Game Theory Evolving, Second Edition

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A Problem-Centered Introduction to Modeling Strategic Interaction

Herbert Gintis

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To Marci and Dan

> Riverrun, past Eve and Adam's, from swerve of shore to bend of bay, brings us by a Commodius vicus of recirculation back
>
> James Joyce

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Preface

> Was sich sagen läßt, läßt sich klar sagen, und wovon man nicht sprechen kann, darüber muß man schweigen.
>
> Ludwig Wittgenstein

This book is a problem-centered introduction to classical and evolutionary game theory. For most topics, I provide just enough in the way of definitions, concepts, theorems, and examples to begin solving problems. Learning and insight come from grappling with and solving problems. I provide extensive answers to some problems, sketchy and suggestive answers to most others. Students should consult the answers in the back of the book only to check their work. If a problem seems too difficult to solve, the student should come back to it a day, a week, or a month later, rather than peeking at the solution.

Game theory is logically demanding, but on a practical level, it requires surprisingly few mathematical techniques. Algebra, calculus, and basic probability theory suffice. However, game theory frequently requires considerable notation to express ideas clearly. The reader should commit to memory the precise definition of every term, and the precise statement of most theorems.

Clarity and precision do not imply rigor. I take my inspiration from physics, where sophisticated mathematics is common, but mathematical rigor is considered an impediment to creative theorizing. I stand by the truth and mathematical cogency of the arguments presented in this book, but not by their rigor. Indeed, the stress placed on game-theoretic rigor in recent years is misplaced. Theorists could worry more about the empirical relevance of their models and take less solace in mathematical elegance.

For instance, if a proposition is proved for a model with a finite number of agents, it is completely irrelevant whether it is true for an infinite number of agents. There are, after all, only a finite number of people, or even bacteria. Similarly, if something is true in games in which payoffs are finitely divisible (e.g., there is a minimum monetary unit), it does not matter whether it is true when payoffs are infinitely divisible. There are no payoffs in the universe, as far as we know, that are infinitely divisible. Even time, which is continuous in principle, can be measured only by devices with a finite number of quantum states. Of course, models based on the real and complex numbers can be hugely useful, but they are just approximations, because there are only a finite number of particles in the universe, and we can construct only a finite set of numbers, even in principle. There is thus no intrinsic value of a theorem that is true for a continuum of agents on a Banach space, if it is also true for a finite number of agents on a finite choice space.

Evolutionary game theory is about the emergence, transformation, diffusion, and stabilization of forms of behavior. Traditionally, game theory has been seen as a theory of how rational actors behave. Ironically, game theory, which for so long was predicated upon high-level rationality, has shown us, by example, the limited capacity of the concept of rationality alone to predict human behavior. I explore this issue in depth in *Bounds of Reason* (Princeton, 2009), which develops themes from epistemic game theory to fill in where classical game theory leaves off. Evolutionary game theory deploys the Darwinian notion that good strategies diffuse across populations of players rather than being learned by rational agents.

The treatment of rationality as preference consistency, a theme that we develop in chapter 2, allows us to assume that agents choose best responses, and otherwise behave as good citizens of game theory society. But they may be pigs, dung beetles, birds, spiders, or even wild things like Trogs and Klingons. How do they accomplish these feats with their small minds and alien mentalities? The answer is that the *agent* is displaced by the *strategy* as the dynamic game-theoretic unit.

This displacement is supported in three ways. First, we show that many static optimization models are stable equilibria of dynamic systems in which agents do not optimize, and we reject models that do not have attractive stability properties. To this end, after a short treatment of evolutionary stability, we develop dynamical systems theory (chapter 11) in sufficient depth to allow students to solve dynamic games with replicator dynamics (chapter 12). Second we provide animal as well as human models. Third, we provide agent-based computer simulations of games, showing that really stupid critters can evolve toward the solution of games previously thought to require "rationality" and high-level information processing capacity.

The Wittgenstein quote at the head of the preface means "What can be said, can be said clearly, and what you cannot say, you should shut up about." This adage is beautifully reflected in the methodology of game theory, especially epistemic game theory, which I develop in *The Bounds of Reason* (2009),and which gives us a language and a set of analytical tools for modeling an aspect of social reality with perfect clarity. Before game theory, we had no means of speaking clearly about social reality, so the great men and women who created the behavioral sciences from the dawn of the Enlightenment to the mid-twentieth century must be excused for the raging ideological battles that inevitably accompanied their attempt to talk about what could not be said clearly. If we take Wittgenstein seriously, it may be that those days are behind us.

Game Theory Evolving, Second Edition does not say much about how game theory applies to fields outside economics and biology. Nor does this volume evaluate the empirical validity of game theory, or suggest why rational agents might play Nash equilibria in the absence of an evolutionary dynamic with an asymptotically stable critical point. The student interested in these issues should turn to the companion volume, *The Bounds of Reason.*

Game Theory Evolving, Second Edition was composed on a word processor that I wrote in Borland Pascal, and the figures and tables were produced by a program that I wrote in Borland Delphi. The simulations are in Borland Delphi and C++Builder, and the results are displayed using SigmaPlot. I used NormalSolver, which I wrote in Delphi, to check solutions to many of the normal and extensive form games analyze herein. *Game Theory Evolving, Second Edition* was produced by LaTeX.

The generous support of the European Science Foundation, as well as the intellectual atmospheres of the Santa Fe Institute and the Central European University (Budapest) afforded me the time and resources to complete this book. I would like to thank Robert Axtell, Ken Binmore, Samuel Bowles, Robert Boyd, Songlin Cai, Colin Camerer, Graciela Chichilnisky, Catherine Eckel, Yehuda Elkana, Armin Falk, Ernst Fehr, Alex Field, Urs Fischbacher, Daniel Gintis, Jack Hirshleifer, David Laibson, Michael Mandler, Larry Samuelson, Rajiv Sethi, E. Somanathan, and Lones Smith for helping me with particular points. Special thanks go to Yusuke Narita and Sean Brocklebank, who read and corrected the whole book. I am grateful to Tim Sullivan, Seth Ditchik, and Peter Dougherty, my editors at Princeton University Press, who had the vision and faith to make this volume possible.

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1

Probability Theory

Doubt is disagreeable, but certainty is ridiculous.

Voltaire

1.1 Basic Set Theory and Mathematical Notation

A *set* is a collection of objects. We can represent a set by enumerating its objects. Thus,

$$A = \{1, 3, 5, 7, 9, 34\}$$

is the set of single digit odd numbers plus the number 34. We can also represent the same set by a formula. For instance,

$$A = \{x | x \in \mathbf{N} \wedge (x < 10 \wedge x \text{ is odd}) \vee (x = 34)\}.$$

In interpreting this formula, **N** is the set of natural numbers (positive integers), "|" means "such that," "∈" means "is a element of," $\wedge$ is the logical symbol for "and," and $\vee$ is the logical symbol for "or." See the table of symbols in chapter 14 if you forget the meaning of a mathematical symbol.

The subset of objects in set X that satisfy property p can be written as

$$\{x \in X | p(x)\}.$$

The *union* of two sets $A, B \subset X$ is the subset of X consisting of elements of X that are in *either* A or B:

$$A \cup B = \{x | x \in A \vee x \in B\}.$$

The *intersection* of two sets $A, B \subset X$ is the subset of X consisting of elements of X that are in *both* A or B:

$$A \cap B = \{x | x \in A \wedge x \in B\}.$$

If $a \in A$ and $b \in B$, the *ordered pair* (a, b) is an entity such that if $(a, b) = (c, d)$, then $a = c$ and $b = d$. The set $\{(a, b) | a \in A \wedge b \in B\}$

is called the *product* of A and B and is written $A \times B$. For instance, if $A = B = \mathbf{R}$, where **R** is the set of real numbers, then $A \times B$ is the real plane, or the real two-dimensional vector space. We also write

$$\prod_{i=1}^{n} A_i = A_1 \times \ldots \times A_n.$$

A *function* f can be thought of as a set of ordered pairs $(x, f(x))$. For instance, the function $f(x) = x^2$ is the set

$$\{(x, y)|(x, y \in \mathbf{R}) \wedge (y = x^2)\}$$

The set of arguments for which f is defined is called the *domain* of f and is written dom(f). The set of values that f takes is called the *range* of f and is written range(f). The function f is thus a subset of dom$(f)\times$range(f). If f is a function defined on set A with values in set B, we write $f{:}A{\rightarrow}B$.

1.2 Probability Spaces

We assume a finite *universe* or *sample space* Ω and a set $\mathcal{X}$ of subsets $A, B, C, \ldots$ of Ω, called *events*. We assume $\mathcal{X}$ is closed under finite unions (if $A_1, A_2, \ldots A_n$ are events, so is $\cup_{i=1}^{n} A_i$), finite intersections (if $A_1, \ldots, A_n$ are events, so is $\cap_{i=1}^{n} A_i$), and complementation (if A is an event so is the set of elements of Ω that are not in A, which we write A^c). If A and B are events, we interpret $A \cap B = AB$ as the event "A and B both occur," $A \cup B$ as the event "A or B occurs," and A^c as the event "A does not occur."

For instance, suppose we flip a coin twice, the outcome being HH (heads on both), HT (heads on first and tails on second), TH (tails on first and heads on second), and TT (tails on both). The sample space is then $\Omega = \{HH, TH, HT, TT\}$. Some events are $\{HH, HT\}$ (the coin comes up heads on the first toss), $\{TT\}$ (the coin comes up tails twice), and $\{HH, HT, TH\}$ (the coin comes up heads at least once).

The *probability* of an event $A \in \mathcal{X}$ is a real number P$[A]$ such that $0 \le$ P$[A] \le 1$. We assume that P$[\Omega] = 1$, which says that with probability 1 *some* outcome occurs, and we also assume that if $A = \cup_{i=1}^{n} A_i$, where $A_i \in \mathcal{X}$ and the $\{A_i\}$ are disjoint (that is, $A_i \cap A_j = \emptyset$ for all $i \neq j$), then P$[A] = \sum_{i=1}^{n} P[A_i]$, which says that probabilities are additive over finite disjoint unions.

1.3 De Morgan's Laws

Show that for any two events A and B, we have

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c.$$

These are called *De Morgan's laws*. Express the meaning of these formulas in words.

Show that if we write p for proposition "event A occurs" and q for "event B occurs," then

$$\text{not } (p \text{ or } q) \Leftrightarrow (\text{ not } p \text{ and not } q),$$

$$\text{not } (p \text{ and } q) \Leftrightarrow (\text{ not } p \text{ or not } q).$$

The formulas are also De Morgan's laws. Give examples of both rules.

1.4 Interocitors

An interocitor consists of two kramels and three trums. Let A_k be the event "the kth kramel is in working condition," and B_j is the event "the jth trum is in working condition." An interocitor is in working condition if at least one of its kramels and two of its trums are in working condition. Let C be the event "the interocitor is in working condition." Write C in terms of the A_k and the B_j.

1.5 The Direct Evaluation of Probabilities

THEOREM 1.1 *Given $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, all distinct, there are $n \times m$ distinct ways of choosing one of the a_i and one of the b_j. If we also have $c_1, \ldots, c_r$, distinct from each other, the a_i and the b_j, then there are $n \times m \times r$ distinct ways of choosing one of the a_i, one of the b_j, and one of the c_k.*

Apply this theorem to determine how many different elements there are in the sample space of

a. the double coin flip

b. the triple coin flip
c. rolling a pair of dice

Generalize the theorem.

1.6 Probability as Frequency

Suppose the sample space Ω consists of a finite number n of equally probable elements. Suppose the event A contains m of these elements. Then the *probability of the event* A is m/n.

A second definition: Suppose an experiment has n distinct outcomes, all of which are equally likely. Let A be a subset of the outcomes, and $n(A)$ the number of elements of A. We define the *probability* of A as $\mathrm{P}[A] = n(A)/n$.

For example, in throwing a pair of dice, there are $6 \times 6 = 36$ mutually exclusive, equally likely events, each represented by an ordered pair (a, b), where a is the number of spots showing on the first die and b the number on the second. Let A be the event that both dice show the same number of spots. Then $n(A) = 6$ and $\mathrm{P}[A] = 6/36 = 1/6$.

A third definition: Suppose an experiment can be repeated any number of times, each outcome being independent of the ones before and after it. Let A be an event that either does or does not occur for each outcome. Let $n_t(A)$ be the number of times A occurred on all the tries up to and including the t^{th} try. We define the *relative frequency* of A as $n_t(A)/t$, and we define the *probability of* A as

$$\mathrm{P}[A] = \lim_{t \to \infty} \frac{n_t(A)}{t}.$$

We say two events A and B are *independent* if $\mathrm{P}[A]$ does not depend on whether B occurs or not and, conversely, $\mathrm{P}[B]$ does not depend on whether A occurs or not. If events A and B are independent, the probability that both occur is the product of the probabilities that either occurs: that is,

$$\mathrm{P}[A \text{ and } B] = \mathrm{P}[A] \times \mathrm{P}[B].$$

For example, in flipping coins, let A be the event "the first ten flips are heads." Let B be the event "the eleventh flip is heads." Then the two events are independent.

For another example, suppose there are two urns, one containing 100 white balls and 1 red ball, and the other containing 100 red balls and 1 white ball. You do not know which is which. You choose 2 balls from the first urn. Let A be the event "The first ball is white," and let B be the event "The second ball is white." These events are not independent, because if you draw a white ball the first time, you are more likely to be drawing from the urn with 100 white balls than the urn with 1 white ball.

Determine the following probabilities. Assume all coins and dice are "fair" in the sense that H and T are equiprobable for a coin, and $1, \ldots, 6$ are equiprobable for a die.

a. At least one head occurs in a double coin toss.
b. Exactly two tails occur in a triple coin toss.
c. The sum of the two dice equals 7 or 11 in rolling a pair of dice.
d. All six dice show the same number when six dice are thrown.
e. A coin is tossed seven times. The string of outcomes is HHHHHHH.
f. A coin is tossed seven times. The string of outcomes is HTHHTTH.

1.7 Craps

A roller plays against the casino. The roller throws the dice and wins if the sum is 7 or 11, but loses if the sum is 2, 3, or 12. If the sum is any other number (4, 5, 6, 8, 9, or 10), the roller throws the dice repeatedly until either winning by matching the first number rolled or losing if the sum is 2, 7, or 12 ("crapping out"). What is the probability of winning?

1.8 A Marksman Contest

In a head-to-head contest Alice can beat Bonnie with probability p and can beat Carole with probability q. Carole is a better marksman than Bonnie, so $p > q$. To win the contest Alice must win at least two in a row out of three head-to-heads with Bonnie and Carole and cannot play the same person twice in a row (that is, she can play Bonnie-Carole-Bonnie or Carole-Bonnie-Carole). Show that Alice maximizes her probability of winning the contest playing the better marksman, Carole, twice.

1.9 Sampling

The mutually exclusive outcomes of a random action are called *sample points*. The set of sample points is called the *sample space*. An *event* A is a subset of a sample space Ω. The event A is *certain* if $A = \Omega$ and *impossible* if $A = \emptyset$ (that is, A has no elements). The *probability* of an event A is $\mathrm{P}[A] = n(A)/n(\Omega)$, if we assume Ω is finite and all $\omega \in \Omega$ are equally likely.

a. Suppose six dice are thrown. What is the probability all six die show the same number?
b. Suppose we choose r object in succession from a set of n distinct objects $a_1, \ldots, a_n$, each time recording the choice and returning the object to the set before making the next choice. This gives an ordered sample of the form $(b_1, \ldots, b_r)$, where each b_j is some a_i. We call this *sampling with replacement*. Show that, in sampling r times with replacement from a set of n objects, there are n^r distinct ordered samples.
c. Suppose we choose r objects in succession from a set of n distinct objects $a_1, \ldots, a_n$, without returning the object to the set. This gives an ordered sample of the form $(b_1, \ldots, b_r)$, where each b_j is some unique a_i. We call this *sampling without replacement* . Show that in sampling r times without replacement from a set of n objects, there are

$$n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

distinct ordered samples, where $n! = n \times (n-1) \times \ldots \times 2 \times 1$.

1.10 Aces Up

A deck of 52 cards has 4 aces. A player draws 2 cards randomly from the deck. What is the probability that both are aces?

1.11 Permutations

A linear ordering of a set of n distinct objects is called a *permutation* of the objects. It is easy to see that the number of distinct permutations of $n > 0$ distinct objects is $n! = n \times (n-1) \times \ldots \times 2 \times 1$. Suppose we have a deck of cards numbered from 1 to $n > 1$. Shuffle the cards so their new order is a random permutation of the cards. What is the average number of cards that appear in the "correct" order (that is, the k^{th} card is in the k^{th} position) in the shuffled deck?

1.12 Combinations and Sampling

The number of *combinations* of n distinct objects taken r at a time is the number of subsets of size r, taken from the n things without replacement. We write this as $\binom{n}{r}$. In this case, we do not care about the order of the choices. For instance, consider the set of numbers {1,2,3,4}. The number of samples of size two without replacement = 4!/2! = 12. These are precisely {12,13,14,21,23,24,31,32,34,41,42,43}. The combinations of the four numbers of size two (that is, taken two at a time) are {12,13,14,23,24,34}, or six in number. Note that $6 = \binom{4}{2} = 4!/2!2!$. A set of n elements has $n!/r!(n-r)!$ distinct subsets of size r. Thus, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

1.13 Mechanical Defects

A shipment of seven machines has two defective machines. An inspector checks two machines randomly drawn from the shipment, and accepts the shipment if neither is defective. What is the probability the shipment is accepted?

1.14 Mass Defection

A batch of 100 manufactured items is checked by an inspector, who examines 10 items at random. If none is defective, she accepts the whole batch. What is the probability that a batch containing 10 defective items will be accepted?

1.15 House Rules

Suppose you are playing the following game against the house in Las Vegas. You pick a number between one and six. The house rolls three dice, and pays you \$1,000 if your number comes up on one die, \$2,000 if your number comes up on two dice, and \$3,000 if your number comes up on all three dice. If your number does not show up at all, you pay the house \$1,000. At first glance, this looks like a *fair game* (that is, a game in which the expected payoff is zero), but in fact it is not. How much can you expect to win (or lose)?

1.16 The Addition Rule for Probabilities

Let A and B be two events. Then $0 \leq \mathrm{P}[A] \leq 1$ and

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[AB].$$

If A and B are disjoint (that is, the events are mutually exclusive), then

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B].$$

Moreover, if $A_1, \ldots, A_n$ are mutually disjoint, then

$$\mathrm{P}[\cup_i A_i] = \sum_{i=1}^{n} \mathrm{P}[A_i].$$

We call events $A_1, \ldots, A_n$ a *partition* of the sample space Ω if they are mutually disjoint and exhaustive (that is, their union is Ω). In this case for any event B, we have

$$\mathrm{P}[B] = \sum_i \mathrm{P}[BA_i].$$

1.17 A Guessing Game

Each day the call-in program on a local radio station conducts the following game. A number is drawn at random from $\{1, 2, \ldots, n\}$. Callers choose a number randomly and win a prize if correct. Otherwise, the station announces whether the guess was high or low and moves on to the next caller, who chooses randomly from the numbers that can logically be correct, given the previous announcements. What is the expected number $f(n)$ of callers before one guesses the number?

1.18 North Island, South Island

Bob is trying to find a secret treasure buried in the ground somewhere in North Island. According to local custom, if Bob digs and finds the treasure, he can keep it. If the treasure is not at the digging point, though, and Bob happens to hit rock, Bob must go to South Island. On the other hand, if Bob hits clay on North Island, he can stay there and try again. Once on South Island, to get back to North Island, Bob must dig and hit clay. If Bob hits rock on South Island, he forfeits the possibility of obtaining the treasure.

On the other hand, if Bob hits earth on South Island, he can stay on South Island and try again. Suppose q_n is the probability of finding the treasure when digging at a random spot on North Island, r_n is the probability of hitting rock on North Island, r_s is the probability of hitting rock on South Island, and e_s is the probability of hitting earth on South Island. What is the probability, P_n, that Bob will eventually find the treasure before he forfeits, if we assume that he starts on North Island?

1.19 Conditional Probability

If A and B are events, and if the probability $\mathrm{P}[B]$ that B occurs is strictly positive, we define the *conditional probability* of A given B, denoted $\mathrm{P}[A|B]$, by

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[AB]}{\mathrm{P}[B]}.$$

We say $B_1, \ldots, B_n$ are a *partition* of event B if $\cup_i B_i = B$ and $B_i B_j = \emptyset$ for $i \neq j$. We have:

a. If A and B are events, $\mathrm{P}[B] > 0$, and B implies A (that is, $B \subseteq A$), then $\mathrm{P}[A|B] = 1$.
b. If A and B are contradictory (that is, $AB = \emptyset$), then $\mathrm{P}[A|B] = 0$.
c. If $A_1, \ldots, A_n$ are a partition of event A, then

$$\mathrm{P}[A|B] = \sum_{i=1}^{n} \mathrm{P}[A_i|B].$$

d. If $B_1, \ldots, B_n$ are a partition of the sample space Ω, then

$$\mathrm{P}[A] = \sum_{i=1}^{n} \mathrm{P}[A|B_i]\,\mathrm{P}[B_i].$$

1.20 Bayes' Rule

Suppose A and B are events with $\mathrm{P}[A], \mathrm{P}[B], \mathrm{P}[B^c] > 0$. Then we have

$$\mathrm{P}[B|A] = \frac{\mathrm{P}[A|B]\,\mathrm{P}[B]}{\mathrm{P}[A|B]\,\mathrm{P}[B] + \mathrm{P}[A|B^c]\,\mathrm{P}[B^c]}.$$

This follows from the fact that the denominator is just $\mathrm{P}[A]$, and is called *Bayes' rule*.

More generally, if $B_1, \ldots, B_n$ is a partition of the sample space and if $\mathrm{P}[A], \mathrm{P}[B_k] > 0$, then

$$\mathrm{P}[B_k|A] = \frac{\mathrm{P}[A|B_k]\,\mathrm{P}[B_k]}{\sum_{i=1}^{n} \mathrm{P}[A|B_i]\,\mathrm{P}[B_i]}.$$

To see this, note that the denominator on the right-hand side is just $\mathrm{P}[A]$, and the numerator is just $\mathrm{P}[AB_k]$ by definition.

1.21 Extrasensory Perception

Alice claims to have ESP. She says to Bob, "Match me against a series of opponents in picking the high card from a deck with cards numbered 1 to 100. I will do better than chance in either choosing a higher card than my opponent or choosing a higher card on my second try than on my first." Bob reasons that Alice will win on her first try with probability 1/2, and beat her own card with probability 1/2 if she loses on the first round. Thus, Alice should win with probability $(1/2) + (1/2)(1/2) = 3/4$. He finds, to his surprise, that Alice wins about 5/6 of the time. Does Alice have ESP?

1.22 Les Cinq Tiroirs

You are looking for an object in one of five drawers. There is a 20% chance that it is not in any of the drawers, but if it is in a drawer, it is equally likely to be in each one. Show that as you look in the drawers one by one, the probability of finding the object in the next drawer rises if not found so far, but the probability of not finding it at all also rises.

1.23 Drug Testing

Bayes' rule is useful because often we know $\mathrm{P}[A|B]$, $\mathrm{P}[A|B^c]$ and $\mathrm{P}[B]$, and we want to find $\mathrm{P}[B|A]$. For example, suppose 5% of the population uses drugs, and there is a drug test that is 95% accurate: it tests positive on a drug user 95% of the time, and it tests negative on a drug nonuser 95% of the time. Show that if an individual tests positive, the probability of his being a drug user is 50%. *Hint*: Let A be the event "is a drug user," let "Pos" be the event "tests positive," let "Neg" be the event "tests negative," and apply Bayes' rule.

1.24 Color Blindness

Suppose 5% of men are color-blind and 0.25% of women are color-blind. A person is chosen at random and found to be color-blind. What is the probability the person is male (assume the population is 50% female)?

1.25 Urns

A collection of $n + 1$ urns, numbered from 0 to n, each contains n balls. Urn k contains k red and $n - k$ white balls. An urn is chosen at random and n balls are randomly chosen from it, the ball being replaced each time before another is chosen. Suppose all n balls are found to be red. What is the probability the next ball chosen from the urn will be red? Show that when n is large, this probability is approximately $n/(n+2)$. *Hint*: For the last step, approximate the sum by an integral.

1.26 The Monty Hall Game

You are a contestant in a game show. The host says, "Behind one of those three doors is a new automobile, which is your prize should you choose the right door. Nothing is behind the other two doors. You may choose any door." You choose door A. The game host then opens door B and shows you that there is nothing behind it. He then asks, "Now would you like to change your guess to door C, at a cost of $1?" Show that the answer is no if the game show host randomly opened one of the two other doors, but yes if he simply opened a door he knew did not have a car behind it. Generalize to the case where there are n doors with a prize behind one door.

1.27 The Logic of Murder and Abuse

For a given woman, let A be the event "was habitually beaten by her husband" ("abused" for short), let B be the event "was murdered," and let C be the event "was murdered by her husband." Suppose we know the following facts: (a) 5% of women are abused by their husbands; (b) 0.5% of women are murdered; (c) 0.025% of women are murdered by their husbands; (d) 90% of women who are murdered by their husbands had been abused by their husbands; (e) a woman who is murdered but not by her husband is neither more nor less likely to have been abused by her husband than a randomly selected woman.

Nicole is found murdered, and it is ascertained that she was abused by her husband. The defense attorneys for her husband show that the probability that a man who abuses his wife actually kills her is only 4.50%, so there is a strong presumption of innocence for him. The attorneys for the prosecution show that there is in fact a 94.74% chance the husband murdered his wife, independent from any evidence other than that he abused her. Please supply the arguments of the two teams of attorneys. You may assume that the jury was well versed in probability theory, so they had no problem understanding the reasoning.

1.28 The Principle of Insufficient Reason

The principle of insufficient reason says that if you are "completely ignorant" as to which among the states $A_1, \ldots, A_n$ will occur, then you should assign probability $1/n$ to each of the states. The argument in favor of the principle is strong (see Savage 1954 and Sinn 1980 for discussions), but there are some interesting arguments against it. For instance, suppose A_1 itself consists of m mutually exclusive events $A_{11}, \ldots, A_{1m}$. If you are "completely ignorant" concerning which of these occurs, then if $\mathrm{P}[A_1] = 1/n$, we should set $\mathrm{P}[A_{1i}] = 1/mn$. But are we not "completely ignorant" concerning which of $A_{11}, \ldots, A_{1m}, A_2, \ldots, A_n$ occurs? If so, we should set each of these probabilities to $1/(n + m - 1)$. If not, in what sense were we "completely ignorant" concerning the original states $A_1, \ldots, A_n$?

1.29 The Greens and the Blacks

The game of bridge is played with a normal 52-card deck, each of four players being dealt 13 cards at the start of the game. The Greens and the Blacks are playing bridge. After a deal, Mr. Brown, an onlooker, asks Mrs. Black: "Do you have an ace in your hand?" She nods yes. After the next deal, he asks her: "Do you have the ace of spades?" She nods yes again. In which of the two situations is Mrs. Black more likely to have at least one other ace in her hand? Calculate the exact probabilities in the two cases.

1.30 The Brain and Kidney Problem

A mad scientist is showing you around his foul-smelling laboratory. He motions to an opaque, formalin-filled jar. "This jar contains either a brain or a kidney, each with probability 1/2," he exclaims. Searching around his workbench, he finds a brain and adds it to the jar. He then picks one blob randomly from the jar, and it is a brain. What is the probability the remaining blob is a brain?

1.31 The Value of Eyewitness Testimony

A town has 100 taxis, 85 green taxis owned by the Green Cab Company and 15 blue taxies owned by the Blue Cab Company. On March 1, 1990, Alice was struck by a speeding cab, and the only witness testified that the cab was blue rather than green. Alice sued the Blue Cab Company. The judge instructed the jury and the lawyers at the start of the case that the reliability of a witness must be assumed to be 80% in a case of this sort, and that liability requires that the "preponderance of the evidence," meaning at least a 50% probability, be on the side of the plaintiff.

The lawyer for Alice argued that the Blue Cab Company should pay, because the witness's testimonial gives a probability of 80% that she was struck by a blue taxi. The lawyer for the Blue Cab Company argued as follows. A witness who was shown all the cabs in town would incorrectly identify 20% of the 85 green taxis (that is, 17 of them) as blue, and correctly identify 80% of the 15 blue taxis (that is, 12 of them) as blue. Thus, of the 29 identifications of a taxi as blue, only twelve would be correct and seventeen would be incorrect. Thus, the preponderance of the evidence is in favor of the defendant. Most likely, Alice was hit by a green taxi.

Formulate the second lawyer's argument rigorously in terms of Bayes' rule. Which argument do you think is correct, and if neither is correct, what is a good argument in this case?

1.32 When Weakness Is Strength

Many people have criticized the Darwinian notion of "survival of the fittest" by declaring that the whole thing is a simple tautology: whatever survives is "fit" by definition! Defenders of the notion reply by noting that we can measure fitness (e.g., speed, strength, resistance to disease, aerodynamic stability) independent of survivability, so it becomes an empirical proposition that the fit survive. Indeed, under some conditions it may be simply false, as game theorist Martin Shubik (1954) showed in the following ingenious problem.

Alice, Bob, and Carole are having a shootout. On each round, until only one player remains standing, the current shooter can choose one of the other players as target and is allowed one shot. At the start of the game, they draw straws to see who goes first, second, and third, and they take turns repeatedly in that order. A player who is hit is eliminated. Alice is a perfect shot, Bob has 80% accuracy, and Carole has 50% accuracy. We assume that players are not required to aim at an opponent and can simply shoot in the air on their turn, if they so desire.

We will show that Carole, the least accurate shooter, is the most likely to survive. As an exercise, you are asked to show that if the player who gets to shoot is picked randomly in each round, then the survivability of the players is perfectly inverse to their accuracy.

There are six possible orders for the three players, each occurring with probability 1/6. We abbreviate Alice as a, Bob as b, and Carole as c, and we write the order of play as xyz, where x,y,z $\in$ {a,b,c}. We let $\pi_i(\text{xyz})$ be the survival probability of player $i \in$ {a,b,c}. For instance, $\pi_{\text{a}}(\text{abc})$ is the probability Alice wins when the shooting order is abc. Similarly, if only two remain, let $\pi_i(\text{xy})$ be the probability of survival for player i =x,y when only x and y remain, and it is x's turn to shoot.

If Alice goes first, it is clear that her best move is to shoot at Bob, whom she eliminates with probability 1. Then, Carole's best move is to shoot at Alice, whom she eliminates with probability 1/2. If she misses Alice, Alice eliminates Carole. Therefore, we have $\pi_{\text{a}}(\text{abc}) = 1/2$, $\pi_{\text{b}}(\text{abc}) = 0$, $\pi_{\text{c}}(\text{abc}) = 1/2$, $\pi_{\text{a}}(\text{acb}) = 1/2$, $\pi_{\text{b}}(\text{acb}) = 0$, and $\pi_{\text{c}}(\text{acb}) = 1/2$.

Suppose Bob goes first, and the order is bac. If Bob shoots in the air, Alice will then eliminate Bob. If Bob shoots at Carole and eliminates her, Alice will again eliminate Bob. If Bob shoots at Alice and misses, then the order is effectively acb, and we know Alice will eliminate Bob. However, if Bob shoots at Alice and eliminates her, then the game is cb. We have

$$p_{\text{c}}(\text{cb}) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{5} p_{\text{c}}(\text{cb}).$$

The first term on the right is the probability Carole hits Bob and wins straight off, and the second term is the probability that she misses Bob (1/2) times the probability Bob misses her (1/5) times the probability that she eventually wins if it is her turn to shoot. We can solve this equation, getting $p_{\text{c}}(\text{cb}) = 5/9$, so $p_{\text{b}}(\text{cb}) = 4/9$. It follows that Bob will indeed shoot at Alice, so

$$p_{\mathrm{b}}(\mathrm{bac}) = \frac{4}{5} \times \frac{4}{9} = \frac{16}{45}.$$

Similarly, we have $p_{\mathrm{b}}(\mathrm{bca}) = 16/45$. Also,

$$p_{\mathrm{a}}(\mathrm{bac}) = \frac{1}{5} p_{\mathrm{a}}(\mathrm{ca}) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10},$$

because we clearly have $p_{\mathrm{a}}(\mathrm{ca}) = 1/2$. Similarly, $p_{\mathrm{a}}(\mathrm{bca}) = 1/10$. Finally,

$$p_{\mathrm{c}}(\mathrm{bac}) = \frac{1}{5} p_{\mathrm{c}}(\mathrm{ca}) + \frac{4}{5} \times p_{\mathrm{c}}(\mathrm{cb}) = \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{5}{9} = \frac{49}{90},$$

because $p_{\mathrm{c}}(\mathrm{ca}) = 1/2$. Similarly, $p_{\mathrm{c}}(\mathrm{bca}) = 49/90$. As a check on our work, note that $p_{\mathrm{a}}(\mathrm{bac}) + p_{\mathrm{b}}(\mathrm{bac}) + p_{\mathrm{c}}(\mathrm{bac}) = 1$.

Suppose Carole gets to shoot first. If Carole shoots in the air, her payoff from cab is $p_{\mathrm{c}}(\mathrm{abc}) = 1/2$, and from cba is $p_{\mathrm{c}}(\mathrm{bac}) = 49/90$. These are also her payoffs if she misses her target. However, if she shoots Alice, her payoff is $p_{\mathrm{c}}(\mathrm{bc})$, and if she shoots Bob, her payoff is $p_{\mathrm{c}}(\mathrm{ac}) = 0$. We calculate $p_{\mathrm{c}}(\mathrm{bc})$ as follows.

$$p_{\mathrm{b}}(\mathrm{bc}) = \frac{4}{5} + \frac{1}{5} \times \frac{1}{2} p_{\mathrm{b}}(\mathrm{bc}),$$

where the first term is the probability he shoots Carole (4/5) plus the probability he misses Carole (1/5) times the probability he gets to shoot again (1/2, because Carole misses) times $p_{\mathrm{b}}(\mathrm{bc})$. We solve, getting $p_{\mathrm{b}}(\mathrm{bc}) = 8/9$. Thus, $p_{\mathrm{c}}(\mathrm{bc}) = 1/9$. Clearly, Carole's best payoff is to shoot in the air. Then $p_{\mathrm{c}}(\mathrm{cab}) = 1/2$, $p_{\mathrm{b}}(\mathrm{cab}) = p_{\mathrm{b}}(\mathrm{abc}) = 0$, and $p_{\mathrm{a}}(\mathrm{cab}) = p_{\mathrm{a}}(\mathrm{abc}) = 1/2$. Also, $p_{\mathrm{c}}(\mathrm{cba}) = 49/50$, $p_{\mathrm{b}}(\mathrm{cba}) = p_{\mathrm{b}}(\mathrm{bac}) = 16/45$, and $p_{\mathrm{a}}(\mathrm{cba}) = p_{\mathrm{a}}(\mathrm{bac}) = 1/10$.

The probability that Alice survives is given by

$$\begin{aligned} p_{\mathrm{a}} &= \frac{1}{6}(p_{\mathrm{a}}(\mathrm{abc}) + p_{\mathrm{a}}(\mathrm{acb}) + p_{\mathrm{a}}(\mathrm{bac}) + p_{\mathrm{a}}(\mathrm{bca}) + p_{\mathrm{a}}(\mathrm{cab}) + p_{\mathrm{a}}(\mathrm{cba})) \\ &= \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{10} + \frac{1}{10} + \frac{1}{2} + \frac{1}{10}\right) = \frac{3}{10}. \end{aligned}$$

The probability that Bob survives is given by

$$\begin{aligned} p_{\mathrm{b}} &= \frac{1}{6}(p_{\mathrm{b}}(\mathrm{abc}) + p_{\mathrm{b}}(\mathrm{acb}) + p_{\mathrm{b}}(\mathrm{bac}) + p_{\mathrm{b}}(\mathrm{bca}) + p_{\mathrm{b}}(\mathrm{cab}) + p_{\mathrm{b}}(\mathrm{cba})) \\ &= \frac{1}{6}\left(0 + 0 + \frac{16}{45} + \frac{16}{45} + 0 + \frac{16}{45}\right) = \frac{8}{45}. \end{aligned}$$

The probability that Carole survives is given by

$$p_C = \frac{1}{6}(p_C(\text{abc}) + p_C(\text{acb}) + p_C(\text{bac}) + p_C(\text{bca}) + p_C(\text{cab}) + p_C(\text{cba}))$$
$$= \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2} + \frac{49}{90} + \frac{49}{90} + \frac{1}{2} + \frac{49}{90}\right) = \frac{47}{90}.$$

You can check that these three probabilities add up to unity, as they should. Note that Carole has a 52.2% chance of surviving, whereas Alice has only a 30% chance, and Bob has a 17.8% chance.

1.33 The Uniform Distribution

The *uniform distribution* on [0, 1] is a random variable that is uniformly distributed over the unit interval. Therefore if $\tilde{x}$ is uniformly distributed over [0, 1] then

$$P[\tilde{x} < x] = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}.$$

If $\tilde{x}$ is uniformly distributed on the interval $[a, b]$, then $(\tilde{x} - a)/(b - a)$ is uniformly distributed on [0, 1], and a little algebra shows that

$$P[\tilde{x} < x] = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \end{cases}.$$

Figure 1.1 depicts this problem.

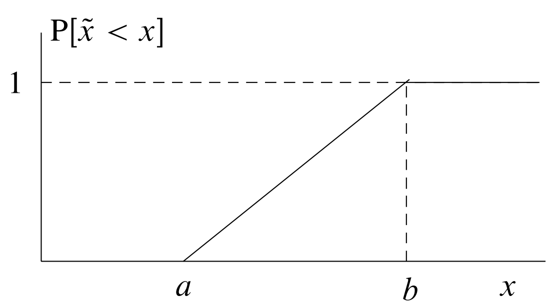


Figure 1.1. Uniform distribution

Suppose $\tilde{x}$ is uniformly distributed on $[a, b]$ and we learn that in fact $\tilde{x} \leq c$, where $a < c < b$. Then $\tilde{x}$ is in fact uniformly distributed on $[a, c]$.

To see this, we write

$$\begin{aligned}
\mathrm{P}[\tilde{x} < x|\tilde{x} \le c] &= \frac{\mathrm{P}[\tilde{x} < x \text{ and } \tilde{x} \le c]}{\mathrm{P}[\tilde{x} \le c]} \\
&= \frac{\mathrm{P}[\tilde{x} < x \text{ and } \tilde{x} \le c]}{(c-a)/(b-a)}.
\end{aligned}$$

We evaluate the numerator as follows:

$$\begin{aligned}
\mathrm{P}[\tilde{x} < x \text{ and } \tilde{x} \le c] &= \begin{cases} 0 & x \le a \\ \mathrm{P}[\tilde{x} < x] & a \le x \le c \\ \mathrm{P}[\tilde{x} \le c] & c \le x \end{cases} \\
&= \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a \le x \le c \\ \frac{c-a}{b-a} & c \le x \end{cases}.
\end{aligned}$$

Therefore,

$$\mathrm{P}[\tilde{x} < x|\tilde{x} \le c] = \begin{cases} 0 & x \le 0 \\ \frac{x-a}{c-a} & a \le x \le c \\ 1 & c \le x \end{cases}.$$

This is just the uniform distribution on $[a, c]$.

1.34 Laplace's Law of Succession

An urn contains a large number n of white and black balls, where the number of white balls is uniformly distributed between 0 and n. Suppose you pick out m balls with replacement, and r are white. Show that the probability of picking a white ball on the next draw is approximately $(r+1)/(m+2)$.

1.35 From Uniform to Exponential

Bob tells Alice to draw repeatedly from the uniform distribution on $[0, 1]$ until her current draw is less than some previous draw, and he will pay her $\$n$, where n is the number of draws. What is the average value of this game for Alice?

2

Bayesian Decision Theory

> In a formal model the conclusions are derived from definitions and assumptions. . . But with informal, verbal reasoning . . . one can argue until one is blue in the face . . . because there is no criterion for deciding the soundness of an informal argument.
>
> Robert Aumann

2.1 The Rational Actor Model

In this section we develop a set of behavioral properties, among which consistency is the most prominent, that together ensure that we can model the individual as maximizing a preference function over outcomes, subject to constraints.

A *binary relation* $\odot_A$ on a set A is a subset of $A \times A$. We usually write the proposition $(x, y) \in \odot_A$ as $x \odot_A y$. For instance, the arithmetical operator "less than" ($<$) is a binary relation, where $(x, y) \in <$ is normally written $x < y$.[1] A *preference ordering* $\succeq_A$ on A is a binary relation with the following three properties, which must hold for all $x, y, z \in A$ and any set B:

1. **Complete**: $x \succeq_A y$ or $y \succeq_A x$,
2. **Transitive**: $x \succeq_A y$ and $y \succeq_A z$ imply $x \succeq_A z$,
3. **Independence from Irrelevant Alternatives**: For $x, y \in B$, $x \succeq_B y$ if and only if $x \succeq_A y$.

Because of the third property, we need not specify the choice set and can simply write $x \succeq y$. We also make the behavioral assumption that given any choice set A, the individual chooses an element $x \in A$ such that for all $y \in A$, $x \succeq y$. When $x \succeq y$, we say "x is weakly preferred to y."

[1]Additional binary relations over the set **R** of real numbers include "$>$,""$<$,""$\leq$," "$=$," "$\geq$," and "$\neq$," but "$+$" is not a binary relation, because $x + y$ is not a proposition.

Completeness implies that any member of A is weakly preferred to itself (for any x in A, $x \succeq x$). In general, we say a binary relation $\odot$ is *reflexive* if, for all x, $x \odot x$. Thus, completeness implies reflexivity. We refer to $\succeq$ as "weak preference" in contrast to $\succ$ as "strong preference." We define as $x \succ y$ to mean "it is false that $y \succeq x$." We say x and y are *equivalent* if $x \succeq y$ and $y \succeq x$, and we write $x \simeq y$. As an exercise, you may use elementary logic to prove that if $\succeq$ satisfies the completeness condition, then $\succ$ satisfies the following *exclusion* condition: if $x \succ y$, then it is false that $y \succ x$.

The second condition is *transitivity*, which says that $x \succeq y$ and $y \succeq z$ imply $x \succeq z$. It is hard to see how this condition could fail for anything we might like to call a "preference ordering."[2] As an exercise, you may show that $x \succ y$ and $y \succeq z$ imply $x \succ z$, and $x \succeq y$ and $y \succ z$ imply $x \succ z$. Similarly, you may use elementary logic to prove that if $\succeq$ satisfies the completeness condition, then $\simeq$ is transitive (that is, satisfies the transitivity condition).

When these three conditions are satisfied, we say the preference relation $\succeq$ is *consistent*. If $\succeq$ is a consistent preference relation, then there always exists a preference function such that the individual behaves as if maximizing this preference function over the set A from which he or she is constrained to choose. Formally, we say that a preference function $u: A \rightarrow \mathbf{R}$ *represents* a binary relation $\succeq$ if, for all $x, y \in A$, $u(x) \geq u(y)$ if and only if $x \succeq y$. We have:

THEOREM 2.1 *A binary relation $\succeq$ on the finite set A of payoffs can be represented by a preference function $u: A \rightarrow \mathbf{R}$ if and only if $\succeq$ is consistent.*

It is clear that $u(\cdot)$ is not unique, and indeed, we have the following.

THEOREM 2.2 *If $u(\cdot)$ represents the preference relation $\succeq$ and $f(\cdot)$ is a strictly increasing function, then $v(\cdot) = f(u(\cdot))$ also represents $\succeq$. Conversely, if both $u(\cdot)$ and $v(\cdot)$ represent $\succeq$, then there is an increasing function $f(\cdot)$ such that $v(\cdot) = f(u(\cdot))$.*

The first half of the theorem is true because if f is strictly increasing, then $u(x) > u(y)$ implies $v(x) = f(u(x)) > f(u(y)) = v(y)$, and conversely.

[2]The only plausible model of intransitivity with some empirical support is *regret theory* (Loomes 1988; Sugden 1993). This analysis applies, however, to only a narrow range of choice situations.

For the second half, suppose $u(\cdot)$ and $v(\cdot)$ both represent $\succeq$, and for any $y \in \mathbf{R}$ such that $v(x) = y$ for some $x \in X$, let $f(y) = u(v^{-1}(y))$, which is possible because v is an increasing function. Then $f(\cdot)$ is increasing (because it is the composition of two increasing functions) and $f(v(x)) = u(v^{-1}(v(x))) = u(x)$, which proves the theorem.

2.2 Time Consistency and Exponential Discounting

The central theorem on choice over time is that time consistency results from assuming that *utility be additive across time periods and the instantaneous utility function be the same in all time periods, with future utilities discounted to the present at a fixed rate* (Strotz 1955). This is called *exponential discounting* and is widely assumed in economic models. For instance, suppose an individual can choose between two consumption streams $x = x_0, x_1, \ldots$ or $y = y_0, y_1, \ldots$ According to exponential discounting, he has a utility function $u(x)$ and a constant $\delta \in (0, 1)$ such that the total utility of stream x is given by[3]

$$U(x_0, x_1, \ldots) = \sum_{k=0}^{\infty} \delta^k u(x_k). \tag{2.1}$$

We call δ the individual's *discount factor*. Often we write $\delta = e^{-r}$ where we interpret $r > 0$ as the individual's one-period, continuous-compounding *interest rate*, in which case (2.1) becomes

$$U(x_0, x_1, \ldots) = \sum_{k=0}^{\infty} e^{-rk} u(x_k). \tag{2.2}$$

This form clarifies why we call this "exponential" discounting. The individual strictly prefers consumption stream x over stream y if and only if $U(x) > U(y)$. In the simple compounding case, where the interest accrues at the end of the period, we write $\delta = 1/(1 + r)$, and (2.2) becomes

$$U(x_0, x_1, \ldots) = \sum_{k=0}^{\infty} \frac{u(x_k)}{(1 + r)^k}. \tag{2.3}$$

[3]Throughout this text, we write $x \in (a, b)$ for $a < x < b$, $x \in [a, b)$ for $a \le x < b$, $x \in (a, b]$ for $a < x \le b$, and $x \in [a, b]$ for $a \le x \le b$.

