$. f is continuous at x_0 iff for every neighborhood V of $f(x_0)$, there is a neighborhood U of x_0 such that $f(U) \subseteq V$.*

It is evident that every topological space is a pretopological space (cf. [1947, 1940, 1955]), and that the convergence space notion of continuity and the topological space notion of continuity coincide for topological spaces. As indicated in the introduction, the spaces and functions of interest to us are naturally organized as categories. The main categories of interest in this paper are:

CONV	the category of convergence spaces and continuous functions
PreTOP	the category of pretopological spaces and continuous functions
TOP	the category of topological spaces and continuous functions
ReRe	the category of reflexive digraphs (i.e. directed graphs with a loop at each vertex) and edge-preserving functions
PostD	the full subcategory of CONV whose objects are the postdiscrete (see below) convergence spaces

TOP is a full subcategory of **PreTOP**, which, in turn, is a full subcategory of **CONV**. Both of these full inclusions are *reflective*, via induced pretopology and induced topology operations, respectively. **ReRe** is isomorphic to **PostD**, which, in turn, embeds into **PreTOP**.[1]

The reflection functor PreT from **CONV** to **PreTOP** can be realized by letting the carrier of $\mathrm{PreT}(X)$ be the carrier of X, and and letting a filter $\mathcal{F}$ converge to a point x in $\mathrm{PreT}(X)$ iff the collection of all neighborhoods of x in X is a subcollection of $\mathcal{F}$.

Similarly, the reflection functor T from **PreTOP** to **TOP** can be realized by letting the carrier of $\mathrm{T}(X)$ be the carrier of X, and defining the topology on $\mathrm{T}(X)$ as $\{U \subseteq X \mid U$ is a neighborhood in X of each point of $U\}$.

ReRe can be embedded, in more than one way, as a full subcategory of **CONV**.

Definition 6. *A convergence space X will be said to be* postdiscrete *if and only if every convergent proper filter is a point filter.*

Proposition 3. *The postdiscrete pretopological spaces are precisely the discrete topological spaces.*

Definition 7. *Let (V, E) be a reflexive digraph. Induce a convergence structure on V by letting a proper filter $\mathcal{F}$ converge to a vertex x iff $\mathcal{F} = [y]$ for some vertex y with an edge in E from x to y.*

It is readily verified that if (V_1, E_1) and (V_2, E_2) are reflexive digraphs, then a function $f : V_1 \longrightarrow V_2$ is continuous (with respect to the induced convergence structures on V_1 and V_2) iff, for all edges (x, y) in E_1, the edge $(f(x), f(y))$ is present in E_2.

Proposition 4. *The construction in Definition 7 embeds **ReRe** as a full subcategory of **CONV**, namely the full subcategory whose objects are the postdiscrete spaces where this embedding is coreflective. The coreflection functor* $\mathrm{ReR} : \boldsymbol{CONV} \longrightarrow \boldsymbol{ReRe}$ *can be obtained letting the vertices of* $\mathrm{ReR}(X)$ *be the members of X, and letting an ordered pair (x, y) be an edge of* $\mathrm{ReR}(X)$ *iff* $[y] \downarrow x$ *in X.*

Alternatively, **ReRe** can be embedded as a full subcategory of **PreTOP**, and thence as a full subcategory of **CONV** [2001, 2003], by letting a filter $\mathcal{F}$ converge to a vertex x iff $\{\, y \mid (x, y) \in E \,\}$ is a member of $\mathcal{F}$.

In general, this embedding of **ReRe** into **PreTOP** imposes a weaker convergence structure on reflexive digraphs than the embedding in Definition 7.

Proposition 5. *The embedding of **ReRe** into **PreTOP** [2001, 2003] is the composite of the embedding in Definition 7 of **ReRe** into **CONV** with the reflection functor from **CONV** to **PreTOP**, and embeds **ReRe** as a full, coreflective subcategory of **PreTOP**, and thence as a full, coreflective subcategory of **CONV**. The coreflection functor from **PreTOP** to **ReRe** (and the coreflection functor from **CONV** to **ReRe** via **PreTOP**) can be obtained in precisely the same way as in Proposition 4.*

[1] The embedding is the restriction of the induced pretopology reflection from **CONV** to **PostD**.

