Hypersequent Calculus for Intuitionistic Logic with Classical Atoms

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Abstract. We introduce a hypersequent calculus for intuitionistic logic with classical atoms, i.e. intuitionistic logic augmented with a special class of propositional variables for which we postulate the decidability property. This system combines classical logical reasoning with constructive and computationally oriented intuitionistic logic in one system. Our main result is the cut-elimination theorem with the subformula property for this system. We show this by a semantic method, namely via proving the completeness theorem of the hypersequent calculus without the cut rule. The cut-elimination theorem gives a semantic completeness of the system, decidability, and some form of the disjunction property.

1 Introduction

Combining logics is a widely discussed topic in logical studies these days, and combining intuitionistic logic and classical logic is no exception. Several ways of combining the two logics have been proposed. One way is to introduce two (intuitionistic and classical) implications or negations in one system, [10], [6] and [13], and another is to introduce a two-sorted language of propositional logic with two different kinds (intuitionistic and classical) of propositional variables, and the law of excluded middle is postulated only for classical variables, [16], [18], [17] and [12].[1] Combining intuitionistic and classical logic in the second way above can be motivated by the following consideration: even in constructive mathematics some formulas are decidable, so we may need some logic that can have both decidable propositions and not necessarily decidable ones anyway. Also, our two-sorted approach has some connection to the proof theory of basic intuitionistic logic of proofs ($iBLP$) [1]. $iBLP$ has formulas of the form "$x : F$" that read "x is a proof of F," which are decidable.

In this paper, we present a Gentzen-style sequent calculus in which we can combine intuitionistic propositional logic (IPC) and classical propositional logic

[1] Strictly speaking, [16] does not discuss a two-sorted language, but it is obvious that we can expand the language of their logic to a two-sorted one to obtain a combined system for IPC and CPC. Also, there are yet other ways of combining logics. See [9],[14] and [8].

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(CPC) in the second way. It is not entirely trivial to formulate such a system especially if you want to formulate a system where cut-elimination and the full subformula property holds. We extend the framework of sequent calculus to a hypersequent calculus, which has already been used in many contexts in non-classical logics. We show cut-elimination by a semantical method and some other properties of the hypersequent system for IPC_{CA}.

2 Hypersequent Calculus

First, we give a specification of our language and fix some notational conventions. The language of IPC_{CA}, $\mathcal{L}_{IPC_{CA}}$, consists of the usual intuitionistic propositional connectives and two sets of propositional variables: intuitionistic propositional variables, $Var_I := \{p_1, \ldots, p_n, ...\}$, and classical propositional variables $Var_C := \{X_1, \ldots, X_n, \ldots\}$. The latter variables will satisfy an additional constraint that will provide their classical behavior. A formula F in $\mathcal{L}_{IPC_{CA}}$ is specified as follows.[2] $F ::= p_i | X_i | \bot | F_1 \to F_2 | F_1 \wedge F_2 | F_1 \vee F_2$.

We use the following notational convention: 1) B° is a formula containing only classical variables. 2) A,B,C,... (without any extra symbol) can be any formula.

We adopt a multi-conclusion intuitionistic sequent calculus where only $R \to$ lacks symmetry. We assume that our sequents are sets of formulas and our hypersequents are multisets of sequents. Here is the system of mLIC (multi-conclusion logical calculus for intuitionistic logic with classical atoms).

1) **Axioms**: $A \Rightarrow A$ $\qquad \bot \Rightarrow$

2) **External structural rules**:

EW $$\frac{G}{G|H}$$ **EC** $$\frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta|H}{G|\Gamma \Rightarrow \Delta|H}$$

3) **Internal structural rules**:

LW $$\frac{G|\Gamma \Rightarrow \Delta}{G|A, \Gamma \Rightarrow \Delta}$$ **RW** $$\frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta, A}$$

4)**Logical rules**

L$\wedge$ $$\frac{G|A, B, \Gamma \Rightarrow \Delta}{G|A \wedge B, \Gamma \Rightarrow \Delta}$$ **R**$\wedge$ $$\frac{G|\Gamma \Rightarrow \Delta, A \qquad G|\Gamma \Rightarrow \Delta, B}{G|\Gamma \Rightarrow \Delta, A \wedge B}$$

L$\vee$ $$\frac{G|A, \Gamma \Rightarrow \Delta \qquad G|B, \Gamma \Rightarrow \Delta}{G|A \vee B, \Gamma \Rightarrow \Delta}$$ **R**$\vee$ $$\frac{G|\Gamma \Rightarrow \Delta, A, B}{G|\Gamma \Rightarrow \Delta, A \vee B}$$

L $\to$ $$\frac{G|\Gamma \Rightarrow \Delta, A \quad G|B, \Gamma \Rightarrow \Delta}{G|A \to B, \Gamma \Rightarrow \Delta}$$ **R** $\to$ $$\frac{G|A, \Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \to B, \Delta}$$

[2] We take $\neg\varphi$ as an abbreviation of $\varphi \to \bot$.

5) **Classical Splitting** $$\frac{G|\Gamma_1, \Gamma_2^\circ \Rightarrow \Delta_1, \Delta_2^\circ|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2^\circ \Rightarrow \Delta_2^\circ|H}$$

6) **Cut** $$\frac{G_1|\Gamma_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|A, \Gamma_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2}$$

Remark 1. Hypersequents are introduced by Avron to formulate cut-free sequent calculi for various non-classical logics, in particular Gödel-Dummett logic.

2. This Splitting would give classical logic if we did not have any restriction of the language of formulas. (See [4].) Without Classical Splitting, the hypersequent calculus would be a hypersequent for IPC in $\mathcal{L}_{IPC_{CA}}$ ([5]).