The derivation of (2.2) is a bit tedious, and except for the exponential discounting part, is intuitively obvious. So let us assume utility $u(x)$ is additive and has the same shape across time, and show that exponential discounting must hold. I will construct a very simple case that is easily generalized. Suppose the individual has an amount of money M that he can either invest or consume in periods $t = 0, 1, 2$. Suppose the interest rate is r, and interest accrues continually, so \$1 put in the bank at time $k = 0$ yields e^{rk} at time k. Thus, by putting an amount $x_k e^{-rk}$ in the bank today, the individual will be able to consume x_k in period k. By the additivity and constancy across periods of utility, the individual will maximize some objective function

$$V(x_0, x_1, x_2) = u(x_0) + au(x_1) + bu(x_2), \tag{2.4}$$

subject to the income constraint

$$x_0 + e^{-r}x_1 + e^{-2r}x_2 = M.$$

where r is the interest rate. We must show that $b = a^2$ if and only if the individual is time consistent. We form the Lagrangian

$$\mathcal{L} = V(x_0, x_1, x_2) + \lambda(x_0 + e^{-r}x_1 + e^{-2r}x_2 - M),$$

where λ is the Lagrangian multiplier. The first-order conditions for a maximum are then given by $\partial\mathcal{L}/\partial x_i = 0$ for $i = 0, 1, 2$. Solving these equations, we find

$$\frac{u'(x_1)}{u'(x_2)} = \frac{be^r}{a}. \tag{2.5}$$

Now, time consistency means that after consuming x_0 in the first period, the individuals will still want to consume x_1 in the second period and x_2 in the third. But now his objective function is

$$V(x_1, x_2) = u(x_1) + au(x_2), \tag{2.6}$$

subject to the (same) income constraint

$$x_1 + e^{-r}x_2 = (M - x_0)e^{-r},$$

We form the Lagrangian

$$\mathcal{L}_1 = V(x_1, x_2) + \lambda(x_1 + e^{-r}x_2 - (M - x_0)e^{-r}),$$

where λ is the Lagrangian multiplier. The first-order conditions for a maximum are then given by $\partial\mathcal{L}_1/\partial x_i = 0$ for $i = 1, 2$. Solving these equations, we find

$$\frac{u'(x_1)}{u'(x_2)} = ae^r. \tag{2.7}$$

Now, time consistency means that (2.5) and (2.7) are equal, which means $a^2 = b$, as required.

2.3 The Expected Utility Principle

What about decisions in which a stochastic event determines the payoffs to the players? Let X be a set of "prizes." A *lottery* with payoffs in X is a function $p{:}X \rightarrow [0, 1]$ such that $\sum_{x\in X} p(x) = 1$. We interpret $p(x)$ as the probability that the payoff is $x \in X$. If $X = \{x_1, \ldots, x_n\}$ for some finite number n, we write $p(x_i) = p_i$.

The *expected value* of a lottery is the sum of the payoffs, where each payoff is weighted by the probability that the payoff will occur. If the lottery l has payoffs $x_1 \ldots x_n$ with probabilities $p_1, \ldots, p_n$, then the expected value $\mathbf{E}[l]$ of the lottery l is given by

$$\mathbf{E}[l] = \sum_{i=1}^{n} p_i x_i.$$

The expected value is important because of the law of large numbers (Feller 1950), which states that as the number of times a lottery is played goes to infinity, the average payoff converges to the expected value of the lottery with probability 1.

Consider the lottery l_1 in pane (a) of figure 2.1, where p is the probability of winning amount a and $1-p$ is the probability of winning amount b. The expected value of the lottery is then $\mathbf{E}[l_1] = pa + (1-p)b$. Note that we model a lottery a lot like an extensive form game, except there is only one player.

Consider the lottery l_2 with the three payoffs shown in pane (b) of figure 2.1. Here p is the probability of winning amount a, q is the probability of winning amount b, and $1 - p - q$ is the probability of winning amount c. The expected value of the lottery is $\mathbf{E}[l_2] = pa + qb + (1 - p - q)c$.

A lottery with n payoffs is given in pane (c) of figure 2.1. The prizes are now $a_1, \ldots, a_n$ with probabilities $p_1, \ldots, p_n$, respectively. The expected value of the lottery is now $\mathbf{E}[l_3] = p_1a_1 + p_2a_2 + \ldots + p_na_n$.

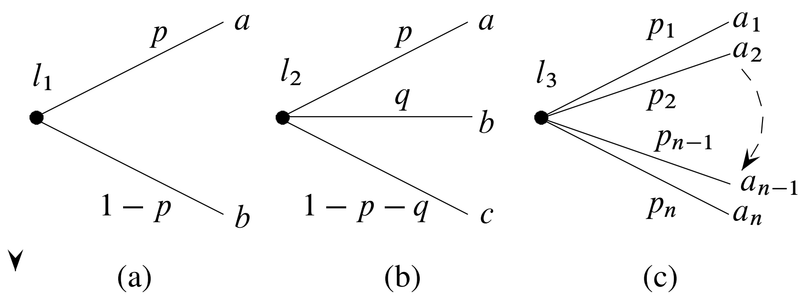


Figure 2.1. Lotteries with two, three, and n potential outcomes

In this section we generalize the previous argument, developing a set of behavioral properties that yield both a utility function over outcomes and a probability distribution over states of nature, such that the expected utility principle, defined in theorem 2.3 holds. Von Neumann and Morgenstern (1944), Friedman and Savage (1948), Savage (1954), and Anscombe and Aumann (1963) showed that the expected utility principle can be derived from the assumption that individuals have consistent preferences over an appropriate set of lotteries. We outline here Savage's classic analysis of this problem.

For the rest of this section, we assume $\succeq$ is a preference relation (§2.1). To ensure that the analysis is not trivial, we also assume that $x \succeq y$ is false for at least some $x, y \in X$. Savage's accomplishment was to show that if the individual has a preference relation over *lotteries* that has some plausible properties, then not only can the individual's preferences be represented by a utility function, but we can infer the probabilities the individual implicitly places on various events, and the expected utility principle (theorem 2.3) holds for these probabilities. These probabilities are called the individuals *subjective prior*.

To see this, let Ω be a finite set of *states of nature*. We call $A \subseteq \Omega$ *events*. Also, let $\mathcal{L}$ be a set of "lotteries," where a *lottery* is a function $\pi : \Omega \to X$ that associates with each state of nature $\omega \in \Omega$ a payoff $\pi(\omega) \in X$. Note that this concept of a lottery does not include a probability distribution over the states of nature. Rather, the Savage axioms allow us to associate a subjective prior over each state of nature ω, expressing the decision maker's personal assessment of the probability that ω will occur. We suppose that the individual chooses among lotteries without knowing the state of nature, after which "Nature" chooses the state $\omega \in \Omega$ that obtains, so that if the individual chose lottery $\pi \in \mathcal{L}$, his payoff is $\pi(\omega)$.

Now suppose the individual has a preference relation $\succ$ over $\mathcal{L}$ (we use the same symbol $\succ$ for preferences over both outcomes and lotteries). We seek a set of plausible properties of $\succ$ over lotteries that together allow us to deduce (a) a utility function $u: X \to \mathbf{R}$ corresponding to the preference relation $\succ$ over outcomes in X; (b) a probability distribution $p: \Omega \to \mathbf{R}$ such that the expected utility principle holds with respect to the preference relation $\succ$ over lotteries and the utility function $u(\cdot)$; that is, if we define

$$\mathbf{E}_{\pi}[u; p] = \sum_{\omega \in \Omega} p(\omega) u(\pi(\omega)), \tag{2.8}$$

then for any $\pi, \rho \in \mathcal{L}$,

$$\pi \succ \rho \Longleftrightarrow \mathbf{E}_{\pi}[u; p] > \mathbf{E}_{\rho}[u; p].$$

Our first condition is that $\pi \succ \rho$ depends only on states of nature where π and ρ have different outcomes. We state this more formally as

A1. For any $\pi, \rho, \pi', \rho' \in \mathcal{L}$, let $A = \{\omega \in \Omega | \pi(\omega) \neq \rho(\omega)\}$. Suppose we also have $A = \{\omega \in \Omega | \pi'(\omega) \neq \rho'(\omega)\}$. Suppose also that $\pi(\omega) = \pi'(\omega)$ and $\rho(\omega) = \rho'(\omega)$ for $\omega \in A$. Then $\pi \succ \rho \Leftrightarrow \pi' \succ \rho'$.

This axiom allows us to define a *conditional preference* $\pi \succ_A \rho$, where $A \subseteq \Omega$, which we interpret as "π is strictly preferred to ρ, conditional on event A," as follows. We say $\pi \succ_A \rho$ if, for some $\pi', \rho' \in \mathcal{L}$, $\pi(\omega) = \pi'(\omega)$ and $\rho(\omega) = \rho'(\omega)$ for $\omega \in A$, $\pi'(\omega) = \rho'(\omega)$ for $\omega \notin A$, and $\pi' \succ \rho'$. Because of A1, this is well defined (that is, $\pi \succ_A \rho$ does not depend on the particular $\pi', \rho' \in \mathcal{L}$). This allows us to define $\succeq_A$ and $\sim_A$ in a similar manner. We then define an event $A \subseteq \Omega$ to be *null* if $\pi \sim_A \rho$ for all $\pi, \rho \in \mathcal{L}$.

Our second condition is then the following, where we write $\pi = x|A$ to mean $\pi(\omega) = x$ for all $\omega \in A$ (that is, $\pi = x|A$ means π is a lottery that pays x when A occurs).

A2. If $A \subseteq \Omega$ is not null, then for all $x, y \in X$, $\pi = x|A \succ_A \pi = y|A \Leftrightarrow x \succ y$.

This axiom says that a natural relationship between outcomes and lotteries holds: if π pays x given event A and ρ pays y given event A, and if $x \succ y$, then $\pi \succ_A \rho$, and conversely.

Our third condition asserts that the probability that a state of nature occurs is independent from the outcome one receives when the state occurs. The difficulty in stating this axiom is that the individual cannot choose probabilities, but only lotteries. But, if the individual prefers x to y, and if $A, B \subseteq \Omega$ are events, then the individual treats A as "more probable" than B if and only if a lottery that pays x when A occurs and y when A does not occur will be preferred to a lottery that pays x when B occurs and y when B does not. However, this must be true for any $x, y \in X$ such that $x \succ y$, or the individual's notion of probability is incoherent (that is, it depends on what particular payoffs we are talking about. For instance, some people engage in "wishful thinking," where if the prize associated with an event increases, the individual thinks it is more likely to occur). More formally, we have the following, where we write $\pi = x, y|A$ to mean "$\pi(\omega) = x$ for $\omega \in A$ and $\pi(\omega) = y$ for $\omega \notin A$."

A3. Suppose $x \succ y$, $x' \succ y'$, $\pi, \rho, \pi', \rho' \in \mathcal{L}$, and $A, B \subseteq \Omega$. Suppose that $\pi = x, y|A$, $\rho = x', y'|A$, $\pi' = x, y|B$, $\rho' = x', y'|B$. Then $\pi \succ \pi' \Leftrightarrow \rho \succ \rho'$.

The fourth condition is a weak version of *first-order stochastic dominance*, which says that if one lottery has a higher payoff than another for any event, then the first is preferred to the second.

A4. For any event A, if $x \succ \rho(\omega)$ for all $\omega \in A$, then $\pi = x|A \succ_A \rho$. Also, for any event A, if $\rho(\omega) \succ x$ for all $\omega \in A$, then $\rho \succ_A \pi = x|A$.

In other words, if for any event A, $\pi = x$ on A pays more than the best ρ can pay on A, the $\pi \succ_A \rho$, and conversely.

Finally, we need a technical property to show that a preference relation can be represented by a utility function. It says that for any $\pi, \rho \in \mathcal{L}$, and any $x \in X$, we can *partition* Ω into a number of disjoint subsets $A_1, \ldots A_n$ such that $\cup_i A_i = \Omega$, and for each A_i, if we change π so that its payoff is x on A_i, then π is still preferred to ρ. Similarly, for each A_i, if we change ρ so that its payoff is x on A_i, then π is still preferred to ρ. This means that no payoff is "supergood," so that no matter how unlikely an event A is, a lottery with that payoff when A occurs is always preferred to a lottery with a different payoff when A occurs. Similarly, no payoff can be "superbad." The condition is formally as follows:

A5. For all $\pi, \pi', \rho, \rho' \in \mathcal{L}$ with $\pi \succ \rho$, and for all $x \in X$, there are disjoint subsets $A_1, \ldots, A_n$ of Ω such that $\cup_i A_i = \Omega$ and for any A_i (a) if $\pi'(\omega) = x$ for $\omega \in A_i$ and $\pi'(\omega) = \pi(\omega)$ for $\omega \notin A_i$, then $\pi' \succ \rho$, and (b) if $\rho'(\omega) = x$ for $\omega \in A_i$ and $\rho'(\omega) = \rho(\omega)$ for $\omega \notin A_i$, then $\pi \succ \rho'$.

We then have Savage's theorem.

THEOREM 2.3 *Suppose* **A1–A5** *hold. Then there is a probability function* p *on* Ω *and a utility function* $u : X \to \mathbf{R}$ *such that for any* $\pi, \rho \in \mathcal{L}$, $\pi \succ \rho$ *if and only if* $\mathbf{E}_\pi[u; p] > \mathbf{E}_\rho[u; p]$.

The proof of this theorem is somewhat tedious (it is sketched in Kreps 1988).

We call the probability p the individual's *Bayesian prior*, or *subjective prior*, and say that A1–A5 imply *Bayesian rationality*, because they together imply Bayesian probability updating.

2.4 Risk and the Shape of the Utility Function

If $\succeq$ is defined over X, we can say nothing about the *shape* of a utility function $u(\cdot)$ representing $\succeq$, because by theorem 2.2, any increasing function of $u(\cdot)$ also represents $\succeq$. However, if $\succeq$ is represented by a utility function $u(x)$ satisfying the expected utility principle, then $u(\cdot)$ is determined up to an arbitrary constant and unit of measure.[4]

THEOREM 2.4 *Suppose the utility function* $u(\cdot)$ *represents the preference relation* $\succeq$ *and satisfies the expected utility principle. If* $v(\cdot)$ *is another utility function representing* $\succeq$, *then there are constants* $a, b \in \mathbf{R}$ *with* $a > 0$ *such that* $v(x) = au(x) + b$ *for all* $x \in X$.

[4]Because of this theorem, the difference between two utilities means nothing. We thus say utilities over outcomes are *ordinal*, meaning we can say that one bundle is preferred to another, but we cannot say by how much. By contrast, the next theorem shows that utilities over lotteries are *cardinal*, in the sense that, up to an arbitrary constant and an arbitrary positive choice of units, utility is numerically uniquely defined.

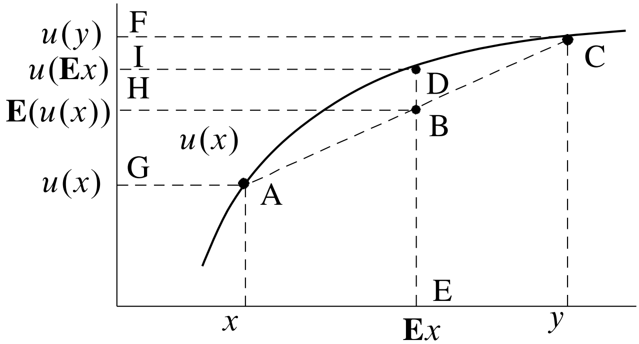


Figure 2.2. A concave utility function

see Mas-Colell, Whinston and Green (1995):173 prove this theorem.

If $X = \mathbf{R}$, so the payoffs can be considered to be "money," and utility satisfies the expected utility principle, what shape do such utility functions have? It would be nice if they were linear in money, in which case expected utility and expected value would be the same thing (why?). But generally utility will be *strictly concave*, as illustrated in figure 2.2. We say a function $u : X \to \mathbf{R}$ is strictly concave, if for any $x, y \in X$, and any $p \in (0, 1)$, we have $pu(x) + (1 - p)u(y) < u(px + (1 - p)y)$. We say $u(x)$ is *weakly concave*, or simply *concave* if, for any $x, y \in X$, $pu(x) + (1 - p)u(y) \leq u(px + (1 - p)y)$.

If we define the lottery π as paying x with probability p and y with probability $1 - p$, then the condition for strict concavity says that *the expected utility of the lottery is less than the utility of the expected value of the lottery*, as depicted in figure 2.2. To see this, note that the expected value of the lottery is $E = px + (1 - p)y$, which divides the line segment between x and y into two segments, the segment xE having length $(px + (1 - p)y) - x = (1 - p)(y - x)$, and the segment Ey having length $y - (px + (1 - p)y) = p(y - x)$. Thus, E divides $[x, y]$ into two segments whose lengths have ratio $(1 - p)/p$. From elementary geometry, it follows that B divides segment $[A, C]$ into two segments whose lengths have the same ratio. By the same reasoning, point H divides segments $[F, G]$ into segments with the same ratio of lengths. This means the point H has the coordinate value $pu(x) + (1 - p)u(y)$, which is the expected utility of the lottery. But by definition, the utility of the expected value of the lottery is at D, which lies above H. This proves that the utility of the expected value is greater than the expected value of the lottery for a strictly concave utility function. This is know as *Jensen's inequality*.

What *are* good candidates for $u(x)$? It is easy to see that strict concavity means $u''(x) < 0$, providing $u(x)$ is twice differentiable (which we assume). But there are lots of functions with this property. According to the famous *Weber-Fechner law* of psychophysics, for a wide range of sensory stimuli, and over a wide range of levels of stimulation, a just-noticeable change in a stimulus is a constant fraction of the original stimulus. If this holds for money, then the utility function is logarithmic.

We say an individual is *risk averse* if the individual prefers the expected value of a lottery to the lottery itself (provided, of course, the lottery does not offer a single payoff with probability 1, which we call a "sure thing"). We know, then, that an individual with utility function $u(\cdot)$ is risk averse if and only if $u(\cdot)$ is concave.[5] Similarly, we say an individual is *risk loving* if he prefers any lottery to the expected value of the lottery, and *risk neutral* if he considers a lottery and its expected value to be equally desirable. Clearly, an individual is risk neutral if and only if he has linear utility.

Does there exist a measure of risk aversion that allows us to say when one individual is more risk averse than another, or how an individual's risk aversion changes with changing wealth? We may define individual A to be *more risk averse* than individual B if whenever A prefers a lottery to an amount of money x, B will also prefer the lottery to x. We say A is *strictly* more risk averse than B if he is more risk averse, and there is some lottery that B prefers to an amount of money x, but such that A prefers x to the lottery.

Clearly, the degree of risk aversion depends on the curvature of the utility function (by definition the *curvature* of $u(x)$ at x is $u''(x)$), but because $u(x)$ and $v(x) = au(x) + b$ $(a > 0)$ describe the same behavior, but $v(x)$ has curvature a times that of $u(x)$, we need something more sophisticated. The obvious candidate is $\lambda_u(x) = -u''(x)/u'(x)$, which does not depend on scaling factors. This is called the *Arrow-Pratt coefficient of absolute risk*

[5]One may ask why people play government-sponsored lotteries, or spend money at gambling casinos, if they are generally risk averse. The most plausible explanation is that people enjoy the act of gambling. The same woman who will have insurance on her home and car, both of which presume risk aversion, will gamble small amounts of money for recreation. An excessive love for gambling, of course, leads an individual either to personal destruction or to wealth and fame (usually the former).

aversion, and it is exactly the measure that we need. We have the following theorem.

THEOREM 2.5 *An individual with utility function $u(x)$ is strictly more risk averse than an individual with utility function $v(x)$ if and only if $\lambda_u(x) > \lambda_v(x)$ for all x.*

For example, the logarithmic utility function $u(x) = \ln(x)$ has Arrow-Pratt measure $\lambda_u(x) = 1/x$, which decreases with x; that is, as the individual becomes wealthier, he becomes less risk averse. Studies show that this property, called *decreasing absolute risk aversion*, holds rather widely (Rosenzweig and Wolpin 1993; Saha, Shumway and Talpaz 1994; Nerlove and Soedjiana 1996). Another increasing concave function is $u(x) = x^a$ for $a \in (0, 1)$, for which $\lambda_u(x) = (1-a)/x$, which also exhibits decreasing absolute risk aversion. Similarly, $u(x) = 1 - x^{-a}$ $(a > 0)$ is increasing and concave, with $\lambda_u(x) = -(a + 1)/x$, which again exhibits decreasing absolute risk aversion. If utility is unbounded, it is easy to show that there is a lottery that you would be willing to give all your wealth to play, no matter how rich you are. This is not plausible behavior. However, this utility has the additional attractive property that *utility is bounded:* no matter how rich you are, $u(x) < 1$. Yet another candidate for a utility function is $u(x) = 1 - e^{-ax}$, for some $a > 0$. In this case $\lambda_u(x) = a$, which we call *constant absolute risk aversion*.

Another commonly used term is the *coefficient of relative risk aversion*, $\mu_u(x) = \lambda_u(x)/x$. Note that for any of the utility functions $u(x) = \ln(x)$, $u(x) = x^a$ for $a \in (0, 1)$, and $u(x) = 1 - x^{-a}$ $(a > 0)$, $\mu_u(x)$ is constant, which we call *constant relative risk aversion*. For $u(x) = 1-e^{-ax}$ $(a > 0)$, we have $\mu_u(x) = a/x$, so we have *decreasing relative risk aversion*.

As an exercise, consider the utility function

$$u(x) = 1 - e^{-ax^b},$$

where $b < 1$ and $ab \geq 0$.

a. Show that $u(x)$ is increasing and concave.
b. Find the Arrow-Pratt measures of absolute and relative risk aversion for this utility function.
c. Show that $u(x)$ exhibits decreasing absolute risk aversion for $a < 0$, constant relative risk aversion for $a = 0$, and increasing relative risk aversion for $a > 0$.

2.5 The Scientific Status of the Rational Actor Model

The extent to which the rational actor model is supported by the facts is dealt with extensively in Gintis (2009), so I give only a brief overview of the issues here.

The three principles of preference consistency developed in section 2.1 are sufficient to support a model of individual choice as maximizing a preference function subject to constraints. Studies of market behavior by economists generally support preference consistency, finding that individual choices are broadly consistent (Mattei 1994; Sippel 1997; Harbaugh, Krause and Berry 2001; Andreoni and Miller 2002). For instance, Mattei (1994) found that 50% of consumers conform perfectly to preference consistency, and almost all deviations were very small (due to the difficulty of maximizing perfectly) or involved consumers making infrequent purchases. Moreover, as suggested by Becker (1962), aggregate demand may exhibit consistency even when individuals are inconsistent. This observation has been tested and validated by Forsythe, Nelson, Neumann and Wright (1992), Gode and Sunder (1993), and List and Millimet (2007).

The rational actor model is, of course, basically a *psychological* model, and it has been widely tested by psychologists, most notably Amos Tversky, Daniel Kahneman, and their co-workers (Kahneman, Slovic and Tversky 1982). Psychologists have been generally critical of the rational actor model, and it is not widely used in psychological research. The reason for this discrepancy between economic and psychological data has a fairly straightforward explanation.

Psychologists study *delibertive decision making*, including the formation of long-term personal goals and the evaluation of inherently unknowable uncertainties. Complex human decisions tend to involve problems that arise infrequently, such as choice of career, whether to marry and to whom, how many children to have, how to deal with a health threat, and how to evaluate and choose when there are novel products.

By contrast, the rational actor model applies to *routine choice* situations, where such ambiguities and complications are absent. In the economic model, the choice set is clearly delineated, and the payoffs, or at least their probabilities, are known. Most psychologists working on decision making in fact do accept the rational actor model as the appropriate model of choice behavior in the realm of routine choice, amended often by making an individual's current state an argument of the preference function. However, they have not found a way to extend the model to the more complex situations involved in deliberative decision making.

The expected utility aspect of the rational actor model (§2.3) is, of course, a far more powerful tool than simple preference consistency, especially because it requires that the individual evaluate complex lotteries appropriately. This is a feat that even advanced students of probability theory have difficulty in accomplishing. This should be clear to you if you had problems with the material in chapter 1! Indeed, there are systematic ways individuals deviate from expected utility theory, revealed in the famous Allais and Ellsberg paradoxes, among others. However, it is reasonable to accept the expected utility theorem and treat the experimentally derived exceptions as equivalent to optical illusions: misjudgment does occur and can be very dramatic, but the existence of optical illusions does not undermine our understanding of vision as generally veridical.

Even when applied to routine decision making, the archetypal rational actor model is a considerable abstraction from individual choice behavior. For one thing, preferences are ineluctably a function of the current *state* of the actor, including physiological state, income and wealth, and everchanging developmental history. Moreover, the subjective prior derived from observed choices over lotteries is a function of *beliefs*, which are socially constructed and deeply dependent upon cultural experience and social interaction. Finally, most humans are inextricably *social creatures* whose preferences are affected by moral considerations that are situationally specific. In particular, people tend to conform to *social norms* that are reflected in the priorities represented in their preference orderings. Recognizing these dimensions of rational action dramatically complicates the analytical representation of rational action, but there is no alternative, if one's aim is the explanation of human behavior.

3

Game Theory: Basic Concepts

> Each discipline of the social sciences rules comfortably within its own chosen domain . . . so long as it stays largely oblivious of the others.
>
> Edward O. Wilson

3.1 Big John and Little John

Big John and Little John eat coconuts, which dangle from a lofty branch of a palm tree. Their favorite coconut palm produces only one fruit per tree. To get the coconut, at least one of them must climb the tree and knock the coconut loose so that it falls to the ground. Careful energy measurements show that a coconut is worth 10 Kc (kilocalories) of energy, the cost of running up the tree, shaking the coconut loose, and running back down to the ground costs 2 Kc for Big John, but is negligible for Little John, who is much smaller. Moreover, if both individuals climb the tree, shake the coconut loose, then climb down the tree and eat the coconut, Big John gets 7 Kc and Little John gets only 3 Kc, because Big John hogs most of it; if only Big John climbs the tree, while Little John waits on the ground for the coconut to fall, Big John gets 6 Kc and Little John gets 4 Kc (Little John eats some before Big John gets back down from the tree); if only Little John climbs the tree, Big John gets 9 Kc and Little John gets 1 Kc (most of the food is gone by the time Little John gets there).

What will Big John and Little John do if each wants to maximize net energy gain? There is one crucial issue that must be resolved: who decides first what to do, Big John or Little John? There are three possibilities: (a) Big John decides first; (b) Little John decides first; (c) both individuals decide simultaneously. We will go through the three cases in turn.

Assuming Big John decides first, we get the situation depicted in figure 3.1. We call a figure like this a *game tree*, and we call the game it defines an extensive form game. At the top of the game tree is the *root node* (the little dot labeled "Big John") with two *branches*, labeled w (wait) and

c (climb). This means Big John gets to choose and can go either left (w) or right (c). This brings us to the two nodes labeled "Little John," in each of which Little John can wait (w) or climb (c).

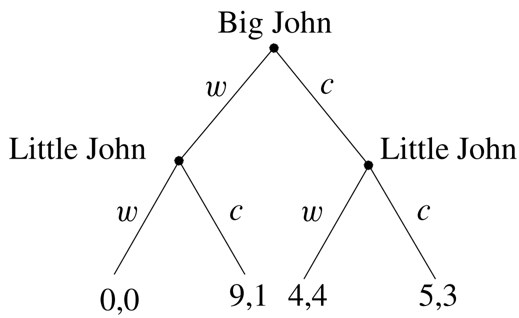


Figure 3.1. Big John and Little John: Big John chooses first.

While Big John has only two strategies, Little John actually has four:

a. Climb no matter what Big John does (cc).
b. Wait no matter what Big John does (ww).
c. Do the same thing Big John does (wc).
d. Do the opposite of what Big John does (cw).

The first letter in parenthesis indicates Little John's move if Big John waits, and the second is Little John's move if Big John climbs.

We call a move taken by a player at a node an *action*, and we call a series of actions that fully define the behavior of a player a *pure strategy* (we define "mixed" and "behavioral" strategies later). Thus, Big John has two strategies, each of which is simply an action, whereas Little John has four strategies, each of which is two actions; one to be used when Little John goes left, and one when Little John goes right.

At the bottom of the game tree are four nodes, which we variously call *leaf* or *terminal nodes*. At each terminal node is the payoff to the two players, Big John (player 1) first and Little John (player 2) second, if they choose the strategies that take them to that particular leaf. You should check that the payoffs correspond to our preceding description. For instance, at the leftmost leaf when both wait, with John neither expending nor ingesting energy, the payoff is (0,0). At the rightmost leaf both climb the tree, costing Big John 2 Kc, after which Big John gets 7 Kc and Little John gets 3 Kc. Their net payoffs are thus (5,3). And similarly for the other two leaves.

How should Big John decide what to do? Clearly, Big John should figure out how Little John will react to each of Big John's two choices, w and c. If Big John chooses w, then Little John will choose c, because this pays 1 Kc as opposed to 0 Kc. Thus, Big John gets 9 Kc by moving left. If Big John chooses c, Little John will choose w, because this pays 4 Kc as opposed to 3 Kc for choosing c. Thus Big John gets 4 Kc for choosing c, as opposed to 9 Kc for choosing w. We now have answered Big John's problem: choose w.

What about Little John? Clearly, Little John must choose c on the left node, but what should he choose on the right node? Of course, it does not really matter, because Little John will never *be* at the right node. However, we must specify not only what a player does "along the path of play" (in this case, the left branch of the tree), but at *all possible nodes on the game tree*. This is because we can say for sure that Big John is choosing a best response to Little John only if we know what Little John does, and conversely. If Little John makes a wrong choice at the right node, in some games (though not this one) Big John would do better by playing c. In short, Little John must choose one of the four strategies listed previously. Clearly, Little John should choose cw (do the opposite of Big John), because this maximizes Little John's payoff no matter what Big John does.

Conclusion: the only reasonable solution to this game is for Big John to wait on the ground, and Little John to do the opposite of what Big John does. Their payoffs are (9,1). We call this a Nash equilibrium (named after John Nash, who invented the concept in about 1950). A Nash equilibrium in a two-player game is a pair of strategies, each of which is a *best response* to the other; that is, each gives the player using it the highest possible payoff, given the other player's strategy.

There is another way to depict this game, called its *strategic form* or *normal form*. It is common to use both representations and to switch back and forth between them, according to convenience. The normal form corresponding to figure 3.1 is in figure 3.2. In this example we array strategies of player 1 (Big John) in rows and the strategies of player 2 (Little John) in columns. Each entry in the resulting matrix represents the payoffs to the two players if they choose the corresponding strategies.

We find a Nash equilibrium from the normal form of the game by trying to pick out a row and a column such that the payoff to their intersection is the highest possible for player 1 down the column, and the highest possible for

		Little John			
		cc	cw	wc	ww
Big John	w	9,1	9,1	0,0	0,0
	c	5,3	4,4	5,3	4,4

Figure 3.2. Normal form of Big John and Little John when Big John moves first

player 2 across the row (there may be more than one such pair). Note that (w, cw) is indeed a Nash equilibrium of the normal form game, because 9 is better than 4 for Big John down the cw column, and 1 is the best Little John can do across the w row.

Can we find any other Nash equilibria to this game? Clearly (w, cc) is also a Nash equilibrium, because w is a best reply to cc and conversely. But the (w, cc) equilibrium has the drawback that if Big John should happen to make a mistake and play c, Little John gets only 3, whereas with cw Little John gets 4. We say cc is *weakly dominated* by cw, meaning that cw pays off at least as well for Little John no matter what Big John does, but for at least one move of Big John, cw has a higher payoff than cc for Little John (§4.1).

But what if Little John plays ww? Then Big John should play c, and it is clear that ww is a best response to c. So this gives us another Nash equilibrium, (c, ww), in which Little John does much better, getting 4 instead of 1, and Big John does much worse, getting 4 instead of 9. Why did we not see this Nash equilibrium in our analysis of the extensive form game? The reason is that (c, ww) involves Little John making an *incredible threat* (see section 4.2 for a further analysis of Little John's incredible threat).

"I do not care what you do," says Little John; "I'm waiting here on the ground no matter what." The threat is "incredible" because Big John knows that if he plays w, then when it is Little John's turn to carry out the threat to play w, Little John will not in fact do so, simply because 1 is better than 0.[1] We say a Nash equilibrium of an extensive form game is *subgame perfect* if, at any point in the game tree, the play dictated by the Nash equilibrium *remains* a Nash equilibrium of the subgame. The strategy (c, ww) is not

[1]This argument fails if the individuals can condition their behavior in one day on their behavior in previous days (see chapter 9). We assume the players cannot do this.

subgame perfect because in the subgame beginning with Little John's choice of w on the left of figure 3.1 is not a best response. Nice try, anyway, Little John!

But what if Little John gets to choose first? Perhaps now Little John can force a better split than getting 1 compared to Big John's 9. This is the extensive form game (figure 3.3). We now call Little John player 1 and Big John player 2. Now Big John has four strategies (the strategies that belonged to Little John in the previous version of the game) and Little John has only two (the ones that belonged to Big John before). Little John notices that Big John's best response to w is c, and Big John's best response to c is w. Because Little John gets 4 in the first case and only 1 in the second, Little John chooses w. Big John's best choice is then cw, and the payoffs are (4,4). Note that *by going first, Little John is able to precommit to a strategy that is an incredible threat when going second.*

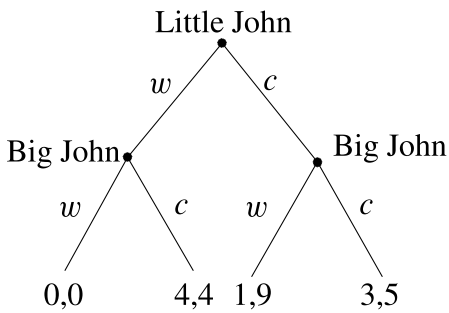


Figure 3.3. Big John and Little John: Little John chooses first

The normal form for the case when Little John goes first is illustrated in figure 3.4. Again we find the two Nash equilibria (w, cc) and (w, cw), and again we find another Nash equilibrium not evident at first glance from the game tree: now it is Big John who has an incredible threat, by playing ww, to which Little John's best response is c.

The final possibility is that the players choose simultaneously, or, equivalently, each player chooses an action without seeing what the other player chooses. In this case, each player has two options: climb the tree (c), or wait on the ground (w). We then get the situation in figure 3.5. Note the new element in the game tree: the dotted line connecting the two places where Little John chooses. This is called an *information set*. Roughly speaking, an information set is a set of nodes at which (a) the same player chooses, and (b) the player choosing does not know which particular node represents

Little John \ Big John	cc	cw	wc	ww
w	4,4	4,4	0,0	0,0
c	3,5	1,9	3,5	1,9

Figure 3.4. Normal form of Big John and Little John game when Little John moves first.

the actual choice node. Note also that we could just as well interchange Big John and Little John in the diagram, reversing their payoffs at the terminal nodes, of course. This illustrates an important point: there may be more than one extensive form game representing the same real strategic situation.

Even though there are fewer strategies in this game, it is hard to see what an equilibrium might be by looking at the game tree. This is because what Little John does cannot depend on which choice Big John makes, because Little John does not see Big John's choice. So let's look at the normal form game, in figure 3.6. From this figure, it is easy to see that both (w, c) and (c, w) are Nash equilibria, the first obviously favoring Big John and the second favoring Little John. In fact, there is a third Nash equilibrium that is more difficult to pick out. In this equilibrium Big John randomizes by choosing c and w with probability $1/2$, and Little John does the same. This is called a *mixed strategy Nash equilibrium*; you will learn how to find and analyze it in section 3.7. In this equilibrium Big John has payoff 4.5 and Little John has payoff 2. The reason for this meager total payoff is that with probability $1/4$, both wait and get zero reward, and sometimes both climb the tree!

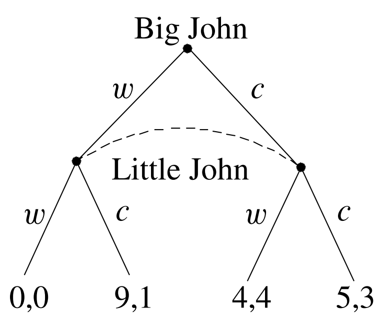


Figure 3.5. Big John and Little John choose simultaneously.

		Little John	
		c	w
Big John	c	5,3	4,4
	w	9,1	0,0

Figure 3.6. Big John and Little John: normal form in the simultaneous-move case

3.2 The Extensive Form

An *extensive form game* $\mathcal{G}$ consists of a number of *players*, a *game tree*, and a set of *payoffs*. A game tree consists of a number of *nodes* connected by *branches*. Each branch connects a *head node* to a distinct *tail node* If b is a branch of the game tree, we denote the head node of b by b^h, and the tail node of b by b^t.

A *path* from node a to node a' in the game tree is a connected sequence of branches starting at a and ending at a'.[2] If there is a path from node a to a', we say a is an *ancestor* of a', and a' is a *successor* to a. We call k the *length* of the path. If a path from a to a' has length one, we call a the *parent* of a', and a' is a *child* of a.

For the game tree, we require a unique node r, called the *root node*, that has no parent, and a set T of nodes, called *terminal nodes* or *leaf nodes*, that have no children. We associate with each terminal node $t \in T$ ($\in$ means "is an element of"), and each player i, a *payoff* $\pi_i(t) \in \mathbf{R}$ ($\mathbf{R}$ is the set of real numbers). We say the game is *finite* if it has a finite number of nodes. We assume all games are finite, unless otherwise stated.

For the graph $\mathcal{G}$, we also require the following *tree property*. There must be *exactly one path* from the root node to any given terminal node in the game tree. Equivalently, *every node except the root node has exactly one parent*.

Players relate to the game tree as follows. Each nonterminal node is assigned to a player who moves at that node. Each branch b with head node b^h represents a particular *action* that the player assigned to b^h can

[2]Technically, a path is a sequence $b_1, \ldots, b_k$ of branches such that $b_1^h = a$, $b_i^t = b_{i+1}^h$ for $i = 1, \ldots k-1$, and $b_k^t = a'$; that is, the path starts at a, the tail of each branch is the head of the next branch, and the path ends at a'.

take there, and hence determines either a terminal node or the next point of play in the game—the particular child node b^t to be visited next.[3]

If a stochastic event occurs at a node a (for instance, the weather is Good or Bad, or your partner is Nice or Nasty), we assign the fictitious player "Nature" to that node, which constitutes the actions Nature takes representing the possible outcomes of the stochastic event, and we attach a *probability* to each branch of which a is the head node, representing the probability that Nature chooses that branch (we assume all such probabilities are strictly positive).

The tree property thus means that there is a *unique* sequence of moves by the players (including Nature) leading from the root node to any specific node of the game tree, and for any two nodes there is *at most one* sequence of player moves leading from the first to the second.

A player may know the exact node in the game tree when it is his turn to move (e.g., the first two cases in Big John and Little John), but he may know only that he is at one of several possible nodes. This is the situation Little John faces in the simultaneous choice case (fig. 3.6). We call such a collection of nodes an *information set*. For a set of nodes to form an information set, the same player must be assigned to move at each of the nodes in the set and have the same array of possible actions at each node.

We also require that if two nodes a and a' are in the same information set for a player, the moves that player made up to a and a' must be the same. This criterion is called *perfect recall*, because if a player never forgets his moves, he cannot make two different choices that subsequently land him in the same information set.[4]

Suppose each player $i = 1, \ldots, n$ chooses strategy s_i. We call $s = (s_1, \ldots, s_n)$ a *strategy profile* for the game, and we define the *payoff* to player i, given strategy profile s, as follows. If there are no moves by Nature, then s determines a unique path through the game tree, and hence

[3]Thus if $\mathbf{p} = (b_1, \ldots, b_k)$ is a path from a to a', then starting from a, if the actions associated with the b_j are taken by the various players, the game moves to a'.

[4]Another way to describe perfect recall is to note that the information sets $\mathcal{N}_i$ for player i are the nodes of a graph in which the children of an information set $\nu \in \mathcal{N}_i$ are the $\nu' \in \mathcal{N}_i$ that can be reached by one move of player i, plus some combination of moves of the other players and Nature. Perfect recall means that this graph has the tree property.

a unique terminal node $t \in T$. The payoff $\pi_i(s)$ to player i under strategy profile s is then defined to be simply $\pi_i(t)$.

Suppose there are moves by Nature, by which we mean that at one or more nodes in the game tree, there is a lottery over the various branches emanating from that node, rather than a player choosing at that node. For every terminal node $t \in T$, there is a unique path $\mathbf{p}_t$ in the game tree from the root node to t. We say $\mathbf{p}_t$ is *compatible* with strategy profile s if, for every branch b on $\mathbf{p}_t$, if player i moves at b^h (the head node of b), then s_i chooses action b at b^h. If $\mathbf{p}_t$ is not compatible with s, we write $p(s,t) = 0$. If $\mathbf{p}_t$ is compatible with s, we define $p(s,t)$ to be the product of all the probabilities associated with the nodes of $\mathbf{p}_t$ at which Nature moves along $\mathbf{p}_t$, or 1 if Nature makes no moves along $\mathbf{p}_t$. We now define the payoff to player i as

$$\pi_i(s) = \sum_{t \in T} p(s,t)\pi_i(t). \tag{3.1}$$

Note that this is the expected payoff to player i given strategy profile s, if we assume that Nature's choices are independent, so that $p(s,t)$ is just the probability that path $\mathbf{p}_t$ is followed, given strategy profile s. We generally assume in game theory that players attempt to maximize their expected payoffs, as defined in (3.1).

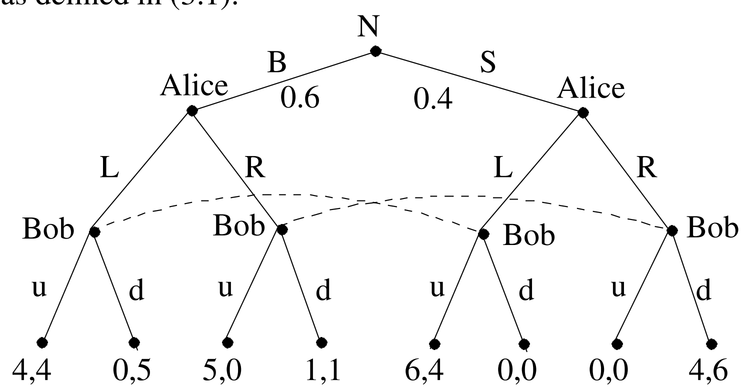


Figure 3.7. Evaluating payoffs when there is a move by Nature

For example, consider the game depicted in figure 3.7. Here, Nature moves first and, with probability $p_l = 0.6$, goes to B where the game between Alice and Bob is known as the prisoner's dilemma (§3.11), and with probability $p_l = 0.4$ goes S, where the game between Alice and Bob is known as the battle of the sexes (§3.9). Note that Alice knows Nature's move, because she has separate information sets on the two branches where

Nature moves, but Bob does not, because when he moves, he does not know whether he is on the left- or right-hand branch.

Alice's strategies can be written LL, LR, RL, and RR, where LL means choose L whatever Nature chooses, RR means choose R whatever Nature chooses, LR means chose L when Nature chooses B, and choose R when Nature chooses S, and finally, RL means choose R when Nature chooses B and choose L when Nature chooses S. Similarly we can write Bob's choices as uu, ud, du, and dd, where uu means choose u whatever Alice chooses, dd means choose d whatever Alice chooses, ud means chose u when Alice chooses L, and choose d when Alice chooses R, and finally, and du means choose d when Alice chooses L and choose u when Alice chooses R.

Let us write, $\pi_A(x,y,z)$ and $\pi_B(x,y,z)$ for the payoffs to Alice and Bob, respectively, when Alice plays $x \in \{LL, LR, RL, RR\}$, Bob plays $y \in \{uu, ud, du, dd\}$ and Nature plays $z \in \{B, S\}$. Then, using the above parameter values, (3.1) gives the following equations.

$$\begin{aligned}
\pi_A(LL,uu) &= p_u\pi_A(LL,uu,B) + p_r\pi_A(LL,uu,S)\\
&= 0.6(4) + 0.4(6) = 4.8;\\
\pi_B(LL,uu) &= p_u\pi_B(LL,uu,B) + p_r\pi_B(LL,uu,S)\\
&= 0.6(4) + 0.4(4) = 4.0;
\end{aligned}$$

The reader should fill in the payoffs at the remaining nodes.

3.3 The Normal Form

The *strategic form* or *normal form* game consists of a number of players, a set of strategies for each of the players, and a payoff function that associates a payoff to each player with a choice of strategies by each player. More formally, an n-player normal form game consists of:

a. A set of *players* $i = 1,\ldots,n$.
b. A set S_i of *strategies* for player $i = 1,\ldots,n$. We call $s = (s_1,\ldots,s_n)$, where $s_i \in S_i$ for $i = 1,\ldots,n$, a *strategy profile* for the game.[5]
c. A function $\pi_i : S \rightarrow \mathbf{R}$ for player $i = 1,\ldots,n$, where S is the set of strategy profiles, so $\pi_i(s)$ is player i's payoff when strategy profile s is chosen.

[5]Technically, these are *pure strategies*, because later we will consider *mixed strategies* that are probabilistic combinations of pure strategies.