The reflexive digraphs whose induced pretopologies are topological are precisely those in which the underlying binary relation is transitive as well as reflexive. [2001, 2003] Unlike **TOP** and **PreTOP**, **CONV** is a Cartesian closed category ([1971, 1975, 1990, 2001, 1965]):

Definition 8. *[1965]*
Let X and Y be convergence spaces. The function space Y^X *is the set of all continuous functions from X to Y, equipped with the convergence structure $\downarrow$ defined as follows: For each $\mathcal{H} \in \Phi(Y^X)$ and each $f_0 \in Y^X$, let $\mathcal{H} \downarrow f_0$ if, and only if, for each $x_0 \in X$ and each $\mathcal{F} \downarrow x_0$, $\{\, \{\, f(x) \mid f \in H, x \in F \,\} \mid H \in \mathcal{H}, F \in \mathcal{F} \,\}$ is a base for a filter which converges to $f(x_0)$ in Y.*

4 Translation Groups and Homogeneous Convergence Spaces

Definition 9. *An* automorphism *of a convergence space X is a homeomorphism $f : X \longrightarrow X$.*

Definition 10. *A* translation group *on a convergence space X is a group T of automorphisms of X such that, for each pair of points p and q of X, there is at most one member of T which maps p to q. In general, we will denote this unique member of T (if it exists) by $(q - p)$.*

Notation: *The group operation of a translation group T on a convergence space X will be written additively, whether or not T is Abelian. Furthermore, for all $\tau \in T$ and all $x \in X$, we will write $\tau(x)$ as $x + \tau$.*

In this notation, the requirement that the translation $(q-p)$ (if it exists) maps p to q becomes the familiar requirement that if $(q-p)$ exists, then $p+(q-p) = q$.

A full translation group *on a convergence space X is a translation group on X which contains a translation $(q - p)$ for each pair of points p and q.*

Proposition 6

i. *Every convergence space X can be embedded as a subspace of a convergence space HX which has a full translation group.*
ii. *X and HX have the same cardinality if and only if the cardinality of X is either zero or infinite.*
iii. *The embedding of X into HX is onto HX if and only if X is empty.*
iv. *If X and Y are arbitrary convergence spaces then every continuous function $f : X \longrightarrow Y$ can be be extended to a continuous function $Hf : HX \longrightarrow HY$.*
v. *If f is a homeomorphism, then so is Hf.*

An immediate consequence of (ii) in Proposition 6 is that, if X is a non-empty finite space with (or without) a full translation group, then HX cannot be homomeomorphic to X. It should also be noted that, given a particular f, the continuous extention Hf need not be unique.

Definition 11. *A convergence space X is* homogeneous *iff for each pair of points x_1 and x_2 of X, there is an automorphism of X which maps x_1 to x_2*

Observation 1. *A convergence space which has a full translation group must be homogeneous. Furthermore, a full translation group on a nonempty convergence space X must have the same cardinality as X.*

5 Differential Calculi

Definition 12. *A* differential calculus *is a category $\mathcal{D}$ in which*

i. every object of $\mathcal{D}$ is a triple $\mathcal{X} = (X, 0, T)$ such that X is a convergence space, 0 is a point of X (called the origin *of $\mathcal{X}$), and T is a full translation group on X.*
ii. every arrow in $\mathcal{D}$ from an object $(X, 0_X, T_X)$ to an object $(Y, 0_Y, T_Y)$ is a continuous function from X to Y which maps 0_X to 0_Y
iii. composition of arrows in $\mathcal{D}$ is function composition.
iv. for every object $\mathcal{X} = (X, 0_X, T_X)$, the identity function on X is an arrow in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{X}$
v. for each pair of objects $\mathcal{X} = (X, 0_X, T_X)$ and $\mathcal{Y} = (Y, 0_Y, T_Y)$, the constant function mapping every point of X to 0_Y is an arrow in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$.