3. $(X \to p) \vee (p \to X)$ is valid w.r.t. the class of Kripke models for IPC$_{CA}$ (cf.[11]). However, it seems difficult to add a rule to an ordinary cut-free sequent calculus so that the above formula can be derived in it. Due to Classical Splitting, we have a cut-free proof of the formula in mLIC.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X \Rightarrow X}{X \Rightarrow | \Rightarrow X}\text{ Classical Splitting}}{X \Rightarrow p|p \Rightarrow X}\; LW,\, RW}{\Rightarrow X \to p, p \to X| \Rightarrow X \to p, p \to X}\; R\to}{\Rightarrow (X \to p) \vee (p \to X)| \Rightarrow (X \to p) \vee (p \to X)}\; R\vee}{\Rightarrow (X \to p) \vee (p \to X)}\; EC$$

3 Kripke Models for Intuitionistic Logic with Classical Atoms

Now we state the main theorem. mLIC$^-$ means mLIC without Cut.

Theorem 1. *If mLIC* $\vdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$, *then mLIC*$^-$ $\vdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$.

We show this theorem by a semantic method, using finite tree Kripke models for IPC$_{CA}$. A Kripke model $\mathcal{K}$ for the language of IPC$_{CA}$ is defined as an ordered triple $(K, \leq, \Vdash)$, where K is a nonempty set and called a set of states, $\leq$ is a partial order of the states, and $\Vdash$ is a forcing relation. The ordered pair of the first two components $(K, \leq)$ is called a Kripke frame. A forcing relation satisfies the condition of "monotonicity" propositional variables: for any $s, t \in K$ and for any propositional variable p, if $s \leq t$ and $s \Vdash p$, then $t \Vdash p$.

Also, $\Vdash$ satisfies the standard inductive clauses for logical connectives $\to$, $\wedge$,$\vee$ for intuitionistic logic.[3] Without loss of generality, we only think about finite tree Kripke models. In addition to these, we have the following condition.

The new condition for classical atoms (Stability) : Let s_0 be the root node of a finite Kripke tree model. For each $X_i \in Var_C$, one of the following holds: $s_j \Vdash X_i$ for all $s_j \geq s_0$ or $s_j \nVdash X_i$ for all $s_j \geq s_0$

[3] We have $\perp$ in the language, and $\perp$ is never forced at any state.

Here $\mathcal{K}, s \Vdash \psi$ means that a formula ψ is forced at state s in a Kripke model $\mathcal{K}$. Also, $\mathcal{K} \Vdash \psi$ means that a formula ψ is valid in $\mathcal{K}$, which means that ψ is forced in all the states in $\mathcal{K}$. A formula ψ is "valid" if it is valid in all Kripke models. We extend our forcing relation and the notion of validity to hypersequents.

Definition 1. *1. $\mathcal{K}, s \nVdash \Gamma \Rightarrow \Delta$, if $\exists s' \in K, s' \geq s$, s.t. for any $\varphi \in \Gamma$, $\mathcal{K}, s' \Vdash \varphi$ and for any $\psi \in \Delta$, $\mathcal{K}, s' \nVdash \psi$. 2. $\mathcal{K}, s \Vdash \Gamma \Rightarrow \Delta$, otherwise.*

Definition 2. *$\mathcal{K}, s \Vdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ iff $\mathcal{K}, s \Vdash \Gamma_1 \Rightarrow \Delta_1$ or ... or $\mathcal{K}, s \Vdash \Gamma_n \Rightarrow \Delta_n$. Also, a hypersequent $G = \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ is valid if for any $\mathcal{K}$ and any $s \in K$, $\mathcal{K}, s \Vdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$.*

4 The Completeness Theorem for the Multi-conclusion Hypersequent Calculus Without the Cut-Rule

Cut-elimination is obtained as a consequence of the following theorem.

Theorem 2. *$mLIC^- \vdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ if $\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ is valid in any Kripke model of mLIC.*

To show cut-elimination, we also need to show the soundness theorem "mLIC$\vdash$ $\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ only if $\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ is valid." However, since we already proved that Hilbert-style system of IPC_{CA} is sound with respect to the relevant class of Kripke models in [11], our proof of soundness is done, provided that the provability of a hypersequent in mLIC implies that of the translated formula in IPC_{CA}.[4] Now we prove the contrapositive of the completeness theorem of hypersequent calculus mLIC^-.

4.1 Saturation Lemma

Definition 3. *A saturated hypersequent $G' = \Gamma'_1 \Rightarrow \Delta'_1 | \ldots | \Gamma'_n \Rightarrow \Delta'_n$ for $mLIC^-$ is a hypersequent satisfying the following conditions.*[5]

1. *For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \vee B \in \Gamma'_i$, then $A \in \Gamma'_i$ or $B \in \Gamma'_i$.*
2. *For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \vee B \in \Delta'_i$, then $A \in \Delta'_i$ and $B \in \Delta'_i$.*
3. *For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \wedge B \in \Gamma'_i$, then $A \in \Gamma'_i$ and $B \in \Gamma'_i$.*

[4] This fact, i.e. the translation is sound, has to be shown. The proof is given by induction on the length of proof. The crucial case is Classical Splitting. We have to show that the soundness of the translation has to be preserved under the application of the rule. Assuming that $\mathrm{IPC}_{CA} \vdash (\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2^\circ) \to (\bigvee \Delta_1 \vee \bigvee \Delta_2^\circ)$, we want to show that $\mathrm{IPC}_{CA} \vdash (\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee (\bigwedge \Gamma_2^\circ \to \bigvee \Delta_2^\circ)$. The inference here is obviously valid with respect to the semantics of IPC_{CA}, so by soundness and completeness of IPC_{CA}, the proof is essentially done and the translation is sound.

[5] In general, we call each sequent $\Gamma_i \Rightarrow \Delta_i$ of a hypersequent G a *component* of G. Here i is an index for a component in a hypersequent. $'$ means that the sequent is saturated.