Two extensive form games are said to be *equivalent* if they correspond to the same normal form game, except perhaps for the labeling of the actions and the naming of the players. But given an extensive form game, how exactly do we form the corresponding normal form game? First, the players in the normal form are the same as the players in the extensive form. Second, for each player i, let S_i be the set of strategies of that player, each strategy consisting of a choice of an action at each information set where i moves. Finally, the payoff functions are given by equation (3.1). If there are only two players and a finite number of strategies, we can write the payoff function in the form of a matrix, as in figure 3.2.

As an exercise, you should work out the normal form matrix for the game depicted in figure 3.7.

3.4 Mixed Strategies

Suppose a player has pure strategies $s_1, \ldots, s_k$ in a normal form game. A *mixed strategy* for the player is a probability distribution over $s_1, \ldots, s_k$; that is, a mixed strategy has the form

$$\sigma = p_1 s_1 + \ldots + p_k s_k,$$

where $p_1, \ldots p_k$ are all nonnegative and $\sum_1^n p_j = 1$. By this we mean that the player chooses s_j with probability p_j, for $j = 1, \ldots, k$. We call p_j the *weight* of s_j in σ. If all the p_j's are zero except one, say $p_l = 1$, we say σ is a *pure strategy*, and we write $\sigma = s_l$. We say that pure strategy s_j is *used* in mixed strategy σ if $p_j > 0$. We say a strategy is *strictly mixed* if it is not pure, and we say that it is *completely mixed* if it uses all pure strategies. We call the set of pure strategies used in a mixed strategy σ_i the *support* of σ_i.

In an n-player normal form game where, for $i = 1, \ldots, n$, player i has pure-strategy set S_i, a *mixed-strategy profile* $\sigma = (\sigma_1, \ldots, \sigma_n)$ is the choice of a mixed strategy σ_i by each player. We define the *payoffs* to σ as follows. Let $\pi_i(s_1, \ldots, s_n)$ be the payoff to player i when players use the pure-strategy profile $(s_1, \ldots, s_n)$, and if s is a pure strategy for player i, let p_s be the weight of s in σ_i. Then we define

$$\pi_i(\sigma) = \sum_{s_1 \in S_1} \ldots \sum_{s_n \in S_n} p_{s_1} p_{s_2} \ldots p_{s_n} \pi_i(s_1, \ldots, s_n).$$

This is a formidable expression, but the idea behind it is simple. We assume the players' choices are made independently, so the probability that the particular pure strategies $s_1 \in S_1,\ldots,s_n \in S_n$ will be used is simply the product $p_{s_1} \ldots p_{s_n}$ of their weights, and the payoff to player i in this case is just $\pi_i(s_1,\ldots,s_n)$. We get the expected payoff by multiplying and adding up over all n-tuples of mixed strategies.

3.5 Nash Equilibrium

The concept of a Nash equilibrium of a game is formulated most easily in terms of the normal form. Suppose the game has n players, with strategy sets S_i and payoff functions $\pi_i : S \to \mathbf{R}$, for $i = 1,\ldots,n$, where S is the set of strategy profiles. We use the following very useful notation. Let ΔS_i be the set of mixed strategies for player i, and let $\Delta^* S = \Delta S_1 \times \ldots \times \Delta S_n$. If $\sigma \in \Delta^* S$, we write σ_i for the ith component of σ (that is, σ_i is player i mixed strategy in σ). If $\sigma \in \Delta^* S$, and $\tau_i \in \Delta S_i$, we write

$$(\sigma_{-i},\tau_i) = (\tau_i,\sigma_{-i}) = \begin{cases} (\tau_1,\sigma_2,\ldots,\sigma_n) & \text{if } i = 1 \\ (\sigma_1,\ldots,\sigma_{i-1},\tau_i,\sigma_{i+1},\ldots,\sigma_n) & \text{if } 1 < i < n \\ (\sigma_1,\ldots,\sigma_{n-1},\tau_n) & \text{if } i = n \end{cases}.$$

Thus, (σ_{-i},τ_i) is the strategy profile obtained by replacing σ_i with τ_i.

We say a strategy profile $\sigma^* = (\sigma_1^*,\ldots,\sigma_n^*) \in \Delta^* S$ is a *Nash equilibrium* if, for every player $i = 1,\ldots,n$, and every $\sigma_i \in \Delta S_i$, we have $\pi_i(\sigma^*) \geq \pi_i(\sigma_{-i}^*,\sigma_i)$; that is, choosing σ_i^* is at least as good for player i as choosing any other σ_i given that the other players choose σ_{-i}^*. Note that in a Nash equilibrium, the strategy of each player is a *best response* to the strategies chosen by all the other players. Finally, notice that a player could have responses that are *equally good* as the one chosen in the Nash equilibrium; there just cannot be a strategy that is strictly better.

The Nash equilibrium concept is important because in many cases we can accurately (or reasonably accurately) predict how people will play a game by assuming they will choose strategies that implement a Nash equilibrium. It will also turn out that, in dynamic games that model an evolutionary process whereby successful strategies drive out unsuccessful ones over time, stable stationary states are always Nash equilibria. Conversely, we will see that Nash equilibria that seem implausible are actually *unstable* equilibria of an evolutionary process, so we would not expect to see them in the real world. Where people appear to deviate systematically from implementing Nash equilibria, we will sometimes find that they do not understand the game, or that we have misspecified the game they are playing or the payoffs we attribute to them. But, in important cases, as we shall see, people simply do not play Nash equilibria at all, and they are the better for it. In no sense is the failure to play a Nash equilibrium and indication if irrationality, bounded rationality, or any other cognitive deficit on the part of the players.

3.6 The Fundamental Theorem of Game Theory

John Nash (1950) showed that every finite game has a Nash equilibrium in mixed strategies. More concretely, we have

THEOREM 3.1 Nash Existence Theorem. If each player in an n-player game has a finite number of pure strategies, then the game has a (not necessarily unique) Nash equilibrium in (possibly) mixed strategies.

The following fundamental theorem of mixed-strategy equilibrium develops the principles for finding Nash equilibria. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a mixed-strategy profile for an n-player game. For any player $i = 1, \dots, n$, let σ_{-i} represent the mixed strategies used by all the players other than player i. The *fundamental theorem of mixed-strategy Nash equilibrium* says that σ is a Nash equilibrium if and only if, for any player $i = 1, \dots, n$ with pure-strategy set S_i,

a. If $s, s' \in S_i$ occur with positive probability in σ_i, then the payoffs to s and s', when played against σ_{-i}, are equal.
b. If s occurs with positive probability in σ_i and s' occurs with zero probability in σ_i, then the payoff to s' is less than or equal to the payoff to s, when played against σ_{-i}.

The proof of the fundamental theorem is straightforward. Suppose σ is the player's mixed strategy in a Nash equilibrium that uses s with probability $p > 0$ and s' with probability $p' > 0$. If s has a higher payoff than s' when played against σ_{-i}, then i's mixed strategy that uses s with probability $(p + p')$, does not use s', and assigns the same probabilities to the other pure strategies as does σ has a higher payoff than σ, so σ is not a best response to σ_{-i}. This is a contradiction, which proves the assertion. The rest of the proof is similar.

3.7 Solving for Mixed-Strategy Nash Equilibria

	L	R
U	a_1, a_2	b_1, b_2
D	c_1, c_2	d_1, d_2

This problem asks you to apply the general method of finding mixed-strategy equilibria in normal form games. Consider the game in the figure. First, of course, you should check for pure-strategy equilibria. To check for a completely mixed-strategy equilibrium, we use the fundamental theorem (3.6). Suppose the column player uses the strategy $\sigma = \alpha L + (1-\alpha)R$ (that is, plays L with probability α). Then, if the row player uses both U and D, they must both have the same payoff against σ. The payoff to U against σ is $\alpha a_1 + (1-\alpha)b_1$, and the payoff to D against σ is $\alpha c_1 + (1-\alpha)d_1$. Equating these two, we find

$$\alpha = \frac{d_1 - b_1}{d_1 - b_1 + a_1 - c_1}.$$

For this to make sense, the denominator must be nonzero, and the right-hand side must lie between zero and one. Note that *column* player's strategy is determined by the requirement that *row* player's two strategies be equal.

Now suppose the row player uses strategy $\tau = \beta U + (1-\beta)D$ (that is, plays U with probability β). Then, if the column player uses both L and R, they must both have the same payoff against τ. The payoff to L against τ is $\beta a_2 + (1-\beta)c_2$, and the payoff to R against τ is $\beta b_2 + (1-\beta)d_2$. Equating these two, we find

$$\beta = \frac{d_2 - c_2}{d_2 - c_2 + a_2 - b_2}.$$

Again, for this to make sense, the denominator must be nonzero, and the right-hand side must lie between zero and one. Note that now *row* player's strategy is determined by the requirement that *column* player's two strategies are equal.

a. Suppose the preceding really is a mixed-strategy equilibrium. What are the payoffs to the two players?
b. Note that to solve a 2×2 game, we have checked for five different "configurations" of Nash equilibria, four pure and one mixed. But there are four more possible configurations, in which one player uses a pure

strategy and the second player uses a mixed strategy. Show that if there is a Nash equilibrium in which the row player uses a pure strategy (say UU) and the column player uses a completely mixed strategy, then *any* strategy for the column player is a best response to UU.

c. How many different configurations are there to check for in a 2×3 game? In a 3×3 game?

d. Can you generalize to the number of possible configurations of Nash equilibria in an $n \times m$ normal form game?

3.8 Throwing Fingers

	c_1	c_2
c_1	$1, -1$	$-1, 1$
c_2	$-1, 1$	$1, -1$

Alice and Bob each throws one (c_1) or two (c_2) fingers, simultaneously. If they are the same, Alice wins; otherwise, Bob wins. The winner takes \$1 from the loser. The normal form of this game is depicted in the accompanying diagram. There are no pure-strategy equilibria, so suppose Bob uses the mixed strategy σ that consists of playing c_1 (one finger) with probability α and c_2 (two fingers) with probability $1-\alpha$. We write this as $\sigma = \alpha c_1 + (1-\alpha)c_2$. If Alice uses both c_1 (one finger) and c_2 (two fingers) with positive probability, they both must have the same payoff against σ, or else Alice should drop the lower-payoff strategy and use only the higher-payoff strategy. The payoff to c_1 against σ is $\alpha \cdot 1 + (1-\alpha) \cdot -1 = 2\alpha - 1$, and the payoff to c_2 against σ is $\alpha \cdot -1 + (1-\alpha) \cdot 1 = 1 - 2\alpha$. If these are equal, then $\alpha = 1/2$. A similar reasoning shows that Alice chooses each strategy with probability 1/2. The expected payoff to Alice is then $2\alpha - 1 = 1 - 2\alpha = 0$, and the same is true for Bob.

3.9 Battle of the Sexes

Alfredo \ Violetta	g	o
g	2,1	0,0
o	0,0	1,2

Violetta and Alfredo love each other so much that they would rather be together than apart. But Alfredo wants to go gambling, and Violetta wants to go to the opera. Their payoffs are described in the diagram. There are two pure-strategy equilibria and one mixed-strategy equilibrium for this game. We will show that Alfredo and Violetta would be better off if they stuck to either of their pure-strategy equilibria.

Let α be the probability of Alfredo going to the opera, and let β be the probability of Violetta going to the opera. Because in a mixed-strategy equilibrium, the payoff to gambling and opera must be equal for Alfredo, we must have $\beta = 2(1-\beta)$, which implies $\beta = 2/3$. Because the payoff to gambling and opera must also be equal for Violetta, we must have $2\alpha = 1-\alpha$, so $\alpha = 1/3$. The payoff of the game to each is then

$$\frac{2}{9}(1,2) + \frac{5}{9}(0,0) + \frac{2}{9}(2,1) = \left(\frac{2}{3},\frac{2}{3}\right),$$

because both go gambling $(1/3)(2/3) = 2/9$ of the time, both go to the opera $(1/3)(2/3) = 2/9$ of the time, and otherwise they miss each other.

Both players do better if they can coordinate, because (2,1) and (1,2) are both better than (2/3,2/3).

We get the same answer if we find the Nash equilibrium by finding the intersection of the players' best-response functions. To see this, note that the payoffs to the two players are

$$\begin{aligned}\pi_A &= \alpha\beta + 2(1-\alpha)(1-\beta) = 3\alpha\beta - 2\alpha - 2\beta + 2\\ \pi_V &= 2\alpha\beta + (1-\alpha)(1-\beta) = 3\alpha\beta - \alpha - \beta + 1.\end{aligned}$$

Thus,

$$\frac{\partial \pi_A}{\partial \alpha} = 3\beta - 2 \begin{cases} > 0 & \text{if } \beta > 2/3 \\ = 0 & \text{if } \beta = 2/3 \\ < 0 & \text{if } \beta < 2/3 \end{cases},$$

so the optimal α is given by

$$\alpha = \begin{cases} 1 & \text{if } \beta > 2/3 \\ [0,1] & \text{if } \beta = 2/3 \\ 0 & \text{if } \beta < 2/3 \end{cases}.$$

Similarly,

$$\frac{\partial \pi_V}{\partial \beta} = 3\alpha - 1 \begin{cases} > 0 & \text{if } \alpha > 1/3 \\ = 0 & \text{if } \alpha = 1/3 \\ < 0 & \text{if } \alpha < 1/3 \end{cases},$$

so the optimal β is given by

$$\beta = \begin{cases} 1 & \text{if } \alpha > 1/3 \\ [0,1] & \text{if } \alpha = 1/3 \\ 0 & \text{if } \alpha < 1/3 \end{cases}.$$

This gives the diagram depicted in figure 3.8. Note that the three Nash equilibria are the three intersections of the two best-response curves, marked by large dots in the figure.

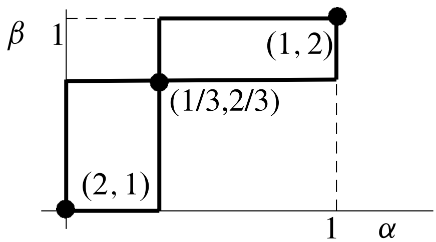


Figure 3.8. Nash equilibria in the battle of the sexes

3.10 The Hawk-Dove Game

	H	D
H	z, z	v, v
D	0,0	$v/2, v/2$

Consider a population of birds that fight over valuable territory. There are two possible strategies. The hawk (H) strategy is to escalate battle until injured or your opponent retreats. The dove (D) strategy is to display hostility but retreat before sustaining injury if your opponent escalates. The payoff matrix is given in the figure, where $v > 0$ is the value of territory, $w > v$ is the cost of injury, and $z = (v - w)/2$ is the payoff when two hawks meet. The birds can play mixed strategies, but they cannot condition their play on whether they are player 1 or player 2, and hence both players must use the same mixed strategy.

As an exercise, explain the entries in the payoff matrix and show that there are no symmetric pure-strategy Nash equilibria. The pure strategy pairs (H,D) and (D,H) are Nash equilibria, but they are not symmetric, so cannot be attained assuming, as we do, that the birds cannot which is player 1 and which is player 2. There is only one symmetric Nash equilibrium, in which players do not condition their behaviors on whether they are player 1 or player 2. This is the game's unique mixed-strategy Nash equilibrium, which we will now analyze.

Let α be the probability of playing hawk. The payoff to playing hawk is then $\pi_h = \alpha(v - w)/2 + (1 - \alpha)v$, and the payoff to playing dove is $\pi_d = \alpha(0) + (1 - \alpha)v/2$. These two are equal when $\alpha^* = v/w$, so the unique symmetric Nash equilibrium occurs when $\alpha = \alpha^*$. The payoff to each player is thus

$$\pi_d = (1-\alpha)\frac{v}{2} = \frac{v}{2}\left(\frac{w-v}{w}\right). \tag{3.2}$$

It is instructive to solve this problem by using the fact that a Nash equilibrium consists of best responses for all players. Now, let α be the probability of playing hawk for one player and let β be the probability of playing hawk for the other (these will turn out to be equal, of course). The payoffs to the two players are

$$\pi_1 = \alpha\beta(v-w)/2 + \alpha(1-\beta)v + (1-\alpha)\beta(0) + (1-\alpha)(1-\beta)v/2$$
$$\pi_2 = \alpha\beta(v-w)/2 + \alpha(1-\beta)(0) + (1-\alpha)\beta v + (1-\alpha)(1-\beta)v/2,$$

which simplifies to

$$\pi_1 = \frac{1}{2}(v(1+\alpha-\beta) - w\alpha\beta)$$
$$\pi_2 = \frac{1}{2}(v(1-\alpha+\beta) - w\alpha\beta).$$

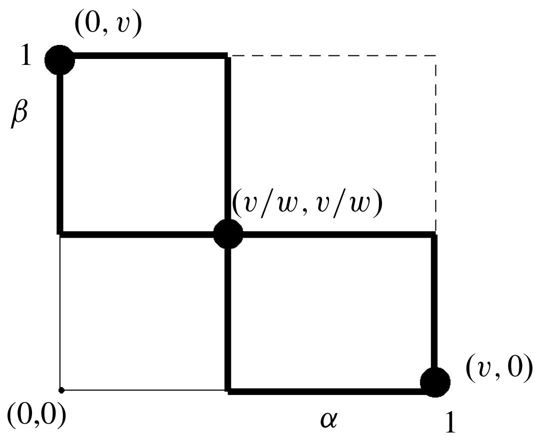


Figure 3.9. Nash equilibria in the hawk-dove game

Thus,

$$\frac{\partial \pi_1}{\partial \alpha} = (v - w\beta)/2 \begin{cases} > 0 & \text{if } \beta < v/w \\ = 0 & \text{if } \beta = v/w \\ < 0 & \text{if } \beta > v/w \end{cases},$$

so the optimal α is given by

$$\alpha = \begin{cases} 1 & \text{if } \beta < v/w \\ [0,1] & \text{if } \beta = v/w \\ 0 & \text{if } \beta > v/w \end{cases}.$$

Similarly,

$$\frac{\partial \pi_2}{\partial \beta} = (v - w\alpha)/2 \begin{cases} > 0 & \text{if } \alpha < v/w \\ = 0 & \text{if } \alpha = v/w \\ < 1 & \text{if } \alpha > v/w \end{cases},$$

so the optimal β is given by

$$\beta = \begin{cases} 0 & \text{if } \alpha > v/w \\ [0,1] & \text{if } \alpha = v/w \\ 1 & \text{if } \alpha < v/w \end{cases}.$$

This gives the diagram depicted in figure 3.9. The best-response functions intersect in three places, each of which is a Nash equilibrium. However, the only symmetric Nash equilibrium, in which the players cannot condition their move on whether they are player 1 or player 2, is the mixed strategy Nash equilibrium $(v/w, v/w)$.

Note that (3.2) implies that when w is close to v, almost all the value of the territory is dissipated in fighting, while for very high w, very little value is lost. This is known as "mutually assured destruction" in military parlance. Of course, if there is some possibility of error, where each player plays hawk by mistake with positive probability, then you can easily show that mutually assured destruction may have a very poor payoff.

3.11 The Prisoner's Dilemma

	C	D
C	R,R	S,T
D	T,S	P,P

Alice and Bob can each earn a profit R if they both work hard (pure strategy C). However, either can shirk by working secretly on private jobs (pure strategy D), earning $T > R$, but the other player will earn only $S < R$. If both shirk, however, they will each earn P, where $S < P < R$. Each must decide independently of the other whether to choose C or D. The game tree is depicted in the figure. The payoff T stands for the 'temptation' to defect on a partner, S stands for "sucker" (for cooperating when your partner defected), P stands for "punishment" (for both shirking), and R stands for "reward" (for both cooperating). We usually assume also that $S + T < 2R$, so there is no gain from "taking turns" playing C (cooperate) and D (defect).

Let α be the probability of playing C if you are Alice, and let β be the probability of playing C if you are Bob. To simplify the algebra, we assume

$P = 1$, $R = 0$, $T = 1 + t$, and $S = -s$, where $s, t > 0$. It is easy to see that these assumptions involve no loss of generality, because adding a constant to all payoffs, or multiplying all payoffs by a positive constant does not change the Nash equilibria of the game. The payoffs to Alice and Bob are now

$$\pi_A = \alpha\beta + \alpha(1-\beta)(-s) + (1-\alpha)\beta(1+t) + (1-\alpha)(1-\beta)(0)$$
$$\pi_B = \alpha\beta + \alpha(1-\beta)(1+t) + (1-\alpha)\beta(-s) + (1-\alpha)(1-\beta)(0),$$

which simplify to

$$\pi_A = \beta(1+t) - \alpha(s(1-\beta) + \beta t)$$
$$\pi_B = \alpha(1+t) - \beta(s(1-\alpha) + \alpha t).$$

It is clear from these equations that π_A is maximized by choosing $\alpha = 0$, no matter what Bob does, and similarly π_B is maximized by choosing $\beta = 0$, no matter what Alice does. This is the mutual defect equilibrium.

This is not how many people play this game in the experimental laboratory. Rather, people very often prefer to cooperate, provided their partners cooperate as well (Kiyonari, Tanida and Yamagishi 2000). We can capture this phenomenon by assuming that there is a psychic gain $\lambda_A > 0$ for Alice and $\lambda_B > 0$ for Bob when both players cooperate, above the temptation payoff $T = 1 + t$. If we rewrite the payoffs using this assumption, we get

$$\pi_A = \alpha\beta(1 + t + \lambda_A) + \alpha(1-\beta)(-s) + (1-\alpha)\beta(1+t)$$
$$\pi_B = \alpha\beta(1 + t + \lambda_B) + \alpha(1-\beta)(1+t) + (1-\alpha)\beta(-s)$$

which simplify to

$$\pi_A = \beta(1+t) - \alpha(s - \beta(s + \lambda_A))$$
$$\pi_B = \alpha(1+t) - \beta(s - \alpha(s + \lambda_B)).$$

The first equation shows that if $\beta > s/(s + \lambda_A)$, then Alice plays C, and if $\alpha > s/(s + \lambda_B)$, then Bob plays C. If the opposite equalities hold, then both play D.

4

Eliminating Dominated Strategies

Um so schlimmer für die Tatsache
(So much the worse for the facts)
Georg Wilhelm Friedrich Hegel

4.1 Dominated Strategies

Suppose S_i is a finite set of pure strategies for players $i = 1, \ldots, n$ in normal form game $\mathcal{G}$, so that $S = S_1 \times \ldots \times S_n$ is the set of pure-strategy profiles for $\mathcal{G}$, and $\pi_i(s)$ is the payoff to player i when strategy profile $s \in S$ is chosen by the players. We denote the set of mixed strategies with support in S as $\Delta^* S = \Delta S_1 \times \ldots \times \Delta S_n$, where ΔS_i is the set of mixed strategies for player i with support in S_i, or equivalently, the set of *convex combinations* of members of S_i. We denote the set of vectors of mixed strategies of all players but i by $\Delta^* S_{-i}$. We say $s'_i \in S_i$ is *strongly dominated* by $s_i \in S_i$ if, for every $\sigma_{-i} \in \Delta^* S_{-i}$, $\pi_i(s_i, \sigma_{-i}) > \pi_i(s'_i, \sigma_{-i})$. We say s'_i is *weakly dominated* by s_i if for every $\sigma_{-i} \in \Delta^* S_{-i}$, $\pi_i(s_i, \sigma_{-i}) \geq \pi_i(s'_i, \sigma_{-i})$, and for at least one choice of σ_{-i} the inequality is strict. Note that a strategy may fail to be strongly dominated by any pure strategy but may nevertheless be strongly dominated by a mixed strategy.

Suppose s_i is a pure strategy for player i such that every $\sigma'_i \neq s_i$ for player i is weakly (respectively strongly) dominated by s_i. We call s_i a *weakly* (respectively *strongly*) *dominant* strategy for i. If there is a Nash equilibrium in which all players use a dominant strategy, we call this a *dominant-strategy equilibrium.*

Once we have eliminated dominated strategies for each player, it often turns out that a pure strategy that was not dominated at the outset is now dominated. Thus, we can undertake a second round of eliminating dominated strategies. Indeed, this can be repeated until pure strategies are no longer eliminated in this manner. In a finite game, this will occur after a finite number of rounds and will always leave at least one pure strategy remaining for each player. If strongly (resp. weakly) dominated strategies are eliminated, we call this the *iterated elimination of strongly (resp. weakly)* *dominated strategies*. We will refer to strategies that are eliminated through the iterated elimination of strongly (resp. weakly) dominated strategies as *recursively strongly* (resp. *weakly*) *dominated*.

In two-player games, the pure strategies that remain after the elimination of strongly dominated strategies correspond to the *rationalizable* strategies. The reader is invited to show that no pure strategy that is recursively strongly dominated can be part of a Nash equilibrium. Weakly dominated strategies can be part of a Nash equilibrium, so the iterated elimination of weakly dominated strategies may discard one or more Nash equilibria of the game. The reader is invited to show that at least one Nash equilibrium survives the iterated elimination of weakly dominated strategies.

There are games, such as the prisoner's dilemma (§3.11), that are "solved" by eliminating recursively strongly dominated strategies in the sense that after the elimination these strategies, a single pure strategy for each player remains and, hence, is the unique Nash equilibrium of the game.

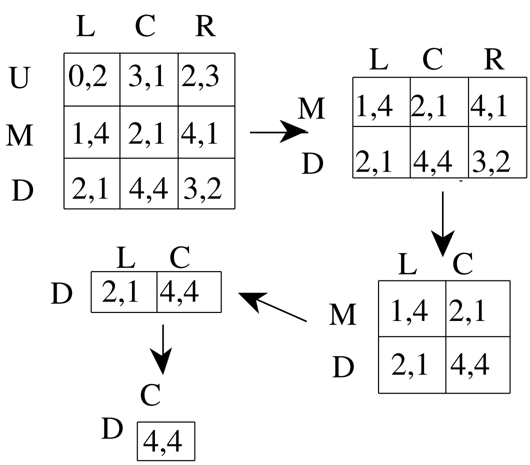


Figure 4.1. The iterated elimination of strongly dominated strategies

Figure 4.1 illustrates the iterated elimination of strongly dominated strategies. First, U is strongly dominated by D for player 1. Second, R is strongly dominated by 0.5L+0.5C for player 2 (note that a pure strategy in this case is not dominated by any other pure strategy but is strongly dominated by a mixed strategy). Third, M is strongly dominated by D. Finally, L is strongly dominated by C. Note that {D,C} is indeed the unique Nash equilibrium of the game.

4.2 Backward Induction

We can eliminate weakly dominated strategies in extensive form games with *perfect information* (that is, where each information set is a single node) as follows. Choose any terminal node $t \in T$ and find the parent node of this terminal node, say node a. Suppose player i chooses at a, and suppose i's highest payoff at a is attained at node $t' \in T$. Erase all the branches from a so a becomes a terminal node, and attach the payoffs from t' to the new terminal node a. Also record i's move at a, so you can specify i's equilibrium strategy when you have finished the analysis. Note that if the player involved moves at one or more other nodes, you have eliminated a *weakly* dominated strategy. Repeat this procedure for all terminal nodes of the original game. When you are done, you will have an extensive form game that is one level less deep than the original game. Now repeat the process as many times as is possible. If the resulting game tree has just one possible move at each node, then when you reassemble the moves you have recorded for each player, you will have a Nash equilibrium.

We call this *backward induction*, because we start at the end of the game and move backward. Note that backward induction can eliminate Nash equilibria that use weakly dominated strategies. Moreover, backward induction is *not* an implication of player Bayesian rationality, as we shall show below.

For an example of backward induction, consider figure 4.2, the Big John (BJ) and Little John (LJ) game where BJ goes first. We start with the terminal node labeled (0,0) and follow it back to the LJ node on the left. At this node, w is dominated by c because $1 > 0$, so we erase the branch where LJ plays w and its associated payoff. We locate the next terminal node in the original game tree, (4,4) and follow back to the LJ node on the right. At this node, c is dominated by w, so we erase the dominated node and its payoff. Now we apply backward induction to this smaller game tree; this time, of course, it is trivial. We find the first terminal node, (9,1), which leads back to BJ. Here c is dominated, so we erase that branch and its payoff. We now have our solution: BJ chooses w, LJ chooses cw, and the payoffs are (9,1).

You also can see from this example that by using backward induction and hence eliminating weakly dominated strategies, we have eliminated the Nash equilibrium c, ww (see figure 3.2). This is because when we assume LJ plays c in response to BJ's w, we have eliminated the weakly dominated

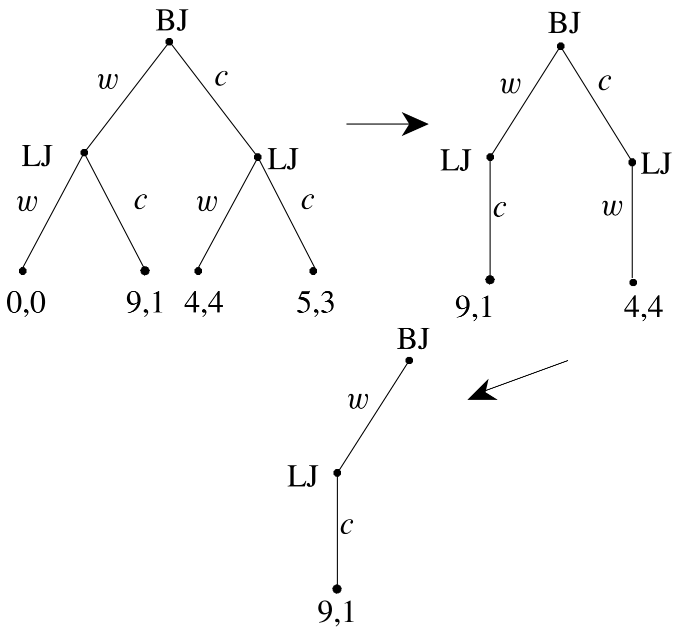


Figure 4.2. An example of backward induction

strategies ww and wc for LJ. We have called c, ww an incredible threat. Backward induction eliminates incredible threats.

4.3 Exercises in Eliminating Dominated Strategies

Apply the iterated elimination of dominated strategies to the following normal form games. Note that in some cases there may remain more that one strategy for each player. Say *exactly* in what order you eliminated rows and columns. Verify that the resulting solution is a Nash equilibrium of the game. Remember: You must eliminate a *whole row* or a *whole column*, not a single cell entry. When eliminating rows, compare the *first* of the two entries, and when eliminating columns, compare the *second* of the two entries.

(a)

	a	b	c
A	2,12	1,10	1,11
B	0,12	0,10	0,11
C	0,12	0,10	0,13

(b)

	a	b	c
A	1,1	$-2,0$	$4,-1$
B	0,3	3,1	5,4
C	1,5	4,2	6,2

(c)

	N_2	C_2	J_2
N_1	73,25	57,42	66,32
C_1	80,26	35,12	32,54
J_1	28,27	63,31	54,29

(d)

	a	b	c	d	e
A	63, −1	28, −1	−2, 0	−2, 45	−3, 19
B	32, 1	2, 2	2, 5	33, 0	2, 3
C	54, 1	95, −1	0, 2	4, −1	0, 4
D	1, −33	−3, 43	−1, 39	1, −12	−1, 17
E	−22, 0	1, −13	−1, 88	−2, −57	−3, 72

(e)

	a	b	c	d	e
A	0, 1	0, 1	0, 1	0, 1	0, 1
B	0.81, 0, 19	0.20, 0.80	0.20, 0.80	0.20, 0.80	0.20, 0.80
C	0.81, 0.19	0.49, 0.51	0.40, 0.60	0.40, 0.60	0.40, 0.60
D	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.60, 0.40	0.60, 0.40
E	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.09, 0.91	0.80, 0.20
F	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.09, 0.91	0.01, 0.99

(f)

	a	b	c	d	e	f	g	h
A	−1, 1	−1, 1	−1, 1	−1, 1	1, −1	1, −1	1, −1	1, −1
B	1, −1	0, 0	1, −1	0, 0	1, −1	0, 0	1, −1	0, 0
C	−1, 1	−1, 1	0, 0	0, 0	−1, 1	−1, 1	0, 0	0, 0

(g)

	0	1	2	3	4	5
0	4, 5	4, 14	4, 13	4, 12	4, 11	4, 10
1	13, 5	3, 4	3, 13	3, 12	3, 11	3, 10
2	12, 5	12, 4	2, 3	2, 12	2, 11	2, 10
3	11, 5	11, 4	11, 3	1, 2	1, 11	1, 10
4	10, 5	10, 4	10, 3	10, 2	0, 1	0, 10

4.4 Subgame Perfection

Let h be a node of an extensive form game $\mathcal{G}$ that is also an information set (that is, it is a *singleton* information set). Let $\mathcal{H}$ be the smallest collection of nodes including h such that if h' is in $\mathcal{H}$, then all of the successor nodes of h' are in $\mathcal{H}$ and all nodes in the same information set as h' are in $\mathcal{H}$. We endow $\mathcal{H}$ with the information set structure, branches, and payoffs inherited from $\mathcal{G}$, the players in $\mathcal{H}$ being the subset of players of $\mathcal{G}$ who move at some information set of $\mathcal{H}$. The reader is invited to show that $\mathcal{H}$ is an extensive form game. We call $\mathcal{H}$ a *subgame* of $\mathcal{G}$.

If $\mathcal{H}$ is a subgame of $\mathcal{G}$ with root node h, then every pure strategy profile s_G of $\mathcal{G}$ that reaches h has a counterpart s_H in $\mathcal{H}$, specifying that players in $\mathcal{H}$ make the same choices with s_H at nodes in $\mathcal{H}$ as they do with s_G at the same node in $\mathcal{G}$. We call s_H the *restriction* of s_G to the subgame $\mathcal{H}$. Suppose $\sigma_G = \alpha_1 s_1 + \ldots + \alpha_k s_k$ ($\sum_i \alpha_i = 1$) is a mixed strategy of $\mathcal{G}$ that reaches the root node h of $\mathcal{H}$, and let $I \subseteq \{1, \ldots, k\}$ be the set of indices such that $i \in I$ iff s_i reaches h. Let $\alpha = \sum_{i \in I} \alpha_i$. Then, $\sigma_H = \sum_{i \in I} (\alpha_i / \alpha) s_i$ is a mixed strategy of $\mathcal{H}$, called the *restriction* of σ_G to $\mathcal{H}$. We are assured that $\alpha > 0$ by the assumption that σ_G reaches h, and the coefficient α_i / α represents the probability of playing s_i, conditional on reaching h.

We say a Nash equilibrium of an extensive form game is *subgame perfect* if its restriction to every subgame is a Nash equilibrium of the subgame. It is clear that if s_G is a Nash equilibrium for a game $\mathcal{G}$, and if $\mathcal{H}$ is a subgame of $\mathcal{G}$ whose root node is reached with positive probability using s_G, then the restriction s_H of s_G to $\mathcal{H}$ must be a Nash equilibrium in $\mathcal{H}$. But if $\mathcal{H}$ is *not* reached using s_G, then it does not matter what players do in $\mathcal{H}$; their choices in $\mathcal{H}$ can be completely arbitrary, so long as these choices do not alter the way people play in parts of the game that *are* reached with positive probability. But how, you might ask, could this make a difference in the larger game? To see how, consider the following Microsoft-Google game.

Microsoft and Google are planning to introduce a new type of Web browser. They must choose between two platforms, Java and ActiveX. If they introduce different platforms, their profits are zero. If they introduce the same platform, their profits are 1, plus Microsoft gets 1 if the platform is ActiveX and Google gets 1 if the platform is Java. The game tree is as shown in figure 4.3, where we assume that Microsoft chooses first.

Because Microsoft has one information set and two choices at that information set, we can write its pure-strategy set as {ActiveX,Java}. Because

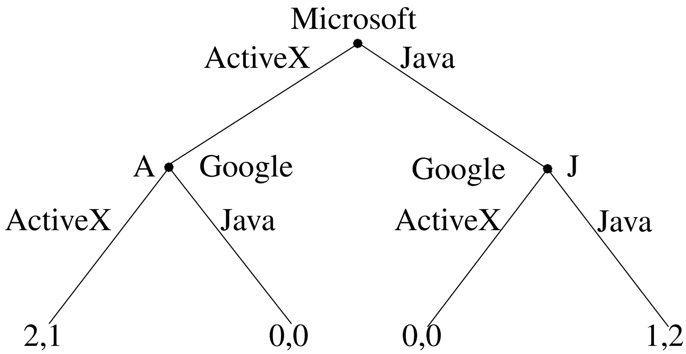


Figure 4.3. The Microsoft-Google game

Google has two information sets and two choices at each, its pure-strategy set has four elements $\{JJ, JA, AJ, AA\}$. Here, the first letter says what Google does if Microsoft plays ActiveX, and the second says what Google does if Microsoft plays Java. Thus, "AJ" means "play ActiveX if Microsoft plays ActiveX, and play Java if Microsoft plays Java."

It is easy to see that $\{A, AA\}$, $\{A, AJ\}$, and $\{J, JJ\}$ are all Nash equilibria (prove this formally!), but only the second is subgame perfect, because the first is not a Nash equilibrium when restricted to the subgame starting at J, and the third is not a Nash equilibrium when restricted to the subgame at A. The outcomes from $\{A, AA\}$ and $\{A, AJ\}$ are the same, because ActiveX is chosen by both players in either equilibrium. But the outcome from $\{J, JJ\}$ is that Java is chosen by both players. This is obviously to the benefit of Google, but it involves an *incredible threat:* Google threatens to choose Java *no matter what Microsoft does*, but Microsoft knows that when it actually comes time to choose at A, Google will in fact choose ActiveX, not Java.

It is easy to see that a simultaneous-move game has no proper subgames (a game is always a subgame of itself; we call the whole game an *improper* subgame), because all the nodes are in the same information set for at least one player. Similarly, a game in which Nature makes the first move and the outcome is not known by at least one other player also has no proper subgames.

At the other extreme, in a game of perfect information (that is, for which all information sets are singletons), *every* nonterminal node is the root node of a subgame. This allows us to find the subgame perfect Nash equilibria of such games by backward induction, as described in section 4.2. This line of reasoning shows that in general, backward induction consists of the iterated elimination of weakly dominated strategies.

4.5 Stackelberg Leadership

A Stackelberg leader is a player who can precommit to following a certain action, so other players effectively consider the leader as "going first," and they predicate their actions on the preferred choices of the leader. Stackelberg leadership is a form of power flowing from the capacity to precommit. To see this, note that a form of behavior that would be an incredible threat or promise without the capacity to precommit becomes part of a subgame perfect Nash equilibrium when precommitment is possible. Formally, consider the normal form game in the matrix.

	t_1	t_2
s_1	0,2	3,0
s_2	2,1	1,3

a. Write an extensive form game for which this is the corresponding normal form, and find all Nash equilibria.
b. Suppose the row player chooses first and the column player sees the row player's choice, but the payoffs are the same. Write an extensive form game, list the strategies of the two players, write a normal form game for this new situation, and find the Nash equilibria.
c. On the basis of these two games, comment on the observation "The capacity to precommit is a form of power."

4.6 The Second-Price Auction

A single object is to be sold at auction. There are $n > 1$ bidders, each submitting a single bid b_i, simultaneously, to the seller. The value of the object to bidder i is v_i. The winner of the object is the highest bidder, but the winner pays only the next highest bid. Thus if i wins and j has made the next highest bid, then i's payoff is $\pi_i = v_i - b_j$, and the payoff for every other bidder is zero.

Show clearly and carefully that a player cannot gain by deviating from truth telling, no matter what the other players do.

a. Does this answer depend on whether other players use this strategy?
b. Does this answer still hold if players are allowed to place a higher bid if someone bids higher than their current bid?

c. Show that if you know every other bid, you still cannot do better than bidding $b_i = v_i$, but there are weakly dominated best responses that affect the payoff to other players.
d. Show that if the bids arrive in a certain order, and if later bidders condition their bids on previous bids, then bidding $b_i = v_i$ may not be a dominant strategy. *Hint*: consider the case where bidder i sets $b_i = 0$ if any previous bidder has placed a bid higher than v_i.
e. Consider the following *increasing price auction*: There are n bidders. The value of the vase to bidder i is $v_i > 0$, $i = 1, \ldots, n$. The auctioneer begins the bidding at zero, and raises the price at the rate of f$1 per ten seconds. All bidders who are willing to buy the vase at the stated price put their hands up simultaneously. This continues until there is only one arm raised. This last and highest bidder is the winner of the auction and must buy the vase for $1 less than the stated price. Show that this auction has the same normal form as the second-price auction. Can you think of some plausible Nash equilibria distinct from the truth-telling equilibrium?
f. What are some reasons that real second-price auctions might not conform to the assumptions of this model? You might want to check with eBay participants in second-price auctions. *Hint*: think about what happens if you do not know your own v_i, and you learn about the value of the prize by tracing the movement over time of the currently high bid.

4.7 The Mystery of Kidnapping

It is a great puzzle as to why people are often willing to pay ransom to save their kidnapped loved ones. The problem is that the kidnappers are obviously not honorable people who can be relied upon to keep their promises. This implies that the kidnapper will release rather than kill the victim *independent* of whether the ransom is paid. Indeed, assuming the kidnapper is *self-regarding* (that is, he cares only about his own payoffs), whether the victim should be released or killed should logically turn on whether the increased penalty for murdering the victim is more or less than offset by the increased probability of being identified if the victim is left alive. Viewed in this light, the threat to kill the victim if and only if the ransom is not paid is an incredible threat. Of course, if the kidnapper is a known group that has a reputation to uphold, the threat becomes credible. But often this is not the case. Here is a problem that formally analyzes this situation.

Suppose a ruthless fellow kidnaps a child and demands a ransom r from the parents, equal to \$1 million, to secure the release of the child, whom the parents value the amount v. Because there is no way to enforce a contractual agreement between kidnapper and parents to that effect, the parents must pay the ransom and simply trust that the kidnapper will keep his word. Suppose the cost to the kidnapper of freeing the child is f and the cost of killing the child is k. These costs include the severity of the punishment if caught, the qualms of the kidnapper, and whatever else might be relevant.

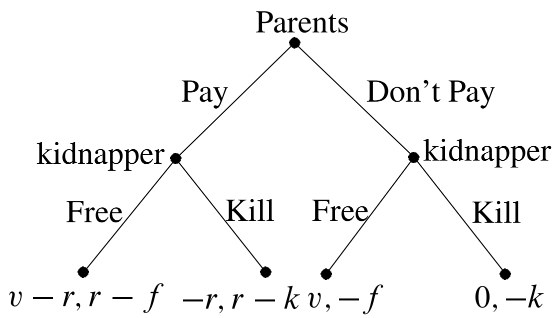


Figure 4.4. The mystery of kidnapping

Because the parents are the first movers, we have the extensive form game shown in figure 4.4. Using backward induction, we see that if $k > f$ the kidnapper will free the child, and if $k < f$ the kidnapper will kill the child. Thus, the parents' payoff is zero if they do not pay, and $-r$ if the pay, so they will not pay.

Parents \ Kidnapper	ff	fk	kf	kk
Pay	$v-r, r-f$	$v-r, r-f$	$-r, r-k$	$-r, r-k$
Don't Pay	$v, -f$	$0, -k$	$v, -f$	$0, -k$

Figure 4.5. Normal form of the mystery of kidnapping game

The corresponding normal form of the game is shown in figure 4.5. Here, ff means "Free no matter what," fk means "Free if pay, kill if not pay, kf means "Kill if pay, free if not pay," and kk means "kill no matter what."

We may assume $v > r$, so the parents will pay the ransom if that ensures that their child will be released unharmed. We now find the Nash equilibria we found by applying backward induction to the extensive form game. Moreover, it is easy to see that if $k < f$, there is no additional Nash equilibrium where the child is released. The threat of killing the child is not credible if $f < k$, and the promise of freeing the child is not credible if $k > f$.

Clearly, our model is inaccurate, because people often pay the ransom. What might be incorrectly modeled? One possibility is that the kidnapper is *vindictive*. We could model that by saying that the costs of freeing versus killing the child if the ransom is paid are f_r and k_r where $f_r < k_r$ whereas the costs of freeing versus killing the child if the ransom is *not* paid are f_n and k_n where $f_n > k_n$. If you plug these assumptions back into the extensive and/or normal forms you will find that there is a unique Nash equilibrium in which the parents pay and the kidnapper frees. In a similar vein, we may follow Harsanyi (1967) in positing that there are two "types" of kidnappers, vindictive and self-regarding. If p_v is the probability a kidnapper is vindictive, we can again solve the problem and find that for some range of values of p_v, the ransom should be paid.

Another possibility is that the parents incur an additional cost (psychic or reputational) if they do not pay the ransom and the child is killed. Specifically, suppose this cost is $v_k > r$. You can now show that, once again, the parents will pay the ransom. In this case, however, the kidnapper will free the child only if $f < k$.

4.8 The Eviction Notice

A landlord has three tenants, Mr. A, Ms. B, and Msgr. C, in a rent-controlled apartment building in New York City. A new law says that the landlord has the right to evict one tenant per building. The landlord calculates that the value of a vacant apartment is $15,000, both to the tenant and to her. She sends the following letter to each of the tenants: "Tomorrow I will be visiting your building. I will offer Mr. A $1,000 if he agrees to vacate his apartment voluntarily; otherwise, I will evict him. If Mr. A agrees to vacate voluntarily, I will then offer Ms. B $1,000, and if she refuses, I will evict her. If she accepts, I will evict Msgr. C." Show that the landlord's plan is consistent with the elimination of strongly dominated strategies.

4.9 Hagar's Battles

There are ten battlefields with military values $a_1 < \ldots < a_{10}$. Each player is endowed with $n_i < 10$ soldiers ($i = 1,2$). A player's strategy is a decision to send his soldiers to these various battlefields. A player can send at most one soldier to a given battlefield. When the fighting begins, each player wins a_j for each battlefield where he has a soldier but his opponent does not. If both or neither combatants occupy a battlefield, the payoffs at that battlefield are zero for both players. The winner of the war is the army whose occupied territory has the highest total military value.