In view of Proposition 6, the requirement that each object have a full translation group is not unduly restrictive. We are now in a position to define the differentiability relation. Let $a \in A \subseteq X$ and let $B \subseteq Y$, where $\mathcal{X} = (X, 0_X, T_X)$ and $\mathcal{Y} = (Y, 0_Y, T_Y)$ are objects of a differential calculus $\mathcal{D}$. Let $f : A \longrightarrow B$ be an arbitrary function.

Let $L \in \mathcal{D}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ is the set of all arrows in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$, equipped with the subspace convergence structure inherited from the function space Y^X in **CONV**.

Definition 13. *L is a* differential *of f at a iff*
for every $\mathcal{F} \downarrow a$ in A, there is some $\mathcal{H} \downarrow L$ in $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ such that

i. $\mathcal{H} \subseteq [L]$, and
ii. for every $H \in \mathcal{H}$, there is some $F \in \mathcal{F}$ such that
for every point $x \in F$, there is at least one function $t \in H$ such that

$$t(x - a) = f(x) - f(a))$$

In Definition 13, $(f(a) - 0_Y) \circ t \circ (0_X - a)$ is called an *extrapolant* of f through $(a, f(a))$ and $(x, f(x))$.

Definition 14. *A function from A to B is* differentiable *(respectively,* uniquely differentiable*) at a point a iff it has* at least *one (respectively,* precisely *one) differential at a. A function from A to B is* differentiable *(respectively,* uniquely differentiable*) iff it is differentiable (respectively, uniquely differentiable) at each point of A.*

We next obtain the chain rule. As we indicated in the introduction, the chain rule plays a central role in differential calculi. In elementary real and complex analysis, for example, the product rule follows from the chain rule after obtaining the differential of the multiplication operation.

Example 1. Expressed in terms of differentials, the product rule for real-valued functions of a real variable reduces to matrix multiplication (i.e. composition of linear functions).

$$\begin{aligned} D_x(\text{mult} \circ (f,g)) &= D_{(f,g)(x)}\text{mult} \circ D_x(f,g) \\ &= D_{(f(x),g(x))}\text{mult} \circ (D_x f, D_x g) \\ &= [g(x) f(x)] \begin{bmatrix} D_x f \\ D_x g \end{bmatrix} \\ &= g(x) D_x f + f(x) D_x g \end{aligned}$$

Returning to our more general setting, let $a \in A \subseteq X$, let $B \subseteq Y$, and let $C \subseteq Z$, where $\mathcal{X} = (X, 0_X, T_X)$, $\mathcal{Y} = (Y, 0_Y, T_Y)$, and $\mathcal{Z} = (Z, 0_Z, T_Z)$ are objects of a differential calculus $\mathcal{D}$. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be arbitrary functions. Let $K : \mathcal{X} \longrightarrow \mathcal{Y}$ and $L : \mathcal{Y} \longrightarrow \mathcal{Z}$ be arrows of $\mathcal{D}$.

Theorem 2. ***(Chain Rule)*** *Suppose that f is continuous at a. Also suppose that K is a differential of f at a, and L is a differential of g at $f(a)$. Then $L \circ K$ is a differential of $g \circ f$ at a.*

Proof: Let $\mathcal{F}$ be a filter converging to a in X. Since K is a differential of f at a, there is some $\mathcal{G} \downarrow K$ in $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{G} \subseteq [K]$ and, for every $G \in \mathcal{G}$, there is some $F_{1,G} \in \mathcal{F}$ such that for each point $x \in F_{1,G}$ there is some function $s_{G,x} \in G$ such that

$$s_{G,x}(x - a) = f(x) - f(a) \tag{1}$$

On the other hand, since f is continuous at a, we have $f(\mathcal{F}) \downarrow f(a)$ in B. Since L is a differential of g at $f(a)$, there is some filter $\mathcal{H} \downarrow L$ in $\mathcal{D}(\mathcal{Y}, \mathcal{Z})$ such that $\mathcal{H} \subseteq [L]$ and, for every $H \in \mathcal{H}$, there is some $N_H \in f(\mathcal{F})$ such that for each point $y \in N_H$, there is some function $t_{H,y} \in H$ such that