4. For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \wedge B \in \Delta'_i$, then $A \in \Delta'_i$ or $B \in \Delta'_i$.
5. For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \to B \in \Gamma'_i$, then $A \in \Delta'_i$ or $B \in \Gamma'_i$.
6. For any component $\Gamma'_i \Rightarrow \Delta'_i$ of G', if $A \to B \in \Delta'_i$, then there exists a component $\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}$ of G', s.t. $A \in \Gamma'_{i\sigma}$ and $B \in \Delta'_{i\sigma}$.[6]

Also, if the conditions 1-5 are satisfied for a component, we call the component "saturated component."[7]

Definition 4. *A hypersequent $\Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_i \Rightarrow \Delta_i|\Gamma_i \cup \{A\} \Rightarrow B|\ldots|\Gamma_n \Rightarrow \Delta_n$ is an associated hypersequent of a hypersequent $\Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_i \Rightarrow \Delta_i|\ldots|\Gamma_n \Rightarrow \Delta_n$ with respect to $\Gamma_i \Rightarrow \Delta_i$, s.t. $A \to B \in \Delta_i$. Also, we call this $\Gamma_i \cup \{A\} \Rightarrow B$ an associated component of $\Gamma_i \Rightarrow \Delta_i$, s.t. $A \to B \in \Delta_i$.*

Lemma 1 (Saturation Lemma)
For any hypersequent $G = \Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_n \Rightarrow \Delta_n$, s.t. $mLIC^- \nvdash \Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_n \Rightarrow \Delta_n$, there exists a saturated hypersequent[8] *$G' = \Gamma'_1 \Rightarrow \Delta'_1|\ldots|\Gamma'_{n\sigma_n} \Rightarrow \Delta'_{n\sigma_n}$ satisfying the following conditions.*

(α) For each component $\Gamma_i \Rightarrow \Delta_i$ and its associated components $\Gamma_{i\sigma} \Rightarrow \Delta_{i\sigma}$,[9]
(α_1) $\Gamma_{i\sigma} \subseteq \Gamma'_{i\sigma}$; ($\alpha_2$) $\Delta_{i\sigma} \subseteq \Delta'_{i\sigma}$; ($\alpha_3$) $\Gamma'_{i\sigma} \cap \Delta'_{i\sigma} = \emptyset$ and $\perp \notin \Gamma'_{i\sigma}$.
(β) Let P and N be the following.
$P := \bigcup\{\Gamma'_{i\sigma}|(\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma})$ is a component of saturated hypersequent $G'\}$
$N := \bigcup\{\Delta'_{i\sigma}|(\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma})$ is a component of saturated hypersequent $G'\}$.
Then $\{X \in Var_C|X \in P\} \cap \{X \in Var_C|X \in N\} = \emptyset$.

Proof. The proof is done by describing the saturation procedure.

1) We start from the leftmost component $\Gamma_1 \Rightarrow \Delta_1$ of the original hypersequent $G = \Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_n \Rightarrow \Delta_n$, and we continue to construct saturated components and associated components for "$\to$" formulas on the succedent of components until we develop all the saturated components. The following rules give us how to construct a saturated *component* of a saturated hypersequent. For $\Gamma_i \Rightarrow \Delta_i$ in G, go through all of the following steps.

1. If $A \vee B \in \Gamma_i$, then put A into Γ_i or put B into Γ_i.
2. If $A \vee B \in \Delta_i$ then put A into Δ_i and put B into Δ_i.
3. If $A \wedge B \in \Gamma_i$, then put A into Γ_i and put B into Γ_i.
4. If $A \wedge B \in \Delta_i$, then put A into Δ_i or put B into Δ_i.

[6] σ stands for a finite sequence of numbers. When we have an implication on the succedent, we will have a new sequent and we name it by an appropriate number. So, we have some sequence of number. The particular construction of a new component will be given below. σ's are convenient labels. The construction does not essentially depends on the labels.
[7] We call this "saturated component" regardless of whether it has $A \to B$ in Δ'_i or not.
[8] Here σ_n stands for a sequence for n-th component.
[9] We identify $\Gamma_i \Rightarrow \Delta_i$ with $\Gamma_{i0} \Rightarrow \Delta_{i0}$, and once saturated, it becomes $\Gamma'_{i0} \Rightarrow \Delta'_{i0}$. An index of an associated component starts with 1. Also, the notion of "associatedness" is transitive.

5. If $A \to B \in \Gamma_i$, then put A into Δ_i or put B into Γ_i.
6. If $\Gamma_i \cap \Delta_i \neq \emptyset$ or $\perp \in \Gamma_i$, then backtrack.

2) We go through the procedure until we can no longer parse any formula in the set,[10] except for implicational formulas on the succedent. If we obtain a sequent that satisfies all the conditions, then we can terminate.[11] If we have the case 6, then we backtrack. For $\Gamma_1 \Rightarrow \Delta_1$, we may terminate with "success," i.e. terminate by constructing a saturated component without any application of 1-6 anymore, or terminate with "failure," i.e. terminate with no possibility of backtracking any more and have only the cases where $\Gamma_1' \cap \Delta_1' \neq \emptyset$ or $\perp \in \Gamma_1'$. If $\Gamma_1' \Rightarrow \Delta_1'$ does not have $A \to B$ in Δ_1', then we are done with $\Gamma_1' \Rightarrow \Delta_1'$.

3) If there is a saturated component $\Gamma_1' \Rightarrow \Delta_1'$ for $\Gamma_1 \Rightarrow \Delta_1$ satisfying all the conditions listed above and that $A_1 \to B_1, \dots, A_k \to B_k \in \Delta_1'$, then we construct k-many associated components for $\Gamma_1 \Rightarrow \Delta_1$. Let $\Gamma_{1j} \Rightarrow \Delta_{1j}$ be as $\Gamma_{1j} = \Gamma \cup \{A_j\}$ and $\Delta_{1j} = \{B_j\}$ $(1 \leq j \leq k)$. So, our new hypersequent looks like $\Gamma_1' \Rightarrow \Delta_1' | \Gamma_{11} \Rightarrow \Delta_{11} | \dots | \Gamma_{1k} \Rightarrow \Delta_{1k} | \dots | \Gamma_n \Rightarrow \Delta_n$.