Show that this game has a unique equilibrium in which each side plays a weakly dominant strategy, in which troups are deployed to the most valuable battlefields.

4.10 Military Strategy

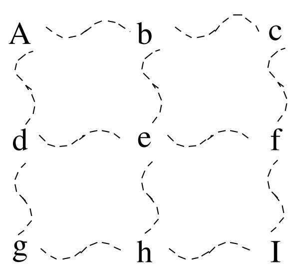


Figure 4.6. Navigational routes between A and I

Country A and country I are at war. The two countries are separated by a series of rivers, illustrated in figure 4.6. Country I's fleet must stop for the night at intersections (e.g., if the fleet takes the path IhebA, it must stop the first night at h, the second at e, and the third at b). Country I sends a naval fleet with just enough supplies to reach A (that is, it has enough supplies to last three nights). If unhindered, on the fourth day the fleet will reach A and destroy Country A. Country A can send a single fleet to prevent this. Country A's fleet also has enough supplies to last three nights (e.g., Abcf). If A intercepts I's fleet (that is, if both countries stop for the night at the same intersection), A destroys I's fleet and wins the war. But, if I's fleet reaches A without being intercepted, I wins the war. List the strategies of the two countries and make a payoff matrix for these strategies, assuming the winner gets 1 and the loser -1.

Show that after the elimination of weakly dominated strategies that never arrive at the other country, there remain six pure strategies for each country. Write out the normal form matrix for the remaining pure strategies, so that the elimination of weakly dominated strategies leaves two strategies for each player and the remaining game is equivalent to throwing fingers (§3.8).

4.11 The Dr. Strangelove Game

In the Cold War days, the United States and the Soviet Union had both conventional ground and nuclear forces. The Soviets had superior conventional forces. If the Soviets launched a ground attack on NATO countries in Europe, the United States could decide to use either nuclear or conventional ground forces to retaliate. A conventional retaliation would leave the Soviet Union better off and the United States worse off by an equal amount. If the United States retaliated with nuclear force, a nuclear war would ensue and the United States would be 100 times worse off than in the conventional case. The Soviet Union would suffer just as much as the United States in the nuclear case.

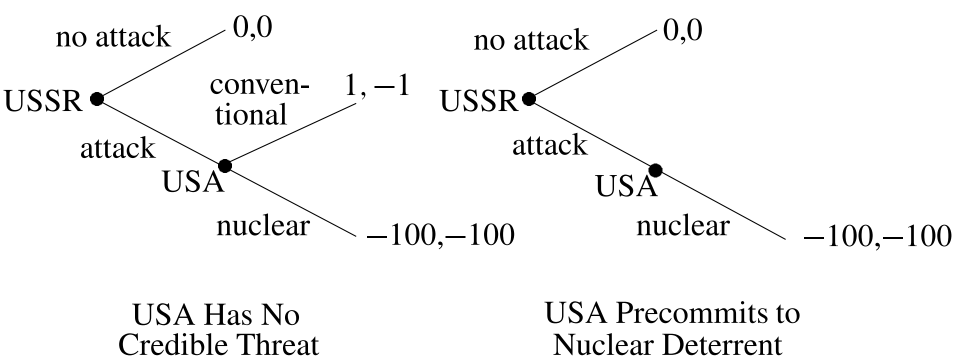


Figure 4.7. The Dr. Strangelove game

Figure 4.7 shows the game trees for this problem for the two cases, with and without precommitment. As an exercise, find the backward induction solution in each case.

4.12 Strategic Voting

Three legislators, Alice, Bob, and Ted, are voting on whether to give themselves a pay raise. The raise is worth b, but each legislator who votes for the raise incurs a cost of voter resentment equal to $c < b$. The outcome is decided by majority rule. Alice votes first, then Bob sees Alice's choice and votes, and finally Ted sees both Alice's and Bob's choice, and votes.

Find a Nash equilibrium for this game by backward induction. Show that there is another Nash equilibrium in which Ted votes no, whatever Alice and Bob do, and this equilibrium favors Ted. Why is this equilibrium eliminated by backward induction?

4.13 Nuisance Suits

Suppose Alice contemplates suing Bob over some purported ill Bob perpetrated upon her. Suppose Alice's court cost for initiating a suit is c, her legal costs for going to trial are p, and Bob's cost of defending himself is d. Suppose both sides know these costs and also share the knowledge that the probability that Alice will win the suit is γ and the expected amount of the settlement is x. We assume $\gamma x < p$, so the suit is a frivolous "nuisance suit" that Alice would not pursue if her goal were win the case. Finally, suppose that before the case goes to trial (but after the suit is initiated), the parties can settle out of court for an amount s. We assume s is given, though clearly we could derive this value as well with the appropriate assumptions.

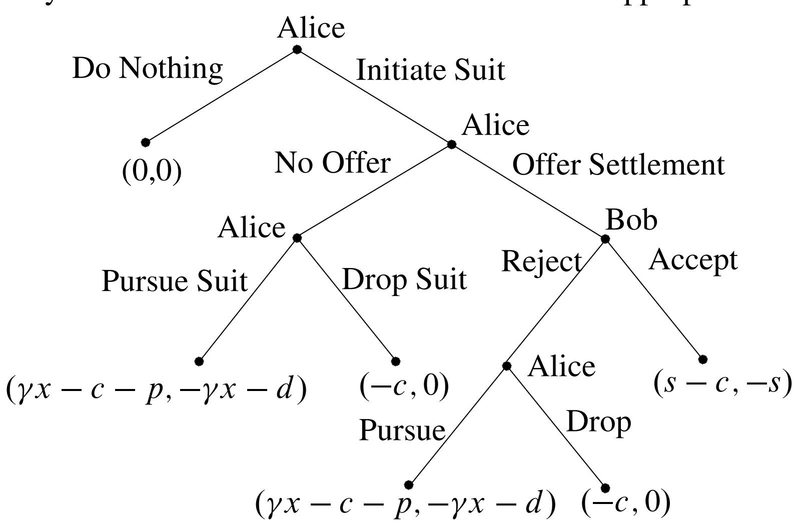


Figure 4.8. Nuisance suits

A consideration of the game tree for the problem, which is depicted in figure 4.8, shows that the only subgame perfect equilibrium is for the plaintiff to do nothing, because the threat of carrying through with the trial is not credible. For rolling back the game tree, we see that Drop Suit dominates

Pursue Suit, so Reject Settlement dominates Accept Settlement, in which case No Offer and Offer Settlement are both inferior to Do Nothing.

The plaintiff's problem is that the threat to sue is not credible, because the suit is frivolous. But suppose the plaintiff puts his lawyer on *retainer*, meaning that he pays the amount p *in advance*, whether or not the suit is taken to trial. We show this in figure 4.9.

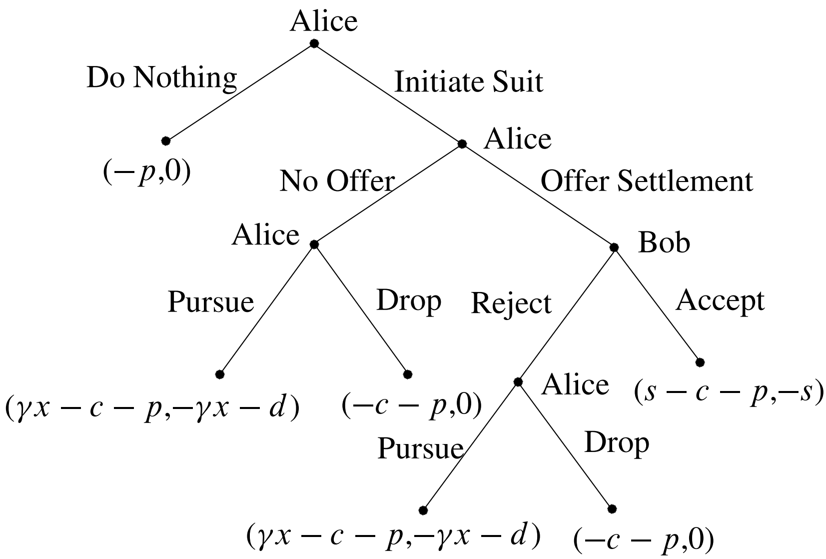


Figure 4.9. Nuisance suit with lawyer on retainer

We can eliminate dominated strategies from this extensive form game by assuming common knowledge of rationality, because each player moves at most once. The payoffs at the Drop Suit leaf are $(-c - p, 0)$, so now Pursue Suit dominates Drop Suit. If $s > \gamma x$, plaintiff will Offer Settlement rather than No Offer, and if $\gamma x + d > s$, defendant prefers to Accept rather than Reject. Thus, the settlement is made and the payoffs are $(s - c - p, -s)$, because the plaintiff has to pay his lawyer's retainer fee p. Thus, this solution is better than Do Nothing for the plaintiff if $s > c + p$.

Note that if $\gamma = 0$ this is a perfect nonsense suit: the plaintiff has no chance of winning the suit, but he can offer to settle for $s = d$, which the defendant will accept, and this works as long as $d > c + p$.

Note that the plaintiff never gets to use the lawyer, so he can sue lots of people, settle all out of court, and still pay the single retainer p. If he can carry out n suits, the condition for profitability if $\gamma = 0$ is then $d > c + p/n$, which is a less demanding constraint.

How can the defendant protect himself against such nuisance suits? Pay his lawyer the retainer d before the fact. Then there is no gain from settling out of court. But of course if you face only *one* plaintiff, this is not a profitable solution. If you potentially face *many* plaintiffs, having a lawyer on retainer is a good idea, because otherwise you will settle out of court many times.

4.14 An Armaments Game

A fighter command has four strategies, and its opponent bomber command has three counterstrategies. Figure 4.10 shows the probability that the fighter destroys the bomber, which is the payoff to player 1. The payoff to player 2 is minus this amount (that is, this is a zero-sum game). Use the elimination of strongly dominated strategies to determine a Nash equilibrium to this game.

	Bomber Command		
Fighter Command	Full Fire Low Speed	Partial Fire Medium Speed	No Fire High Speed
Guns	0.30, −0.30	0.25, −0.25	0.15, −0.15
Rockets	0.18, −0.18	0.14, −0.14	0.16, −0.16
Toss-bombs	0.35, −0.35	0.22, −0.22	0.17, −0.17
Ramming	0.21, −0.21	0.16, −0.16	0.10, −0.10

Figure 4.10. An armaments game

4.15 Football Strategy

In a football game, the offense has two strategies, run or pass. The defense has three strategies, counter run, counter pass, or blitz the quarterback. After studying many games, the statisticians came up with the following table, giving the expected number of yards gained when the various strategies in the figure are followed. Use the elimination of dominated strategies to find a solution to this game.

	Defense		
Offense	Counter Run	Counter Pass	Blitz
Run	3, −3	7, −7	15, −15
Pass	9, −9	8, −8	10, −10

4.16 Poker with Bluffing

Ollie and Stan decide to play the following game of poker. Each has a deck consisting of three cards, labeled H (high), M (medium), and L (low). Each puts \$1 in the pot, chooses a card randomly from his deck, and does not show the card to his friend. Ollie (player 1) either stays, leaving the pot unchanged, or raises, adding \$1 to the pot. Stan simultaneously makes the same decision. If both raise or both stay, the player with the higher card wins the pot (which contains \$2 if they stayed and \$4 if they raised), and if they tie, they just take their money back. If Ollie raises and Stan stays, then Ollie gets the \$3 pot. However, if Stan raise and Ollie stays, Ollie gets another chance. He can either drop, in which case Stan wins the \$3 pot (only \$1 of which is Ollie's), or he can call, adding \$1 to the pot. Then, as before, the player with the higher card wins the pot, and with the same card, they take their money back. A game tree for poker with bluffing is depicted in figure 4.11 (the "?" in the figure means that the payoff depends on who has the higher card).

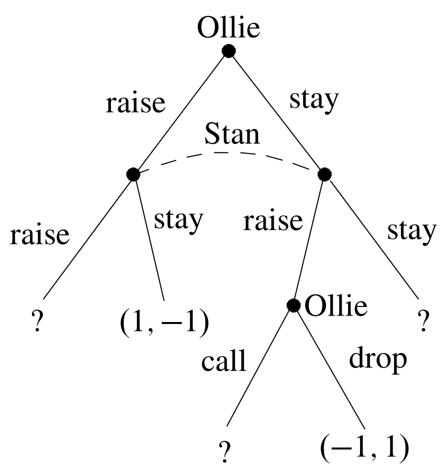


Figure 4.11. Poker with bluffing

a. Show that Ollie has 64 pure strategies and Stan has 8 pure strategies.

b.* Find the normal form game. Note that although poker with bluffing is a lot simpler than real poker, the normal form is nevertheless a 64×8 matrix! If you know computer programming, solving this is not a hard task, however. (Recall that starred questions are more challenging and/or more time-consuming than others).

c.* Show that the iterated elimination of one round of weakly dominated strategies for each player leaves one Nash equilibrium. Ollie raises with either H or M, and stays with L. If Stan raises when Ollie stays, Ollie drops on the second round.

d.* Show that the payoff to the game for Ollie is zero.

e.* Suppose Stan sees Ollie's first move before deciding to stay or raise (that is, the two nodes where Stan moves are separate information sets). Now find the normal form game. Note that this matrix is 64×64, so calculating this by hand is quite prohibitive.

f.* Show that the game can still be solved by the iterated elimination of dominated strategies. Show that both players now raise for all cards.

g.* Show that the payoff to Ollie is zero.

4.17 The Little Miss Muffet Game

While eating her curds and whey, Miss Muffet confronted a spider sitting on her tuffet. Unfazed, she informed the spider that they would engage in a bargaining game in which she would offer the spider a certain share x of the curds and whey. If the spider accepts, they will divide the food accordingly, and proceed on their merry ways. If the spider rejects, neither gets any of the food. Miss Muffet knows that spiders are both rational and benevolent: they reject offers only if they can gain something by doing so. Show that Muffet gets all of the curds and whey.

Now suppose the spider has enough of a scary countenance to force another game: if he rejects the first offer, he gets to make a counteroffer to Miss Muffet, under the same conditions. He knows Miss Muffet is rational and benevolent as well, and hence will accept any offer unless she can do better by rejecting it. *But* the sun is hot, and bargaining takes time. By the time the second offer is accepted or rejected, the curds and whey have melted to half their original size. Show that Miss Muffet offers a 50-50 split, and the spider accepts immediately.

Now suppose there are a maximum of three rounds of bids and the food shrinks by one-third for each rejected offer. Show that Miss Muffet will offer the spider 1/3 of the food, and the spider will accept.

Now suppose there are an even number n of rounds, and the curds and whey shrink by $1/n$ per rejected offer. Show that Miss Muffet still offers the spider 1/2 of the food, and the spider will accept. But if there is an odd number n of rounds, and the curds and whey shrink by $1/n$ per rejected offer, then Miss Muffet offers the spider a share $1/2 - 1/2n$ of the food, the spider accepts, and Miss Muffet keeps the rest. As the number of periods increases, they converge to the just solution, sharing equally.

4.18 Cooperation with Overlapping Generations

Consider a society in which there is a public good to which each member can contribute either zero or one unit of effort each period. The value of the public good to each member is the total amount of effort contributed to it.

Suppose the cost of contributing a unit of effort is α. It is clear that if $0 < \alpha < 1$, a dominant strategy for each member is to contribute a unit of effort, but if $\alpha > 1$, then a dominant strategy is to contribute no effort. Suppose there are N members of this society. Then, if $1 < \alpha < N$, the unique Nash equilibrium is inefficient, but that if $\alpha > N$ the equilibrium is efficient. Suppose for the rest of this problem that $1 < \alpha < N$, so members would benefit from erecting incentives that induced people to contribute to the public good.

At a town meeting, one member observed: "By the end of a period, we know which members contributed to the public good that period. Why do we not exclude a member from sharing in the public good in the next period if he failed to contribute in the current period?" Another said: "That seems like a good idea, but as we all know, we each live exactly T years, and with your plan, the eldest generation will certainly not contribute." A third person responded: "Well, we revere our elders anyway, so why not tolerate some indolence on their part? Indeed, let us agree to an 'age of veneration' T^* such that any member who fails to contribute in the first T^* periods of life is banished from ever enjoying the public good, but members who are older than T^* need not contribute." The first speaker queried: "How shall we determine the proper age of veneration?" The proposer responded: "We should choose T^* to maximize our net lifetime gain from the public good." They all agree.

Suppose there are n members of each generation, so $N = nT$, and they do not discount the future. Suppose also that $n(T-1) \geq \alpha$. We can then show that

$$T^2 \geq \frac{4(\alpha - 1)}{n} \tag{4.1}$$

is necessary and sufficient for there to be a T^* such that the strategy in which each agent contributes up to age T^* and shirks thereafter is a Nash subgame perfect equilibrium.

To see this, let T^* be the age of veneration. Assume all members but one cooperate, and test whether the final player will cooperate. If this final player is of age $t \leq T^*$, the gains from cooperating are $(T - t + 1)nT^* - (T^* - t + 1)\alpha$, and the gains from defecting are $nT^* - 1$. Thus, the net gains from cooperating are

$$f(t, T^*) = (T - t)nT^* - (T^* - t + 1)\alpha + 1.$$

Then T^* supports a Nash subgame perfect equilibrium of the desired form if and only if $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T^*$. In particular, we must have $f(T^*, T^*) \geq 0$. But $f(t,t) = (T-t)nt - (\alpha - 1)$ is a parabola with a maximum at $t = T/2$, and $f(T/2, T/2) = nT^2/4 - (\alpha - 1)$. Because this must be nonnegative, we see that

$$T^2 \geq \frac{4(\alpha - 1)}{n} \tag{4.2}$$

is necessary.

Now suppose (4.2) holds, and choose T^* such that $f(T^*, T^*) \geq 0$. Because $f_1(t, T^*) = -nT^* + \alpha$, if $\alpha \leq nT^*$ then f is decreasing in t, so $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T^*$. If $\alpha > nT^*$, $f(t, T^*)$ is increasing in t, so we must ensure that $f(1, T^*) \geq 0$. But $f(1, T^*) = T^*(n(T-1)-\alpha)+1$, which is strictly positive by assumption. Thus, (4.2) is sufficient.

Note that now the optimal T^* is $T^* = T$, because the total utility from the public good for a member is $T^*(nT-\alpha)$, which is an increasing function of T^*.

4.19 Dominance-Solvable Games

Let $S = S_1 \times \ldots \times S_n$ be the set of strategy profiles in a finite game with payoffs $\pi_i : S \to \mathbf{R}$, $i = 1, \ldots, n$. Let $S^* = S_1^* \times \ldots \times S_n^*$ be the set of strategy profiles remaining in a game after the iterated elimination of weakly dominated strategies. We say the game is *dominance solvable* if each $\pi_i(\cdot)$ is constant on S^*. Suppose the game is of perfect information (§4.2) and satisfies the following "one-to-one" assumption (Moulin 1979): for all $s,t \in S$, and all $i,j = 1,\ldots,n$, if $\pi_i(s) = \pi_i(t)$, then $\pi_j(s) = \pi_j(t)$. Show that the game is dominance solvable. Give examples of games that are and are not dominance solvable. *Hint*: Chess, checkers, and Go are dominance solvable. For a counterexample, look at two-player games where the first player has two moves, the second player has two moves at each node where he chooses, his payoffs are the same for all four moves, but the first player's payoff depends on what the second player does.

4.20 Agent-based Modeling

In evolutionary game theory, as developed later in the book, we will see that a simple evolutionary process in which agents who do well in a game are imitated by others who have had less success. This evolutionary process often converges to a Nash equilibrium. Thus, very little intelligence is required of the agents who play the game (§12.2). This dynamical process can be modeled on the computer in what is known as *agent-based modeling*.

Agent-based models are computer programs that use game theory to create artificial strategic actors, set up the structure of their interactions and their payoffs, and display and/or record the dynamics of the ensuing social order. An *evolutionary* agent-based model has, in addition, a *reproduction phase*, in which agents reproduce in proportion to their average success, old agents die off, and the new agents inherit the behavior of their parents, perhaps with some mutation. Thus, in an evolutionary agent-based model, there is a tendency for the more successful strategies to increase in frequency at the expense of the less successful.

Agent-based modeling is important because many games are too complicated to admit closed-form analytical solutions. Moreover, sometimes the assumptions we make to get explicit algebraic solutions to games are sufficiently unrealistic (e.g., continuous time, infinite numbers of potential agents) that the agent-based model performs differently from the analytical solutions, and perhaps more like the real-life situations we are trying to model.

Virtually any computer language can be used for agent-based modeling. My favorite is Pascal, as implemented in Borland Delphi, but C++ is almost as good, especially Borland's C++Builder. Both are very fast, whereas most other languages (e.g. Visual Basic and Mathematica) are painfully slow. C++ is more difficult to learn, and compilation times are slow compared to Delphi. Some modelers prefer Java (especially Borland's JBuilder), but I have not used this language extensively.

When I do serious agent-based modeling (for a journal article, for instance), I try to program in two very different languages (e.g., Delphi and Mathematica) and ensure that the output of the two are the same, although this is infeasible for very complex models. It is even better to have someone else write the model in the second language, as this way one discovers the many implicit design choices in one's program to which one never even gave a passing thought. Some researchers have their agent-based models written by professional programmers. I do not advise this, unless the programmer has an excellent understanding of the theory behind the model. So, if one wants to use agent-based models, one must either learn programming or have a coauthor adequately trained in one's behavioral discipline who does the programming.

Figure 4.12 shows the programming structure of a typical evolutionary agent-based model. In the figure, the "Number of Generations" specifies how many rounds of reproduction you want to take place. This may be as small as 10 or as large as 10 million. The "Number of Rounds/Generation" refers to the speed of play as compared to the speed of reproduction. By the law of large numbers, the more rounds per generation, the more accurately the actual success of agents reflects the expected payoff of the strategies they represent. "Randomly Pair All Agents" captures the evolutionary nature of the model, as well as the notion that a strategy in a game reflects a social practice *occurring in the population*, rather than the carefully thought-out *optimizing strategy* of the brainy actor of classical game theory. Note that in some situations, we will want some structure to this stage of the model. For instance, "neighbors" may meet more frequently than "strangers," or agents who play similar strategies may meet more frequently than agents with different strategies. It's all up to the modeler. The "Agents Reproduce" box is expanded in figure 4.13.

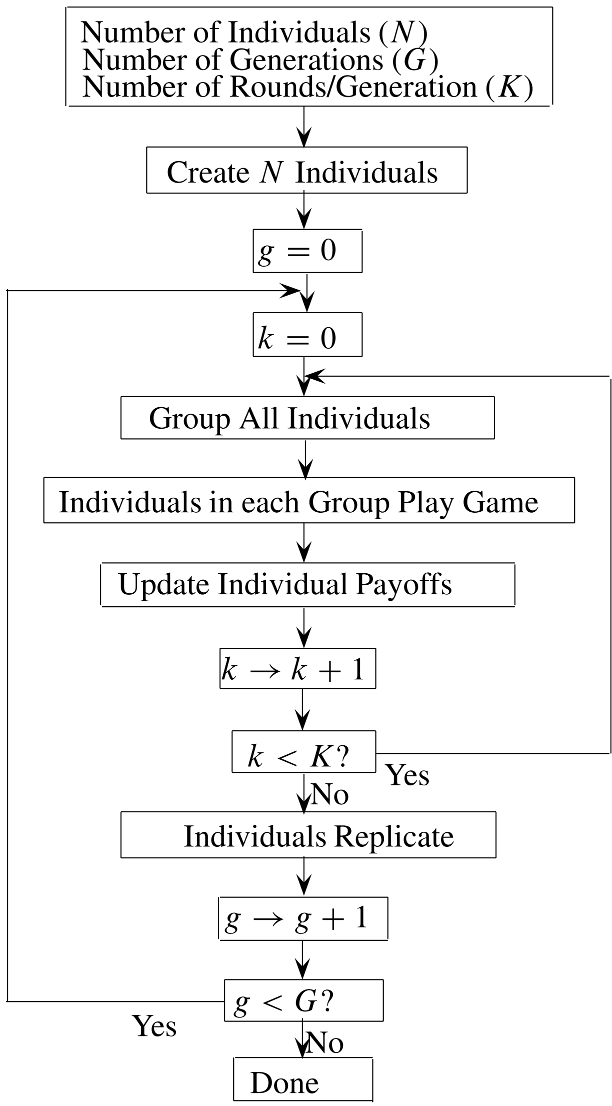


Figure 4.12. Structure of an evolutionary agent-based model

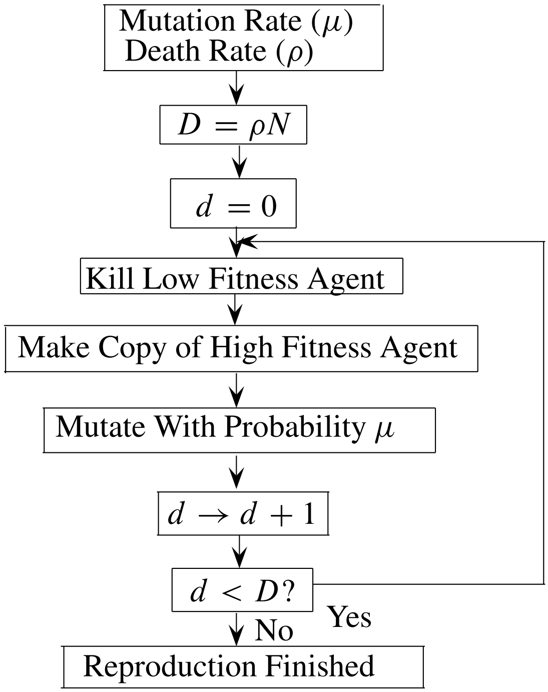


Figure 4.13. Structure of reproduction process

4.21 Why Play a Nash Equilibrium?

Suppose Alice and Bob play the prisoner's dilemma, one stage of which is shown in the diagram, 100 times. Common sense tells us that players will cooperate for at least 95 rounds, and this intuition is supported by experimental evidence (Andreoni and Miller 1993). However, a backward induction argument indicates that players will defect on the very first round. To see this, note that the players will surely defect on round 100. But then, nothing they do on round 99 can help prolong the game, so they will both defect on round 99. Repeating this argument 99 times, we see that they will both defect on round 1.

	C	D
C	3,3	0,4
D	4,0	1,1

This example points out a general problem in classical game theory that is generally glossed over in the textbooks, leading to a general opinion among game theorists that if people fail to play a Nash equilibrium, or if they play a Nash equilibrium that is not subgame perfect, they must be "irrational." This example, and there are many, many more, shows that this opinion is quite unfounded. Perfectly rational individuals fail to play the Nash equilibrium in this case because it is really stupid for Alice to defect on every round. Indeed, with a little bit of thought, Alice will cooperate on the first couple of rounds, just to see what Bob will do. Moreover, unless Bob is really stupid, he will be thinking the same thing. Thus, they might cooperate for many rounds.

Another way to see the problem with the Nash equilibrium criterion, and more generally the iterated elimination of dominated strategies, is to treat Alice's and Bob's strategic interaction as a problem in decision theory for each player. Suppose that Alice conjectures that Bob will cooperate up to round k, and then defect forever, with probability g_k. Then, Alice will choose a round m to defect that maximizes the expression

$$\pi_m = \sum_{i=1}^{m-1}((i-1)R+S)g_i + ((m-1)R+P)g_m + ((m-1)R+T)(1-G_m), \tag{4.3}$$

where $G_m = g_1 + \ldots + g_m$, $R = 3$, $S = 0$, $T = 4$, and $P = 1$. The first term in this expression represents the payoff if Bob defects first, the second if both defect on the same round, and the final term if Alice defects first. In many cases, maximizing this expression will suggest cooperating for many rounds for all plausible probability distributions. For instance, suppose g_k is uniformly distributed on the rounds $m = 1, \ldots, 99$. Then, you can check by using equation (4.3) that it is a best response to cooperate up to round 98. Indeed, suppose you expect your opponent to defect on round 1 with probability 0.95, and otherwise defect with equal probability on any round from 2 to 99. Then it is still optimal to defect on round 98. Clearly, the backward induction assumption is not plausible unless you think your opponent is highly likely to be an obdurate backward inductor.

Bayesian decision theory works here precisely because the players do *not* backward induct. For instance, suppose Alice said to herself, "given my prior over Bob's behavior, I should defect on round 98. But, if Bob has the same prior, he will also defect on round 98. Thus, I should defect on round 97. But, Bob must be thinking this as well, so he will defect on round 97. But then I should defect on round 96." If Alice continues like this, she will defect on round 1! The fallacy in her reasoning is that there is no reason for Bob to have the same prior as Alice. After a few turns of thinking like Sherlock Holmes and Professor Moriarty, Alice would rightly conclude that small differences in their priors prohibit further backward induction.

The general question as to when rational individuals will play a Nash equilibrium is deep, very important, and only partially understood. It is a central question explored, using epistemic game theory, in my book, *The Bounds of Reason* (2009).

4.22 Modeling the Finitely-Repeated Prisoner's Dilemma

As we will show later, the fundamental theorem of evolutionary game theory says that every stable equilibrium of an evolutionary dynamic is a Nash equilibrium of the stage game (§12.6). Suppose, then, that we had a large population of agents repeatedly paired to play the 100-round repeated prisoner's dilemma. Suppose each agent has the strategy of defecting for the remainder of the game the first round after his partner defected, and otherwise cooperating up to some round k, defecting thereafter. We treat k as a heritable genetic characteristic of the individual. Then, if we start with a random distribution of k's in the population and allow agents to reproduce according to their payoffs in playing the 100-round prisoner's dilemma *stage game*, the fundamental theorem allows us to predict that if the process ends up at an equilibrium, this equilibrium will involve all agents defecting on the first round, every time they play the game.

If we do an agent-based simulation of this evolutionary dynamic, we must ensure that no genotype ever becomes extinct, or we will not get this result. This is because the dynamical system underlying the fundamental theorem of evolutionary game theory is a system of differential equations, which assumes an infinite population of agents, so if players of type k have positive frequency in one period, they always have positive frequency, however much reduced percentage-wise, in the succeeding period.

The way we ensure nonextinction in an agent-based model is to assume there is a small probability of mutation in the process of the birth of new agents. Figure 4.14 shows a simulation in which two players play the 100-stage prisoner's dilemma with the temptation payoff $T = 5$, the reward payoff $R = 2$, the punishment payoff $P = 1$, and the sucker payoff $S = -2$. The program creates 200 agents, each randomly assigned an integer between 1 and 101, which represents the first round on which the agent defects (an agent with number 101 never defects). The agents are randomly paired in each of 1 million periods and play the 100-round prisoner's dilemma game. Every 10 periods the agents reproduce. This implies that

5% of poorly performing agents are replaced by copies of well-performing agents. Then 2.5% of the newly created agents undergo a mutation process. There are two types of mutation, *random* and *incremental*. Under random mutation, the agent is given a random defection point between 1 and 101, whereas under incremental mutation, his mutation point is incremented by one or decremented by one, with equal probability, but the defection point is not permitted to fall below 1 or above 101.

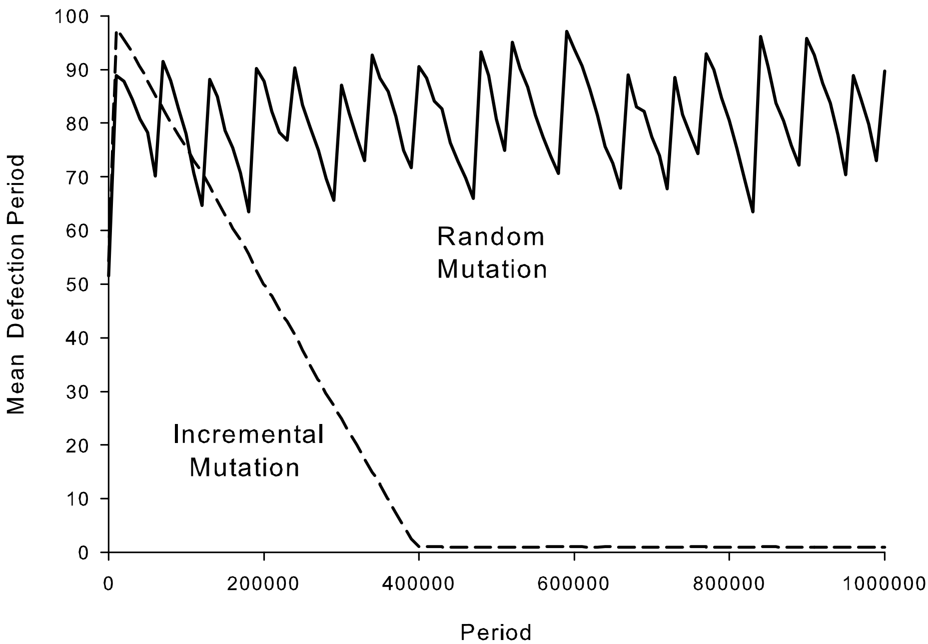


Figure 4.14. An agent-base model of 100-round prisoner's dilemma with $T = 5$, $R = 2$, $P = 1$, and $S = -2$

The "incremental mutation" plot in figure 4.14 shows that although cooperation at first attains a very high frequency, in the long run the Nash equilibrium solution predominates. To see why cooperation declines, note that the highest cooperators are relatively less fit than the next highest, so the latter displace the highest cooperators. They are next replaced by cooperators who defect just a little before they do, and so on, until there is no cooperation. Although mutation may slow down this passage to the Nash equilibrium, it does not reverse it.

With random mutation, we have a completely different dynamic. Here, minimum cooperation is about 63%, maximum cooperation is about 96%, and average cooperation is about 81%. In this case, the intuition behind the dynamic is as follows. As in the previous case, evolution drives the highest level of cooperators downward. When this maximum is about 80%, approximately 20% of mutants are higher cooperators than the current maximum, by an average of 10 points; that is, the average mutant with a higher defect point than 80 has a defect point 90. When two of these mutant cooperators meet, they do better than the current majority, so they reproduce at a fast rate, leading to a rapid increase in cooperation. This cycle repeats indefinitely. This dynamic does not contradict the fundamental theorem of evolutionary game theory, because this theorem does not say what happens if an evolutionary dynamic does not converge to an equilibrium.

We conclude that our intuition concerning the absurdity of the backward induction solution is thus likely due to the fact that we do not face the relentless process of "weeding out" cooperators in daily life that is depicted in the model, and/or we do not face incremental mutation. If we did face both of these forces, we would not find the backward induction solution strange, and we would experience life in our world, to paraphrase Thomas Hobbes, as solitary, poor, brutish, nasty, and short.

4.23 Review of Basic Concepts

Carefully answer the following questions.

a. Describe *backward induction* in an extensive form game.
b. Define the notion of a strategy being *strongly dominated* in a normal or extensive form game. Explain in words why a strongly dominated strategy can never be used by a player in a Nash equilibrium.
c. Describe the *iterated elimination of strongly dominated strategies* and explain why a strategy eliminated in this manner cannot be used by a player in a Nash equilibrium.
d. Explain why, if the iterated elimination of weakly dominated strategies leaves exactly one strategy for each player, the resulting strategy profile is a Nash equilibrium.

5

Pure-Strategy Nash Equilibria

> Competition among agents . . . has merit solely as a device to extract information optimally. Competition per se is worthless.
>
> Bengt Holmström

A *pure strategy* Nash equilibrium of a game is a Nash equilibrium in which each player uses a pure strategy, but not necessarily one determined by the iterated elimination of dominated strategies (§4.1). Not every game has a pure-strategy Nash equilibrium. Indeed, there are even very simple 2×2 normal form games with no pure-strategy Nash equilibria—for instance throwing fingers (§3.8), where the Nash equilibrium consists in each player throwing one or two fingers, each with probability 1/2.

This chapter explores some of the more interesting applications of games with pure-strategy equilibria. As you will see, we obtain extremely deep results in various branches of economic theory, including altruism (§5.16,§5.17), the tragedy of the commons (§5.5), the existence of pure-strategy equilibria in games of perfect information (§5.6), the real meaning of competition (it is probably not what you think) (§5.3), honest signaling equilibria (§5.9) and (§5.18). Another feature of this chapter is its use of agent-based modeling, in no-draw, high-low poker (§5.7), to give you a feel for the dynamic properties of games for which the Nash equilibrium concept is a plausible description of reality.

5.1 Price Matching as Tacit Collusion

Bernie and Manny both sell DVD players and both have unit costs of 250. They compete on price: the low-price seller gets all the market and they split the market if they have equal prices. Explain why the only Nash equilibrium has both firms charging 250, splitting the market and making zero profit.

Suppose that the monopoly price for DVD players (the price that maximizes the sum of the profits of both firms) is 300. Now suppose Bernie advertises that if a customer buys a DVD player from him for 300 and discovers he or she can buy it cheaper at Manny's, Bernie will refund the full purchase price. Suppose Manny does the same thing. Show that it is now Nash for both stores to charge 300. Conclusion: pricing strategies that seem to be supercompetitive can in fact be anticompetitive!

5.2 Competition on Main Street

The residents of Pleasantville live on Main Street, which is the only road in town. Two residents decide to set up general stores. Each can locate at any point between the beginning of Main Street, which we will label 0, and the end, which we will label 1. The two decide independently where to locate and they must remain there forever (both can occupy the same location). Each store will attract the customers who are closest to it and the stores will share equally customers who are equidistant between the two. Thus, for instance, if one store locates at point x and the second at point $y > x$, then the first will get a share $x + (y - x)/2$ and the second will get a share $(1 - y) + (y - x)/2$ of the customers each day (draw a picture to help you see why). Each customer contributes \$1.00 in profits each day to the general store it visits.

a. Define the actions, strategies, and daily payoffs to this game. Show that the unique pure-strategy Nash equilibrium where both players locate at the midpoint of Main Street;
b. Suppose there are three General Stores, each independently choosing a location point along the road (if they all choose the same point, two of them share a building). Show that there is no pure-strategy Nash equilibrium. *Hint*: First show that there is no pure-strategy Nash equilibrium where all three stores locate on one half of Main Street. Suppose two stores locate on the left half of Main Street. Then, the third store should locate a little bit to the right of the rightmost of the other two stores. But, then the other two stores are not best responses. Therefore, the assumption is false. Now finish the proof.

5.3 Markets as Disciplining Devices: Allied Widgets

In *The Communist Manifesto* of 1848, Karl Marx offered a critique of the nascent capitalist order that was to resound around the world and fire the imagination of socialists for nearly a century and a half.

> The bourgeoisie, wherever it has got the upper hand, has put an end to all feudal, patriarchal, idyllic relations. It has pitilessly torn asunder the motley feudal ties that bound man to his "natural superiors," and has left no other nexus between man and man than naked self-interest, than callous "cash payment" ... It has resolved personal worth into exchange value, and in place of the numberless indefeasible chartered freedoms, has set up that single, unconscionable freedom: Free Trade. (Marx 1948)

Marx's indictment covered the two major institutions of capitalism: market competition and private ownership of businesses. Traditional economic theory held that the role of competition was to set prices, so supply equals demand. If this were correct, a socialist society that took over ownership of the businesses could replace competition by a central-planning board that sets prices using statistical techniques to assess supply and demand curves.

The problem with this defense of socialism is that traditional economic theory is wrong. The function of competition is to reveal private information concerning the shape of production functions and the effort of the firm's managers and use that information to reward hard work and the efficient use of resources by the firm. Friedrich von Hayek (1945) recognized this error and placed informational issues at the heart of his theory of capitalist competition. By contrast, Joseph Schumpeter (1942), always the bitter opponent of socialism, stuck to the traditional theory and predicted the inevitable victory of the system he so hated (Gintis 1991).

This problem pins down analytically the notion that competition is valuable because it reveals otherwise private information. In effect, under the proper circumstances, market competition subjects firms to a prisoner's dilemma in which it is in the interest of each producer to supply high effort, even in cases where consumers and the planner cannot observe or contract for effort itself. This is the meaning of Bengt Holmström's quotation at the head of this chapter.

If Holmström is right, and both game-theoretic modeling and practical experience indicate that he is, the defense of competitive markets in neoclassical economics is a great intellectual irony. Because of Adam Smith, supporters of the market system have defended markets on the grounds that they allocate goods and services efficiently. However, empirical estimates of the losses from monopoly, tariffs, quotas, and the like indicate that misallocation has little effect on per capita income or the rate of economic growth

(Hines 1999). By contrast, the real benefits of competition, which include its ability to turn private into public information, have come to light only through game-theoretic analysis. The following problem is a fine example of such analysis.

Allied Widgets has two possible constant returns to scale production techniques: fission and fusion. For each technique, Nature decides in each period whether marginal cost is 1 or 2. With probability $\theta \in (0, 1)$, marginal cost is 1. Thus, if fission is high cost in a given production period, the manager can use fusion, which will be low cost with probability θ. However, it is costly for the manager to inspect the state of Nature and if he fails to inspect, he will miss the opportunity to try fusion if the cost of fission is high.

Allied's owner cannot tell whether the manager inspected or not, but he does know the resulting marginal cost and can use this to give an incentive wage to the manager. Figure 5.1 shows the manager's decision tree, which assumes the manager is paid a wage w_1 when marginal costs are low and w_2 when marginal costs are high, the cost of inspecting is α and the manager has a logarithmic utility function over income: $u(w) = \ln w$.[1]

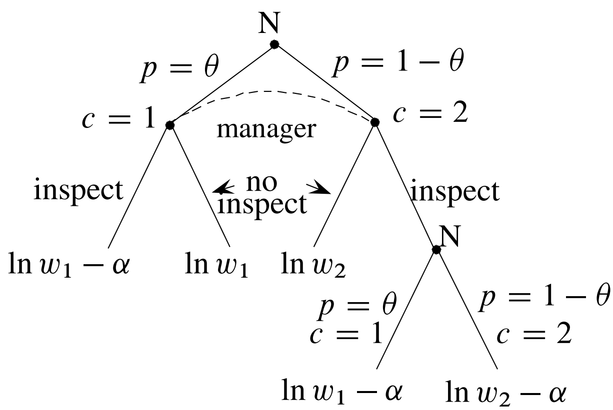


Figure 5.1. The Allied Widgets problem

To induce the manager to inspect the fission process, the owner decides to pay the manager a wage w_1 if marginal cost is low and $w_2 < w_1$ if marginal cost is high. But how should the owner choose w_1 and w_2 to

[1]The logarithmic utility function is a reasonable choice, because it implies constant relative risk aversion; that is, the fraction of wealth an agent desires to put in a particular risky security is independent of wealth.

maximize profits? Suppose the manager's payoff is $\ln w$ if he does not inspect, $\ln w - \alpha$ if he inspects and $\ln w_o$ if he does not take the job at all. In this case, w_o is called the manager's *reservation wage* or *fallback position*.

The expression that must be satisfied for a wage pair (w_1, w_2) to induce the manager to inspect the fission process is called the *incentive compatibility constraint*. To find this expression, note that the probability of using a low-cost technique if the manager does not inspect is θ, so the payoff to the manager from not inspecting (by the expected utility principle) is

$$\theta \ln w_1 + (1-\theta) \ln w_2.$$

If the manager inspects, both techniques will turn out to be high cost with probability $(1-\theta)^2$, so the probability that at least one of the techniques is low cost is $1-(1-\theta)^2$. Thus, the payoff to the manager from inspecting (again by the expected utility principle) is

$$[1-(1-\theta)^2] \ln w_1 + (1-\theta)^2 \ln w_2 - \alpha.$$

The incentive compatibility constraint is then

$$\theta \ln w_1 + (1-\theta) \ln w_2 \le [1-(1-\theta)^2] \ln w_1 + (1-\theta)^2 \ln w_2 - \alpha.$$

Because there is no reason to pay the manager more than absolutely necessary to get him to inspect, we can assume this is an equality,[2] in which case the constraint reduces to $\theta(1-\theta)\ln[w_1/w_2] = \alpha$, or

$$w_1 = w_2 e^{\frac{\alpha}{\theta(1-\theta)}}.$$

For instance, suppose $\alpha = 0.4$ and $\theta = 0.8$. Then $w_1 = 12.18 w_2$; that is, the manager must be paid more than twelve times as much in the good state as in the bad!

But the owner must also pay the manager enough so that taking the job, compared to taking the fallback w_o, is worthwhile. The expression that must be satisfied for a wage pair (w_1, w_2) to induce the manager to take the job is called the *participation constraint*. In our case, the participation constraint is

$$[1-(1-\theta)^2] \ln w_1 + (1-\theta)^2 \ln w_2 - \alpha \ge \ln w_o.$$

[2]Actually, this point may not be obvious and is false in the case of repeated principal-agent models. This remark applies also to our assumption that the participation constraint, defined in the text, is satisfied as an equality.

If we assume that this is an equality and using the incentive compatibility constraint, we find $w_o = w_2 e^{\alpha/(1-\theta)}$, so

$$w_2 = w_o e^{-\frac{\alpha}{(1-\theta)}}, \qquad w_1 = w_o e^{\frac{\alpha}{\theta}}.$$

Using the above illustrative numbers and if we assume $w_o = 1$, we get

$$w_2 = 0.14, \qquad w_1 = 1.65.$$

The expected cost of the managerial incentives to the owner is

$$[1-(1-\theta)^2]w_1 + (1-\theta)^2 w_2 = w_o\left[\theta(2-\theta)e^{\frac{\alpha}{\theta}} + (1-\theta)^2 e^{-\frac{\alpha}{(1-\theta)}}\right].$$

Again, using our illustrative numbers, we get expected cost

$$0.96(1.65) + 0.04(0.14) = 1.59.$$

So where does competition come in? Suppose Allied has a competitor, Axis Widgets, subject to the same conditions of production. In particular, whatever marginal-cost structure Nature imposes on Allied, Nature also imposes on Axis. Suppose also that the managers in the two firms cannot collude. We can show that Allied's owner can write a Pareto-efficient contract for the manager using Axis's marginal cost as a signal, satisfying both the participation and incentive compatibility constraints and thereby increasing profits. They can do this by providing incentives that subject the managers to a prisoner's dilemma, in which the dominant strategy is to defect, which in this case means to inspect fission in search of a low-cost production process.