$$t_{H,y}(y - f(a)) = g(f(x)) - g(f(a)) \tag{2}$$

Consider such a set N_H. By definition, $N_H \in f(\mathcal{F})$, i.e. there is some $F_{2,H} \in \mathcal{F}$ such that

$$f(F_{2,H}) \subseteq N_H$$

By (2), for each point $x \in F_{2,H}$, we have

$$t_{H,f(x)}(f(x) + (0_Y - f(a))) = g(f(x)) + (0_Z - g(f(a))) \tag{3}$$

Next, note that $\{ \{ h_2 \circ h_1 \mid h_1 \in G, h_2 \in H \} \mid G \in \mathcal{G}, H \in \mathcal{H} \}$ is a basis for a filter $\mathcal{J}$ on $\mathcal{D}(\mathcal{X}, \mathcal{Z})$, and that $\mathcal{J} \subseteq [L \circ K]$.

By joint continuity of composition, $\mathcal{J} \downarrow L \circ L$ in $\mathcal{D}(\mathcal{X}, \mathcal{Z})$.

Let J be an arbitrary member of $\mathcal{J}$. There exist $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that

$$\{\, h_2 \circ h_1 \mid h_1 \in G, h_2 \in H \,\} \subseteq J$$

Let $F = F_{1,G} \cap F_{2,H}$. Then $F \in \mathcal{F}$. For each point $x \in F$, we have $s_{G,x} \in G$ and $t_{H,f(x)} \in H$, and therefore $t_{H,f(x)} \circ s_{G,x} \in J$. Furthermore, by (1) and (3),

$$t_{H,f(x)}(s_{G,x}(x+(0_X-a))) = t_{H,f(x)}(f(x)+(0_Y-f(a))) = g(f(x))+(0_Z-g(f(a))) \tag{4}$$

Thus, $(g(f(a))-0_Z)\circ t_{H,f(x)} \circ s_{G,x} \circ (0_X - a)$ is the required extrapolant of $g \circ f$ through $(a, g(f(a))$ and $(x, g(f(x))$.

Since the selection of x is arbitrary (once G and H have been chosen), $L \circ K$ is indeed a differential of $g \circ f$ at a.

6 Examples

Throughout, let $\mathbf{R}$ be the real line (equipped with its Euclidean topology), and let $\mathbf{N}$ be the set of all natural numbers.

Example 2. **The classical differential calculus of real variables:** The objects of this differential calculus are the spaces $\mathbf{R}^n$ ($n \in \mathbf{N}$), equipped with their respective Euclidean topologies, with their respective zero vectors as origins, and with their usual translation groups. The arrows of this calculus are the $\mathbf{R}$-linear functions. In this calculus, differentiability and unique differentiability are equivalent, and a function f has a differential at a point p iff f is differentiable (in the usual sense) at p.

Example 3. **The directional calculus of real variables:** Again, the objects are the sets $\mathbf{R}^n$ ($n \in \mathbf{N}$), with their respective zero vectors as origins, and with their usual translation groups. Again, $\mathbf{R}$ is equipped with its Euclidean topology.

However, for $n > 1$, the convergence structure imposed on $\mathbf{R}^n$ is stronger than the (Euclidean) product structure. In the directional calculus of real variables, a filter $\mathcal{F}$ will be said to converge to a point p iff there is some unit vector q such that $\{\, p + \alpha q \mid \alpha \in \mathbf{R}, |\alpha| < \epsilon \,\} \in \mathcal{F}$ for every real number $\epsilon > 0$.

The arrows of this calculus are the $\mathbf{R}$-homogeneous functions of degree one.

In this calculus, differentiabilty and unique differentiabilty are equivalent, but a function f has a differential at a point p iff f has directional derivatives in all directions at p.

Example 4. **Boolean differential calculus** (cf. Boolean derivatives [1954, 1959, 1990]): Let $\mathbf{B}$ be a complete digraph with two vertices, F and T, and equipped with the induced postdiscrete convergence structure (cf. Definition 7).

Both of the point filters on $\mathbf{B}$ converge *both* to F and to T. (This convergence structure differs from the induced pretopological structure (namely, the indiscrete structure) in that the filter $\{F, T\}$ does *not* converge in $\mathbf{B}$, but converges to both points in the indiscrete structure.)