4) By going through from 1) to 3) for $\Gamma_{11} \Rightarrow \Delta_{11}$, we get $\Gamma_{11}' \Rightarrow \Delta_{11}'$. If there is any $A \to B \in \Delta_{11}'$, then we have to construct an associated component $\Gamma_{111} \Rightarrow \Delta_{111}$ (possibly $\Gamma_{112} \Rightarrow \Delta_{112}, \dots, \Gamma_{11l} \Rightarrow \Delta_{11l}$) for $\Gamma_{11} \Rightarrow \Delta_{11}$. We also saturate $\Gamma_{111} \Rightarrow \Delta_{111}, \dots, \Gamma_{11l} \Rightarrow \Delta_{11l}$.[12] So, we get $\Gamma_1' \Rightarrow \Delta_1' | \Gamma_{11}' \Rightarrow \Delta_{11}' | \Gamma_{111}' \Rightarrow \Delta_{111}' | \dots | \Gamma_{1k}' \Rightarrow \Delta_{1k}' | \Gamma_{1k1}' \Rightarrow \Delta_{1k1}' | \dots | \Gamma_n \Rightarrow \Delta_n$.

5) We go through all the steps in 1)-4) until we finish saturating all the associated components for $\Gamma_n \Rightarrow \Delta_n$. Since the number of $A \to B$ formulas on the succedents is finite, our procedure terminates in a finite number of steps.[13]

6) After we systematically construct all the saturated components (including all associated ones), we take the union of $\Gamma_{i\sigma}'$ and the union of $\Delta_{i\sigma}'$ (P and N) and check whether the condition of disjointness of the classical variables (β) in P and N. If (β) is violated, then we backtrack. We have the two possible cases of termination: (1) We terminate with "success," i.e. by constructing a saturated component without violating (α_3) or (β); (2) We terminate with "failure," i.e. violating (α_3) or (β) with no case of backtracking.

If we have a successful case, then the other conditions of (α) are indeed satisfied with respect to any component $\Gamma_{i\sigma}' \Rightarrow \Delta_{i\sigma}'$. In all the steps except the case $A \to B \in \Delta_{i\tau}'$ we simply add some new formulas. So, once $\Gamma_{i\sigma} \Rightarrow \Delta_{i\sigma}$ is constructed, we only add formulas on $\Gamma_{i\sigma}$ and $\Delta_{i\sigma}$. Hence, the condition (α_1) and (α_2) of the lemma are satisfied for all the successful cases of components.

On the other hand, the case (2) is impossible.

[10] After a formula is used once, it becomes unavailable in one component.

[11] The procedure for the component $\Gamma_i \Rightarrow \Delta_i$ will terminate because we only go through the subformulas of the component, and the complexity of them strictly goes down at each step.

[12] For any implication formula on any succedent, we unfold all associated components.

[13] We may have a repetition of having the same formula every time we construct a new associated sequent. But then we can stop whenever we get into the repetition of the same step.

Claim. If a component (or any pair of components) in a candidate of a saturated hypersequent G' violate(s) the condition (α_3) or the condition (β) of the disjointness of classical variables or both without any possibility of backtracking, then we can construct a cut-free proof of the hypersequent G.

Proof. Essentially by tracing the saturation procedure backwards (from the rightmost component to the leftmost one), we first construct a cut-free derivation of the original hypersequent G from all the cases of saturation in which the condition (α_3) or (β) is violated. Assume that we have violations of (α_3) or (β) with implications on the succedent developed and we have no case of backtracking.

We arrange all those possible alternatives of saturated hypersequents which violate the condition (α_3) or (β)[14] so that we can construct a derivation tree of G whose leaves are saturated hypersequents. We first construct a tree labeled by hypersequents (at this point, not necessarily a derivation tree yet) by the rules: 1) The root is the original hypersequent G; 2)-1. if a saturation step is deterministic, then put the resulting G_2 above G_1; 2)-2. if a saturation step is non-deterministic, then put the two alternatives G_2 and G_3 above the previous one G_1; 2)-3. if a saturation step is $A \to B \in \Delta'_{i\sigma}$,[15] then put the hypersequents as follows.

$$\frac{\Gamma'_1 \Rightarrow \Delta'_1 | \dots | \Gamma'_i \Rightarrow \Delta'_i | \Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma} | \Gamma'_{i\sigma 1}, A \Rightarrow B | \dots | \Gamma_n \Rightarrow \Delta_n}{\Gamma'_1 \Rightarrow \Delta'_1 | \dots | \Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma} | \dots | \Gamma_n \Rightarrow \Delta_n.}$$

3) Leaf nodes are saturated hypersequents with (α_3) or (β) violated.

Subclaim 1: With some minor modifications, our finite tree of saturated hypersequents becomes a derivation of the original hypersequent G from all the alternative saturated hypersequents with violation of either (α_3) or (β).

Proof. About 1) and 3): first, the construction traces saturation of G backwards, so we end with G. By assumption, all the topmost saturated hypersequents satisfy the following conditions: in case of (α_3), all such saturated hypersequents have a component $\Gamma'_{i\sigma}, A \Rightarrow \Delta'_{i\sigma}, A$ or $\Gamma'_{i\sigma}, \bot \Rightarrow \Delta'_{i\sigma}$; in case of (β) (possibly with (α_3), all such saturated hypersequents have at least one pair of components $\Gamma'_{i\sigma}, X \Rightarrow \Delta'_{i\sigma}$ and $\Gamma'_{j\tau} \Rightarrow \Delta'_{j\tau}, X$ (or a component $\Gamma'_{i\sigma}, A \Rightarrow \Delta'_{i\sigma}, A$ or $\Gamma'_{i\sigma}, \bot \Rightarrow \Delta'_{i\sigma}$).

About 2): (Outline) We show inductively (on the number of applications of logical rules) that the constructed tree with some modifications is the desired derivation. For 2)-1,2, by IH, we have a derivation up to G_2 (and G_3) from the failed cases of saturated hypersequents. We can take this as a derivation

[14] Even in a case with violation of (β), in some alternatives, there may be a violation of (α_3).