To see this, consider the following payment scheme, used by both the Axis and the Allied owners, where $\phi = 1-\theta+\theta^2$, which is the probability that both managers choose equal-cost technologies when one manager inspects and the other does not (or, in other words, one minus the probability that the first choice is high and the second low). Moreover, we specify the parameters β and γ so that $\gamma < -\alpha(1-\theta+\theta^2)/\theta(1-\theta)$ and $\beta > \alpha(2-\phi)/(1-\phi)$. This gives rise to the payoffs to the manager shown in the table, where the example uses $\alpha = 0.4$, $\theta = 0.8$, and $w_0 = 1$.

Allied Cost	Axis Cost	Allied Wage	Example
$c = 1$	$c = 1$	$w^* = w_o e^{\alpha}$	$w^* = 1.49$
$c = 2$	$c = 2$	$w^* = w_o e^{\alpha}$	$w^* = 1.49$
$c = 1$	$c = 2$	$w^+ = w_o e^{\beta}$	$w^+ = 54.60$
$c = 2$	$c = 1$	$w^- = w_o e^{\gamma}$	$w^- = 0.02$

We will show that the manager will always inspect and the owner's expected wage payment is w^*, which merely pays the manager the equivalent of the fallback wage. Here is the normal form for the game between the two managers.

	Inspect	Shirk
Inspect	$\ln w^* - \alpha$ $\ln w^* - \alpha$	$\phi \ln w^* + (1-\phi) \ln w^+ - \alpha$ $\phi \ln w^* + (1-\phi) \ln w^-$
Shirk	$\phi \ln w^* + (1-\phi) \ln w^-$ $\phi \ln w^* + (1-\phi) \ln w^+ - \alpha$	$\ln w^*$ $\ln w^*$

Why is this so? The inspect/inspect and shirk/shirk entries are obvious. For the inspect/shirk box, with probability ϕ the two managers have the same costs, so they each get $\ln w^*$ and with probability $1-\phi$ the Allied manager has low costs and the Axis manager has high costs, so the former gets $\ln w^+$ and the latter gets $\ln w^-$.

To show that this is a prisoner's dilemma, we need only show that

$$\ln w^* - \alpha > \phi \ln w^* + (1-\phi) \ln w^-$$

and

$$\phi \ln w^* + (1-\phi) \ln w^+ - \alpha > \ln w^*.$$

The first of these becomes

$$\ln w_o > \phi \ln w_o + \phi\alpha + (1-\phi) \ln w_o + (1-\phi)\gamma,$$

or $\gamma < -\phi\alpha/(1-\phi)$, which is true by assumption. The second becomes

$$\ln w^+ > \frac{\alpha}{1-\phi} + \ln w^*,$$

or $\beta > \alpha\frac{2-\phi}{1-\phi}$, which is also true by assumption.

Note that in our numerical example the cost to the owner is $w^* = 1.49$ and the incentives for the managers are given by the normal form matrix

	Inspect	Shirk
Inspect	0,0	0.58,-0.30
Shirk	$-0.30, 0.58$	0.4,0.4

This example shows that markets may be disciplining devices in the sense that they reduce the cost involved in providing the incentives for agents to act in the interests of their employers or clients, even where enforceable contracts cannot be written. In this case, there can be no enforceable contract for managerial inspecting. Note that in this example, even though managers are risk averse, imposing a structure of competition between the managers means each inspects and the cost of incentives is no greater than if a fully specified and enforceable contract for inspecting could be written.

Of course, if we weaken some of the assumptions, Pareto-optimality will no longer be attainable. For instance, suppose when a technique is low cost for one firm, it is not necessarily low cost for the other, but rather is low cost with probability $q > 1/2$. Then competition between managers has an element of uncertainty and optimal contracts will expose the managers to a positive level of risk, so their expected payoff must be greater than their fallback.

5.4 The Tobacco Market

The demand for tobacco is given by

$$q = 100000(10 - p),$$

where p is the price per pound. However, there is a government price support program for tobacco that ensures that the price cannot go under \$0.25 per pound. Three tobacco farmers have each harvested 600,000 pounds of tobacco. Each must make an independent decision on how much to ship to the market and how much to discard.

a. Show that there are two Nash equilibria, one in which each farmer ships the whole crop and a second in which each farmer ships 250,000 pounds and discards 350,000 pounds.
b. Are there any other Nash equilibria?

5.5 The Klingons and the Snarks

Two Klingons are eating from a communal cauldron of snarks. There are 1,000 snarks in the cauldron and the Klingons decide individually the rate r_i, $(i = 1, 2)$ at which they eat per eon. The net utility from eating snarks, which depends on both the amount eaten and the rate of consumption (too slow depletes the Klingon Reservoir, too fast overloads the Klingon Kishkes) is given by

$$u_i = 4q_i + 50r_i - r_i^2,$$

where q_i is the total number of snarks Klingon i eats. Since the two Klingons eventually eat all the snarks, $q_i = 1000r_i/(r_1 + r_2)$.

a. If they could agree on an optimal (and equal) rate of consumption, what would that rate be?
b. When they choose independently, what rate will they choose?
c. This problem illustrates the tragedy of the commons (Hardin 1968), in which a community (in this case the two Klingons, though it usually involves a larger number of individuals) overexploits a resource (in this case the bowl of snarks) because its members cannot control access to the resource. Some economists believe the answer is simple: the problem arises because no one owns the resource. So give an individual the right to control access to the resource and let that individual sell the right to extract resources at a rate r to the users. To see this, suppose the cauldron of snarks is given to a third Master Klingon and suppose the Master Klingon charges a diner a fixed number of drecks (the Klingon monetary unit), chosen to maximize his profits, for the right to consume half the cauldron. Show that this will lead to an optimal rate of consumption.

This "create property rights in the resource" solution is not always satisfactory, however. First, it makes the new owner rich and everyone else poor. This could possibly be solved by obliging the new owner to pay the community for the right to control the resource. Second, it may not be possible to write a contract for the rate of resource use; the community as a whole may be better at controlling resource use than a single owner (Ostrom, Walker and Gardner 1992). Third, if there is unequal ability to pay among community members, the private property solution may lead to an unequal distribution of resources among community members.

5.6 Chess: The Trivial Pastime

A *finite game* is a game with a finite number of nodes in its game tree. A game of *perfect information* is a game where every information set is a single node and Nature has no moves. In 1913 the famous mathematician

Ernst Zermelo proved that in chess either the first mover has a winning pure strategy, the second mover has a winning pure strategy, or either player can force a draw. This proof was generalized by Harold Kuhn (1953), who proved that every finite game of perfect information has a pure-strategy Nash equilibrium. In this problem you are asked to prove a special case of this, the game of chess.

Chess is clearly a game of perfect information. It is also a finite game, because one of the rules is that if the board configuration is repeated three times, the game is a draw. Show that in chess, either Black has a winning strategy, or White has a winning strategy, or both players have strategies that can force a draw.

Of course, just because there exists an optimal strategy does not imply that there is a feasible way to find one. There are about 10^{47} legal positions in chess, give or take a few orders of magnitude, implying a game-tree complexity that is almost the cube of this number. This is far more than the number of atoms in the universe.

5.7 No-Draw, High-Low Poker

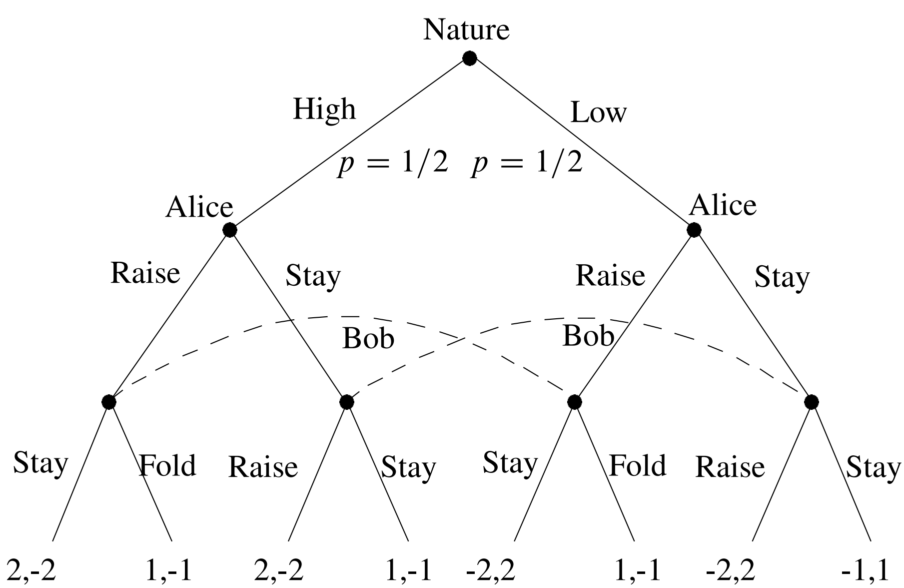


Figure 5.2. Game Tree for no-draw, high-low poker

Alice and Bob are playing cards. The deck of cards has only two types of card in equal numbers: high and low. Each player each puts 1 in the pot.

Alice is dealt a card (by Nature). After viewing the card, which Bob cannot see, she either raises or stays. Bob can stay or fold if Alice raises and can raise or stay if Alice stays. If Alice raises, she puts an additional 1 in the pot. If Bob responds by folding, he looses and if he responds by staying, he must put an additional 1 in the pot. If Alice stays and Bob raises, both must put an additional 1 in the pot. If the game ends without Bob folding, Alice wins the pot if she has a high card and loses the pot if she has a low card. Each player's objective is to maximize the expected value of his or her winnings. The game tree is in figure 5.2

We now define strategies for each of the players. Alice has two information sets (each one a node) and two choices at each. This gives four strategies, which we label RR, RS, SR, and SS. These mean "raise no matter what," "raise with high, stay with low," "stay with high, raise with low," and "stay no matter what." Bob also has two information sets, one where Alice raises and one where Alice stays. We denote his four strategies SR, SS, FR, and FS. These mean "stay if Alice raises, raise if Alice stays," "stay no matter what," "fold if Alice raises, raise if Alice stays," and "fold if Alice raises, stay if Alice stays." To find the normal form game, we first assume Nature gives Alice a high card and compute the normal form matrix. We then do the same assuming Nature plays low. This give the following payoffs:

Nature Plays High

	SR	SS	FR	FS
RR	2,-2	2,-2	1,-1	1,-1
RS	2,-2	2,-2	1,-1	1,-1
SR	2,-2	1,-1	2,-2	1,-1
SS	2,-2	1,-1	2,-2	1,-1

Nature Plays Low

	SR	SS	FR	FS
RR	-2,2	-2,2	1,-1	1,-1
RS	-2,2	-1,1	-2,2	-1,1
SR	-2,2	-2,2	1,-1	1,-1
SS	-2,2	-1,1	-2,2	-1,1

The expected values of the payoffs for the two players are simply the averages of these two matrices of payoffs, because Nature chooses high or Low each with probability 1/2. We have:

Expected Value Payoffs

	SR	SS	FR	FS
RR	0,0	0,0	1,-1	1,-1
RS	0,0	0.5,-0.5	-0.5,0.5	0,0
SR	0,0	-0.5,0.5	1.5,-1.5	1,-1
SS	0,0	0,0	0,0	0,0

Some strategies in this game can be dropped because they are recursively dominated. For Alice, RR weakly dominates SS and for Bob, SR weakly dominates FS. It is quite straightforward to check that $\{RR,SR\}$ is a Nash equilibrium. Note that the game is fair: Alice raises no matter what and Bob stays if Alice raises and raises if Alice stays. A box-by-box check shows that there is another pure-strategy equilibrium, $\{SS,SR\}$, in which Alice uses a weakly dominated strategy. There are also some mixed strategy equilibria for which you are invited to search.

5.8 An Agent-based Model of No-Draw, High-Low Poker

The heavy emphasis on finding Nash equilibria in evolutionary game theory flows from two assertions. First, the equilibria of dynamic evolutionary games are always Nash equilibria (§12.8). Second, the evolutionary process does not require high-level rationality from the agents who populate dynamic evolutionary games. We can illustrate both points by modeling the dynamics of no-draw, high-low poker on the computer. In this agent-based model, I created 100 player 1 types and 100 player 2 types, each programmed to play exactly one pure strategy, assigned randomly to them. In each round of play, player 1s and player 2s are randomly paired and they play no-draw high-low poker once. Every 100 rounds we allow reproduction to take place. Reproduction consisted in killing off 5% of the players of each type with the lowest scores and allowing the top 5% of players with the highest score to reproduce and take the place of the defunct low scorers. However with a 1% probability, a newly-born player ‘mutates’ by using some randomly-chosen other strategy. The simulation ran for 50,000 rounds. The results of a typical run of the simulations for the distribution of player 2 types in the economy are shown in figure 5.3. Note that the Nash strategy for player 2 slowly but surely wins out and the other strategies remain at very low levels, though they cannot disappear altogether because mutations constantly occur to replenish their ranks. Although this is not shown in figure 5.3, player 1 uses RR rather than the weakly dominated SS.

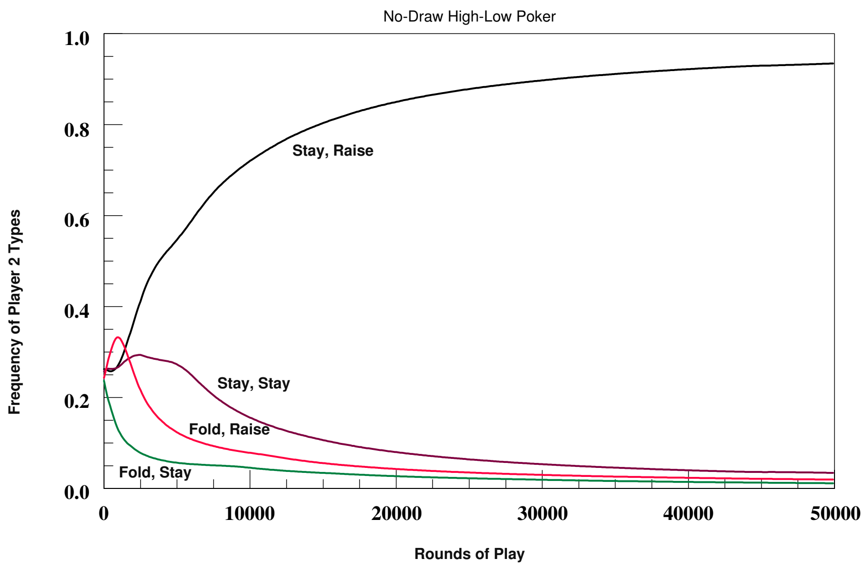


Figure 5.3. An agent-based model of no-draw, high-low poker

5.9 The Truth Game

Bob wants a used car and asks the sales representative, Alice, "Is this a good car for the money?" Alice wants to sell the car, but does not want to ruin her reputation by lying. We can model the strategies followed by Alice and Bob as follows.

Nature flips a coin and it comes out H with probability $p = 0.8$ and T with probability $p = 0.2$. Alice sees the result, but Bob does not. Alice announces to Bob that the coin came out either H or T. Then Bob announces either h or t. The payoffs are as follows: Alice receives 1 for telling the truth and 2 for inducing Bob to choose h. Bob receives 1 for making a correct guess and 0 otherwise.

The game tree for the problem is depicted in figure 5.4. Alice has strategy set {HH,HT,TH,TT}, where HH means announce H if you see H, announce H if you see T, HT means announce H if you see H, announce T if you see T, and so on. Thus, HH means "always say H," HT means "tell the truth," TH means "always lie," TT means "always say T." Bob has strategy

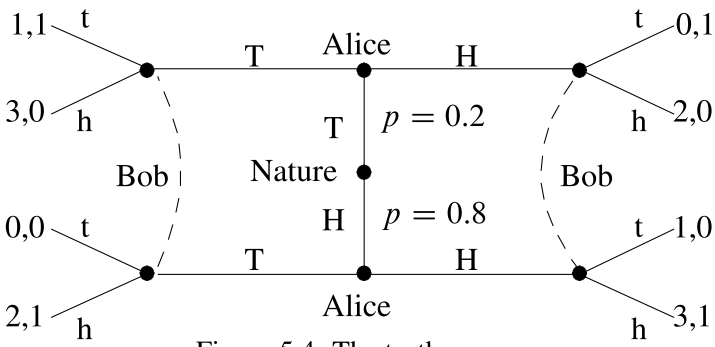


Figure 5.4. The truth game

set {hh,ht,th,tt}, where hh means "say h if you are told h and say h if you are told t;" ht means "say h if you are told h and say t if you are told t;" and so on. Thus, hh means "always say h," ht means "trust Alice" th means "distrust Alice," and tt means "always say t." The payoffs to the two cases, according to Nature's choice, are listed in figure 5.5.

	hh	ht	th	tt
HH	2,0	2,0	0,1	0,1
HT	3,0	1,1	3,0	1,1
TH	2,0	2,0	0,1	0,1
TT	3,0	1,1	3,0	1,1

Payoff when coin is T

	hh	ht	th	tt
HH	3,1	3,1	1,0	1,0
HT	3,1	3,1	1,0	1,0
TH	2,1	0,0	2,1	0,0
TT	2,1	0,0	2,1	0,0

Payoff when coin is H

Figure 5.5. Payoffs for the truth game

The actual payoff matrix is 0.2 $\times$ first matrix $+$ 0.8 $\times$ second, which is shown in figure 5.6

	hh	ht	th	tt
HH	2.8,0.8	2.8,0.8	0.8,0.2	0.8,0.2
HT	3.0,0.8	2.6,1.0	1.4,0.0	1.0,0.2
TH	2.0,0.8	0.4,0.0	1.6,1.0	0.0,0.2
TT	2.2,0.8	0.2,0.8	2.2,0.8	0.2,0.2

Figure 5.6. Consolidated payoffs for the truth game

It is easy to see that tt is dominated by hh, but there are no other strategies dominated by pure strategies. (TT,th) and (HH,ht) are both Nash equilibria. In the former, Alice always says T and Bob assumes Alice lies; in the second, Alice always says H and Bob always believes Alice. The first equilibrium is Pareto-inferior to the second.

It is instructive to find the effect of changing the probability p and/or the cost of lying on the nature of the equilibrium. You are encouraged so show that if the cost of lying is sufficiently high, Alice will always tell the truth and in equilibrium, Bob will believe her.

5.10 The Rubinstein Bargaining Model

Suppose Bob and Alice bargain over the division of a dollar (Rubinstein 1982). Bob goes first and offers Alice a share x of the dollar. If Alice accepts, the payoffs are $(1-x, x)$ and the game is over. If Alice rejects, the pie "shrinks" to $\delta < 1$ and Alice offers Bob a share y of this smaller pie. If Bob accepts, the payoffs are $(y, \delta - y)$. Otherwise, the pie shrinks to δ^2 and it is once again Bob's turn to make an offer. The game continues until they settle, or if they never settle, the payoff is (0,0). The game tree is shown in figure 5.7.

Clearly, for any $x \in [0, 1]$ there is a Nash equilibrium in which the payoffs are $(1-x, x)$, simply because if Alice accepts nothing less than x, then it is Nash for Bob to offer x to Alice and conversely. But these equilibria are not necessarily credible strategies, because, for instance, it is not credible to demand more than δ, which is the total size of the pie if Alice rejects the offer. What are the plausible Nash equilibria? In this case, equilibria are implausible because they involve incredible threats, so in this case forward induction on the part of Bob and Alice lead them both to look for subgame perfect equilibria.

For the subgame perfect case, let $1-x$ be the *most* Bob can possibly get in any subgame perfect Nash equilibrium. Then, the most Bob can get on his second turn to offer is $\delta^2(1-x)$, so on Alice's first turn, the most she must offer Bob is $\delta^2(1-x)$, so the least Alice gets when it is her turn to offer is $\delta - \delta^2(1-x)$. But then, on his first turn, Bob must offer Alice at least this amount, so his payoff is at most $1 - \delta + \delta^2(1-x)$. But this must equal $1-x$, so we have $x = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta}$.

Now let $1-x$ be the *least* Bob can possibly get in any subgame perfect Nash equilibrium. Then, the least Bob can get on his second turn to offer is $\delta^2(1-x)$, so on Alice's first turn, the least she must offer Bob is $\delta^2(1-x)$, so the most Alice gets when it is her turn to offer is $\delta - \delta^2(1-x)$. But then, on his first turn, the most Bob must offer Alice is this amount, so his

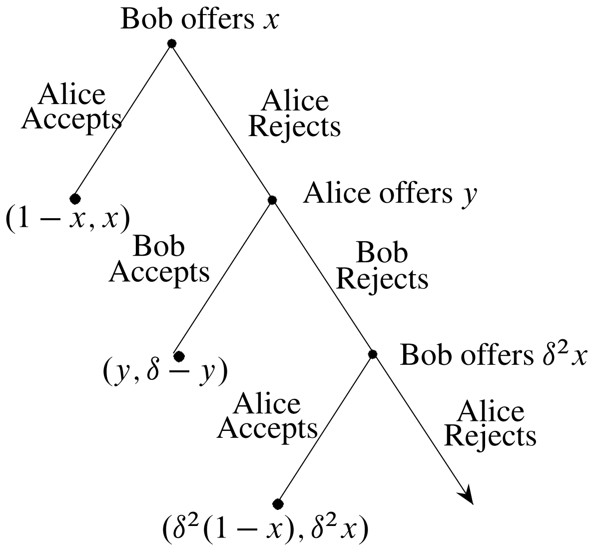


Figure 5.7. Payoffs for the Rubinstein bargaining model

payoff is at least $1-\delta+\delta^2(1-x)$. But this must equal $1-x$, so we have $x = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta}$.

Because the *least* Bob can earn and the *most* Bob can earn in a subgame perfect Nash equilibrium are equal, there is a *unique* subgame perfect Nash equilibrium, in which the payoffs are

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right).$$

Note that there is a small first-mover advantage, which disappears as $\delta \to 1$.

Do people actually find this subgame perfect equilibrium? Because it requires only two levels of backward induction, we might expect people to do so, despite that fact that it involves the iterated elimination of weakly dominated strategies (note that if the players are Bayesian rational, we never get to Bob's second move). But in fact, the game tree is *infinite* and our trick to reducing backward induction to two steps is purely formal. Thus, we might *not* expect people to settle on the Rubinstein solution. Indeed, experimental evidence indicates that they do not (Neelin, Sonnenschein and Spiegel 1988; Babcock, Loewenstein and Wang 1995). Part of the reason is that *fairness* issues enter into many bargaining situations. These issues are usually not important in the context of the current game, because unless the discount factor is very low, the outcome is almost a fifty-fifty split. But if we complicated the model a bit—for instance, by giving players unequal "outside options" that occur with positive probability after each rejected offer—very unequal outcomes become possible. Also, the basic Rubinstein model predicts that all bargaining will be efficient, because the first offer is in fact never refused. In real-world bargaining, however, breakdowns often occur (strike, war, divorce). Generally, we need models of bargaining with asymmetric information or outside options to have breakdowns with positive probability (see section 5.13 and section 6.42).

5.11 Bargaining with Heterogeneous Impatience

Suppose in the Rubinstein bargaining model we assume Bob has discount factor δ_B and Alice has discount factor δ_A, so δ_B and δ_A represent the *level of impatience* of the two players. For instance, if $\delta_B > \delta_A$, then Bob does not get hurt as much by delaying agreement until the next rounds. We would expect that *a more impatient player would get a small share of the pie in a subgame perfect Nash equilibrium* and that is what we are about to see.

We must revise the extensive form game so that the payoffs are relative to the point at which the game ends, not relative to the beginning of the game. When it is Bob's second time to offer, he will again offer x and if this is accepted, he will receive $1-x$. If Alice is to induce Bob to accept her y offer on the previous round, we must have $y \geq \delta_B(1-x)$, so to maximize her payoff, Alice offers Bob $y = \delta_B(1-x)$. This means Alice gets $1-y = 1-\delta_B(1-x)$. But then for Bob to induce Alice to accept on the first round, Bob must offer $x \geq \delta_A(1-y)$ and to maximize his payoff, he then sets $x = \delta_A(1-y)$. We then have

$$x = \delta_A(1-y) = \delta_A(1-\delta_B(1-x)),$$

which gives

$$x = \frac{\delta_A(1-\delta_B)}{1-\delta_B\delta_A} \qquad 1-x = \frac{1-\delta_A}{1-\delta_B\delta_A}.$$

To see what this means, let's set $\delta_A = 0.9$ and vary δ_B from 0 (Bob is infinitely impatient) to 1 (Bob is infinitely patient). We get figure 5.8

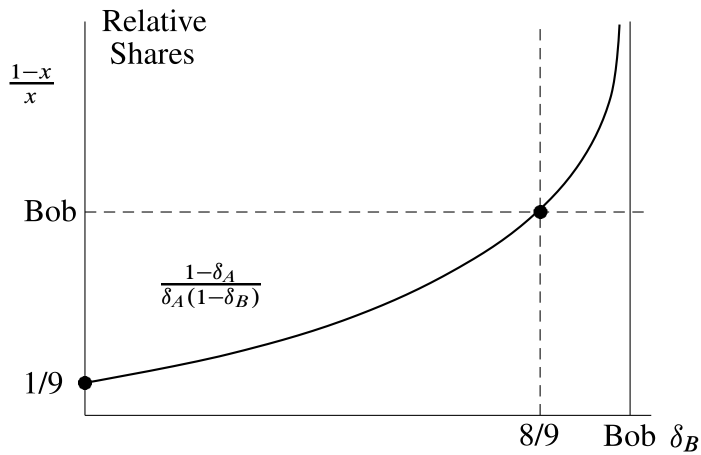


Figure 5.8. How impatience affects relative shares

5.12 Bargaining with One Outside Option

Suppose in the Rubinstein bargaining model (§5.10) that if Alice rejects Bob's offer, he receives an outside option of amount $s_2 > 0$ with probability $p > 0$, with no delay. If he accepts this option, he receives s_2, Bob receives nothing and the game is over.

To analyze this problem, let $1 - x$ be the maximum Bob can get in any subgame perfect Nash equilibrium, if we assume that Alice accepts the outside offer when it is available and that $1-x \geq 0$. Then, when it is Alice's turn to offer, she must offer at least $\delta(1-x)$, so her maximum payoff when rejecting x is $ps_2 + (1-p)\delta(1-\delta(1-x))$. The most Bob must offer Alice is this amount, so the most he can make satisfies the equation

$$1 - x = 1 - (ps_2 + (1-p)\delta(1 - \delta(1-x))),$$

which gives

$$x = \frac{ps_2 + (1-p)\delta(1-\delta)}{1-(1-p)\delta^2}.$$

A similar argument shows that x is the minimum Bob can get in a subgame perfect Nash equilibrium, if we assume that Alice accepts the outside offer when it is available and that $1-x \geq 0$. This shows that such an x is unique. Our assumption that Alice accepts the outside offer requires

$$s_2 \geq \delta(1 - \delta(1-x)),$$

which is Alice's payoff if she rejects the outside option. It is easy to show that this inequality holds exactly when $s_2 \geq \delta/(1+\delta)$. We also must have $1-x \geq 0$, or Bob will not offer x. It is easy to show that this is equivalent to $s_2 \leq (1-\delta(1-p))/p$.

It follows that if $s_2 < \delta/(1+\delta)$, there is a unique subgame perfect Nash equilibrium in which the players use the same strategies and receive the same payoffs, as if the outside option did not exist. In particular, Alice rejects the outside option when it is available. Moreover, if

$$\frac{\delta}{1+\delta} \leq s_2 \leq \frac{1-(1-p)\delta}{p}, \tag{5.1}$$

then there is a unique subgame perfect Nash equilibrium in which Alice would accept s_2 if it became available, but Bob offers Alice the amount

$$x = \frac{ps_2 + (1-p)\delta(1-\delta)}{1-(1-p)^2\delta^2},$$

which Alice accepts.

Note that Bob's payoff decreases from $1/(1+\delta)$ to zero as s_2 increases over the interval (5.1).

Finally, if

$$s_2 > \frac{1-(1-p)\delta}{p},$$

then there is a subgame perfect Nash equilibrium in which Alice simply waits for the outside option to become available (that is, he accepts no offer $x \leq 1$ and offers a strictly negative amount).

5.13 Bargaining with Dual Outside Options

Alice and Bob bargain over splitting a dollar using the Rubinstein alternating-offer bargaining model with common discount factor δ (§5.10). Alice makes the first offer. Bob, as respondent, can accept or reject the offer. If Bob rejects Alice's offer, an outside option worth $s_B > 0$ becomes available to Bob with probability $p > 0$. If available, Bob can accept or reject the option. If Bob accepts the option, his payoff is s_B and Alice's payoff is zero. Otherwise, the proposer and the respondent exchange roles and, after a delay of one time period, the game continues. Each game delay decreases both the amount to be shared and the outside options by a factor of δ.

The following reasoning specifies the unique subgame perfect Nash equilibria of this bargaining game. The game tree is depicted in figure 5.9, where a means "accept" and r means "reject." Note that this is not a complete game tree, because we do not represent the players' decisions concerning accepting or rejecting the outside offer.

First, if $s_A, s_B < \delta/(1+\delta)$, then there is a unique subgame perfect Nash equilibrium in which Bob offers $\delta/(1+\delta)$ and Alice accepts. To see this, note that under these conditions, the subgame perfect Nash equilibrium of the Rubinstein bargaining model without outside options is a subgame perfect Nash equilibrium of this game, with the added proviso that neither player takes the outside option when it is available. To see that rejecting the outside option is a best response, note that when Alice has the outside option, she also knows that if she rejects it, her payoff will be $\delta/(1+\delta)$, so she should reject it. A similar argument holds for Bob.

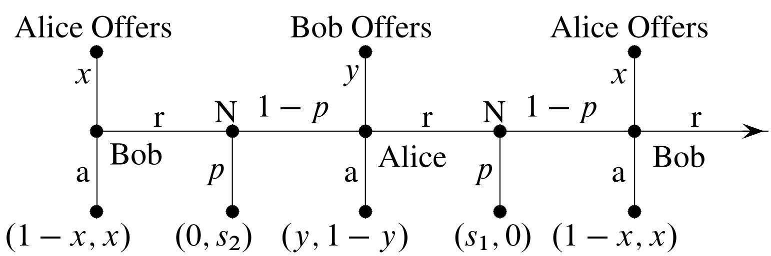


Figure 5.9. Bargaining with dual outside options

Second, suppose the following inequalities hold:

$$p(1-p)\delta s_A - (1-p)\delta(1-(1-p)\delta) < ps_B < p(1-p)\delta s_A + 1 - (1-p)\delta,$$

$$s_A(1-(1-p)\delta^2) + \delta p s_B > \delta(1-\delta(1-p)),$$

$$s_B(1-(1-p)\delta^2) + \delta p s_A > \delta(1-\delta(1-p)).$$

Then, there is a subgame perfect Nash equilibrium in which Alice offers

$$x = \frac{ps_B + (1-p)\delta(1-ps_A) - (1-p)^2\delta^2}{1-(1-p)^2\delta^2}$$

and Bob accepts. In this subgame perfect Nash equilibrium, both players accept the outside option if it is available and Alice makes Bob an offer that

Bob accepts (note that the subgames at which these actions are carried out do not lie on the game path). Show also that in this case, s_A and s_B exist satisfying the preceding inequalities and all satisfy $s_A, s_B > \delta/(1+\delta)$. To see this if we assume that both agents take the outside option when it is available, we have the recursive equation

$$x = ps_B + \delta(1-p)(1-(ps_A + (1-p)\delta(1-x))),$$

and the preceding inequalities ensure that $x \in [0,1]$, that $s_A > \delta(1-x)$, so Alice takes the outside option when available and $s_B > \delta(1 - ps_A - (1 - p)\delta(1-x))$, so Bob takes the outside option when available. This justifies our assumption.

Third, suppose the following inequalities hold:

$$p(1-p)\delta s_A - ps_B > (1-p)\delta(1-(1-p)\delta),$$

$$ps_A + ps_B(1-(1-p)\delta) < 1 - \delta^2(1-p)^2.$$

Then, there is a subgame perfect Nash equilibrium in which Bob rejects Alice's offer, Bob takes the outside option if it is available and if not, Bob offers Alice

$$\frac{ps_A}{1-(1-p)^2\delta^2}$$

and Alice accepts. In this subgame perfect Nash equilibrium, Alice also accepts the outside option if it is available. What are the payoffs to the two players? Show also that there are s_A and s_B that satisfy the preceding inequalities and we always have $s_B > \delta/(1+\delta)$. To see this, first show that if Alice could make an acceptable offer to Bob, then the previous recursion for x would hold, but now $x < 0$, which is a contradiction. Then, either Alice accepts an offer from Bob, or Alice waits for the outside option to become available. The payoff to waiting is

$$\pi_A = \frac{(1-p)ps_A\delta}{1-(1-p)^2\delta^2},$$

but Alice will accept $\delta(ps_A + (1-p)\delta\pi_A)$, leaving Bob with $1-\delta(ps_A+(1-p)\delta\pi_A)$. This must be better for Bob than just waiting for the outside option to become available, which has value, at the time Bob is the proposer,

$$\frac{(1-p)ps_B\delta}{1-(1-p)^2\delta^2}.$$

Show that the preceding inequalities imply Bob will make an offer to Alice. Then, use the inequalities to show that Alice will accept. The remainder of the problem is now straightforward.

Show that this case applies to the parameters $\delta = 0.9$, $p = 0.6$, $s_A = 1.39$, and $s_B = 0.08$. Find Bob's offer to Alice, calculate the payoffs to the players and show that Alice will not make an offer that Bob would accept.

Fourth, suppose

$$ps_A > 1 - (1-p)^2\delta > p(1-p)\delta s_A + ps_B.$$

Then, there is a subgame perfect Nash equilibrium where Alice offers Bob

$$\frac{ps_B}{1-(1-p)^2\delta^2}$$

and Bob accepts. In this equilibrium, Alice accepts no offer and both players accept the outside option when it is available. It is easy to show also that there are s_A and s_B that satisfy the preceding inequalities and we always have $s_A > \delta/(1+\delta)$. Note that in this case Alice must offer Bob

$$\pi_2 = \frac{ps_B}{1-(1-p)^2\delta^2},$$

which is what Bob can get by waiting for the outside option. Alice minus this quantity must be greater than π_A, or Alice will not offer it. Show that this holds when the preceding inequalities hold. Now, Alice will accept $ps_A + (1-p)\delta\pi_A$, but you can show that this is greater than 1, so Bob will not offer this. This justifies our assumption that Alice will not accept anything that Bob is willing to offer.

This case applies to the parameters $\delta = 0.9$, $p = 0.6$, $s_A = 1.55$, and $s_B = 0.88$. Find Alice's offer to Bob, calculate the payoffs to the players and show that Bob will not make an offer that Alice would accept.

Finally, suppose the following inequalities hold:

$$1 - \delta^2(1-p)^2 < ps_B + ps_A\delta(1-p),$$

$$1 - \delta^2(1-p)^2 < ps_B + ps_A\delta(1-p).$$

Then, the only subgame perfect Nash equilibrium has neither agent making an offer acceptable to the other, so both agents wait for the outside option to become available. Note that by refusing all offers and waiting for the outside option to become available, Alice's payoff is π_A and Bob's payoff is π_2. Show that the inequalities imply $1-\pi_2 < \pi_A$, so Alice will not make an acceptable offer to Bob and $1 - \pi_A/\delta(1-p) < \delta(1-p)\pi_2$, so Bob will not make an acceptable offer to Alice.

5.14 Huey, Dewey, and Louie Split a Dollar

Huey, Dewey, and Louie have a dollar to split. Huey gets to offer first, and offers shares d and l to Dewey and Louie, keeping h for himself (so $h + d + l = 1$). If both accept, the game is over and the dollar is divided accordingly. If either Dewey or Louie rejects the offer, however, they come back the next day and start again, this time Dewey making the offer to Huey and Louie and if this is rejected, on the third day Louie gets to make the offer. If this is rejected, they come back on the fourth day with Huey again making the offer. This continues until an offer is accepted by both players, or until the universe winks out, in which case they get nothing. However, the present value of a dollar tomorrow for each of the players is $\delta < 1$.

There exists a unique symmetric (that is, all players use the same strategy), stationary (that is, players make the same offers, as a fraction of the pie, on each round), subgame perfect equilibrium (interestingly enough, there exist other, nonsymmetric subgame perfect equilibria, but they are difficult to describe and could not occur in the real world).

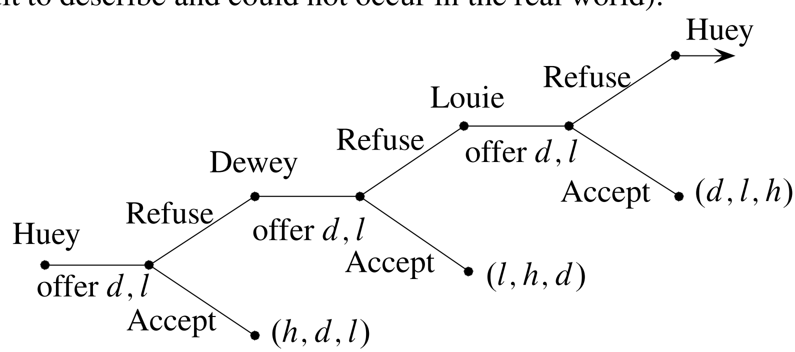


Figure 5.10. Huey, Dewey, and Louie split a dollar

The game tree for this problem appears in figure 5.10, where the equilibrium shares are (h, d, l). We work back the game tree (which is okay, because we are looking only for subgame perfect equilibria). At the second place where Huey gets to offer (at the right side of the game tree), the value of the game to Huey is h, because we assume a stationary equilibrium. Thus, Louie must offer Huey at least δh where Louie gets to offer, to get Huey to accept. Similarly, Louie must offer Dewey at least δd at this node. Thus, the value of the game where Louie gets to offer is $(1-\delta h-\delta d)$.

When Dewey gets to offer, he must offer Louie at least δ times what Louie gets when it is Louie's turn to offer, to get Louie to accept. This amount is just $\delta(1-\delta h-\delta d)$. Similarly, he must offer Huey $\delta^2 h$ to accept, because Huey gets δh when it is Louie's turn to offer. Thus, Dewey gets

$$1-\delta(1-\delta h-\delta d)-\delta^2 h = 1-\delta(1-\delta d)$$

when it is his turn to offer.

Now Huey, on his first turn to offer, must offer Dewey δ times what Dewey can get when it is Dewey's turn to offer, or $\delta(1-\delta(1-\delta d))$. But then we must have

$$d = \delta(1-\delta(1-\delta d)).$$

Solving this equation for d, we find

$$d = \frac{\delta}{1+\delta+\delta^2}.$$

Moreover, Huey must offer Louie δ times what Dewey would offer Louie in the next period or $\delta^2(1-\delta h-\delta d)$. Thus, Huey offers Dewey and Louie together

$$\delta(1-\delta(1-\delta d))+\delta^2(1-\delta h-\delta d) = \delta-\delta^3 h,$$

so Huey gets $1-\delta+\delta^3 h$ and this must equal h. Solving, we get

$$h = \frac{1}{1+\delta+\delta^2},$$

so we must have

$$l = 1-d-h = \frac{\delta^2}{1+\delta+\delta^2},$$

which is the solution to the problem.

Note that there is a simpler way to solve the problem, just using the fact that the solution is symmetric: we must have $d = \delta h$ and $l = \delta d$, from which the result follows. This does not make clear, however, where subgame perfection comes in.

5.15 Twin Sisters

A mother has twin daughters who live in different towns. She tells each to ask for a certain whole number of dollars, at least 1 and at most 100, as a birthday present. If the total of the two amounts does not exceed 101, each will have her request granted. Otherwise each gets nothing.

a. Find all the pure-strategy Nash equilibria of this game.
b. Is there a *symmetric equilibrium* among these? A symmetric Nash equilibrium is one in which both players use the same strategy.
c. What do you think the twins will most likely do, if we assume that they cannot communicate? Why? Is this a Nash equilibrium?

5.16 The Samaritan's Dilemma

Many conservatives dislike Social Security and other forms of forced saving by means of which the government prevents people from ending up in poverty in their old age. Some liberals respond by claiming that people are too short-sighted to manage their own retirement savings successfully. James Buchanan (1975) has made the insightful point that even if people are perfectly capable of managing their retirement savings, if we are altruistic toward others, we will force people to save more than they otherwise would.[3] Here is a simple model exhibiting his point.

A father and a daughter have current income $y > 0$ and $z > 0$, respectively. The daughter saves an amount s for her schooling next year and receives an interest rate $r > 0$ on her savings. She also receives a transfer t from her father in the second period. Her utility function is $v(s,t) = v_A(z-s) + \delta v_2(s(1+r)+t)$, where $\delta > 0$ is a discount factor, v_A is her first-period utility and v_2 is her second-period utility. Her father has personal utility function $u(y)$, but he has degree $\alpha > 0$ of altruistic feeling for his daughter, so he acts to maximize $U = u(y-t) + \alpha v(s,t)$. Suppose all utility functions are increasing and concave, the daughter chooses s to maximize her utility, the father observes the daughter's choice of s and then chooses t. Let (s^*, t^*) be the resulting equilibrium. Show that the daughter will save too little, in the sense that if the father can precommit to t^*, both she and her father would be better off. Show by example that, if we assume

[3]In this case, by an *altruist* we mean an agent who takes actions that improve the material well-being of other agents at a material cost to himself.

that $u(\cdot) = v_A(\cdot) = v_2(\cdot) = \ln(\cdot)$ and that the father can precommit, he may precommit to an amount less than t^*.

Note first that the father's first-order condition is

$$U_t(t, s) = -u'(y - t) + \alpha\delta v_2'(s(1 + r) + t) = 0,$$

and the father's second-order condition is

$$U_{tt} = u''(y - t) + \alpha\delta v_2''(s(1 + r) + t) < 0.$$

If we treat t as a function of s (one step of backward induction, which is uncontroversial, because each player moves only once), then the equation $U_t(t(s), s) = 0$ is an identity, so we can differentiate it totally with respect to s, getting

$$U_{tt}\frac{dt}{ds} + U_{ts} = 0.$$

But $U_{ts} = \alpha\delta(1 + r)v_2'' < 0$, so $t'(s) < 0$; that is, the more the daughter saves, the less she gets from her father in the second period.

Now, the daughter's first-order condition is

$$v_s(s, t) = -v_A' + \delta v_2'(1 + r + t'(s)) = 0.$$

Suppose the daughter's optimal s is s^*, so the father's transfer is $t^* = t(s^*)$. If the father precommits to t^*, then $t'(s) = 0$ would hold in the daughter's first-order condition. Therefore, in this case $v_s(s^*, t^*) > 0$, so the daughter is better off by increasing s to some $s^{**} > s^*$. Thus, the father is better off as well, because he is a partial altruist.

For the example, if $u(\cdot) = v_A(\cdot) = v_2(\cdot) = \ln(\cdot)$, then it is straightforward to check that

$$t^* = \frac{y(1 + \alpha\delta(1 + \delta)) - \delta(1 + r)z}{(1 + \delta)(1 + \alpha\delta)}$$

$$s^* = \frac{\delta(1 + r)z - y}{(1 + r)(1 + \delta)}.$$

If the father can precommit, solving the two first-order conditions for maximizing $U(t, s)$ gives

$$t^f = \frac{\alpha(1 + \delta)y - (1 + r)z}{1 + \alpha + \alpha\delta},$$

$$s^f = \frac{(1 + r)(1 + \alpha\delta)z - \alpha y}{(1 + r)(1 + \alpha + \alpha\delta)}.$$

We then find

$$t^* - t^f = \frac{y + (1+r)z}{(1+\delta)(1+\alpha\delta)(1+\alpha+\alpha\delta)} > 0,$$
$$s^f - s^* = \frac{y + (1+r)z}{(1+r)(1+\delta)(1+\alpha+\alpha\delta)} > 0.$$

5.17 The Rotten Kid Theorem

This problem is the core of Gary Becker's (1981) famous theory of the family. You might check the original, though, because I'm not sure I got the genders right.

A certain family consists of a mother and a son, with increasing, concave utility functions $u(y)$ for the mother and $v(z)$ for the son. The son can affect both his income and that of the mother by choosing a level of familial work commitment a, so $y = y(a)$ and $z = z(a)$. The mother, however, feels a degree of altruism $\alpha > 0$ toward the son, so given y and z, she transfers an amount t to the son to maximize the objective function

$$u(y - t) + \alpha v(z + t). \tag{5.2}$$

The son, however, is perfectly selfish ("rotten") and chooses the level of a to maximize his own utility $v(z(a) + t)$. However, he knows that his mother's transfer t depends on y and z and hence on a.

We will show that the son chooses a to maximize total family income $y(a) + z(a)$ and t is an increasing function of α. Also, if we write $y = y(a) + \hat{y}$, then t is an increasing function of the mother's exogenous wealth $\hat{y}$. We can also show that for sufficiently small $\alpha > 0$, $t < 0$; that is, the transfer is from the son to the mother.