In this calculus, the carriers of objects are the spaces $\mathbf{B}^n$ ($n \in \mathbf{N}$). Each carrier is equipped with the product convergence structure (which coincides with the postdiscrete convergence structure induced by the complete digraph on the carrier).

For each n, the bit vector $(F, F, \ldots, F)$ of length n is taken as the origin of $\mathbf{B}^n$.

The group generated by the flips $h_1, h_2, \ldots, h_n$ is taken as the translation group of $\mathbf{B}^n$, where (as one would expect) $h_k(\boldsymbol{b})$ is obtained from bit vector $\boldsymbol{b}$ by changing the k^{th} bit of $\boldsymbol{b}$ (and leaving every other bit unchanged).

For each m and n, the arrows from $\mathbf{B}^m$ to $\mathbf{B}^n$ are defined to be *all* origin-preserving functions from $\mathbf{B}^m$ to $\mathbf{B}^n$. (This is compatible with our definition of a differential calculus, since every function between complete digraphs preserves edges.)

In particular, there are precisely two arrows between $\mathbf{B}$ and itself, namely, the identity function, and the constant function which returns F.

In the Boolean differential calculus, every function between $\mathbf{B}$ and itself is uniquely differentiable.

Example 5. **Differentiating a function from 3R to $\mathbf{K}_3^-$:** $\mathbf{K}_3^-$ is the complete directed graph on 3 vertices, but with one edge from one vertex to another removed. It is universal for the all pretopological convergence spaces in the sense that every pretopological space embeds in some Cartesian power of it, (Bourdaud [1976]). The space **3R** is our designation for the set of real numbers equipped with Euclidean filter structure at each real number r, and in addition at r, all filters containing the filters generated by the open intervals whose right end point is r, and all filters containing the filters generated by the open intervals whose left end point is r. Take all functions from **3R** to $\mathbf{K}_3^-$ that are piecewise constant at 0 for differentials. Then $g : \mathbf{3R} \longrightarrow \mathbf{K}_3^-$ is a differential of $f : \mathbf{3R} \longrightarrow \mathbf{K}_3^-$ at x_0 iff f is constant on an open interval whose right end point is r and constant on an open interval whose left end point is r.

7 Differential Calculi with Nonhomogeneous Objects

In the preceding development, convergence spaces without full translation groups are "second class citizens" in the sense that they cannot be the carriers of objects of a differential calculus. A somewhat more general (and slightly more complicated) concept of "differential calculus" permits all convergence spaces to be carriers of objects.

Observation 3. *Let T be a translation group on a convergence space X. For each point x of X, let $[x]_T$ be the T-*orbit *of x, i.e. $[x]_T = \{\, x + \tau \mid \tau \in T \,\}$.*

If X is nonempty, then the set of all T orbits partitions X into homogeneous subspaces. For each T-orbit $[x]_T$, the restrictions of the members of T to $[x]_T$ form a full translation group $T_{[x]}$ on $[x]_T$.

Each $T_{[x]}$ is a quotient group of T.

Definition 15. *A* system of origins *for a convergence space X with respect to a translation group T is a set of representatives of the T-orbits, i.e., a subset $\mathcal{O}$ of X containing precisely one member of each T-orbit.*

For each point x of X, let 0_x be the unique member of $\mathcal{O}$ belonging to the same T-orbit as x.

Definition 16. *Let $f : X \longrightarrow Y$ be a function between convergence spaces. Let $\mathcal{O}_X$ ($\mathcal{O}_Y$, respectively) be a system of origins for X with respect to a translation group S (for Y with respect to a translation group T, respectively).*

i. f will be said to respect orbits *iff, for each pair of points p and q of X, if p and q lie in the same S-orbit, then $f(p)$ and $f(q)$ lie in the same T-orbit.*
ii. f will be said to be preserve origins *iff $f(\mathcal{O}_X) \subseteq \mathcal{O}_Y$.*