[15] If the succedent has more than one implication formula, then we have to deal with all of them by putting one new line of a hypersequent whenever we apply a case of $\to$ in $\Delta'_{i\sigma}$.

of the lower hypersequent G_1 from G_2 (and G_3), where a principal formula is obtained as a result of applying the rule. So, we have a desired derivation of G_1 from the failed saturated hypersequents. For 2)-3: ($A \rightarrow B \in \Delta'_{i\sigma}$) In this case, we need a slight modification. In our saturation, we put some new sequents adjacent to the component that has $\rightarrow$ on the succedent. To accommodate this, we insert one intermediate line that has another copy of $\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}$ to the tree. So,

$$\cfrac{\cfrac{\Gamma'_1 \Rightarrow \Delta'_1|\dots|\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}|\Gamma'_{i\sigma 1}, A \Rightarrow B|\dots|\Gamma_n \Rightarrow \Delta_n}{\Gamma'_1 \Rightarrow \Delta'_1|\dots|\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}|\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}|\dots|\Gamma_n \Rightarrow \Delta_n}\ R\rightarrow}{\Gamma'_1 \Rightarrow \Delta'_1|\dots|\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}|\dots|\Gamma_n \Rightarrow \Delta_n}\ \mathrm{EC}$$

Here $\Gamma'_{i\sigma 1} = \Gamma'_{i\sigma}$, and by IH, we have a derivation up to the top line from the failed saturated hypersequents. Note that the first step is just $R \rightarrow$ and the second is EC. So, we have a derivation of the bottom line. ⊠ (subclaim 1)

Subclaim 2: Any hypersequent of the form $\Gamma_1 \Rightarrow \Delta_1|\dots|A, \Gamma_i \Rightarrow \Delta_i, A|\dots|\Gamma_n \Rightarrow \Delta_n$ or $\Gamma_1 \Rightarrow \Delta_1|\dots|\bot, \Gamma_i \Rightarrow \Delta_i, |\dots|\Gamma_n \Rightarrow \Delta_n$ is provable in mLIC$^-$.
Subclaim 3: Any hypersequent of the form $\Gamma_1 \Rightarrow \Delta_1|\dots|X, \Gamma_i \Rightarrow \Delta_i|\dots|\Gamma_j \Rightarrow \Delta_j, X|\Gamma_n \Rightarrow \Delta_n$ is provable in mLIC$^-$.

Proof. (For 2) The entire hypersequent can be taken as the result of applying LW, RW and EW to an axiom. (For 3) We can have the following proof.

$$\cfrac{\cfrac{\cfrac{X \Rightarrow X}{X \Rightarrow | \Rightarrow X}\ \text{Classical Splitting}}{X, \Gamma_i \Rightarrow \Delta_i|\Gamma_j \Rightarrow \Delta_j, X}\ \text{several LW or RW}}{\Gamma_1 \Rightarrow \Delta_1|\dots|X, \Gamma_i \Rightarrow \Delta_i|\dots|\Gamma_j \Rightarrow \Delta_j, X|\dots|\Gamma_n \Rightarrow \Delta_n}\ \text{several EW}$$

The failed saturated hypersequents have the form of the hypersequents in the above subclaims. So, by putting proofs in the subclaim 2, 3 on top of our cut-free derivation from saturated hypersequents, we can construct a cut-free *proof* of the original hypersequent G based on all the cases of violation of the condition (α_3) or (β) (possibly with those of (α_3), respectively. This completes transforming the cases of failure into a cut-free proof in mLIC$^-$. ⊠ (claim)

By the claim, the existence of a failed case (2) would be contradictory to the assumption of unprovability of the original hypersequent. So, the existence of a saturated hypersequent satisfying the conditions has been proven. ⊠(Lemma)

4.2 Constructing a Kripke Countermodel

Let G' be a saturated hypersequent satisfying all the conditions in the lemma. First, list up the classical variables $X_1, \dots, X_p$, "safe variables," that are in the set $P = \bigcup\{\Gamma_{i\sigma}|\Gamma_{i\sigma} \Rightarrow \Delta_{i\sigma}$ is a component of the saturated hypersequent $G'\}$. We consider the hypersequent all of whose components are of the form $\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}$ such that 1. $\Gamma^{+\prime}_{i\sigma} = \Gamma'_{i\sigma} \cup \{X_1, \dots X_p\}$ and 2. $\Delta'_{i\sigma}$ is as before. We call this hypersequent a "modified saturated hypersequent" G^+.

Proposition 1. *Suppose mLIC⁻ $\nvdash \Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_n \Rightarrow \Delta_n$ $(= G)$, and let G' be a saturated hypersequent satisfying the conditions of Saturation Lemma with "safe variables" $X_1, \ldots, X_p$. Then, there is a modified saturated hypersequent G^+, s.t. for each saturated component $\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}$ of G^+, the following are satisfied:*

(α) $\Gamma^{+\prime}_{i\sigma} \cap \Delta'_{i\sigma} = \emptyset$ *and* $\bot \notin \Gamma^{+\prime}_{i\sigma}$.
(β) *Let P^+ and N be the following.*
- $P^+ := \bigcup\{\Gamma^{+\prime}_{i\sigma}|(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma})$ *is a component of* $G^+\}$.
- $N := \bigcup\{\Delta'_{i\sigma}|(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma})$ *is a component of* $G^+\}$.

Then $\{X \in Var_C | X \in P^+\} \cap \{X \in Var_C | X \in N\} = \emptyset$.

Proof. Given a G' and $X_1, \ldots, X_p$, add X_i's to the antecedent of each $\Gamma'_{i\sigma} \Rightarrow \Delta'_{i\sigma}$ in G'. We check that the resulting hypersequent satisfies the conditions of G^+ in the proposition. The hypersequent is still *saturated* even after X_i's are added since these are all variables and inactive in the saturation procedure. The condition (α) is satisfied because, by definition, safe variables X_i's are the classical variables such that $X_i \notin N$. So, adding X_i to the antecedent keeps the antecedent of each saturated component in G' disjoint with its succedent. Similarly, for the condition (β), there are no X_i's such that $X_i \notin N$. Also, it is obvious that this is a modified saturated hypersequent in the sense defined above. ⊠

Based on this modified saturated hypersequent G^+, we construct a Kripke countermodel for G.[16] Except for classical variables, a model to be constructed from a (modified) saturated hypersequent must have almost the same structure as a Kripke model for intuitionistic logic. However, to construct a Kripke model from one saturated hypersequent, we first construct Kripke models for components, and we glue those Kripke models to construct a Kripke model for G.