First, Mom's objective function is

$$V(t, a) = u(y(a) - t) + \alpha v(z(a) + t),$$

so her first-order condition is

$$V_t(t, a) = -u'(y(a) - t) + \alpha v'(z(a) + t) = 0.$$

If we treat t as a function of a in the preceding equation (this is one stage of backward induction, which is uncontroversial), it becomes an identity, so we can differentiate with respect to a, getting

$$-u''(y' - t') + \alpha v''(z' + t') = 0. \tag{5.3}$$

Therefore, $z' + t' = 0$ implies $y' - t' = y' + z' = 0$. Thus, the first-order conditions for the maximization of $z + t$ and $z + y$ have the same solutions.

Note that because a satisfies $z'(a) + y'(a) = 0$, a does not change when α changes. Differentiating the first-order condition $V_t(t(\alpha)) = 0$ totally with respect to α, we get

$$V_{tt}\frac{dt}{d\alpha} + V_{t\alpha} = 0.$$

Now $V_{tt} < 0$ by the second-order condition for a maximum and

$$V_{t\alpha} = v' > 0,$$

which proves that $dt/d\alpha > 0$. Because a does not depend on $\hat{y}$, differentiating $V_t(t(y)) = 0$ totally with respect to $\hat{y}$, we get

$$V_{tt}\frac{dt}{d\hat{y}} + V_{t\hat{y}} = 0.$$

But $V_{t\hat{y}} = -u'' > 0$ so $dt/d\hat{y} > 0$.

Now suppose t remains positive as $\alpha \to 0$. Then v' remains bounded, so $\alpha v' \to 0$. From the first-order condition, this means $u' \to 0$, so $y - t \to \infty$. But y is constant, because a maximizes $y + z$, which does not depend on α. Thus $t \to -\infty$.

5.18 The Shopper and the Fish Merchant

A shopper encounters a fish merchant. The shopper looks at a piece of fish and asks the merchant, “Is this fish fresh?” Suppose the fish merchant knows whether the fish is fresh or not and the shopper knows only that the probability that any particular piece of fish is fresh is 1/2. The merchant can then answer the question either yes or no. The shopper, upon hearing this response, can either buy the fish or wander on.

Suppose both parties are *risk neutral* (that is, they have linear utility functions and hence maximize the expected value of lotteries), with utility functions $u(x) = x$, where x is an amount of money. Suppose the price of the fish is 1, the value of a fresh fish to the shopper is 2 (that is, this is the maximum the shopper would pay for fresh fish) and the value of fish that is not fresh is zero. Suppose the fish merchant must throw out the fish if it is not sold, but keeps the 1 profit if she sells the fish. Finally, suppose the

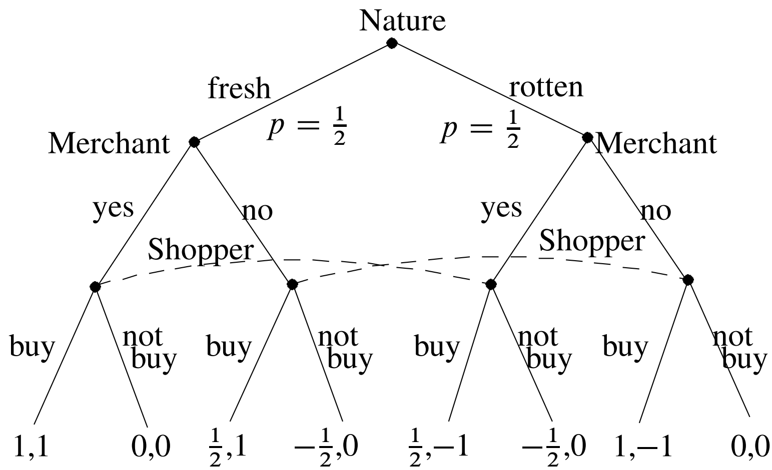


Figure 5.11. Shopper and fish merchant: extensive form

	bb	bn	nb	nn
yy	1,1	1,1	0,0	0,0
yn	1,1	1,1	0,0	0,0
ny	$\frac{1}{2}$,1	$-\frac{1}{2}$,0	$\frac{1}{2}$,1	$-\frac{1}{2}$,0
nn	$\frac{1}{2}$,1	$-\frac{1}{2}$,0	$\frac{1}{2}$,1	$-\frac{1}{2}$,0

Good Fish

	bb	bn	nb	nn
yy	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
yn	$1, -1$	$0, 0$	$1, -1$	$0, 0$
ny	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
nn	$1, -1$	$0, 0$	$1, -1$	$0, 0$

Bad Fish

	bb	bn	nb	nn
yy	$\frac{3}{4}, 0$	$\frac{3}{4}$,0	$-\frac{1}{4}$,0	$-\frac{1}{4}, 0$
yn	1,0	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$	0,0
ny	$\frac{1}{2}, 0$	$0, -\frac{1}{2}$	$0, \frac{1}{2}$	$-\frac{1}{2}, 0$
nn	$\frac{3}{4}$,0	$-\frac{1}{4}$,0	$\frac{3}{4}$,0	$-\frac{1}{4}$,0

$\frac{1}{2}$ Good Fish + $\frac{1}{2}$ Bad Fish

Figure 5.12. Shopper and fish merchant: normal form

merchant has a reputation to uphold and loses 0.50 when she lies, regardless of the shopper's action.

The extensive form for the game is shown in figure 5.11 and the normal form for each of the two cases good fish/bad fish and their expected value is given in figure 5.12. Applying the successive elimination of dominated strategies, we have yn dominates ny, after which bn dominates bb, nb, and nn. But then yy dominates yn and nn. Thus, a Nash equilibrium is

yy/bn: the merchant says the fish is good no matter what and the buyer believes him. Because some of the eliminated strategies were only weakly dominated, there could be other Nash equilibria and we should check for this. We find that another is for the seller to use pure strategy nn and the buyer to use pure strategy nb. Note that this equilibrium works only if the buyer is a "nice guy" in the sense of choosing among equally good responses that maximizes the payoff to the seller. The equilibrium yy/bn does not have this drawback.

5.19 Pure Coordination Games

We say one allocation of payoffs *Pareto-dominates* another, or is *Pareto-superior* to another, if all players are at least as well off in the first as in the second and at least one is better off. We say one allocation of payoffs *strictly Pareto-dominates* another if all players are strictly better off in the first than in the second. We say an allocation is *Pareto-efficient* or *Pareto-optimal* if it is not Pareto-dominated by any other allocation. An allocation is *Pareto-inefficient* if it is not Pareto-efficient. A *pure coordination game* is a game in which there is one pure-strategy Nash equilibrium that strictly Pareto-dominates all other Nash equilibria.

a. Consider the game where you and your partner independently guess an integer between 1 and 5. If you guess the same number, you each win the amount of your guess. Otherwise you lose the amount of your guess. Show that this is a pure coordination game. *Hint*: Write down the normal form of this game and find the pure-strategy Nash equilibria.
b. Consider a two-player game in which each player has two strategies. Suppose the payoff matrices for the two players are $\{a_{ij}\}$ for player 1 and $\{b_{ij}\}$ for player 2, where $i,j = 1,2$. Find the conditions on these payoff matrices for the game to be pure coordination game. *Hint*: First solve this for a 2×2 game, then a 3×3, then generalize.

5.20 Pick Any Number

Three people independently choose an integer between zero and nine. If the three choices are the same, each person receives the amount chosen. Otherwise each person loses the amount the person chose.

a. What are the pure-strategy Nash equilibria of this game?

b. How do you think people will actually play this game?
c. What does the game look like if you allow communication among the players before they make their choices? How would you model such communication and how do you think communication would change the behavior of the players?

5.21 Pure Coordination Games: Experimental Evidence

Your Choice	Smallest value of x chosen (including own)						
of x	7	6	5	4	3	2	1
7	1.30	1.10	0.90	0.70	0.50	0.30	0.10
6		1.20	1.00	0.80	0.60	0.40	0.20
5			1.10	0.90	0.70	0.50	0.30
4				1.00	0.80	0.60	0.40
3					0.90	0.70	0.50
2						0.80	0.60
1							0.70

Table 5.1. An experimental coordination game (Van Huyck, Battalio, and Beil, 1990)

Show that the game in table 5.1 is a pure coordination game and find the number of pure-strategy equilibria. In this game, a number n of players each chooses a number between 1 and 7. Suppose x_i is the choice of player i and the lowest of the numbers is y. Then player i wins $0.60 + 0.10 \times x_i - 0.20 \times (x_i - y)$.

Most people think it is obvious that players will always play the Pareto-optimal $x_i = 7$. As van Huyck, Battalio and Beil (1990) show, this is far from the case. The experimenters recruited 107 Texas A&M undergraduates and the game was played ten times with n varying between 14 and 16 subjects. The results are shown in table 5.2. Note that in the first round, only about 30% of the subjects chose the Pareto-efficient $x = 7$ and because the lowest choice was $y = 1$, they earned only 0.10. Indeed, the subjects who earned the *most* were precisely those who chose $x = 1$. The subjects progressively learn from trial to trial that it is hopeless to choose a high number and in the last round almost all subjects are choosing $x = 1$ or $x = 2$.

Choice	Period									
of x	1	2	3	4	5	6	7	8	9	10
7	33	13	9	4	4	4	6	3	3	8
6	10	11	7	0	1	2	0	0	0	0
5	34	24	10	12	2	2	4	1	0	1
4	17	23	24	18	15	5	3	3	2	2
3	5	18	25	25	17	9	8	3	4	2
2	5	13	17	23	31	35	39	27	26	17
1	2	5	15	25	37	50	47	70	72	77

Table 5.2. Results of ten trials with 107 Subjects. Each entry represents the number of subjects who chose x (row) in period y (column).

Why do people do such a poor job of coordinating in situations like these? A possibility is that not all subjects really want to maximize their payoff. If one subject wants to maximize his payoff relative to the other player, then $x = 1$ is the optimal choice. Moreover, if one or more players *think* that there might be a player who is such a "relative maximizer," such players will play $x = 1$ even if they want to maximize their absolute payoffs. There are also other possible explanations, such as wanting to maximize the minimum possible payoff.

5.22 Introductory Offers

A product comes in two qualities, high and low, at unit costs c_h and c_l, with $c_h > c_l > 0$. Consumers purchase one unit per period and a consumer learns the quality of a firm's product only by purchasing it in the first period. Consumers live for two periods and a firm cannot change its quality between the first and second periods. Thus, a consumer can use the information concerning product quality gained in the first period to decide whether to buy from the firm again in the second period. Moreover, firms can discriminate between first- and second-period consumers and offer different prices in the two periods, for instance, by extending an *introductory offer* to a new customer.

Suppose the value of a high-quality good to the consumer is h, the value of a low-quality good is zero, a consumer will purchase the good only if this does not involve a loss and a firm will sell products only if it makes positive profits. We say that the industry is in a *truthful signaling equilibrium* if the firms' choice of sale prices accurately distinguishes high-quality from low-quality firms. If the firms' choices do not distinguish high from low quality, we have a *pooling equilibrium.* In the current situation, this means that only the high-quality firms will sell.

Let δ be the consumer's discount factor on second-period utility. We show that if $h > c_h + (c_h - c_l)/\delta$, there is a truthful signaling equilibrium and not otherwise. If a high-quality firm sells to a consumer in the first period at some price p_1, then in the second period the consumer will be willing to pay $p_2 = h$, because he knows the product is of high quality. Knowing that it can make a profit $h - c_h$ from a customer in the second period, a high-quality firm might want to make a consumer an "introductory offer" at a price p_1 in the first period that would not be mimicked by the low-quality firm, in order to reap the second-period profit.

If $p_1 > c_l$, the low-quality firm could mimic the high-quality firm, so the best the high-quality firm can do is to charge $p_1 = c_l$, which the low-quality firm will not mimic, because the low-quality firm cannot profit by doing so (it cannot profit in the first period and the consumer will not buy the low-quality product in the second period). In this case, the high-quality firm's profits are $(c_l - c_h) + \delta(h - c_h)$. As long as these profits are positive, which reduces to $h > c_h + \delta(c_h - c_l)$, the high-quality firm will stay in business.

Note that each consumer gains $h - c_l$ in the truthful signaling equilibrium and firms gain $c_l - c_h + \delta(h - c_h)$ per customer.

5.23 Web Sites (for Spiders)

In the spider *Agelenopsis aperta*, individuals search for desirable locations for spinning webs. The value of a web is $2v$ to its owner. When two spiders come upon the same desirable location, the two invariably compete for it. Spiders can be either strong or weak, but it is impossible to tell which type a spider is by observation. A spider may rear onto two legs to indicate that it is strong, or fail to do so, indicating that it is weak. However, spiders do not have to be truthful. Under what conditions will they in fact signal truthfully whether they are weak or strong? Note that if it is in the interest of both the weak and the strong spider to represent itself as strong, we have a "pooling equilibrium," in which the value of the signal is zero and it will be totally ignored; hence, it will probably not be issued. If only the strong spider signals, we have a truthful signaling equilibrium.

Assume that when two spiders meet, each signals the other as strong or weak. Sender sends a signal to Receiver, Receiver simultaneously sends a signal to Sender and they each choose actions simultaneously. Based on the signal, each spider independently decides to attack or retreat. If two strong spiders attack each other, they each incur a cost of c_s and each has a 50% chance of gaining or keeping the territory. Thus, the expected payoff to each is $v - c_s$. If both spiders retreat, neither gets the territory, so their expected payoff is 0 for each. If one spider attacks and the other retreats, the attacker takes the location and there are no costs. So the payoffs to attacker and retreater are $2v$ and 0, respectively. The situation is the same for two weak spiders, except they have a cost c_w. If a strong and a weak spider attack each other, the strong wins with probability 1, at a cost b with $c_s > b > 0$ and the weak spider loses, at a cost $d > 0$. Thus, the payoff to the strong spider against the weak is $2v - b$ and the payoff to the weak against the strong is $-d$. In addition, strong spiders incur a constant cost per period of e to maintain their strength. Figure 5.13 shows a summary of the payoffs for the game.

Type 1,Type 2	Action 1,Action 2	Payoff 1,Payoff 2
strong,strong	attack,attack	$v - c_s, v - c_s$
weak,weak	attack,attack	$v - c_w, v - c_w$
strong,weak	attack,attack	$2v - b, -d$
either,either	attack,retreat	$2v$,0
either,either	retreat,retreat	0,0

Figure 5.13. Web sites for spiders

Each spider has eight pure strategies: signal that it is strong or weak (s/w), attack/retreat if the other spider signals strong (a/r), attack/retreat if the other spider signals weak (a/r). We may represent these eight strategies as saa, sar, sra, srr, waa, war, wra, wrr, where the first indicates the spider's signal, the second indicates the spider's move if the other spider signals strong, and the third indicates the spider's move if the other spider signals weak (for instance, sra means "signal strong, retreat from a strong signal, and attack a weak signal"). This is a complicated game, because the payoff matrix for a given pair of spiders has 64 entries and there are four types of pairs of spiders. Rather than use brute force, let us assume there is a truthful signaling equilibrium and see what that tells us about the relationships among v, b, c_w, c_s, d, e and the fraction p of strong spiders in the population.

Suppose $v > c_s, c_w$ and the proportion p of strong spiders is determined by the condition that the payoffs to the two conditions of being strong and being weak are equal. In a truthful signaling equilibrium, strong spiders use saa and weak spiders use wra. To see this, note that strong spiders say they're strong and weak spiders say they're weak, by definition of a truthful signaling equilibrium. Weak spiders retreat against strong spiders because $d > 0$ and attack other weak spiders because $v - c_w > 0$. Strong spiders attack weak spiders if they do not withdraw, because $2v - b > 2v - c_s > v$.

If p is the fraction of strong spiders, then the expected payoff to a strong spider is $p(v - c_s) + 2(1 - p)v - e$ and the expected payoff to a weak spider is $(1 - p)(v - c_w)$. If these two are equal, then

$$p = \frac{v + c_w - e}{c_w + c_s}, \tag{5.4}$$

which is strictly between 0 and 1 if and only if $e - c_w < v < e + c_s$.

In a truthful signaling equilibrium, each spider has expected payoff

$$\pi = \frac{(v - c_w)(c_s + e - v)}{c_w + c_s}. \tag{5.5}$$

Suppose a weak spider signals that it is strong and all other spiders play the truthful signaling equilibrium strategy. If the other spider is strong, it will attack and the weak spider will receive $-d$. If the other spider is weak, it will withdraw and the spider will gain $2v$. Thus, the payoff to the spider for a misleading communication is $-pd + 2(1 - p)v$, which cannot be greater than (5.5) if truth telling is Nash. Solving for d, we get

$$d \geq \frac{(c_s + e - v)(v + c_w)}{c_w - e + v}.$$

Can a strong spider benefit from signaling that it is weak? To see that it cannot, suppose first that it faces a strong spider. If it attacks the strong spider after signaling that it is weak, it gets the same payoff as if it signaled strong (because its opponent always attacks). If it withdraws against its opponent, it gets 0, which is less than the $v - c_s$ it gets by attacking. Thus, signaling weak against a strong opponent cannot lead to a gain. Suppose the opponent is weak. Then signaling weak means that the opponent will attack. Responding by withdrawing, it gets 0; responding by attacking, it gets $2v - b$, because it always defeats its weak opponent. But if it had signaled strong, it would have earned $2v > 2v - b$. Thus, it never pays a strong spider to signal that it is weak.

Note that as long as both strong and weak spiders exist in equilibrium, an increase in the cost e of being strong leads to an increase in payoff to all spiders, weak and strong alike. This result follows directly from equation (5.5) and is due to the fact that higher e entails a lower fraction of strong spiders, from (5.4). But weak spiders earn $(1 - p)(v - c_w)$, which is decreasing in p and strong spiders earn the same as weak spiders in equilibrium.

6

Mixed-Strategy Nash Equilibria

Leges sine moribus vanae

Horace

This chapter presents a variety of games with mixed-strategy Nash equilibria, many in the form of problems to be solved by the reader. Some mixed-strategy equilibria, such as throwing fingers (§3.8), are intuitively obvious. Others, such as the hawk-dove equilibrium (§3.10) are not intuitive, but the equilibrium depends in plausible ways on the parameters of the problem. For instance, as the cost of injury w increases in the hawk-dove game, the probability of playing hawk declines, and as the value of the territory v increases, the probability of playing hawk also increases. However, the mixed-strategy equilibrium in battle of the sexes (§3.9), is implausible because it suggest that a player's Nash strategy does not depend on the relative strength of preferences for the two pure-strategy equilibrium outcomes. Indeed, you will ascertain later that this mixed-strategy equilibrium is not stable in an evolutionary dynamic (§12.17).

6.1 The Algebra of Mixed Strategies

There is a simple way to do the algebra of mixed strategies. Examples in this case are worth more than formalities, so I will give one. The reader will find it easy to generalize.

Suppose Alice has strategy set $\{L, R\}$ and uses mixed strategy $\sigma = \alpha L + (1-\alpha)R$, whereas Bob has strategy set $\{U, D\}$ and uses mixed strategy $\tau = \beta U + (1-\beta)D$. We can then think of the payoff to Alice, $\pi_1(\sigma, \tau)$, as the value to Alice of the compound lottery in figure 6.1.

We can reduce this compound lottery to a simple lottery with four payoffs, (L, U), (L, D), (R, U), and (R, D), with probabilities $\alpha\beta$, $\alpha(1-\beta)$, $(1-\alpha)\beta$, and $(1-\alpha)(1-\beta)$, respectively. The payoff to this lottery for player i is then

$$\pi_i(\sigma, \tau) = \alpha\beta\pi_i(L, U) + \alpha(1-\beta)\pi_i(L, D)$$

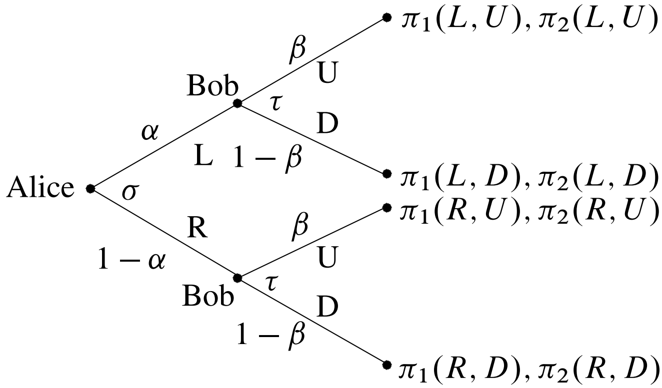


Figure 6.1. Mixed strategies as compound lotteries

$$+ (1-\alpha)\beta\pi_i(R,U) + (1-\alpha)(1-\beta)\pi_i(R,D).$$

Another way to think of this is to define the *product* of mixed strategies by

$$\begin{aligned}\sigma\tau &= (\alpha L + (1-\alpha)R)(\beta U + (1-\beta)D)\\ &= \alpha\beta LU + \alpha(1-\beta)LD + (1-\alpha)\beta RU + (1-\alpha)(1-\beta)RD,\end{aligned}$$

and then the payoff to Alice from the strategy pair (σ, τ) is

$$\begin{aligned}\pi_1(\sigma,\tau) &= \pi_1(\alpha\beta LU + \alpha(1-\beta)LD + (1-\alpha)\beta RU + (1-\alpha)(1-\beta)RD)\\ &= \alpha\beta\pi_1(L,U) + \alpha(1-\beta)\pi_1(L,D)\\ &\qquad + (1-\alpha)\beta\pi_1(R,U) + (1-\alpha)(1-\beta)\pi_1(R,D).\end{aligned}$$

6.2 Lions and Antelope

	BA	LA
BA	$c_b/2$,$c_b/2$	c_b,c_l
LA	c_l,c_b	$c_l/2$,$c_l/2$

Two lions out hunting see Big Antelope and Little Antelope in the distance. They will surely catch whatever prey they chase, whether alone or together. However, if they pick different antelopes to chase, there is no need to share, whereas if they go after the same antelope, each will get only half of the kill and the other antelope will escape. Suppose their decisions are independent, the caloric value of Big Antelope is c_b, the caloric value of Little Antelope is c_l and $0 < c_l < c_b$. Let BA be the strategy "hunt Big Antelope," and let LA be the strategy "hunt Little Antelope." The normal form game is shown in the diagram. The lions cannot distinguish between player 1 and player 2, so only symmetric Nash equilibria are acceptable.

If (BA,BA) is to be a pure-strategy equilibrium, it is necessary that $c_b/2 \geq c_l$ and it is easy to see that this condition is also sufficient. Because $c_b > c_l$, it is easy to see that (LA,LA) is not Nash. To find the mixed-strategy equilibrium, we assume (BA,BA) is not Nash, so $c_b < 2c_l$. Let α be the probability a lion uses BA. Then the payoff to the other lion from using BA is

$$\alpha\frac{c_b}{2} + (1-\alpha)c_b = c_b - \alpha\frac{c_b}{2},$$

and the payoff to using LA is

$$\alpha c_l + (1-\alpha)\frac{c_l}{2} = (1+\alpha)\frac{c_l}{2}.$$

Equating these two, we get

$$\alpha = \frac{2c_b - c_l}{c_b + c_l}.$$

For both lions, the payoff to the mixed strategy is equal to the payoff to playing either one of the strategies in support of the mixed strategy, so this payoff is equal to the payoff to BA, which is

$$c_b - \alpha\frac{c_b}{2} = c_b\left(1 - \frac{\alpha}{2}\right) = c_b\frac{3c_l}{2(c_b + c_l)}. \tag{6.1}$$

It is easy to check that the preceding fraction is greater than 1/2, so they should play the mixed strategy.

One can also calculate the expected payoff using the payoff to LA instead of the payoff to BA:

$$(1+\alpha)\frac{c_l}{2} = \left(\frac{3c_b}{c_b + c_l}\right)\frac{c_l}{2} = c_l\frac{3c_b}{2(c_b + c_l)}, \tag{6.2}$$

which is the same.

6.3 A Patent Race

Rapoport and Amaldoss (1997) set up a "patent race" game in which a "weak" player is given an endowment of 4, any integral amount of which could be invested in a project with a return of 10. However a "strong" player is given an endowment of 5 and both players are instructed that whichever player invests the most will receive the return of 10 for the patent and if there is a tie, neither gets the return of 10. It is clear that the weak player has five pure strategies (invest 0, 1, 2, 3, or 4) and the strong player will choose from 0, 1, 2, 3, 4, or 5. The payoffs to the game are as in section 4.3, matrix (g).

Show that each player has three strategies remaining after the iterated elimination of dominated strategies and then show that the game has a mixed-strategy equilibrium in which each player uses the remaining strategies with probabilities (3/5,1/5,1/5) and (1/5,1/5,3/5), respectively. Show also the the expected payoff to the players is $(4, 10)$. This is in fact the unique Nash equilibrium of the game, although this is a bit harder to show.

6.4 Tennis Strategy

In tennis, the server can serve to either the receiver's backhand or the receiver's forehand. The receiver can anticipate that the ball will come to either the forehand or backhand side. A receiver who anticipates correctly is more likely to return the ball. On the other hand, the server has a stronger backhand than forehand serve. Therefore, the receiver will return a correctly anticipated backhand serve with 60% probability and a correctly anticipated forehand serve with 90% probability. A receiver who wrongly anticipates a forehand hits a good return 20% of the time, whereas a receiver who wrongly anticipates a backhand hits a good return 30% of the time. The normal for game is shown in the diagram. Find the Nash equilibria of this game.

	b_r	f_r
b_s	$0.4, 0.6$	$0.7, 0.3$
f_s	$0.8, 0.2$	$0.1, 0.9$

6.5 Preservation of Ecology Game

Each of three firms (1, 2, and 3) uses water from a lake for production purposes. Each has two pure strategies: purify sewage (strategy 1) or divert it back into the lake (strategy 2). We assume that if zero or one firm diverts its sewage into the lake, the water remains pure, but if two or more firms do, the water is impure and each firm suffers a loss of 3. The cost of purification is 1.

We will show that the Nash equilibria are: (a) One firm always pollutes and the other two always purify, (b) All firms always pollute, (c) Each firm purifies with probability $1/(3+\sqrt{3})$, (d) Each firm purifies with probability $1/(3-\sqrt{3})$, or (e) One firm always purifies and the other two purify with probability 2/3.

The pure-strategy cases (a) and (b) are obvious. For the completely mixed equilibria (c) and (d), suppose x, y and z are the probabilities the three firms purify, $x, y, z > 0$. If firm 3 purifies, its expected payoff is $-xy - x(1-y) - y(1-x) - 4(1-x)(1-y)$. If firm 3 pollutes, its payoff is $-3x(1-y) - 3(1-x)y - 3(1-x)(1-y)$. Because firm 3 uses a completely mixed strategy, these must be equal, so after simplification we have $(1-3x)(1-3y) = 3xy$. Solving and repeating for the other two firms, we get the two desired solutions. Case (e) is derived by assuming one firm choose purify with probability 1 and then finding the completely mixed strategies of the other firms.

6.6 Hard Love

A mother wants to help her unemployed son financially, but she does not want to contribute to his distress by allowing him to loaf around. Therefore, she announces that she *may* help her son in the current period if he does not find a job. The son, however, seeks work only if he cannot depend on his mother for support and may not find work even if he searches. The payoff matrix is as shown. It is clear from the diagram that there are no pure-strategy Nash equilibria. Find the unique mixed-strategy equilibrium.

	Son	
Mom	Seek Work	Watch Soaps
Help Son	3,2	−1,3
Hard Love	−1,1	0,0

6.7 Advertising Game

Three firms (players 1, 2, and 3) put three items on the market and can advertise these products either on morning or evening TV. A firm advertises exactly once per day. If more than one firm advertises at the same time, their profits are 0. If exactly one firm advertises in the morning, its profit is 1 and if exactly one firm advertises in the evening, its profit is 2. Firms must make their daily advertising decisions simultaneously.

There is one set of equilibria in which one firm always chooses morning, another always chooses evening, and the third chooses morning with any probability. Moreover, these are the *only* Nash equilibria in which at least one firm uses a pure strategy. To see this, suppose first that firm 1 chooses the pure strategy M (morning). If both firms 2 and 3 choose mixed strategies, then one of them could gain by shifting to pure strategy E (evening). To see this, let the two mixed strategies be $\alpha M + (1-\alpha)E$ for firm 2 and $\beta M + (1-\beta)E$ for firm 3. Let $\pi_i(s_1 s_2 s_3)$ be the payoff to player i when the three firms use pure strategies $s_1 s_2 s_3$. Then, the payoff to M for firm 2 is

$$\begin{aligned}\pi_2 &= \alpha\beta\pi_2(MMM) + \alpha(1-\beta)\pi_2(MME) + (1-\alpha)\beta\pi_2(MEM)\\ &\qquad + (1-\alpha)(1-\beta)\pi_2(MEE)\\ &= \alpha\beta(0) + \alpha(1-\beta)(0) + (1-\alpha)\beta(2) + (1-\alpha)(1-\beta)(0)\\ &= 2(1-\alpha)\beta.\end{aligned}$$

Because $0 < \beta$ by definition, this is maximized by choosing $\alpha = 0$, so firm 2 should use pure strategy E. This contradicts our assumption that both firms 1 and 2 use mixed strategies.

A similar argument holds if firm 1 uses pure strategy E. We conclude that if firm 1 uses a pure strategy, at least one of the other two firms will use a pure strategy. The firm that does will not use the same pure strategy as firm 1, because this would not be a best response. Therefore, two firms use opposite pure strategies and it does not matter what the third firm does. Now we repeat the whole analysis assuming firm 2 uses a pure strategy, with clearly the same outcome. Then, we do it again for firm 3. This proves that if one firm uses a pure strategy, at least two firms use a pure strategy, which concludes this part of the problem.

To find the mixed-strategy equilibria, let x, y, and z be the probabilities of advertising in the morning for firms 1, 2, and 3. The expected return to 1 of advertising in the morning is $(1-y)(1-z)$ and in the evening it is $2yz$. If these are equal, any choice of x for firm 1 is Nash. But equality means $1-y-z-yz=0$, or $y=(1-z)/(1+z)$. Now repeat for firms 2 and 3, giving the equalities $y=(1-z)/(1+z)$ and $z=(1-x)/(1+x)$. Solving simultaneously, we get $x=y=z=\sqrt{2}-1$. To see this, substitute $y=(1-z)/(1+z)$ in $x=(1-y)/(1+y)$, getting

$$x = \frac{1-y}{1+y} = \frac{1-\frac{1-z}{1+z}}{1+\frac{1-z}{1+z}} = z.$$

Thus, $x=(1-x)/(1+x)$, which is a simple quadratic equation, the only root of which between 0 and 1 is $\sqrt{2}-1$. Thus, this is Nash.

To show that there are no other Nash equilibria, suppose $0 < x < 1$ and $0 < y < 1$. We must show $0 < z < 1$, which reproduces equilibrium (b). But $0 < x < 1$ implies $(1 + y)(1 + z) = 2$ (*why?*) and $0 < y < 1$ implies $(1 + x)(1 + z) = 2$. If $z = 0$, then $x = y = 1$, which we assumed is not the case. If $z = 1$ then $x = y = 0$, which is also not the case. This proves it.

6.8 Robin Hood and Little John

Robin Hood and Little John both want to cross a rope bridge at the same time. There is only room for one. Each has two strategies: go (G) and wait (W). It takes Robin Hood and Little John times τ_r and τ_{lj}, respectively, to cross the bridge. If both go at the same time, they fight it out at cost $\delta > 0$, after which the winner crosses the bridge. The probability of winning is 1/2 for each. If both wait, they play a polite little game of Alphonse and Gaston, at a cost $\epsilon > 0$ and one of them eventually crosses first, again with probability 1/2. We assume $0 < \epsilon < \delta$, while τ_r and τ_{lj} represent the cost of waiting for the other to cross the bridge.

Write the payoff matrix for this game, considering each player's cost as not including the necessary crossing time for himself and find all of the Nash equilibria, writing α_r and α_{lj} for the probabilities of Robin Hood and Little John going. Show that the larger δ, the less likely a go-go situation emerges and find the socially optimal δ. Show that if Robin Hood always waits, he would gain by an appropriate reduction in the costs of fighting but would not gain by an increase in the costs of fighting.

6.9 The Motorist's Dilemma

Alice and Bob, traveling in opposite directions, come to an intersection and each wants to turn left, so one must wait for the other. The time one must wait while the other turns left is the same for both and is equal to $\tau > 0$. The loss if both wait is $\epsilon > 0$ each and then one of the two is randomly chosen to turn, the other incurring the additional cost τ of waiting. If both go at the same time, the loss is $\delta > \epsilon$ each and then one of the two is randomly chosen to turn, the other incurring the additional cost τ of waiting. To the two strategies G (go) and W (wait), we add a third, C (contingent). Playing C means choosing W if the other driver chooses G and choosing G if the other driver chooses W. If both drivers choose C, we treat this as a foul-up equivalent to GG. Find the Nash equilibria of the game.

6.10 Family Politics

Male	Female: Coy	Female: Loose
Faithful	$v - r - w$ $v - r - w$	$v - r$ $v - r$
Philan-derer	0 0	v $v - 2r$

In certain species of bird (actually, this is true of many bird species) males are faithful or philanderers, females are coy or loose. Coy females insist on a long courtship before copulating, while loose females do not. Faithful males tolerate a long courtship and help rear their young, while philanderers do not wait and do not help. Suppose v is the value of having offspring to either a male or a female, $2r > 0$ is the total cost of rearing an offspring and $w > 0$ the cost of prolonged courtship to both male and female. We assume $v > r + w$. This means that if courtship leads to sharing the costs of raising an offspring, then it is worth it to both birds. The normal form matrix is shown in the diagram.

a. Show that if $v > 2r$, there is one Nash equilibrium with only loose females and only philandering males.
b. Show that if $v < 2r$, there is a unique completely mixed strategy for males and females. The fraction q of females who are coy is then given by $q = r/(v - w)$ and the fraction p of males who are philanderers is given by $w/(2r + w - v)$.

6.11 Frankie and Johnny

Frankie must pay Johnny a certain amount of money as compensation for shooting Johnny's lover, but they disagree on the amount. They agree on a negotiator, who will pick whichever of Frankie's bid x_f and Johnny's bid x_j is closer to the negotiator's opinion y. We assume $x_f, x_j \in [0, 1]$. Frankie and Johnny do not know y, but they know it is drawn from a distribution F with a continuous density f, such that $\Pr\{y < \tilde{y}\} = F(\tilde{y})$. Find the equilibrium values of x_f and x_j in terms of f and F. Solve explicitly in case y is drawn from a uniform distribution.

6.12 A Card Game

There are two players, each of whom bets \$1 and receives a number between 0 and 1 (uniformly distributed). Each player observes only his number. Player 1 can either fold or raise \$5. Player 2 can either fold or see. If neither folds, the player with the higher number wins.

The only undominated strategy for each player is to choose a critical level x_i^* and to fold iff $x_i < x_i^*$. Let (x_1^*, x_2^*) be Nash strategies. The payoff to player 1 is

$$\begin{aligned} -1 \cdot \mathrm{P}[x_1 < x_1^*] + 1 \cdot \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] \\ - 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] \\ + 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1]. \end{aligned}$$

Clearly, we have

$$\mathrm{P}[x_1 < x_1^*] = x_1^*, \qquad \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] = (1 - x_1^*)x_2^*.$$

We also know

$$\begin{aligned} \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] + \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] \\ = \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*] \\ = (1 - x_1^*)(1 - x_2^*). \end{aligned}$$

To evaluate $\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, suppose $x_1^* > x_2^*$. Then,

$$\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] = \mathrm{P}[x_1 > x_1^*, x_2 > x_1] = \frac{(1 - x_1^*)^2}{2}.$$

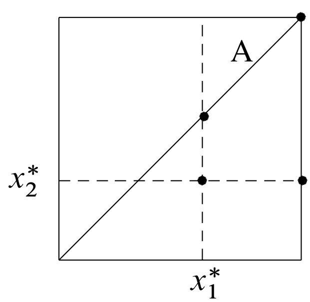


Figure 6.2. A card game

To see this, consider the diagram in figure 6.2. Because x_1 and x_2 are independently distributed, the pair (x_1, x_2) is uniformly distributed in the unit square. The case $P[x_1 > x_1^*, x_2 > x_1]$ is the little triangle labeled "A," which has area $(1 - x_1^*)^2/2$. We thus have

$$P[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}.$$

To evaluate $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$ when $x_1^* < x_2^*$, refer to Figure 6.3. Calculating the area of trapezoid A representing the case $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, we get

$$P[x_1 > x_1^*, x_2 > x_2^*, x_1 < x_2] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}.$$

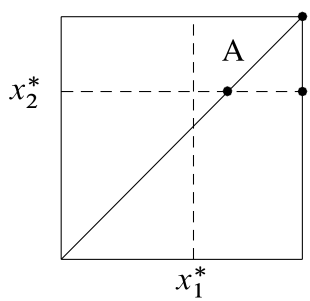


Figure 6.3. A card game II

Suppose $x_1^* > x_2^*$. The payoff to player 1 is then

$$\begin{aligned}\pi &= -x_1^* + (1 - x_1^*)x_2^* - 6\frac{(1 - x_1^*)^2}{2} \\ &\quad + 6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}\right] \\ &= 5x_1^* - 5x_2^* - 6x_1^{*2} + 5x_1^* x_2^*.\end{aligned}$$

The first-order condition on x_2^* is then $-5 + 5x_1^* = 0$, so $x_1^* = 1$. The first-order condition on x_1^* is $5 - 12x_1^* + 5x_2^* = 0$, so $x_2^* = 7/5$, which is impossible.

Thus, we must have $x_1^* < x_2^*$. The payoff to player 1 is then

$$-x_1^* + (1 - x_1^*)x_2^* - 6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}\right] + 6\frac{(1 - x_2^*)^2}{2},$$

which reduces to

$$5x_1^* - 5x_2^* - 7x_1^*x_2^* + 6x_2^{*2}.$$

The first-order condition on x_1^* gives $x_2^* = 5/7$ and the first-order condition on x_2^* then gives $x_1^* = 25/49$. Note that we indeed have $x_1^* < x_2^*$. The payoff of the game to player 1 is then

$$5\frac{25}{49} - 5\frac{5}{7} + 6\left(\frac{5}{7}\right)^2 - 7\left(\frac{25}{49}\right)\left(\frac{5}{7}\right) = -\frac{25}{49}.$$

6.13 Cheater-Inspector

	trust	inspect
cheat	n	$-na$
honest	g_{n-1}	$b + g_{n-1}$

There are n rounds in a game between an inspector and a "taker." The taker can cheat in any round and the inspector can inspect in any round. If the taker cheats without getting inspected, the game stops and she gains 1 in that period and in every remaining period. If the taker is inspected while cheating, the game stops and she is fined a in that period and in every remaining period. If the taker is honest, she receives b in that round from the inspector if inspected and nothing if not inspected and the game goes on. The game is zero-sum (that is, whatever the taker gets the inspector loses). Let g_n be the payoff of game of length $n > 0$ and let $g_o = 0$. Then, for any n, we have the game in the diagram. Find the payoffs to the players in a Nash equilibrium.

6.14 The Vindication of the Hawk

	A	W
A	$-a$,$-a$	2,0
W	0,2	1,1

Chicken (also known as the hawk-dove game) is a two-player game in which each player can either attack (A) or remain peaceful (P). Suppose at the start of the game, each player has one util of good stuff. If both players remain peaceful, they each get to consume their stuff. If one is peaceful and the other attacks, the attacker takes the other's stuff. But if both attack, they each lose $a > 0$ utils. This gives us the normal form matrix in the diagram. Show that there is a unique Nash equilibrium, players use completely mixed strategies and the payoff of the game to the players *increases* when the potential loss from conflict, a, increases.

6.15 Characterizing 2×2 Normal Form Games I

We say a normal form game is *generic* if no two payoffs for the same player are equal. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are the payoff matrices for Alice and Bob, so the payoff to Alice's strategy s_i against Bob's strategy t_j is a_{ij} for Alice and b_{ij} for Bob. We say two generic 2×2 games with payoff matrices (A, B) and (C, D) are *equivalent* if, for all $i, j, k, l = 1, 2$:

$$a_{ij} > a_{kl} \text{ if and only if } c_{ij} > c_{kl}$$

and

$$b_{ij} > b_{kl} \text{ if and only if } d_{ij} > d_{kl}.$$

In particular, if a constant is added to the payoffs to all the pure strategies of one player when played against a given pure strategy of the other player, the resulting game is equivalent to the original.

Show that equivalent 2×2 generic games have the same number of pure Nash equilibria and the same number of strictly mixed Nash equilibria. Show also that every generic 2×2 game is equivalent to either the prisoner's dilemma (§3.11), the battle of the sexes (§3.9), or the hawk-dove (§3.10). Note that this list does not include throwing fingers (§3.8), which is not generic.

	L	R
U	a_1, a_2	b_1, b_2
D	c_1, c_2	d_1, d_2

To solve this problem, we refer to the figure in the diagram. first-order the strategies so the highest payoff for player 1 is a_1. Second, add constants so that $c_1 = d_1 = b_2 = d_2 = 0$. Because the game is generic, $a_1 > 0$ and either $a_2 > 0$ (case I) or $a_2 < 0$ (case II). Third, explain why only the signs of c_2 and b_1, rather than their magnitudes, remain to be analyzed. If either is positive in case I, the game has a unique equilibrium found by the iterated elimination of dominated strategies and is equivalent to the prisoner's dilemma. The same is true in case II if either $b_1 > 0$ or $c_2 < 0$. The only remaining case I situation is $b_1, c_2 < 0$, which is equivalent to the battle of the sexes, with two pure- and one mixed-strategy equilibria. The only remaining case II is $b_1 < 0, c_2 > 0$, which is equivalent to hawk-dove and there is a unique mixed-strategy equilibrium.

6.16 Big John and Little John Revisited

Find the mixed-strategy Nash equilibrium to the simultaneous-move Big John and Little John game discussed at the end of section 3.1.

6.17 Dominance Revisited

Show that if a game has a solution by the iterated elimination of strongly dominated strategies (§4.1), then this solution is the only Nash equilibrium of the game. *Hint*: Use the fundamental theorem to show that each strongly dominated strategy has weight 0 in a mixed-strategy Nash equilibrium.

6.18 Competition on Main Street Revisited

In Competition on Main Street (§5.2), you showed that there is no pure-strategy equilibrium with three agents. Suppose that general stores can only be set up at locations $0, 1/n, \ldots, (n-1)/n, 1$ (multiple stores can occupy the same location).

a. Let $\pi(x, y, z)$ be the payoff to the agent choosing location x when the other two agents choose y and z. Find an expression for $\pi(x, y, z)$.

b. Show that for $n = 4$ there is a mixed-strategy Nash equilibrium in which each agent locates at points $1/4$ and $3/4$ with probability $1/7$ and point $1/2$ with probability $5/7$.

c. Show that for $n = 6$ there is a mixed-strategy Nash equilibrium in which each agent locates at points $1/3$, $1/2$, and $2/3$ each with probability $1/3$.

d.* Show that for $n = 10$ there is no mixed-strategy equilibrium in which all agents locate within one location of the center, but there is one in which they locate within two locations of the center. Show that locating at $3/10$, $2/5$, $1/2$, $4/5$, and $7/10$ with equal probability is such an equilibrium.

e.* If you have the appropriate mathematical software (e.g., Mathematica or Maple), or if you have a long weekend with nothing to do, find mixed-strategy equilibria for $n = 12, 14, 16$. *Hint*: In each case there are five locations that are occupied with nonzero probability and the probabilities are symmetric around $n/2$.

6.19 Twin Sisters Revisited

In section 5.15, a mother tells each of her twin daughters to ask for a certain whole number of dollars, at least 1 and at most 100. If the total of the two amounts does not exceed 101, each will have her request granted. Otherwise each gets nothing. What will the sisters do?

You probably answered that both sisters would ask for \$50, even though this is not a Nash equilibrium. However, if one sister is pretty sure the other will bid \$50, she herself might be tempted to bid \$51. With mixed strategies available, we can find a Nash equilibrium that captures our intuition that bidding \$50 will almost always be the observed behavior.

Let us write s_x for the pure strategy "bid x." Suppose both sisters use the mixed strategy $\sigma = ps_x + (1-p)s_y$, where $p \in (0,1)$ and (σ,σ) is a Nash equilibrium. In any Nash equilibrium we must have $x + y = 101$, so we can assume that $x < y = 101 - x$. Check that because the payoffs to s_x and s_y must be equal, we have $x = py$. Show that any σ satisfying the previous two conditions is indeed a Nash equilibrium; that is, show that no pure strategy s_z has higher payoff against σ that σ has against σ (Hint: consider separately the cases $z < x$, $x < z < 51$, $51 < z < y$, and $y < z$).

It is easy to see that the payoff to the Nash equilibrium with $x < y = 101 - x$ and $x = py$ is simply x per sister. Thus, though there are many Nash equilibria, the highest payoff is the one in which $x = 50$, $y = 51$, and $p = 50/51$, which is practically unity. So both sisters will ask for \$50 most of the time, as our intuition suggested to us.

But why should the highest-payoff Nash equilibrium actually be the one that the sisters choose? Could they not get "locked into" an inferior Nash equilibrium, say where $x = 10$? The answer is: to this point, we have no way of answering this question. But suppose the way that sisters play this game is a sort of "social convention" that people learn and suppose further that there is some sort of social process whereby superior conventions grow in frequency and inferior ones contract. Then, perhaps, the $x = \$50$ solution might come to be established in society.