Definition 17. *A* generalized differential calculus *is a category $\mathcal{D}$ in which*

i. every object of $\mathcal{D}$ is a triple $\mathcal{X} = (X, T, \mathcal{O})$ such that X is a convergence space, T is a translation group on X, and $\mathcal{O}$ is a system of origins for X with respect to T.
ii. every arrow in $\mathcal{D}$ from an object $(X, S, \mathcal{O}_X)$ to an object $(Y, T, \mathcal{O}_Y)$ is a continuous, orbit-respecting, origin-preserving function from X to Y.
iii. composition of arrows in $\mathcal{D}$ is function composition.
iv. for every object $\mathcal{X} = (X, T, \mathcal{O})$, the identity function on X is an arrow in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{X}$
v. for each pair of objects $\mathcal{X} = (X, S, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, T, \mathcal{O}_Y)$ and each ζ in $\mathcal{O}_Y$, the constant function mapping every point of X to ζ is an arrow in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$

A differential calculus (in the sense of Definition 12) is essentially the same notion as a generalized differential calculus in which the translation group of every object is a full translation group.

At the opposite extreme, there are generalized differential calculii in which the translation group of every object is trivial (and hence all orbits are singletons).

Example 6. **CONV as a generalized differential calculus**
The objects of the *trivial generalized differential calculus* are *all* convergence spaces, equipped with trivial translation groups. The arrows from an object X to an object Y are *all* continuous functions from X to Y. (Since all orbits are singletons, *every* function is orbit-respecting and origin-preserving.)

Let $a \in X$ and let $b \in Y$, where $\mathcal{X} = (X, S, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, T, \mathcal{O}_Y)$ are objects of a generalized differential calculus $\mathcal{D}$. Let $f : A \longrightarrow B$ be an arbitrary function.

Let $L \in \mathcal{D}(\mathcal{X}, \mathcal{Y})$, where, again, $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ is the set of all arrows in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$, equipped with the subspace convergence structure inherited from the function space Y^X in **CONV**.

Definition 18. *L is a* differential *of f at a iff for every $\mathcal{F} \downarrow a$ in X, there is some $\mathcal{H} \downarrow L$ in $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ such that*

i. $\mathcal{H} \subseteq [\{L\}]$, and
ii. for every $H \in \mathcal{H}$, there is some $F \in \mathcal{F}$ such that for every point $x \in F$, there is at least one function $t \in H$ such that

$$t(x + (0_x - a)) = f(x) + (0_{f(x)} - f(a))$$

Differentiability and unique differentiability are defined exactly as before.

Example 7. **The classical affine differential calculus of real variables** The objects of this generalized differential calculus are the Euclidean spaces, equipped with trivial translation groups. The arrows from $\mathbf{R}^m$ to $\mathbf{R}^n$ are all affine functions from $\mathbf{R}^m$ to $\mathbf{R}^n$

As in the classical linear differential calculus, a function f has a differential at a point p iff f is differentiable (in the usual sense) at p.

Let $E_p f$ be the differential of f at p in the classical affine differential calculus of real variables. That is, $E_p(f)$ is the affine function which best approximates f in arbitrarily small neighhborhoods of p.

Then the differential of f at p in the classical linear differential calculus is the unique linear function which can be obtained from $E_p f$ by composing it on both sides with translations (in the usual sense), i.e. the function which maps each point q to $f(p) + (E_p f)(q - p)$.

Next, we obtain the chain rule for generalized differential calculi. Let $a \in X$, where $\mathcal{X} = (X, R, \mathcal{O}_X)$, $\mathcal{Y} = (Y, S, \mathcal{O}_Y)$, and $\mathcal{Z} = (Z, T, \mathcal{O}_Z)$ are objects of a generalized differential calculus $\mathcal{D}$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be arbitrary functions. Let $K : \mathcal{X} \longrightarrow \mathcal{Y}$ and $L : \mathcal{Y} \longrightarrow \mathcal{Z}$ be arrows of $\mathcal{D}$.

Theorem 4. ***(Chain Rule)***
Suppose that f is continuous at a. Also suppose that K is a differential of f at a, and L is a differential of g at $f(a)$.

Then $L \circ K$ is a differential of $g \circ f$ at a.

Proof: Similar to the proof of Theorem 2, but with the obvious modifications.

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