Suppose we are given a modified saturated sequent $G^+ = \Gamma^{+\prime}_{10} \Rightarrow \Delta'_{10}|\Gamma^{+\prime}_{11} \Rightarrow \Delta'_{11}|\ldots|\Gamma^{+\prime}_{i0} \Rightarrow \Delta'_{i0}|\ldots|\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}|\ldots|\Gamma^{+\prime}_{n0} \Rightarrow \Delta'_{n0}|\ldots|\Gamma^{+\prime}_{n\tau} \Rightarrow \Delta'_{n\tau}$. We construct Kripke models that falsify the original components $\Gamma_i \Rightarrow \Delta_i$ $(1 \leq i \leq n)$.

Let a Kripke model $\mathcal{K}_i$ be the following triple $(S^+_i, \leq_i, \Vdash_i)$ based on G^+:

1. $S^+_i = \{\Gamma^{+\prime}_{i0} \Rightarrow \Delta'_{i0}, \Gamma^{+\prime}_{i1} \Rightarrow \Delta'_{i1}, \ldots, \Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}\}$ (finite).
2. $\leq_i$ is defined as: $(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}) \leq_i (\Gamma^{+\prime}_{i\tau} \Rightarrow \Delta'_{i\tau})$ iff $\Gamma^{+\prime}_{i\sigma} \subseteq \Gamma^{+\prime}_{i\tau}$
3. $\Vdash_i$ is defined as follows[17]: for any $p, X \in \bigcup_{i\leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$,
1) $(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}) \Vdash_i p$ iff $p \in \Gamma^{+\prime}_{i\sigma}$; 2) $(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}) \Vdash_i X$ iff $X \in \Gamma^{+\prime}_{i\sigma}$.

For any $p, X \notin \bigcup_{i\leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$, $(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}) \nVdash_i p$ and $(\Gamma^{+\prime}_{i\sigma} \Rightarrow \Delta'_{i\sigma}) \nVdash_i X$. By this definition, we can immediately obtain some desirable properties of a model of our logic. First, $\geq_i$ is a partial order, since this is an inclusion. Secondly, for any intuitionistic variable $p \in \bigcup_{i\leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$, monotonicity clearly holds. Thirdly, about classical variables, the following hold.

[16] In the following, we say "model" unless we emphasize that one is a countermodel.
[17] $Sb(\Gamma)$ stands for the set of subformulas contained in the set of formulas Γ.

Claim. For any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $\{X | X \in \Gamma_{i0}^{+\prime}\} = \{X | X \in \Gamma_{i\sigma}^{+\prime}\}$.

Proposition 2. *For any $X \in Var_C$, the stability holds, i.e.,*
For any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \Vdash_i X$ or
for any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \nVdash_i X$.

Proof. If $X \notin \bigcup_{i \leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$, then by definition, for any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma})$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \nVdash_i X$. The statement easily follows. If $X \in \bigcup_{i \leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$, then suppose that for $X \in \bigcup_{i \leq n}(Sb(\Gamma_i) \cup Sb(\Delta_i))$ of G, $\exists(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \nVdash_i X$ and $\exists(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \Vdash_i X$. For particular ρ and τ, $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$ and $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \nVdash_i X$, and $(\Gamma_{i\rho}^{+\prime} \Rightarrow \Delta'_{i\rho}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}^{+\prime})$ and $(\Gamma_{i\rho}^{+\prime} \Rightarrow \Delta'_{i\rho}) \Vdash_i X$. So, by definition, $X \notin \Gamma_{i\tau}^{+\prime}$ and $X \in \Gamma_{i\rho}^{+\prime}$. However, by the claim, $X \in \Gamma_{i0}^{+\prime}$ iff $X \in \Gamma_{i\tau}^{+\prime}$ and $X \in \Gamma_{i0}^{+\prime}$ iff $X \in \Gamma_{i\rho}^{+\prime}$. Then, $X \in \Gamma_{i0}^{+\prime}$ iff $X \notin \Gamma_{i0}^{+\prime}$. Contradiction. ⊠

Proposition 3. *Let $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma})$ be a component of G^+ in S_i^+ and $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$. For any $X \in Sb(\Gamma_i^+) \cup Sb(\Delta_i)$,*[18]

1. $X \in \Gamma_{i\sigma}^{+\prime} \Longrightarrow \forall(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \Vdash_i X$ and
2. $X \in \Delta'_{i\sigma} \Longrightarrow \forall(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \nVdash_i X$

Proposition 4. *For any $\psi \in Sb(\Gamma_i^+) \cup Sb(\Delta_i)$, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \leq_i (\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau})$ and $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \Vdash_i \psi \Longrightarrow (\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \Vdash_i \psi$.*

Proof. By induction on the complexity of formulas. ⊠

For purely classical formulas, we have another statement to show.

Proposition 5. *For any formula $\psi^\circ \in Sb(\Gamma_i^+) \cup Sb(\Delta_i)$ and for any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma'_{i0} \Rightarrow \Delta'_{i0})$, 1. $\psi^\circ \in \Gamma_{i\sigma}^{+\prime} \Longrightarrow \forall(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \Vdash_i \psi^\circ$; 2. $\psi^\circ \in \Delta'_{i\sigma} \Longrightarrow \forall(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$, $(\Gamma_{i\tau}^{+\prime} \Rightarrow \Delta'_{i\tau}) \nVdash_i \psi^\circ$.*

Proof. By induction of the complexity of ψ°.

Lemma 2 (Semantic Lemma).
Let G^+ be a modified saturated hypersequent, and let $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$ be a component of G^+ in S_i^+. For any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq_i (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0})$ and for any formula $\varphi \in Sb(\Gamma_i^+) \cup Sb(\Delta_i)$,
1. $\varphi \in \Gamma_{i\sigma}^{+\prime} \Longrightarrow (\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \Vdash_i \varphi$; 2. $\varphi \in \Delta'_{i\sigma} \Longrightarrow (\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \nVdash_i \varphi$.