6.20 Twin Sisters: An Agent-Based Model

We can use the procedures described in section 4.20 to create an agent-based model of Twin Sisters (§5.15). I wrote a program (in the Pascal programming language) with 200 agents, each of whom is given a random strategy s_i, where i is an integer between 0 and 101, so there were approximately 10 agents with each possible strategy. The were randomly paired for 40,000 generations, each consisting of a single period of play. In each generation, 10 players "died" and were replaced by the "offspring" of 10 other players, the probabilities of dying and of reproducing being proportional to the player's current score in the game. Moreover, 10% of the new agents were given a new, random bid ("mutation"). The results are shown in figure 6.4. Note that at the beginning of the simulation, s_{49}, s_{50}, and s_{51} represent only a few % of the population, but after 10,000 generations, they represent almost the whole population. By the end of the simulation, s_{49} has dwindled, leaving only the two strategies we intuitively expected to be played, s_{50} and s_{51} (plus stray mutations, not shown in the figure). Note also that the relative proportions of the two remaining strategies are approximately 50 to 1, as expected in a Nash equilibrium.

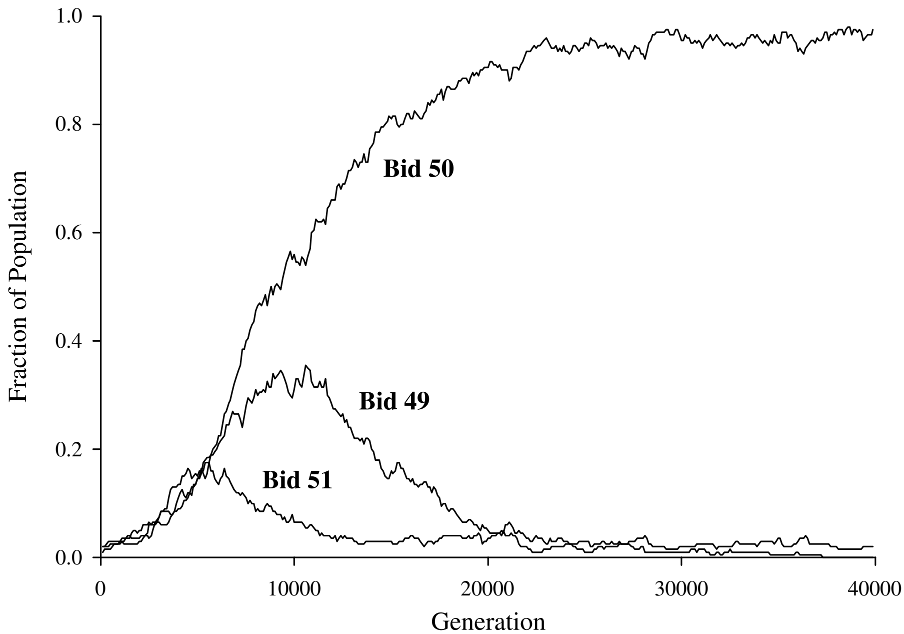


Figure 6.4. Twin sisters: an agent-based model

In chapters 10 and 12 we deal in depth with dynamical and evolutionary models, but some simple calculations suggest that the more unequal the two bids, say s_x and s_{101-x}, the smaller the minimum fraction q of "invaders" using s_{50} need be to displace a population composed of a fraction $1-q$ of agents playing the mixed strategy using s_x and s_{101-x}. As an exercise, you can show that $q = x(51-x)/50(101-2x)$, which increases almost linearly from $x = 0$ to $x = 45$ but then increases sharply to unity as x moves from 45 to 50.

This example supports the evolutionary notion that the Nash concept is justified not by the fact that a few wise, rational agents will play Nash equilibria, but because many intellectually challenged agents (the agents in our model could do absolutely nothing but (a) play their genetically inherited strategies and (b) reproduce) could dynamically settle on one of the many Nash equilibria of the game. As we shall see in chapter 10, the equilibrium points of evolutionary dynamics are always Nash equilibria, although not all Nash equilibria are equilibrium points of an evolutionary dynamic and not all equilibrium points of an evolutionary dynamic are stable in the appropriate sense.

6.21 One-Card, Two-Round Poker with Bluffing

Alice and Bob start by each putting \$2 into the "pot." Alice is dealt a card, which with equal probability is either H (high) or L (low). After looking at her card, which Bob cannot see, she either raises or folds. If she folds, the game is over and Bob takes the pot. If she raises, she must put an additional \$2 into the pot and Bob must now either stay or fold. If Bob folds, the game is over and he loses the pot. If he stays, he must put an additional \$2 into the pot to meet Alice's previous bet and Alice has another turn. Alice must again raise or fold. If she folds, she loses the pot and if she plays, she must put another \$2 into the pot and Bob has a final turn, in which he must either fold or stay. If Bob folds, he loses the pot and the game is over. If he stays, he must put an additional \$2 into the pot and Alice must show her card. If it is H, she wins the pot and if it is L, she loses the pot.

It is easy to see that Bob has three pure strategies: ss (stay,stay), sf (stay,fold), and f (fold). Alice has nine strategies: $rrbb$ (raise,raise on H, and bluff,bluff on L), $rrbf$ (raise,raise on H, and bluff,fold on L), rrf (raise,raise on H, and fold on L), $rfbb$ (raise,fold on H, and bluff,bluff on L), $rfbf$ (raise,fold on H, bluff,fold on L), fbb (fold on H, bluff,bluff on L), fbf (fold on H, bluff,fold on L), rff (raise,fold on H, fold on L), and ff (fold on H, fold on L).

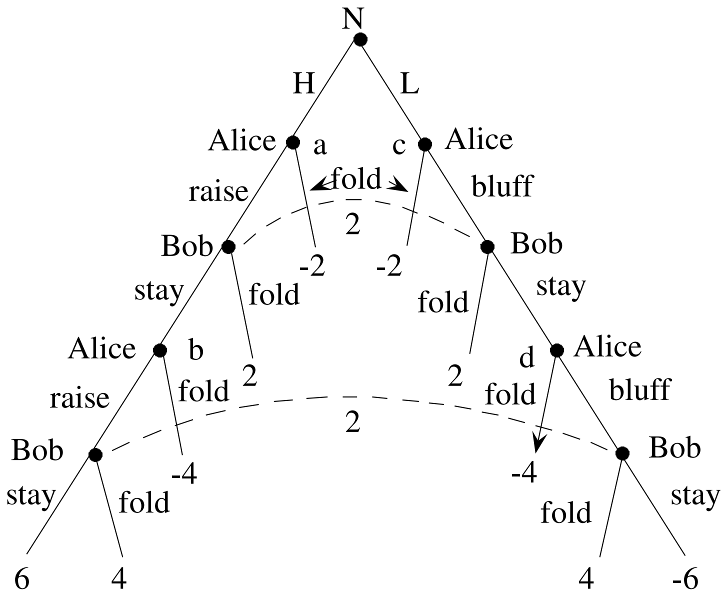


Figure 6.5. One-card two-round poker with bluffing

Show that Alice has only three weakly undominated pure strategies and find her best-response mixed strategy. Then find the best-response mixed strategy for Bob. This gives the Nash equilibrium for the game.

6.22 An Agent-Based Model of Poker with Bluffing

We can create an agent-based model of one-card two-round poker with bluffing by creating in the computer silicon creatures with very little information processing capacity (none, in fact). The creature's genome consists of a mixed strategy (that is, a probability distribution over the three nondominated strategies) for Alice and similarly for Bob. In this model, I created 200 players of each type and assigned them pure strategies randomly. In each period of play, partners are randomly assigned and every 100 periods we allow reproduction to take place. In this model, reproduction consisted in killing off the player with the lowest score and allowing the player with the highest score to reproduce, with mutation in the genome at rate 2%. The simulation ran for 50,000 periods. The results of one run of the simulations for the distribution of Bob types in the economy are shown

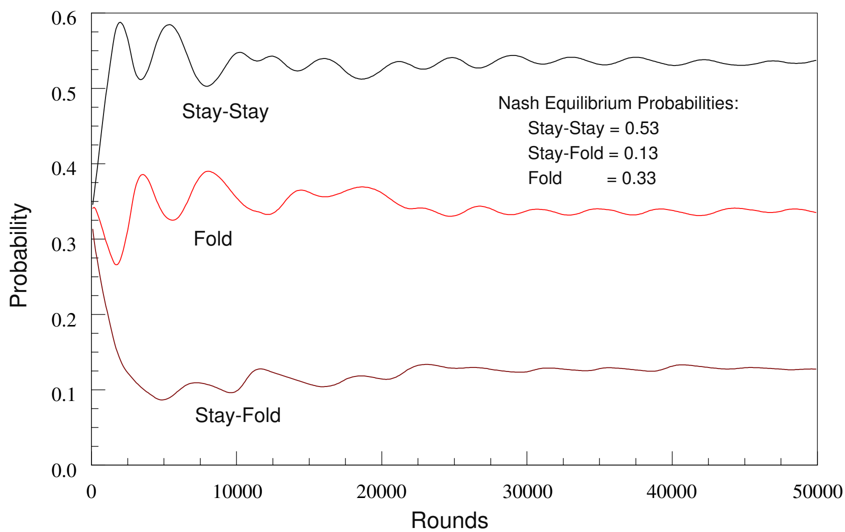


Figure 6.6. An Agent-based model of one-card two-round poker with bluffing

in figure 6.6. Note that after 25,000 periods, the frequency of each strategy has settled down to the theoretically predicted equilibrium value.

6.23 Trust in Networks

Consider a network of many traders who are randomly paired to play a one-shot (that is, played only once) prisoner's dilemma in which each receives -1 if they both defect, each receives 1 if they both cooperate and a defector receives 2 when playing against a cooperator, who receives -2. There are three types of agents: defectors defect unconditionally against all partners; trusters cooperate unconditionally with all partners; and inspectors monitor an imperfect signal indicating whether or not one's current partner defects against cooperators. The signal correctly identifies a defector with probability $p > 1/2$ and correctly identifies a non-defector with the same probability p. The inspector then refuses to trade with a partner who is signalled as a defector and otherwise plays the cooperate strategy. An agent who does not trade has payoff 0. The payoff matrix for a pair of agents has the normal form shown in figure 6.7.

Think of a "strategy" in this network as a fraction α of inspectors, a fraction β of trusters and a fraction $1-\alpha-\beta$ of defectors. A Nash equilibrium is a population composition (α, β) that is a best response to itself.

	Inspect	Trust	Defect
Inspect	p^2 p^2	p p	$-2(1-p)$ $2(1-p)$
Trust	p p	1 1	-2 2
Defect	$2(1-p)$ $-2(1-p)$	2 -2	-1 -1

Figure 6.7. The inspect-trust-defect game

It is easy to show that there are no pure-strategy Nash equilibria and for $p \geq \sqrt{3} - 1 \approx 3/4$, there are no Nash equilibria involving only two types of players. Use section 3.6 to prove that there exists a unique completely mixed Nash equilibrium for $p > 5/7$ and show that it is unique.

6.24 El Farol

In Santa Fe there is nothing to do at night but look at the stars or go to the local bar, El Farol. Let us define the utility of looking at the stars as 0 and let the cost of walking over to the bar be 1. Suppose the utility from being at the bar is 2 if there are fewer than three people at the bar and 1/2 if there are three or more people at the bar. For a given Nash equilibrium, we define the *social surplus* provided by the bar to be the sum of the payoffs to all the residents.

Suppose there are three people in Santa Fe, so there are three pure-strategy Nash equilibria, in each of which two people go to the bar and one watches the stars. The average payoff per player in each of these is 2/3. There is also a unique symmetric mixed-strategy Nash equilibrium (that is, each player uses the same mixed strategy) in which each resident goes to the bar with probability $\sqrt{(2/3)} \approx 81.65\%$. To see this, let p be the probability of going to the bar for each resident. The payoff to not going to the bar and the payoff to going to the bar must be equal. To find p, note that the probability that the other two people go to the bar is p^2, so the expected payoff to going to the bar is

$$2(1 - p^2) + \frac{1}{2}p^2 - 1 = 0,$$

the solution to which is $p = \sqrt{(2/3)}$.

Note that in this equilibrium the average payoff is 0: the bar might as well not exist!

To generalize the problem, suppose for each player $i = 1, 2, 3$ the cost of walking to the bar is c_i, the payoff when there are fewer than three people in the bar is a_i and the payoff otherwise is b_i, where $0 < b_i < c_i < a_i$. Now, if there is any mixed-strategy equilibrium, it is unique and once again bar might as well not exist. To see this, let p_i be the probability of player i going to the bar, for $i = 1, \ldots, 3$. In a mixed strategy-equilibrium, the payoff for each player to going to the bar and staying home must be the same. It is easy to show that this is equivalent to

$$p_i p_j = \frac{a_k - c_k}{a_k - b_k} \qquad i \neq j \neq k \neq i.$$

Let α_k be the right-hand side of this equation. We can solve the resulting three equations, getting $p_i = \sqrt{\alpha_j \alpha_k / \alpha_i}$. The conditions for a mixed-strategy equilibrium are thus $\alpha_i \alpha_j < \alpha_k$ for $i \neq j \neq k \neq i$. We conclude that, if the costs and benefits of the bar are not too dissimilar for the three players, the mixed-strategy equilibrium exists. Otherwise, one resident must always stay home. The only equilibrium in which there is a positive payoff from the bar's existence is if at least one resident stays home.

We can generalize the problem to n people, in which case the bar still might as well not exist, provided the equilibrium is completely mixed. However, you can show that if El Farol charges an appropriate entry fee, the payoff to both the bar and its clients can be strictly positive.

6.25 Decorated Lizards

The side-blotched lizard *Uta stansburiana* has three distinct male types: orange-throats, blue-throats, and yellow-striped. The orange-throats are violently aggressive, keep large harems of females (up to seven) and defend large territories. The blue-throats are less aggressive, keep small harems (usually three females) and defend small territories. The yellow-stripes are very docile but they look like females, so they can infiltrate another male's territory and secretly copulate with the females. Field researchers note that there is regular succession from generation to generation, in which orange-throated males are a majority in one period, followed by a majority of yellow-striped males, who are followed in turn by a majority of blue-throated males and finally by a new majority of orange-throated males, thus completing the cycle.

This cycle occurs because the orange-throats have so large a territory and so large a harem that they cannot guard effectively against the sneaky yellow-striped males, who mix in with the females and because they look a lot like females, go undetected by the orange-throats, who could easily detect the bright blue-throat males. The yellow-striped males thus manage to secure a majority of the copulations and hence sire lots of yellow-striped males, who are very common in the next period. When yellow-stripes are very common, however, the males of the blue-throated variety benefit, because they can detect and eject the yellow-stripes, as the blue-throats have smaller territories and fewer females to monitor. The blue-throat males thus have the greatest number of male offspring in the next period, which is thus dominated by blue-throat males. When the blue-throats predominate, the vigorous orange-throats eject them from their territories and hence they come to dominate the succeeding period, because they acquire the blue-throat harems and territories. Thus there is a recurring three-period cycle in which each type of male dominates in one period, only to be outdone by a different male type in the succeeding period.

	Orange Throat	Yellow Striped	Blue Throat
Orange Throat	0,0	−1,1	1,−1
Yellow Striped	1,−1	0,0	−1,1
Blue Throat	−1,1	1,−1	0,0

The game underlying this is the familiar children's game rock, paper, and scissors, with the payoff structure as in the diagram. Note that just as in the lizard case, each "type" (rock, paper, scissors), receives 0 payoff playing against itself, but is superior to one of its two dissimilar adversaries and is inferior to the other of its dissimilar adversaries (yellow-striped beats orange-throat but is beaten by blue-throat; orange-throat beats blue-throat but is beaten by yellow-striped; blue-throat beats yellow-striped but is beaten by orange-throat). Show that the only Nash equilibrium to this game is the mixed-strategy equilibrium in which each strategy is played with equal probability.

After you have learned how to model game dynamics, we will return to this problem and show that under a replicator dynamic, the male lizard population does indeed cycle among the three forms in successive breeding periods (§12.14).

6.26 Sex Ratios as Nash Equilibria

Most organisms that employ sexual reproduction have two sexes: male and female. The fraction of a female's offspring that are female is determined by genetic factors and hence is heritable. In many species (e.g., most animals), the fraction is almost exactly 1/2, even if the viabilities of males (σ_m) and females (σ_f), the probability that they mature to the point of sexual reproduction, are very different. Why is this the case?

To streamline the process of solving this problem, suppose all females breed simultaneously and their offspring constitute the next generation of birds (that is, birds live for only one breeding period). Unless otherwise stated, you should assume (a) females "choose" a ratio u of sons to daughters born that maximizes the expected number of their genes among their grandchildren; (b) each female produces c offspring; (c) males and females contribute an equal number of genes to their offspring; (d) all males are equally likely to sire an offspring; (e) there is random mating in the next generation; and (f) the next generation is so large that no single female can affect the ratio v of males to females in the next generation.

First we show that $u = 1/2$ in equilibrium; that is, a female produces half sons and half daughters. Call the birds surviving to maturity in the next generation the "breeding pool." Let s and d be the number of sons and daughters in the breeding pool. Then $\alpha = d/s$ is the expected fraction of a female's offspring sired by any given male in the breeding pool. We then have

$$\alpha = \frac{d}{s} = \frac{\sigma_f(1-v)}{\sigma_m v}. \tag{6.3}$$

We now write an expression for $f(u, v)$, the number of grandchildren of a female, in terms of α and the other parameters of the problem (u, v, σ_f, σ_m, and c). We have

$$f(u, v) = \sigma_f(1-u)c^2 + \sigma_m u c^2 \alpha. \tag{6.4}$$

To understand this expression, note that $\sigma_f(1-u)c$ is the number of daughters who survive to maturity and so $\sigma_f(1-u)c^2$ is the number of grandchildren born to daughters. Similarly, $\sigma_m uc$ is the number of sons and $\sigma_m uc(c\alpha)$ is the number of grandchildren sired by sons.

Substituting equation (6.3) into equation (6.4) and simplifying, we get

$$f(u,v) = c^2\sigma_f\left\{1 + u\left(\frac{1-2v}{v}\right)\right\}.$$

If we now choose u to maximize $f(u,v)$, we see that the only Nash equilibrium occurs when $u = v$. Thus, if $v \neq 1/2$, there cannot be a mixed-strategy equilibrium: if the fraction of males in the population is less than 50%, each female should produce all males (that is, set $u = 1$) and if the fraction of males in the population is greater than 50%, each female should produce all females (that is, set $u = 0$). The only possible Nash strategy is therefore $u = v = 1/2$, because such a strategy must be symmetric (the same for all agents) and mixed (because all pure strategies are clearly not Nash).

Suppose now that there are n females and n is sufficiently small that a single female's choice *does* affect the ratio of daughters to sons. We can still show that an equal number of daughters and sons remains a Nash equilibrium. It is easy to check that (6.3) becomes

$$\alpha = \frac{d}{s} = \frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]}.$$

The number of grandchildren as expressed in (6.4) then becomes

$$\begin{aligned} f(u,v) &= \sigma_f(1-u)c^2 + \sigma_m uc^2\frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]} \\ &= \frac{c^2\sigma_f}{(n-1)v+u}\{-2u^2 - u[2(n-1)v-(n+1)] + (n-1)v\}. \end{aligned}$$

The first-order condition on u for maximizing $f(u,v)$ then gives

$$2(n-1)v = n + 1 - 4u.$$

In a symmetric equilibrium, we must have $u = v$, which implies $u = v = 1/2$.

Now suppose that instead of only breeding once, a fraction δ_m of breeding males and δ_f of breeding females die in each period and the rest remain in the mating pool. The expression for the equilibrium ratio of males to females is derived as follows. Let m be the number of males and let n be the number of females in the first period. Then the ratio α of females to males in the breeding pool in the next period is given by

$$\alpha = \frac{d + n(1-\delta_f)}{s + m(1-\delta_m)} = \frac{\sigma_f cn(1-v) + n(1-\delta_f)}{\sigma_m cnv + m(1-\delta_m)}. \tag{6.5}$$

The number of grandchildren of one female who has fraction u of males and $1-u$ of females, when the corresponding fraction for other breeding females is v, is given by

$$f(u,v) = c^2\left[\sigma_f(1-u) + \sigma_m u\alpha\right] = c^2\left\{1 + u\left[\frac{\sigma_m}{\sigma_f}\alpha - 1\right]\right\}.$$

Hence, a mixed-strategy Nash equilibrium requires

$$\alpha = \frac{\sigma_f}{\sigma_m}. \tag{6.6}$$

Solving (6.5) and (6.6) and simplifying, we get

$$v = \frac{1}{2}\left[1 - \frac{\sigma_f\gamma(1-\delta_m) - \sigma_m(1-\delta_f)}{\sigma_m\sigma_f c}\right], \tag{6.7}$$

where we have written $\gamma = m/n$. But in the second period, m is simply the denominator of (6.5) and n is the numerator of (6.5), so (6.6) implies $\gamma = m/n = \sigma_m/\sigma_f$. Substituting this expression for γ in (6.7), we get

$$v = \frac{1}{2}\left[1 - \frac{\delta_f - \delta_m}{\sigma_f c}\right],$$

from which the result follows. Note that this ratio remains 1/2 if $\delta_f = \delta_m$.

Finally, suppose the species is haplodiploid (many bee species are). This means that males have only one copy of each gene, which they get from their mother (that is, males come from unfertilized eggs), whereas females have two copies of each gene, one from each parent. We will find the equilibrium ratio of daughters to sons assuming birds live for one breeding period and females maximize the number of copies of their genes in their grandchildren. For a female who has fraction u of sons and $1-u$ of daughters, when the corresponding fraction for other breeding females is v, the fraction of genes in daughters is $c(1-u)/2$ and the fraction in sons is cu. The number of genes (normalizing the mother's gene complement to unity) in daughters of daughters is $c^2(1-u)(1-v)/4$, the number of genes in sons of daughters is $c^2(1-u)v/2$ and the number of genes in daughters of sons is $c^2u\alpha(1-v)$. None of the female's genes are in sons of sons, because only the mother passes genetic material to her sons. The number of genes in the mother's grandchildren is the sum of these three components, which simplifies to

$$f(u,v) = c^2\left\{\frac{1+v}{4} - u\left[\frac{1+v}{4} - (1-v)\alpha\right]\right\},$$

so we must have

$$\alpha = \frac{1+v}{4(1-v)}. \tag{6.8}$$

But by our assumption that individuals live for only one breeding period, (6.3) still holds. Solving (6.3) and (6.8) simultaneously and defining $\nu = \sigma_f/\sigma_m$, we get

$$v = \frac{1+8\nu \pm \sqrt{32\nu+1}}{2(4\nu-1)},$$

where the sign of the square root is chosen to ensure $0 < v < 1$. This implies that, for instance, if $\sigma_f = \sigma_m$, then $v \approx 0.54$; that is, the ratio of daughters to sons should be only slightly biased toward males.

6.27 A Mating Game

Consider a mating system in which there are males and females, 50% of each sex being *hierarchical* (H) and the other half *egalitarian* (E). When a male meets a female to mate, their sex is visible, but neither knows the other's H/E type. There are two mating strategies: *forward* (F) and *reserved* (R). Females prefer their partners to be reserved, but males prefer to be forward. In addition, when a pair of hierarchicals meet, they both prefer that one be forward and the other reserved, but when a pair of egalitarians meet, they both prefer to play the same strategy, both forward or both reserved. The payoffs are depicted in figure 6.8.

There are four pure strategies: FF (forward if H, forward if E), FR (forward if H, reserved if E), RF (reserved if H, forward if E), RR (reserved if H, reserved if E). A mixed strategy for a female is a pair of probabilities (α_H, α_E), where α_H is the probability of being forward when she is H and α_E is the probability of being forward when she is E. A mixed strategy

	F	R
F	0,0	2,1
R	1,2	0,0

H Meets H

	F	R
F	0,2	2,0
R	1,0	0,1

H Meets E

	F	R
F	1,0	0,1
R	0,2	2,0

E Meets H

	F	R
F	1,2	0,0
R	0,0	2,1

E Meets E

Figure 6.8. Mating game payoffs, where the female is the row player

for a male is a pair of probabilities (β_H, β_E), where β_H is the probability of being forward when he is H and β_E is the probability of being forward when he is E. Find all Nash equilibria of the game.

6.28 Coordination Failure

Find the unique mixed Nash equilibrium of the game in the diagram. Show that if either player adopts any strategy other than his unique Nash strategy, the optimal response by the other player will result in a superior outcome for both. In this case, then, the Nash equilibrium is the worst of all possible worlds.

	L	M	R
U	0,0	50,40	40,50
C	40,50	0,0	50,40
D	50,40	40,50	0,0

6.29 Colonel Blotto Game

Colonel Blotto and his adversary, the Folks' Militia each try to occupy two posts by properly distributing their forces. Colonel Blotto has four regiments and the Militia has three regiments. If Colonel Blotto has more regiments than the enemy at a post, Colonel Blotto receives the enemy's regiments plus one (that is, one is the value of occupying the post). If Colonel Blotto has fewer regiments at a post than the enemy, he loses one plus the number of regiments he has at the post. A draw gives both sides 0. The total payoff is the sum of the payoffs at the two posts. Show that Colonel Blotto has five pure strategies and the Folks' Militia has four pure strategies. Write the payoff matrix and find the Nash equilibria of the game.

6.30 Number Guessing Game

Bob picks a number from 1 to 3. Alice tries to guess the number. Bob responds (truthfully!) by saying "high," "low," or "correct." The game continues until Alice guess correctly. Bob takes a number of dollars from Alice equal to the number of guesses Alice took.

The game is determined in the first two rounds. Let us write Alice's strategies as (g h l), for "first guess g, if high guess h and if low guess l." If a high guess is impossible, we write (1 x l) and if a low guess is impossible, we write (3 h x). For instance, (1x3) means "first choose 1 and if this is low, then choose 3." Write the payoff matrix and find the Nash equilibria of the game.

6.31 Target Selection

There are n targets whose military values are $a_1, \ldots, a_n$, where $a_1 > a_2 > \ldots > a_n > 0$. Attacker has one attacking unit to allocate to one of the n targets and defender has one unit to allocate to the defense of the targets. If target k is attacked and is undefended, it will be captured, with the value a_k going to attacker. If target k is defended, it has a probability p of being successfully held by defender, so the expected payoff to attacker is $(1 - p)a_k$.

Show that there is some m such that targets $1, \ldots, m$ are attacked and defended with positive probability and targets $m + 1, \ldots, n$ are neither attacked nor defended.

6.32 A Reconnaissance Game

Attacker can either attack with all its forces (strategy 1) or attack with part of its forces, leaving the remainder as reserves and rear guards in case its forces are outflanked (strategy 2). Defender has the same two strategy options. The payoff to attacker if attacker uses strategy i and defender uses strategy j is a_{ij}. We assume it is best for attacker to use the same strategy as defender; that is, $a_{11} > a_{21}$ and $a_{22} > a_{12}$.

Attacker can also send out a reconnaissance force, at cost c, to find out defender's strategy. This will surely work, unless defender takes countermeasures at cost d (these countermeasures must be taken without knowing whether Attacker will actually reconnoiter), in which case reconnaissance will fail. Suppose the game is zero-sum, with attacker payoff given by

$$A = [a_{ij}] = \begin{bmatrix} 48 & 24 \\ 12 & 36 \end{bmatrix} \qquad \text{with } c = 9 \text{ and } d = 7.$$

Find the Nash equilibria.

6.33 Attack on Hidden Object

Alice has a bomb that she wants to drop on Bob's country. She can carry the bomb in one of two identical bombers P (protected) and F (flank). Bob can prevent the damage by destroying the bomber containing the bomb.

The two bombers fly in formation, so to attack P, Bob must fly past F, in which case he runs the risk α, with $0 < \alpha < 1$, of being shot down before engaging P. Once Bob has engaged his target (whether F or P), he can destroy it with probability β, with $0 < \beta 1$. Thus, in any attack on P, if F is intact, the probability of destroying P is $\gamma = (1-\alpha)\beta$.

Suppose Bob has enough fuel to attack the bombers twice and hence, two chances to hit a target and destroy the valued object. Alice has two strategies: load the bomb in F and load the bomb in P. Bob has four strategies: attack F both times, attack P both times, attack F the first time and P the second time and vice versa. The understanding is that if the first attack was successful, the second attack is directed against the remaining target, whatever the strategy used. Write normal form and find the Nash equilibria.

6.34 Two-Person, Zero-Sum Games

A *zero-sum game* is, appropriately enough, a game in which the sum of the payoffs to all the players is 0. Von Neumann and Morgenstern (1944), who launched modern game theory, lay great stress on zero-sum games and, indeed, defined equilibrium in a way that works only for two-person zero-sum games. Nash had not yet invented the equilibrium concept that bears his name; that came in 1950.

Suppose the payoff to player 1 is $\pi(\sigma,\tau)$ when player 1 uses σ and player 2 uses τ, so the payoff to player 2 is $-\pi(\sigma,\tau)$. Von Neumann and Morgenstern defined (σ^*,τ^*) to be an equilibrium of the two-person, zero-sum game if σ^* maximizes $\min_\tau \pi(\sigma,\tau)$ and τ^* minimizes $\max_\sigma \pi(\sigma,\tau)$. They showed that this *maximin solution* satisfies

$$\pi(\sigma^*,\tau^*) = \max_\sigma \min_\tau \pi(\sigma,\tau) = \min_\tau \max_\sigma \pi(\sigma,\tau). \tag{6.9}$$

It is easy to show that *a strategy profile in a two-person, zero-sum game is a Nash equilibrium if and only if it is a maximin solution.* This implies, in particular, that *all Nash equilibria of a two-person, zero-sum game have the same payoffs.*

To prove a Nash equilibrium is a maximin solution, suppose (σ^*,τ^*) is a Nash equilibrium. Then, for all σ,τ, we have

$$\begin{aligned}\max_\sigma \pi(\sigma,\tau^*) \geq \min_\tau \max_\sigma \pi(\sigma,\tau) &\geq \pi(\sigma^*,\tau^*)\\ &\geq \max_\sigma \min_\tau \pi(\sigma,\tau) \geq \min_\tau \pi(\sigma^*,\tau).\end{aligned}$$

The first inequality is obvious, the second follows from

$$\min_\tau \max_\sigma \pi(\sigma,\tau) \geq \min_\tau \pi(\sigma^*,\tau) = \pi(\sigma^*,\tau^*),$$

the third follows from

$$\max_\sigma \min_\tau \pi(\sigma,\tau) \leq \max_\sigma \pi(\sigma,\tau^*) = \pi(\sigma^*,\tau^*),$$

and the fourth inequality is obvious. But the first and third terms must then be equal, because (σ^*,τ^*) is Nash and similarly for the third and fifth terms. Thus, they are all equal, so (σ^*,τ^*) is maximin.

To show that a maximin solution is a Nash equilibrium, suppose (σ^*,τ^*) is maximin. We know that the second equation in (6.9) holds because there exists a Nash equilibrium and we have already shown that a Nash equilibrium is maximin satisfying (6.9). But then we have

$$\pi(\sigma^*,\tau^*) \leq \max_\sigma \pi(\sigma,\tau^*) = \min_\tau \pi(\sigma^*,\tau) \leq \pi(\sigma^*,\tau^*).$$

This proves all three terms are equal, so (σ^*,τ^*) is Nash.

6.35 Mutual Monitoring in a Partnership

Two agents share a resource. One agent, whom we call the "taker," gets to use the resource and can either steal (S) or be honest (H) in the amount of resource used. The other agent, the "watcher," can monitor (M) or trust (T). We normalize the payoffs to the two players following the "cooperative" strategy (T, H) to be 0. Let b be the benefit to the taker from stealing and not getting caught, let p be the loss to the taker from getting caught stealing and let α be the probability of getting caught if the watcher monitors. Also, let c be the cost to the watcher of monitoring and let λ be the loss to the watcher if the taker steals and is not caught. We assume $b, p, \alpha, \lambda > 0$. We can normalize $b + p = 1$ (e.g., by dividing all of the payoffs to player 1 by $b + p$). The game matrix is then given in the diagram.

	T	M
H	0,0	0,$-c$
S	b,$-\lambda$	$b-\alpha$,$-\lambda(1-\alpha)-c$

Let μ be the probability of monitoring in the watcher's mixed strategy and let σ be the probability of stealing in the taker's mixed strategy.

a. Prove that if $c < \alpha\lambda$ and $b < \alpha$, then there is a completely mixed-strategy Nash equilibrium with $\mu = b/\alpha$ and $\sigma = c/\alpha\lambda$. Show that the payoff to the taker is 0 and the payoff to the watcher is $-c/\alpha$.
b. Explain why the loss to the watcher depends only on c and α and not, for instance, on λ. Explain why the return to the taker does not depend on any of the parameters of the problem, so long as $c < \alpha\lambda$ and $b < \alpha$.
c. What are the Nash equilibria if one or both of the inequalities $c < \alpha\lambda$ and $b < \alpha$ are violated?

6.36 Mutual Monitoring in Teams

This is a continuation of the previous problem. Now suppose there are $n + 1$ agents, where agent $n + 1$ is the "taker," agents $1, \ldots, n$ being identical "watchers." Suppose each watcher has the same probability $\alpha > 0$ of detecting stealing if monitoring, the same cost $c > 0$ of monitoring and the same loss $\lambda > 0$ from an undetected theft. We care only about *symmetric equilibria*, in which all watchers choose the *same* probability μ of monitoring the taker.

Let $b < 1$ be the gain to the taker from stealing and define

$$\rho = \alpha\lambda(1-b)^{\frac{n-1}{n}}.$$

Answer the following questions, assuming $b < \alpha\lambda$ and $c < \rho$.

a. Show that there is a mixed-strategy Nash equilibrium with the probability σ of stealing and the probability μ of monitoring given by

$$\sigma = \frac{c}{\rho} \qquad \mu = \frac{1-(1-b)^{\frac{1}{n}}}{\alpha}.$$

b. Show that the payoff to a watcher is now $-c/\alpha(1-b)^{\frac{n-1}{n}}$. Why does this not depend on λ?

c. How does this solution change as the group size n increases? Why does the tragedy of the commons (that is, the free rider) result not hold in this case?

d. What would happen as n increases if, for some fixed λ^*, we wrote $\lambda = \lambda^*/n$? This formulation would be reasonable if a dishonest taker imposed a fixed cost on the group no matter what its size, the cost being shared equally by the watchers.

6.37 Altruism(?) in Bird Flocks

This is an application of the results of the previous problem. Consider a flock of n birds eating in a group. A cat can catch a bird if it can sneak up behind a nearby rock without being seen. Each bird has an incentive to let the other birds look out for the cat whereas it conserves all its resources for eating (studies show that birds dissipate a considerable amount of energy and lose a considerable amount of time looking out for enemies). Why then do birds actually look out for predators when they eat in flocks? Are they "altruistic"? Perhaps not.[1]

[1]A similar problem arises in modeling the foraging behavior of flocks of birds in patchy environments, because if one bird finds a patch of food, all get to eat their fill (Motro 1991; Benkman 1988).

Suppose it takes the cat one second out in the open to reach the rock. If seen during that one second by even one bird, the birds will all fly off and the cat will lose $p \in (0, 1)$ in wasted time. If the cat reaches the rock, it catches one of the birds for a gain of $b = 1 - p$.

If each bird looks up from eating every $k \geq 1$ seconds, it will see the cat with probability $1/k$. Thus, we can take $\alpha = 1$ and $\mu = 1/k$ in the previous problem (T corresponds to $k = \infty$). The cost to the bird of being caught is $\lambda = 1$ and the cost of looking up once is c. Prove the following, where we define $\mu = 1 - c^{1/(n-1)}$ and $\eta = 1 - c^{n/(n-1)}$.

THEOREM: There are three types of symmetric equilibria.

a. If $c > 1$, then no bird looks up and the cat stalks the birds with probability 1.
b. If $c < 1$ and $b > \eta$, there is a symmetric Nash equilibrium in which the cat stalks with certainty, and birds look up every $1/\mu$ seconds.
c. If $b < \eta$, then there is a symmetric Nash equilibrium where the cat stalks with probability $\sigma = c(1-b)^{-(n-1)/n}$ and the birds look up with probability $1/\mu$.

6.38 The Groucho Marx Game

Alice and Bob ante an amount $a \geq 0$ and cards numbered from 1 to n are placed in a hat. The players draw one card each, each observing his own but not the other's. They simultaneously and independently decide to stay (s) or raise (r) by betting an additional $b \geq 0$. The high card wins $a + b$, or if one player raises and the other stays, the one raising wins a.

When $n = 3$, if $a \geq b$ a Nash equilibrium involves staying if you pick a 1 and raising otherwise. If $a < b$, a Nash equilibrium is to stay unless you pick the 3. To see this, note that staying if you get the 3 is strongly dominated by raising, so there are four strategies left: rr, rs, sr, and ss, where rr means "raise if you pick a 1, raise if you pick a 2"; rs means "raise if you pick a 1, stay if you pick a 2"; etc. The payoffs, where down the first column, 12 means Alice draws 1, Bob draws 2 etc. are shown in figure 6.9

The conclusion follows directly from the resulting payoff matrix, shown in figure 6.10.

When $n = 4$, we can show that (i) if $b > a = 0$ it is Nash to stay unless you pick the 4; (ii) if $2b > a > 0$, it is Nash to stay unless you get a 3 or a 4; (iii) if $a > 2b > 0$, it is Nash to stay if you get a 1 and raise otherwise.

	rr/rr	rr/rs	rr/sr	rr/ss	rs/rs	rs/sr	rs/ss	sr/sr	sr/ss	ss/ss
12	$-a-b$	a	$-a-b$	a	a	$-a-b$	a	$-a$	0	0
13	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	$a+b$	$a+b$	a	a	$-a$	0	0	a	a	0
23	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$	$-a-b$	$-a-b$	$-a$
31	$a+b$	$a+b$	a	a	$a+b$	a	a	a	a	a
32	$a+b$	a	$a+b$	a	a	$a+b$	a	$a+b$	a	a
	0	$2a$	$-2b$	$2(a-b)$	0	$-a-b$	$a-b$	0	$a-b$	0

Figure 6.9. The Groucho Marx game

	rr	rs	sr	ss
rr	0	$a/3$	$-b/3$	$(a-b)/3$
rs	$-a/3$	0	$-(a+b)/3$	$(a-b)/3$
sr	$b/3$	$(a+b)/3$	0	$(a-b)/3$
ss	$-(a-b)/3$	$-(a-b)/3$	$-(a-b)/3$	0

Figure 6.10. Payoff matrix for Groucho Marx game

To see this, note that staying when you pick the 4 is strongly dominated by raising. This leaves us with eight strategies for each player. Staying with 2 and raising with 1 is weakly dominated by staying with 1 or 2. This generalizes to the conclusion that you can eliminate dominated strategies by staying unless the card you pick is greater than some number between 0 and 3. Thus, four strategies remain: {rrr, srr, ssr, sss}. The payoff of any strategy against itself is clearly 0. Thus, it remains to calculate the table in figure 6.11

Figure 6.12 shows 12 times the payoff matrix for Alice, from which the conclusion follows.

It is possible now to generalize that for any $n > 0$, the only undominated pure strategies take the form of choosing a particular number and raising only if your card is greater than that number. To see this, we represent the strategy of raising if and only if the card chosen is greater than k by s_k. Thus, each player has n pure strategies (eliminating weakly dominated strategies). We must find the payoff to each pure-strategy pair $\{(s_k, s_l)|k, l = 1, \ldots, n\}$. Suppose the pure strategies used are (s_k, s_l) and the cards picked from the hat by Alice and Bob are $\tilde{k}$ and $\tilde{l}$, respectively.

	rrr/srr	rrr/ssr	rrr/sss	srr/ssr	srr/sss	ssr/sss
12	$-a-b$	a	a	0	0	0
13	$-a-b$	$-a-b$	a	$-a$	0	0
14	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	a	a	a	a	a	0
23	$-a-b$	$-a-b$	a	$-a-b$	a	0
24	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$
31	a	a	a	a	a	a
32	$a+b$	a	a	a	a	a
34	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$
41	a	a	a	a	a	a
42	$a+b$	a	a	a	a	a
43	$a+b$	$a+b$	a	$a+b$	a	$a+b$
	$-3b$	$2a-4b$	$6a-3b$	$a-2b$	$4a-2b$	$2a$

Figure 6.11. Groucho Marx with $n = 4$

	rrr	srr	ssr	sss
rrr	0	$-3b$	$2a-4b$	$6a-3b$
srr	$3b$	0	$a-2b$	$4a-2b$
ssr	$-2a+4b$	$-a+2b$	0	$2a$
sss	$6a-3b$	$-4a+2b$	$-2a$	0

Figure 6.12. Payoffs (times 12) for Groucho Marx with $n = 4$

First, suppose $k \geq l$. The probability that Alice wins if both stay is

$$\begin{aligned} \mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l\right] &= \mathrm{P}\left[\tilde{k} \leq l\right] \mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k}, \tilde{l} \leq l\right] \\ &\quad + \mathrm{P}\left[\tilde{k} > l | \tilde{k} \leq k, \tilde{l} \leq l\right] \\ &= \frac{l}{k}\frac{1}{2} + \frac{k-l}{k} = 1 - \frac{l}{2k}. \end{aligned}$$

Because the probability that Alice loses if both stay is one minus the preceding quantity and Alice stands to win or lose a in this case, we find that Alice's expected payoff in this case is

$$\pi_{k \geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l\right] = a\left(1 - \frac{l}{k}\right).$$

By symmetry (interchange k and l and then negate, or you can calculate it out), we have

$$\pi_{k<l}\left[\tilde{k} > \tilde{l} | \tilde{k} \le k, \tilde{l} \le l\right] = -a\left(1 - \frac{k}{l}\right).$$

We also have the following easy payoffs:

$$\pi\left[\tilde{k} > k, \tilde{l} \le l\right] = a$$
$$\pi\left[\tilde{k} \le k, \tilde{l} > l\right] = -a$$

Finally, suppose both players raise. First assume $k \ge l$. Then,

$$\begin{aligned} \mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] &= \mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k}, \tilde{l} > k\right] + \mathrm{P}\left[\tilde{l} \le k | \tilde{l} > l\right] \\ &= \frac{n-k}{n-l}\frac{1}{2} + \frac{k-l}{n-l}. \end{aligned}$$

Because the probability that Alice loses if both raise is one minus the preceding quantity and Alice stands to win or lose $a + b$ in this case, we find that Alice's expected payoff in this case is

$$\pi_{k\ge l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-l}.$$

By symmetry (or you can calculate it out), we have

$$\pi_{k<l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-k}.$$

Now we add everything up:

$$\begin{aligned} \pi_{k\ge l} = {} & \mathrm{P}\left[\tilde{k} \le k\right]\mathrm{P}\left[\tilde{l} \le l\right]\pi_{k\ge l}\left[\tilde{k} > \tilde{l} | \tilde{k} \le k, \tilde{l} \le l\right] \\ & +\mathrm{P}\left[\tilde{k} \le k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi\left[\tilde{k} \le k, \tilde{l} > l\right] \\ & +\mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} \le l\right]\pi\left[\tilde{k} > k, \tilde{l} \le l\right] \\ & +\mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi_{k\ge l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] \\ = {} & \frac{1}{n^2}(l-k)(a(k-l) - b(n-k)). \end{aligned}$$

By symmetry (or calculation if you do not trust your answer. I did it by calculation and checked it by symmetry), we have

$$\pi_{k<l} = \frac{1}{n^2}(k-l)(a(k-l)+b(n-l)).$$

The reader is invited to write the matrix for the normal form game for $n = 5$, and show that when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.125s_1 + 0.375s_3 + 0.5s_4$. Moreover, for $n = 6$, when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.083s_1 + 0.667s_4 + 0.25s_5$. For $n = 7$, when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.063s_1 + 0.937s_5$.

6.39 Games of Perfect Information

Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be the payoffs in a Nash equilibrium of a finite extensive form game G with perfect information (§5.6). Show that there is a pure-strategy, subgame perfect, Nash equilibrium with payoffs π. *Hint*: Use mathematical induction on the number of nonterminal nodes in the game.

A Nash equilibrium s is *strict* if there is a neighborhood of s (considered as a point in n-space) that contains no other Nash equilibrium of the game. Strict Nash equilibria of finite games are extremely well behaved dynamically, as we shall see in later chapters. They are especially well behaved if they are unique. A strict Nash equilibrium is always a pure-strategy equilibrium. Give an example of a pure-strategy equilibrium in a game of perfect information that is not strict.

6.40 Correlated Equilibria

	l	r
u	5,1	0,0
d	4,4	1,5

Consider the up-down/left-right game played by Alice and Bob, with normal form matrix shown in the diagram. There are two Pareto-efficient (§5.3) pure-strategy equilibria: (1,5) and (5,1). There is also a mixed-strategy equilibrium with payoffs (2.5,2.5), in which Alice plays u with probability 0.5, and Bob plays l with probability 0.5.

If the players can jointly observe an event with probability 1/2, they can achieve the payoff (3,3) by playing (u,l) when the event occurs, and (d,r)

when it does not. Note that this is Nash, because if the event occurs and Bob plays l, Alice's best response is u; if the event does not occur and Bob plays r, then Alice's best response is d; and similarly for Bob. This is called a *correlated equilibrium*.

A more general correlated equilibrium for this coordination game can be constructed as follows. Build a device that has three states: a, b, and c, with probability of occurrence α, β, and $1-\alpha-\beta$. Allow Alice to have the information set $[\{a\},\{b,c\}]$, and allow Bob to have the information set $[\{a,b\},\{c\}]$. For what values of α and β is the following Nash: Alice plays u when she sees a and plays d when she sees $\{b,c\}$; Bob plays r when he sees c and plays l when he sees $\{a,b\}$.