Proof. By induction on the complexity of formulas.
Case 1.1) $\varphi = p$. For 1. By definition. For 2, suppose $p \in \Delta'_{i\sigma}$. By the condition 3. of Saturation Lemma , $p \notin \Gamma_{i\sigma}^{+\prime}$. So, by definition, $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \nVdash_i p$.
Case 1.2) $\varphi = X$. This is a corollary of proposition 12.

[18] Γ_i^+ is the antecedent of the original component with safe variables added. We assume we have fixed one saturated hypersequent G^+ to construct a Kripke model.

Case 2) $\varphi = A * B$ ($* = \to, \wedge, \vee$). We have four subcases of combinations of a mixed formula and a classical formula (1) $A = C^\circ$ and $B = D^\circ$, (2) A is mixed and B is mixed, (3) $A = C^\circ$ and B is mixed, and (4) A is mixed and $B = D^\circ$. However, classical formulas are special cases of mixed for intuitionistic formulas, so if the lemma holds for general cases, it obviously holds for classical formulas. It suffices to prove the lemma for generic formulas without specifying whether these are classical or not. The proof is similar to the case of IPC. $\boxtimes$.

4.3 Proof of Completeness and Cut-Elimination

Assuming mLIC$^- \nvdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$, we first construct a Kripke countermodel for a component $\Gamma_i \Rightarrow \Delta_i$. Out of a modified saturated hypersequent G^+, we have constructed Kripke models $\mathcal{K}_i$ ($1 \leq i \leq n$). For each of these, by construction, we have a (root) saturated component ($\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0}$) in S_i^+, s.t. $\{X_1, ..., X_p\} \cup \Gamma_i \subseteq \Gamma_{i0}^{+\prime}$ and $\Delta_i \subseteq \Delta'_{i0}$. By Semantic Lemma, definition $\Vdash_i$ for sequent and reflexivity of $\leq_i$, $\mathcal{K}_i, (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0}) \nVdash_i \Gamma_i \Rightarrow \Delta_i$ ($1 \leq i \leq n$).

Next, we show the completeness theorem of the hypersequent calculus itself. Assume mLIC$^- \nvdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$. We want to show that there is a Kripke model $\mathcal{K} = (K, \leq, \Vdash)$ and there is a state $s_r \in K$ s.t. $\bigwedge_{1 \leq i \leq n} (\mathcal{K}, s_r \nVdash \Gamma_i \Rightarrow \Delta_i)$. We construct the desired Kripke model by "gluing" constructed Kripke models.

For all the safe variables $X_1, ..., X_p$ in G^+, $X_1, \ldots, X_p \in \Gamma_{i0}^{+\prime}$ ($1 \leq i \leq n$). We already have $\mathcal{K}_i = (S_i^+, \geq_i, \Vdash_i)$ s.t. $\mathcal{K}_i, (\Gamma_{i0}^{+\prime} \Rightarrow \Delta'_{i0}) \nVdash_i \Gamma_i \Rightarrow \Delta_i$ ($1 \leq i \leq n$). Then, we first take the disjoint union[19] of the models $\mathcal{K}_i$ ($1 \leq i \leq n$) and add a new node below the roots of the models. Let $s_r = (\Gamma'_r \Rightarrow \Delta'_r)$, $\Gamma'_r = \{X_1, ..., X_p\}$ and $\Delta'_r = \emptyset$. s_r is the new root node. Let $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma})$ be a component of G^+.

Now let $\mathcal{K} = (S^+, \leq, \Vdash)$, where 1. $S^+ = \{s_r\} \cup \biguplus_{1 \leq i \leq n} S_i^+$; 2. $\leq = \{(s_r, s) \in \{s_r\} \times S^+ | \Gamma'_r \subseteq \Gamma_{i\sigma}^{+\prime}$ or $\Gamma'_r \subseteq \Gamma'_r\} \cup \biguplus_{1 \leq i \leq n} (\leq_i)$, where either $s = (\Gamma'_r \Rightarrow \Delta'_r)$ or $s = (\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma})$ s.t. $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \in \biguplus_{1 \leq i \leq n} S_i^+$; 3. $\Vdash = \{(s_r, X_l) | X_l \in \Gamma'_r\} \cup \biguplus_{1 \leq i \leq n} (\Vdash_i)$. Such a model $\mathcal{K}$ must exist, since the only case where the glued model does not exist is the case where we have a conflict among classical variables, but by construction we never have such a case here.

Since the partial order in the new model is the inclusion on the antecedents of the saturated sequents, monotonicity obviously holds for intuitionistic atoms. The new condition for classical variables must be a consequence of the claim: for any $(\Gamma_{i\sigma}^{+\prime} \Rightarrow \Delta'_{i\sigma}) \geq (\Gamma'_r \Rightarrow \Delta'_r)$ in S^+, $\{X | X \in \Gamma'_r\} = \{X | X \in \Gamma_{i\sigma}^{+\prime}\}$. The proof is essentially the same as that given to the nodes of $\mathcal{K}_i$.

Also, the following are the immediate consequences of the definition.

1. $s_r \Vdash X_l$ iff $X_l \in \Gamma'_r$ iff $\forall s \in S^+$, $s \Vdash X_l$.
2. $\forall s \in S_i^+$, $s \Vdash \varphi \iff s \Vdash_i \varphi$ for any formula φ,.

Claim. If mLIC$^- \nvdash \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$, then $\bigwedge_{1 \leq i \leq n} (\mathcal{K}, s_r \nVdash \Gamma_i \Rightarrow \Delta_i)$.

[19] $\biguplus$ stands for the disjoint union operator.

Proof. In $\mathcal{K}_i$, at $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}')$, each component is falsified, respectively. So, for each i $(1 \leq i \leq n)$, at the state $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}') \in S^+$, the following hold: 1) $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}') \geq (\Gamma_r' \Rightarrow \Delta_r')$; 2) $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}') \Vdash \varphi$ for all $\varphi \in \Gamma_i$; 3) $(\Gamma_{i0}^{+\prime} \Rightarrow \Delta_{i0}') \nVdash \psi$ for all $\psi \in \Delta_i$. So, by definition, $\bigwedge_{1\leq i\leq n}(\mathcal{K}, s_r \nVdash \Gamma_i \Rightarrow \Delta_i)$ ⊠(claim)

Hence, mLIC$^-$ $\nvdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$ implies $\bigwedge_{1\leq i\leq n}(\mathcal{K}, s_r \nVdash \Gamma_i \Rightarrow \Delta_i)$. So, if mLIC$^-$ $\nvdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$, then $\mathcal{K}, s_r \nVdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$. So, if $\Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$ is valid, then mLIC$^-$ $\vdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$.

We have shown completeness of mLIC$^-$. On the other hand, the soundness theorem for mLIC itself holds with respect to the same class of Kripke models, as already discussed. Then, if there is a Kripke model $\mathcal{K}$ and $s \in K$, s.t. $\mathcal{K}, s \nVdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$, then mLIC$\nvdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$.

So, if mLIC$\vdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$, then mLIC$^-$ $\vdash \Gamma_1 \Rightarrow \Delta_1|\dots|\Gamma_n \Rightarrow \Delta_n$. This gives a semantic proof of the cut-elimination theorem for mLIC.

As a corollary, the *subformula property* holds for mLIC, since Cut is the only rule that spoils the subformula property in mLIC. Due to this corollary, IPC$_{CA}$ is a *conservative extension* of its intuitionistic and classical fragments, respectively, since for any purely intuitionistic (or classical) formula provable in mLIC, there is a proof using only subformulas of the formula. The model we constructed is finite, so we have *decidability* of mLIC.

We have another corollary of completeness. A sequent system S has the *disjunction property* (DP) if $S \vdash\Rightarrow A \vee B \Longrightarrow S \vdash\Rightarrow A$ or $S \vdash\Rightarrow B$. We have a refinement of DP. To state the proposition, we need a definition of a positive and negative occurrence of a formula in a sequent. We put $+$ and $-$ symbol in front of a formula to state that the given formula has a positive occurrence and a negative occurrence. "$+\varphi$" means φ has a positive occurrence and "$-\varphi$" means φ has a negative occurrence. The occurrences of a subformula of a given formula is determined inductively as follows. For a sequent $\Gamma \Rightarrow \Delta$, 1) If $\varphi \in \Gamma$, then $-\varphi$: if $-\varphi$ and $\varphi = (A \wedge B)$, then $-A$ and $-B$; if $-\varphi$ and $\varphi = (A \vee B)$, then $-A$ and $-B$; if $-\varphi$ and $\varphi = (A \rightarrow B)$, then $+A$ and $-B$; 2) If $\varphi \in \Delta$, then $+\varphi$: if $+\varphi$ and $\varphi = (A \wedge B)$, then $+A$ and $+B$; if $+\varphi$ and $\varphi = (A \vee B)$, then $+A$ and $+B$; if $+\varphi$ and $\varphi = (A \rightarrow B)$, then $-A$ and $+B$.

We say "the *Extended Disjunction Property* (EDP) holds" when the following statement holds, since DP for IPC is a special case of this.

Proposition 6. *In mLIC, if no classical subformula of $A \vee B$ has both negative and positive occurrences in A and B of $\Rightarrow A \vee B$, then $\Rightarrow A$ or $\Rightarrow B$ holds.*

Proof. Proof by contradiction. Suppose that there exists $A \vee B$ such that not $\Rightarrow A$ and not $\Rightarrow B$, but that $\Rightarrow A \vee B$ s.t. there is no classical subformula of $A \vee B$ whose positive and negative occurrences appear in A and B of $A \vee B$.

First, observe that the above inductive characterization of positive and negative occurrences of a subformula of φ in Γ or Δ corresponds to a step in the saturation procedure that puts a subformula of a formula in Γ and Δ. The inductive characterization obviously gives us: $-\varphi$ in G iff φ occurs in the antecedent of some saturated component of some G'; $+\varphi$ in G iff φ occurs in the succedent of some saturated component of some G'.

By assumption, we have a case of $\Rightarrow A \vee B$ where we do not have any occurrence of a classical subformula φ° in A (in B) s.t. $+\varphi^\circ$ and in B (in A) s.t. $-\varphi^\circ$. So, in particular, there is no X such that $+X$ $(-X)$ in mLIC$^-$ $\nvdash\Rightarrow A$ and $-X$ $(+X)$ in mLIC$^-$ $\nvdash\Rightarrow B$. So, we can construct saturated hypersequents G'_A of mLIC$^-$ $\nvdash\Rightarrow A$ and G'_B of mLIC$^-$ $\nvdash\Rightarrow B$ such that there is no classical variable in some succedent (antecedent) of G'_A and in some antecedent (succedent) of G'_B.

By completeness, we can construct countermodels $\mathcal{K}_A$ and $\mathcal{K}_B$ based on G'_A and G'_B s.t. $\mathcal{K}_A, r_A \nVdash\Rightarrow A$ and $\mathcal{K}_B, r_B \nVdash\Rightarrow B$. By construction, there is no classical variable X such that at some s_A of $\mathcal{K}_A$, $s_A \Vdash X$ (or $s_A \nVdash X$) and at some s_B of $\mathcal{K}_B$, $s_B \nVdash X$ (or $s_B \Vdash X$). This is obviously sufficient to use the same gluing method as used in the proof of completeness. So, we can construct a countermodel $\mathcal{K}, r \nVdash\Rightarrow A \vee B$. So, by soundness, $\Rightarrow A \vee B$ is not mLIC-provable, which contradicts our assumption $\Rightarrow A \vee B$.[20] ⊠

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[20] Note that the converse of the proposition does not hold. Even if there is such a subformula in the formula, we may not have to apply Classical Splitting to get a cut-free proof of the formula, and we have an intuitionistic proof of it. Here is a simple example, $\Rightarrow ((X \wedge p) \to p) \vee (p \to X)$.

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