Note that when a occurs, Alice sees a, so she knows that Bob sees $\{a,b\}$, so Bob plays l. Thus, Alice's best response is u. So far, so good. When b occurs, Alice sees $\{b,c\}$, so using Bayes' rule, she knows that Bob sees b with probability $\beta/(1-\alpha)$, and Bob sees c with probability $(1-\alpha-\beta)/(1-\alpha)$. Thus, Alice knows she faces the mixed strategy l played with probability $\beta/(1-\alpha)$ and r played with probability $(1-\alpha-\beta)/(1-\alpha)$. The payoff to u in this case is $5\beta/(1-\alpha)$, and the payoff to d is $4\beta/(1-\alpha)+(1-\alpha-\beta)/(1-\alpha)$. If d is to be a best response, we must thus have $1 \geq \alpha+2\beta$. If c occurs, the same conditions for Alice hold.

What about the conditions for Bob? When c occurs, Alice sees $\{b,c\}$, so she plays d. Bob's best response is r. So far, so good. When a occurs, Bob sees $\{a,b\}$, so his Bayesian posterior for the probability that Alice sees a is then $\alpha/(\alpha+\beta)$. A straightforward argument, parallel to that of the previous paragraph, shows that playing l is a best response if and only if $\alpha \geq \beta$.

Any α and β that satisfy $1 \geq \alpha+2\beta$ and $\alpha \geq \beta$ permit a correlated equilibrium. Another characterization is $\beta \leq 1/3$ and $1-2\beta \geq \alpha \geq \beta$. What are the Pareto-efficient choices of α and β? Because the equilibrium is $a \to (u,l)$, $b \to (d,l)$, and $c \to (d,r)$, the payoffs to (a,b,c) are

$$\alpha(5,1)+\beta(4,4)+(1-\alpha-\beta)(1,5)=(1+4\alpha+3\beta,5-4\alpha-\beta),$$

where $\beta \leq 1/3$ and $1-2\beta \geq \alpha \geq \beta$. This is a linear programming problem. The solution is shown in figure 6.13.

The pair of straight lines connecting (1,5) to (10/3,10/3) to (5,1) is the set of Pareto-efficient points. Note that the symmetric point (10/3,10/3) corresponds to $\alpha=\beta=1/3$.

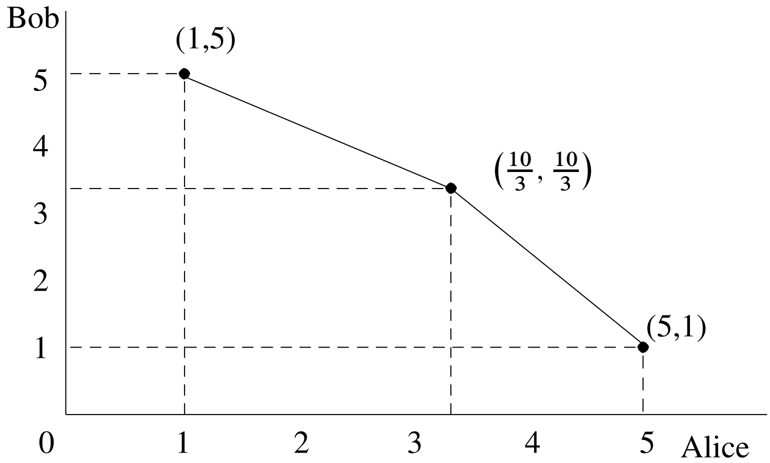


Figure 6.13. Alice and Bob correlate their behavior.

6.41 Territoriality as a Correlated Equilibrium

The hawk-dove game (§3.10) is an inefficient way to allocate property rights, especially if the cost of injury w is not much larger than the value v of the property. To see this, note that players choose hawk with probability v/w, and you can check that the ratio of the payoff to the efficient payoff $v/2$ is

$$1 - \frac{v}{w}.$$

When w is near v, this is close to zero.

Suppose some members of the population decide to play a new strategy based on the fact that whenever two players have a property dispute, one of them must have gotten there first, and the other must have come later. We may call the former the "incumbent" and the latter the "contester." The new strategy, B, called the "bourgeois" strategy, always plays hawk when incumbent and dove when contester. When we add B to the normal form matrix of the game, we get the *hawk, dove, bourgeois game* depicted in figure 6.14. Note that the payoff to bourgeois against bourgeois, $v/2$ is greater than $3v/4 - w/4$, which is the payoff to hawk against bourgeois, and is also greater than $v/4$, which is the payoff to dove against bourgeois. Therefore, bourgeois is a strict Nash equilibrium. It is also efficient, because there is never a hawk-hawk confrontation in the bourgeois equilibrium, so there is never any injury.

	H	D	B
H	$(v-w)/2$	v	$3v/4-w/4$
D	0	$v/2$	$v/4$
B	$(v-w)/4$	$3v/4$	$v/2$

Figure 6.14. The hawks-dove-bourgeois game

The bourgeois strategy is really a correlated equilibrium of the hawk-dove game, with the correlating device being the signal as to who was the first to occupy the territory. We may think of the signal as a moral justification for ownership.

This example can be widely generalized. Indeed, there are excellent grounds for considering the correlated equilibrium, rather than the Nash equilibrium, the fundamental equilibrium concept in game theory, and for identifying correlated equilibria with social norms (Gintis 2009). Moreover, our species developed through a dynamic call gene-culture coevolution (Boyd and Richerson 1985), and cultural values concerning property rights have been important elements in this coevolutionary process (Gintis 2007).

6.42 Haggling at the Bazaar

Consider seller Alice facing potential buyer Bob in a two-period game. In the first period, Alice makes an offer to sell at price p_1, and Bob accepts or rejects. If Bob accepts, the exchange is made, and the game is over. Otherwise, Alice makes another offer p_2, and Bob accepts or rejects. If he accepts in the second period, the exchange is made. Otherwise, no trade occurs. The game tree is depicted in figure 6.15.

Suppose the reservation price of the good to Alice is s and the value to Bob is b. Suppose Bob and Alice have discount factors δ_a and δ_b for trades that are made in the second period. The value b to Bob is unknown to Alice, but Alice believes that with probability π it is b_h and with probability $1-\pi$ it is b_l, where $b_h > b_l > s$, so Alice would gain from transacting with Bob either way. Suppose that the parameters of the problem are such that if Alice did not get a second chance to make an offer, she would charge the lesser amount b_l.

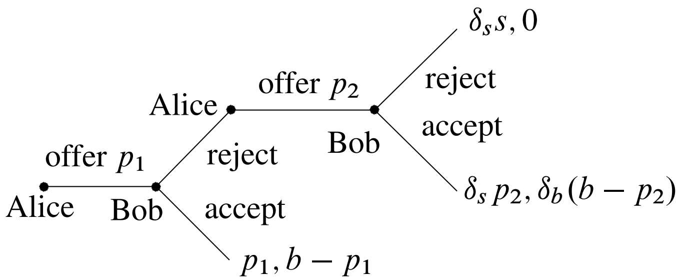


Figure 6.15. Haggling at the bazaar

The payoffs are $(p_1, b-p_1)$, $(\delta_a p_2, \delta_b(b-p_2))$, and $(\delta_a s, 0)$ in case Bob accepts on the first round, the second round, and neither round, respectively. Because, if Alice did not get a second chance to make an offer, she would charge the lesser amount b_l, we must have $\pi b_h + (1-\pi)s \leq b_l$, or simply $\pi \leq (b_l - s)/(b_h - s)$. Suppose $b_h \geq p_1 > b_l$, so there is some chance of getting to the second round. Let $\mu(p_1)$ be Alice's posterior probability that $b = b_h$, given that Bob refused on round 1 at price p_1. Then, $\mu(p_1) = x\pi/(x\pi + 1 - \pi)$ where x is probability that $b = b_h$ and Bob refuses price p_1. To see this, let $\mu = \mathrm{P}(b_h|\text{refuse})$. Then by Bayes' rule, with $x = P(\text{refuse}|b_h)$,

$$\mu = \frac{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h)}{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h) + \mathrm{P}(\text{refuse}|b_l)\mathrm{P}(b_l)}$$

$$= \frac{x\pi}{x\pi + 1 - \pi} = \frac{\pi}{\pi + \frac{1-\pi}{x}} \leq \pi.$$

This implies $\mu(p_1) \leq \pi$; that is, if we get to the second round, Alice's posterior probability of the event $\{b = b_h\}$ cannot increase. Thus, we conclude that if we reach a second round, Alice will offer $p_2 = b_l$.

We can now roll back the game tree to the one-stage game in which Alice offers price p_1, the payoff for Bob's strategy "accept the offer" is $(p_1, b - p_1)$, and the payoff to his strategy "reject the offer" is $(\delta_a b_l, \delta_b(b - b_l))$. Define $p^* = b_h - \delta_b(b_h - b_l)$. Then, the only undominated strategies for Alice are $p_1 = b_l$ and $p_1 = p^*$. To see this, note that Bob accepts $p_1 = b_l$, so $p_1 < b_l$ is dominated. Any player who accepts p_1 for $b_l < p_1 < p^*$ accepts $p_1 = p^*$. Bob will not accept $p_1 > p^*$ because, whether $b = b_h$ or $b = b_l$, Bob prefers to wait until the second round and get $p_2 = b_l$. At $p_1 = p^*$, the payoffs to "accept" and "reject" are equal if $b = b_h$, because then $b_h - p_1 = \delta_b(b_h - b_l)$, so Bob accepts on round one if $b = b_h$.

It follows that Alice chooses $p_1 = b_l$ if b_l is greater than $\pi p^* + (1-\pi)\delta_a b_l$, chooses $p_1 = p^*$ if the opposite inequality holds, and otherwise is indifferent between the two choices. The first case reduces to the inequality

$$b_h > \left(\frac{1-\delta_a}{\pi} - (\delta_b - \delta_a)\right)\frac{b_l}{1-\delta_b}. \tag{6.10}$$

We conclude that there is a unique Nash equilibrium.

Now, suppose the parameters of the problem are such that if Alice did not get a second chance to make an offer, she would charge the *greater* amount b_h. Then, there is no Nash equilibrium in pure strategies. To see this, suppose Alice chooses $p_2 = b_l$. Because Alice's posterior probability for $\{b = b_l\}$ cannot be less than π (for the same reason as in the preceding problem) and because she would charge b_h in the one-shot game, she must charge $p_2 = b_h$. So, suppose Alice chooses $p_2 = b_h$. Then, the only undominated strategies on the first round are $p_1 = \delta_b b_h$ and $p_1 = b_l$. But if Bob rejects $p_1 = \delta_b b_h$, we must have $b = b_l$, so it is not subgame perfect to charge $p_2 = b_h$. The reader is invited to find the mixed-strategy Nash equilibrium of this game.

6.43 Poker with Bluffing Revisited

If you have access to computer software to solve for Nash equilibria of normal form games, you can easily do the following. Doing the problem by hand is not feasible.

a. * Show that there are two Nash equilibria to Poker with Bluffing (§4.16). Ollie uses the same strategy in both, bluffing on the first round with 40% probability (that is, he raises with H or L on the first round, and drops with M). Stan has two mixed-strategy best responses to Ollie, one of which uses two pure strategies and the other uses three. The latter involves bluffing with 40% probability. The expected payoff to the game for Ollie is \$1.20. (It's the same for both equilibria, because it's a zero-sum game; see section 6.34).

b. * Now suppose Stan sees Ollie's first move before raising or staying. Show that there are twenty one Nash equilibria, but that Stan uses only two different mixed strategies, and both involve raising with high and raising with medium or low with 25% probability on the first round, and calling with high or medium and calling with 25% probability on the second round. Stan has lots of mixed strategies, but they entail only two different behaviors at the nodes where he chooses. If Ollie raised, Stan raises with H, raises with 75% probability with medium, and stays with low. If Ollie stayed, Stan raises with H, and raises with 25% probability with low. In one set of strategies, Stan raises with 50% probability with medium, and with 25% probability in the other. In all Nash equilibria, Ollie can expect to lose $0.25.

6.44 Algorithms for Finding Nash Equilibria

In all but the simplest cases, finding the complete set of Nash equilibria of a game can be an error-prone chore. If you have the appropriate computer software, however, and if the game is not too complicated, the process can be completely automated. I use Mathematica for this purpose. Even if you do not have access to such software, the analysis in this section is useful, as it supplies considerable insight into the nature of Nash equilibria.

	l	r
A	1,1	1,1
L	$2,-1$	$-10,-5$
R	$-1,-5$	$0,-1$

Mathematica has a command of the form "Solve[*eqns*,*vars*]" that solves a list of equations (*eqns*) for a list of variables (*vars*). For instance, "Solve[{x+y == 5,2x-3y == 7},{x,y}]" would return $x = 22/5$ and $y = 3/5$. Note that in Mathematica, a list is a set of objects, separated by commas, enclosed in a pair of curly brackets, such as {x,y}. Note also that Mathematica uses double equals signs (==) to indicate equality. I will assume the particular game in the diagram. It is easy to see how this generalizes to any finite game. Let $\alpha = \mathrm{P}[A]$, $\beta = \mathrm{P}[L]$, $\gamma = \mathrm{P}[R]$, $\nu = P[l]$, and $\mu = \mathrm{P}[r]$.

The Nash equilibrium payoffs $\bar{\pi}_1$ for player 1 are then given by $\bar{\pi}_1 = \alpha\pi_A + \beta\pi_L + \gamma\pi_R$, where $\gamma = 1 - \alpha - \beta$ and π_A, π_L, and π_R are the payoffs to A, R, and L, respectively. We then have $\pi_A = \nu\pi_{Al} + \mu\pi_{Ar}$, $\pi_L = \nu\pi_{Ll} + \mu\pi_{Lr}$, and $\pi_R = \nu\pi_{Rl} + \mu\pi_{Rr}$, where $\mu = 1 - \nu$ and π_{xy} is the payoff to player 1 choosing pure strategy x and player 2 choosing pure strategy y. Similarly, for player 2, we have $\bar{\pi}_2 = \nu\pi_l + \mu\pi_r$, where $\pi_l = \alpha\pi_{Al} + \beta\pi_{Ll} + \gamma\pi_{Rl}$ and $\pi_r = \alpha\pi_{Ar} + \beta\pi_{Lr} + \gamma\pi_{Rr}$. From the fundamental theorem (§3.6), all the Nash equilibria are solutions to the three equations

$$\alpha(\pi_A - \bar{\pi}_1) = 0, \tag{6.11}$$
$$\beta(\pi_L - \bar{\pi}_1) = 0, \tag{6.12}$$

$$\nu(\pi_l - \bar{\pi}_2) = 0. \tag{6.13}$$

To see this, note first that if the first two equations are satisfied, then $\gamma(\pi_R - \bar{\pi}) = 0$ as well and if the third equation is satisfied, then $\mu(\pi_r - \bar{\pi}_2) = 0$ as well. Suppose $(\bar{\pi}_1, \bar{\pi}_2)$ form a Nash equilibrium. Then if $\pi_x \neq \bar{\pi}_1$ for $x = A, L, R$, we must have $\alpha = 0$, $\beta = 0$, or $\gamma = 0$, respectively. Thus the first two equations in (6.11) hold. A similar argument holds for the third equation.

It is clear, however, that not all the solutions to (6.11) need be Nash equilibria, because these equations do not preclude that pure strategies that do not appear in the Nash equilibrium have higher payoffs than included strategies. To check for this, we may ask Mathematica to list all the payoffs for all the solutions to (6.11), and we can then visually pick out the non-Nash solutions. The Mathematica command for this is

$$\{\{\pi_A, \pi_L, \pi_R, \bar{\pi}_1\}, \{\pi_l, \pi_r, \bar{\pi}_2\}, \{\alpha, \beta, \nu\}\}/.sol$$

Mathematica's response is

$\{\{\pi_A, \pi_L, \pi_R, \bar{\pi}_1\}$	$\{\pi_l, \pi_r, \bar{\pi}_2\}$	$\{\alpha, \beta, \nu\}\}$
$\{1, -10, 0, 0\}$	$\{-5, -1, -1\}$	$\{0, 0, 0\}$
$\{1, 2, -1, -1\}$	$\{-5, -1, -5\}$	$\{0, 0, 1\}$
$\{1, -\frac{10}{13}, -\frac{10}{13}, -\frac{10}{13}\}$	$\{-3, -3, -3\}$	$\{0, \frac{1}{2}, \frac{10}{13}\}$
$\{1, -10, 0, -10\}$	$\{-1, -5, -5\}$	$\{0, 1, 0\}$
$\{1, 2, -1, 2\}$	$\{-1, -5, -1\}$	$\{0, 1, 1\}$
$\{1, 2(6\nu - 5), -\nu, 1\}$	$\{1, 1, 1\}$	$\{1, 0, \nu\}$

The first solution, for $\alpha = \beta = \nu = 0$, is not Nash because 1, the payoff to π_A is greater than 0, the payoff to $\bar{\pi}_1$. The second is not Nash because $1, 2 > -1$, the third is not Nash because $1 > -\frac{10}{13}$, and the fourth is not Nash because $1 > -10$. The fifth is the Nash equilibrium Ll, with payoffs $(2, -1)$. The last is Nash provided $2(6\nu - 5) \leq 1$. In this Nash equilibrium, player 1 plays A and player 2 uses any strategy in which $0 \leq \nu \leq \frac{11}{12}$.

Although this still appears to be somewhat arduous, with the proper software it is almost completely automated, except for the last step, in which non-Nash solutions are discarded. However, with more powerful algorithms, even this step can be automated. To see how, note that if $\bar{\pi} = \sum_i \alpha_i \pi_i$ is the payoff for a player in a Nash equilibrium, in which pure strategy i is played with probability α_i, and π_i is the payoff to strategy i in this equilibrium, then we must have $0 \leq \alpha_i \leq 1$, $\alpha_i(\pi_i - \bar{\pi}) \geq 0$ and

$(1-\alpha_i)(\pi_i - \bar{\pi}) \leq 0$. This is because, by the fundamental theorem (§3.6), if $\alpha_i > 0$, then $\pi_i = \bar{\pi}$, and if $\alpha_i = 0$, then $\pi_i \leq \bar{\pi}$. The converse is also the case: $0 \leq \alpha_i \leq 1$, $\alpha_i(\pi_i - \bar{\pi}) \geq 0$ and $(1-\alpha_i)(\pi_i - \bar{\pi}) \leq 0$ for all i imply the player is using a best response.

Thus, if player 1 has n pure strategies and player 2 has m pure strategies, we can completely characterize the Nash equilibria by $4(m+n)$ inequalities. Solving these inequalities gives exactly the Nash equilibrium for the problem. Mathematica has an algorithm called "InequalitySolve" that does just this. We must first load this routine with the command <<Algebra'InequalitySolve' . Then, writing

$\gamma = 1 - \alpha - \beta$
$\nu = 1 - \mu$
InequalitySolve[
$\{0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 0 \leq \gamma \leq 1, 0 \leq \nu \leq 1, 0 \leq \mu \leq 1$
$\alpha(\pi_A - \bar{\pi}_1) \geq 0, (1-\alpha)(\pi_A - \bar{\pi}_1) \leq 0,$
$\beta(\pi_L - \bar{\pi}_1) \geq 0, (1-\beta)(\pi_L - \bar{\pi}_1) \leq 0,$
$\gamma(\pi_R - \bar{\pi}_1) \geq 0, (1-\gamma)(\pi_R - \bar{\pi}_1) \leq 0,$
$\nu(\pi_l - \bar{\pi}_2) \geq 0, (1-\nu)(\pi_l - \bar{\pi}_2) \leq 0,$
$\mu(\pi_r - \bar{\pi}_2) \geq 0, (1-\mu)(\pi_r - \bar{\pi}_2) \leq 0\},$
$\{\alpha, \beta, \nu\}]$

Mathematica returns exactly the set of Nash equilibria:

$$\{\alpha == 0, \beta == 1, \nu == 1\} || \{\alpha == 1, \beta == 0, 0 \leq \nu \leq \frac{11}{12}\}$$

A note of warning: Mathematica's InequalitySolve does not always find completely mixed Nash equilibria (e.g., try the hawk-dove game), (§3.10), so you should always use both the Solve and InequalitySolve procedures.

We see that Nash equilibria of simple games can be found by a straightforward algorithm. Games that are too complicated for Mathematica to solve can sometimes be solved by hand using mathematical ingenuity. However, creativity and expertise in game theory do not depend on the capacity to solve systems of equations by hand. Rather, creativity and expertise come from understanding how to translate real-life strategic situations into appropriate game-theoretic format, and how to interpret the mechanically derivable results.

6.45 Why Play Mixed Strategies?

In twin sisters (§6.19), we found that there are no pure-strategy Nash equilibria, but there are many mixed-strategy Nash equilibria. In the Nash equilibrium with the highest payoff, each sister asks for $50 with probability 50/51 and $51 with probability 1/51. However, if both pure strategies have equal payoffs against the mixed strategy of the other player, why should either sister bother randomizing? Indeed, if one sister conjectures that the other will play her Nash strategy, all mixed strategies have equal payoff, so why prefer the (50/51,1/51) strategy over any other? Moreover, if each sister believes the other is thinking the same way, it is irrational for each to conjecture that the other will choose the (50/51,1/51) strategy or any other particular strategy. Therefore, the whole mixed-strategy Nash equilibrium collapses.

Of course, this problem is not limited to twin sisters; it applies to any strictly mixed Nash equilibrium. By the fundamental theorem (§3.6), any mixed-strategy best response consists of equal-payoff pure strategies, so why should a player bother randomizing? The answer is that there is no reason at all. Therefore, no player should expect any other player to randomize or to do anything else in particular.

You may think that this is some sort of trick argument, or a verbal paradox that can be left to the philosophers to sort out. It is not. The argument completely destroys the classical game-theoretic analysis of mixed-strategy Nash equilibria. In *The Bounds of Reason* (2009), I present some ingenious arguments defending the mixed-strategy equilibrium by Nobel Prize winners John Harsanyi (1973) and Robert Aumann (1987), but their constructions do not work for games that model complex social interaction, such as principal-agent models or repeated games.

Of course, evolutionary game theory has a solution to the problem, provided it is socially meaningful to consider the game itself as the stage game in an evolutionary dynamic in which agents repeatedly meet to play the game and higher-payoff strategies expand as a fraction of the population at the expense of lower-payoff agents. In this setting, each agent plays a pure strategy, but in an evolutionary equilibrium the fraction of each strategy represented in the population equals its weight in the Nash equilibrium. Indeed, this is exactly what we saw when we developed an agent-based model of twin sisters (§6.20).

6.46 Reviewing of Basic Concepts

a. Define a *mixed strategy*, and write the expression for the payoff to using a mixed strategy as a function of the payoffs to the underlying pure strategies.
b. Write the condition for a set of mixed strategies to form a Nash equilibrium.
c. We say a Nash equilibrium is *strict* if the strategy used by each player in this equilibrium is the only best response to the strategies used by the other player. Define a *strictly mixed* strategy, and show that if any player in a Nash equilibrium uses a strictly mixed strategy, then the equilibrium is not strict.

7

Principal-Agent Models

> Things are gettin' better
> It's people that are gettin' worse
>
> Mose Allison

In the *principal-agent model*, the payoff to the *principal* depends on an action taken by the *agent*. The principal cannot contract for the action, but can compensate the agent based on some observable signal that is correlated with the action. The principal is first mover, and chooses an incentive scheme for paying the agent that depends on the observed signal. The agent then determines the optimal action to take, given the incentives, then decides whether to accept the principal's offer, based on the expected payment and the subjective cost of performing the action. Upon accepting, the agent chooses an action that maximizes his payoff, and the principal observes the signal correlated with the action, pays the agent according to the incentive scheme, and receives a payoff dependent upon the signal. The incentive scheme is a precommitment by the principal, even though the agent's action is not.

7.1 Gift Exchange

In a famous paper, "Labor Contracts as Partial Gift Exchange," George Akerlof (1982) suggested that sometimes employers pay employees more than they must to attract the labor they need, and employees often reciprocate by working harder or more carefully than they otherwise would. He called this *gift exchange*. This section analyzes a simple model of gift exchange in labor markets. A firm hires N identical employees, each of whom supplies effort level $e(w-z)$, where $e(\cdot)$ is increasing and concave, w is the wage, and z is a benchmark wage such that $w > z$ indicates that the employer is being generous, and conversely $w < z$ indicates that the boss is ungenerous. The firm's revenue is an increasing and concave function $f(eN)$ of total amount of effort supplied by the N employees, so the firm's net profit is given by

$$\pi(w, N) = f(e(w-z)N) - wN.$$

Suppose that the firm chooses w and N to maximize profits. Show that the *Solow condition*

$$\frac{de}{dw} = \frac{e}{w} \tag{7.1}$$

holds (Solow 1979) and that the second-order condition for a profit maximum is satisfied. Then, writing the equilibrium wage w^*, equilibrium effort e^*, and equilibrium profits π^* as a function of the benchmark wage z, show that

$$\frac{de^*}{dz} > 0; \quad \frac{d\pi^*}{dz} < 0; \quad \frac{dw^*}{dz} > 1.$$

7.2 Contract Monitoring

An employer hires supervisors to oversee his employees, docking the pay of any employee who is caught shirking. Employee effort consists of working a fraction e of the time, so if there are N employees, and each employee works at effort level e for one hour, then total labor supplied is eN. The employer's revenue in this case is $q(eN)$, where $q(\cdot)$ is an increasing function. All employees have the same utility function $u(w,e) = (1-e)w$, where w is the wage rate and e is the effort level. An employee, who is normally paid w, is paid $z < w$ if caught shirking.

Suppose that an employee who chooses effort level e is caught shirking with probability $p(e) = 1-e$, so the harder the employee works, the lower the probability of being caught shirking. The game tree for this problem is depicted in figure 7.1.

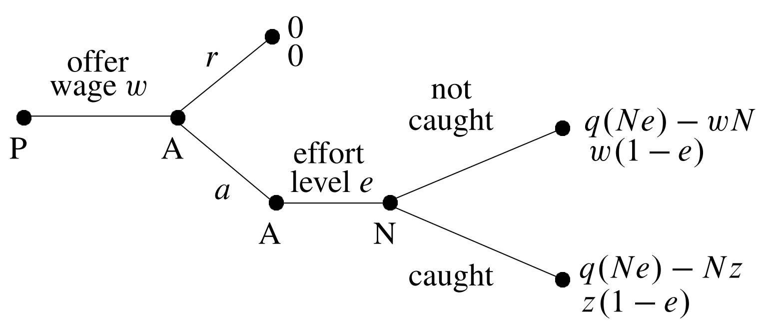


Figure 7.1. Labor discipline with monitoring

a. Show that $w(1-e)e + z(1-e)^2$ is the payoff to an employee who chooses effort level e.

b. Show that if the employer offers wage $w > 2z$, the employee's best response is to choose

$$e(w) = \frac{w - 2z}{2(w - z)}.$$

Show that this employee's *best-response schedule* is increasing and concave, as depicted in figure 7.2.

c. If the employer chooses w and N to maximize profits, show that the choice of w in fact maximizes $e(w)/w$, the amount of effort per dollar of wages, which is the slope of the employer iso-cost line in figure 7.2.

d. Show that Nash equilibrium (w^*, e^*) satisfies the *Solow condition* (Solow 1979),

$$e'(w^*) = \frac{e(w^*)}{w^*}.$$

This is where the employer iso-cost line is tangent to the employee's best-response schedule at (w^*, e^*) in figure 7.2.

e. Show that

$$w^* = (2 + \sqrt{2})z \approx 3.41z, \qquad e^* = \frac{1}{\sqrt{2}(1 + \sqrt{2})} \approx 0.29.$$

f. Suppose the employee's reservation utility is $z_o > 0$, so the employee must be offered expected utility z_o to agree to come to work. Show that the employer will set $z = 2z_o/(1 + \sqrt{2}) \approx 0.83z_o$.

7.3 Profit Signaling

An employer hires an employee to do a job. There are two possible levels of profits for the employer, high (π_H) and low ($\pi_L < \pi_H$). The employee can affect the probability of high profits by choosing to work with either high or low effort. With high effort the probability of high profits is p_h, and with low effort the probability of high profits is p_l, where $0 < p_l < p_h < 1$.

If the employer could see the employee's choice of effort, he could simply write a contract for high effort, but he cannot. The only way he can induce A to work hard is to offer the proper *incentive contract*: pay a wage w_H if profits are high and $w_L < w_H$ if profits are low.

How should the employer choose the incentives w_L and w_H to maximize expected profits? The game tree for this situation is shown in figure 7.3,

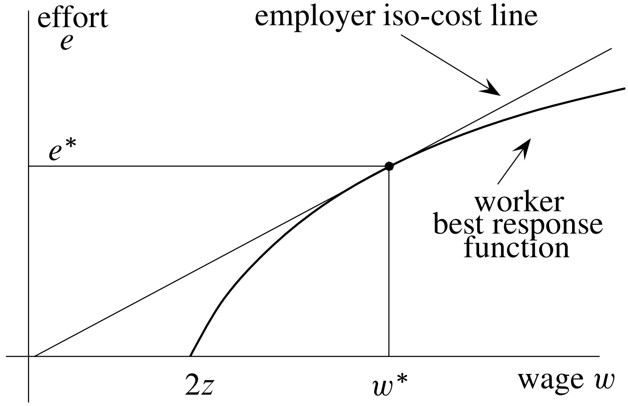


Figure 7.2. Equilibrium in the labor discipline model

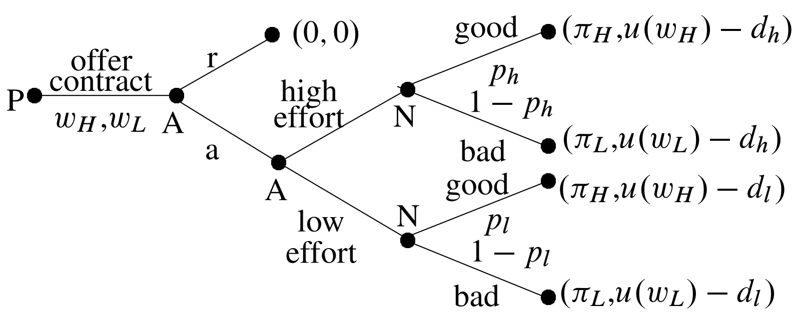


Figure 7.3. Labor incentives

where we assume the utility of the wage is $u(w)$, the cost of high effort to the employee is d_h and the cost of low effort is $d_l < d_h$. By working hard, the employee faces a lottery with payoffs $u(w_H) - d_h, u(w_L) - d_h$ with probabilities $(p_h, 1 - p_h)$, the expected value of which is

$$p_h(u(w_H) - d_h) + (1 - p_h)(u(w_L) - d_h)$$
$$= p_h u(w_H) + (1 - p_h)u(w_L) - d_h.$$

With low effort, the corresponding expression is $p_l u(w_H) + (1 - p_l)u(w_L) - d_l$. Thus, the employee will choose high effort over low effort only if the first of these expressions is at least as great as the second, which gives

$$(p_h - p_l)(u(w_H) - u(w_L)) \geq d_h - d_l. \tag{7.2}$$

This is called the *incentive compatibility constraint* for eliciting high effort.

Now suppose the employee's next-best job prospect has expected value z. Then to get the employee to take the job, the employer must offer the employee at least z. This gives the *participation constraint:*

$$p_h u(w_H) + (1 - p_h)u(w_L) - d_h \geq z, \tag{7.3}$$

if we assume that the principal wants the employee to work hard.

The expected profit of the employer, if we assume that the employee works hard, is given by

$$p_h(\pi_H - w_H) + (1 - p_h)(\pi_L - w_L). \tag{7.4}$$

It is clear that, to minimize the expected wage bill, the employer should choose w_H and w_L so that equation (7.3) is satisfied as an equality. Also, the employee should choose w_H and w_L so that equations (7.2) and (7.3) are satisfied as equalities. This is illustrated in figure 7.4.

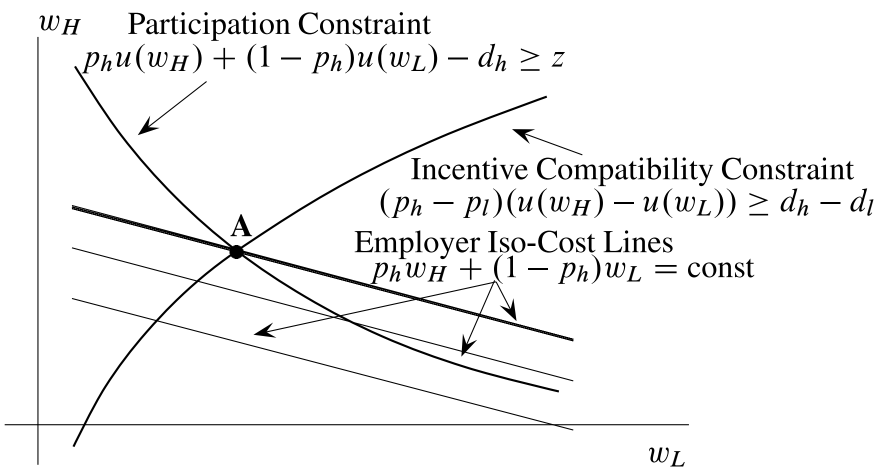


Figure 7.4. Minimizing the cost of inducing an action, given participation and incentive compatibility constraints

Using this figure, we note that the employer's iso-cost lines are of the form $p_h w_H + (1 - p_h)w_L =$ const., and we show that the participation constraint is decreasing and convex. We treat w_H as a function of w_L and differentiate the participation constraint, getting

$$p_h u'(w_H)\frac{dw_H}{dw_L} + (1 - p_h)u'(w_L) = 0.$$

Thus,

$$\frac{dw_H}{dw_L} = -\frac{1-p_h}{p_h}\frac{u'(w_L)}{u'(w_H)} < -\frac{1-p_h}{p_h} < 0. \qquad (7.5)$$

The second inequality (which we use later) holds because $w_L < w_H$, so if the agent is strictly risk averse, u' is decreasing. The participation constraint is thus decreasing. Now take the derivative of equation (7.5), getting

$$\frac{d^2w_H}{dw_L^2} = -\frac{1-p_h}{p_h}\left[\frac{u''(w_H)}{u'(w_H)} - \frac{u'(w_L)u''(w_H)}{u'(w_H)^2}\frac{dw_H}{dw_L}\right] > 0.$$

Thus, the participation constraint is convex.

The incentive compatibility constraint is increasing and cuts the w_L-axis for some $w_L > 0$. If the agent is weakly decreasingly risk averse (that is, if $u''' > 0$), then the incentive compatibility constraint is concave. To see this, we differentiate the incentive compatibility constraint $u(w_H) = u(w_L) +$ constant, getting

$$u'(w_H)\frac{dw_H}{dw_L} = u'(w_L),$$

so $dw_H/dw_L > 1 > 0$, and the incentive compatibility constraint is increasing. Differentiate again, getting

$$u''(w_H)\frac{dw_H}{dw_L} + u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L).$$

Thus

$$u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L) - u''(w_H)\frac{dw_H}{dw_L} < u''(w_L) - u''(w_H) < 0,$$

and the constraint is concave.

If the agent is strictly risk averse (§2.4), the slope of the iso-cost lines is less than the slope of the participation constraint at its intersection A with the incentive compatibility constraint. To see this, note that the slope of the iso-cost line is $|dw_H/dw_L| = (1-p_h)/p_h$, which is less than the slope of the participation constraint, which is

$$|(1-p_h)u'(w_L)/p_hu'(w_H)|,$$

by equation (7.5).

It follows that the solution is at **A** in figure 7.4.

7.4 Properties of the Employment Relationship

The unique Nash equilibrium in the labor discipline model of the previous section is the solution to the two equations $p_h u(w_H) + (1 - p_h)u(w_L) - d_h = z$ and $(p_h - p_l)(u(w_H) - u(w_L)) = d_h - d_l$. Solving simultaneously, we get

$$u(w_L) = z + \frac{p_h d_l - p_l d_h}{p_h - p_l}, \quad u(w_H) = u(w_L) + \frac{d_h - d_l}{p_h - p_l}. \tag{7.6}$$

Note that the employee exactly achieves his reservation position. As we might expect, if z rises, so do the two wage rates w_L and w_H. If d_h rises, you can check that w_H rises and w_L falls. Similar results hold when p_h and p_l vary.

Now that we know the cost to the principal of inducing the agent to take each of the two actions, we can determine which action the principal should ask the agent to choose. If H and L are the expected profits in the good and bad states, respectively, then the return $\pi(a)$ for inducing the agent to take action $a = h, l$ is given by

$$\pi(h) = Hp_h + L(1 - p_h) - \mathbf{E}_h w, \quad \pi(l) = Hp_l + L(1 - p_l) - \mathbf{E}_l w, \tag{7.7}$$

where $\mathbf{E}_h w$ and $\mathbf{E}_l w$ are the expected wage payments if the agent takes actions h and l, respectively; that is, $\mathbf{E}_h w = p_h w_H + (1 - p_h)w_L$ and $\mathbf{E}_l w = p_l w_H + (1 - p_l)w_L$.

Is it worth inducing the employee to choose high effort? For low effort, only the participation constraint $u(w_l) = d_l + z$ must hold, where w_l is the wage paid independent of whether profits are H or L, with expected profit $p_l H + (1 - p_l)L - w_l$. Choose the incentive wage if and only if $p_h(H - w_H) + (1 - p_h)(L - w_L) \geq p_l H + (1 - p_l)L - w_l$. This can be written

$$(p_h - p_l)(H - L) \geq p_h w_H + (1 - p_h)w_L - w_l. \tag{7.8}$$

We will see that, in general, if the employee is risk neutral and it is worth exerting high effort, then the optimum is to make the principal the fixed claimant and the agent the residual claimant (§7.7). To see this for the current example, we can let $u(w) = w$. The participation constraint is then $p_h u(w_H) + (1 - p_h)u(w_L) = p_h w_H + (1 - p_h)w_L = z + d_h$, and the employer's profit is then $A = p_h H + (1 - p_h)L - (z + d_h)$. Suppose we give

A to the employer as a fixed payment and let $w_H = H - A$, $w_L = L - A$. Then the participation constraint holds, because

$$p_h w_H + (1 - p_h)w_L = p_h H + (1 - p_h)L - A = z + d_h.$$

Because high effort is superior to low effort for the employer, 7.8) must hold, giving

$$\begin{aligned}(p_h - p_l)(H - L) &\geq p_h w_H + (1 - p_h)w_L - (z + d_l)\\ &= z + d_h - (z + d_l) = d_h - d_l.\end{aligned}$$

But then,

$$w_H - w_L = H - L \geq \frac{d_h - d_l}{p_h - p_l},$$

which says that the incentive compatibility constraint is satisfied.

Figure 7.5 is a graphical representation of the principal's problem. Note that in this case there are many profit-maximizing contracts. Indeed, any point on the heavy solid line in the figure maximizes profits.

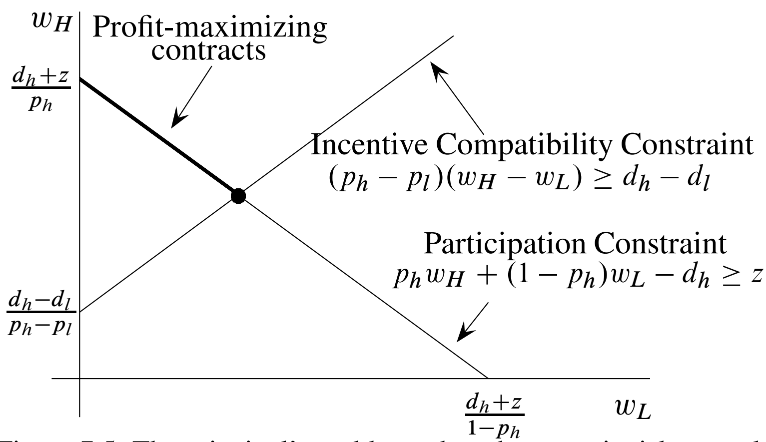


Figure 7.5. The principal's problem when the agent is risk neutral

7.5 Peasant and Landlord

A landlord hires a peasant to tend a cornfield. The landlord's profit is H if the crop is good and $L < H$ if the crop is poor. The peasant can work at either high effort h or low effort l, and the probability p_h of a good crop when he exerts high effort is greater than the probability p_l of a good crop when he expends low effort, with $0 < p_l < p_h < 1$. The landowner cannot observe the peasant's effort.

Suppose the peasant's utility function when the wage is w is given by $u(w) - d_h$ with high effort, and $u(w) - d_l$ with low effort. We assume $d_h > d_l$, so unless given some inducement, the peasant will not work hard and $u' > 0, u'' < 0$, so the peasant has diminishing marginal utility of the wage. The peasant's fallback utility is z.

To induce the peasant to work hard, the landlord chooses a pair of wages w_H and w_L, and pays the peasant w_H if profit is H, and w_L if profit is L. This is called an *incentive wage*.

What should the landlord pay the peasant if he wants to minimize the expected wage $Ew = p_h w_H + (1 - p_h) w_L$, subject to eliciting high effort? First, w_H and w_L must satisfy a *participation constraint*; w_H and w_L must be sufficiently large that the peasant is willing to work at all. Suppose the peasant's next-best alternative gives utility z. Then the landowner must choose w_H and w_L so that the peasant's expected utility is at least z:

$$p_h u(w_H) + (1 - p_h) u(w_L) - d_h \geq z. \qquad \text{(PC)}$$

Second, w_H and w_L must satisfy an *incentive compatibility constraint*: the payoff (that is, the expected return) to the peasant for working hard must be at least as great as the payoff to not working hard. Thus, we must have

$$p_h u(w_H) + (1 - p_h) u(w_L) - d_h \geq p_l u(w_H) + (1 - p_l) u(w_L) - d_l.$$

We can rewrite this second condition as

$$[u(w_H) - u(w_L)](p_h - p_l) \geq d_h - d_l. \qquad \text{(ICC)}$$

We now prove that both the PC and the ICC must hold as equalities. The problem is to minimize $p_h w_H + (1 - p_h) w_L$ subject to PC and ICC. This is the same as maximizing $-p_h w_H - (1 - p_h) w_L$ subject to the same constraints, so we form the Lagrangian

$$\begin{aligned}\mathcal{L}(w_H, w_L, \lambda, \mu) = -p_h w_H - (1 - p_h) w_L& \\ + \lambda[p_h u(w_H) + (1 - p_h) u(w_L) - d_h - z]& \\ + \mu[(u(w_H) - u(w_L))(p_h - p_l) - (d_h - d_l)].&\end{aligned}$$

The first-order conditions can be written:

$$\mathcal{L}_H = 0, \mathcal{L}_L = 0, \qquad \lambda, \mu \geq 0;$$

$$\text{if } \lambda > 0, \text{ then the PC holds with equality;}$$

$$\text{if } \mu > 0, \text{ then the ICC holds with equality.}$$

But we have

$$\begin{aligned} \mathcal{L}_H &= -p_h + \lambda p_h u'(w_H) + \mu u'(w_H)(p_h - p_l) = 0, \\ \mathcal{L}_L &= -1 + p_h + \lambda(1 - p_h)u'(w_L) - \mu u'(w_L)(p_h - p_l) = 0. \end{aligned}$$

Suppose $\lambda = 0$. Then, by adding the two first-order conditions, we get

$$\mu(u'(w_H) - u'(w_L))(p_h - p_l) = 1,$$

which implies $u'(w_H) > u'(w_L)$, so $w_H < w_L$ (by declining marginal utility of income). This, of course, is not incentive compatible, because ICC implies $u(w_H) > u(w_L)$, so $w_H > w_L$. It follows that our assumption that $\lambda = 0$ is contradictory and hence $\lambda > 0$, from which it follows that the participation constraint holds as an equality.

Now suppose $\mu = 0$. Then the first-order conditions $\mathcal{L}_H = 0$ and $\mathcal{L}_L = 0$ imply $u'(w_H) = 1/\lambda$ and $u'(w_L) = 1/\lambda$. Because $u'(w_H) = u'(w_L) = 1/\lambda$, $w_H = w_L$ (because u' is strictly decreasing). This also is impossible by the ICC. Hence $\mu > 0$, and the ICC holds as an equality.

The optimal incentive wage for the landlord is then given by

$$\begin{aligned} u(w_L) &= d_h - p_h(d_h - d_l)/(p_h - p_l) + z \\ u(w_H) &= d_h + (1 - p_h)(d_h - d_l)/(p_h - p_l) + z. \end{aligned}$$

To see this, suppose the landlord has concave utility function v, with $v' > 0$ and $v'' < 0$. The peasant is risk neutral, so we can assume her utility function is $u(w, d) = w - d$, where w is income and d is effort. The assumption that high effort produces a surplus means that the following social optimality (SO) condition holds:

$$p_h H + (1 - p_h)L - d_h > p_l H + (1 - p_l)L - d_l,$$

or

$$(p_h - p_l)(H - L) > d_h - d_l. \qquad \text{(SO)}$$

The landlord wants to maximize

$$p_h v(H - w_H) + (1 - p_h) v(L - w_L)$$

subject to the participation constraint

$$p_h w_H + (1 - p_h) w_L - d_h \geq z \qquad \textbf{(PC)}$$

and the incentive compatibility constraint, which as before reduces to

$$(p_h - p_l)(w_h - w_L) \geq d_h - d_l. \qquad \textbf{(ICC)}$$

We form the Lagrangian

$$\begin{aligned}\mathcal{L} = {} & p_h v(H - w_H) + (1 - p_h) v(L - w_L) \\ & + \lambda (p_h w_H + (1 - p_h) w_L - d_h - z) \\ & + \mu ((p_h - p_l)(w_h - w_L) - (d_h - d_l)),\end{aligned}$$

so the first-order conditions are $\partial\mathcal{L}/\partial w_H = \partial\mathcal{L}/\partial w_L = 0$, $\lambda, \mu \geq 0$, $\lambda > 0$ or the PC holds as an equality and $\mu > 0$ or the ICC holds as an equality. The conditions $\partial\mathcal{L}/\partial w_H = \partial\mathcal{L}/\partial w_L = 0$ can be written as

$$\frac{\partial\mathcal{L}}{\partial w_H} = -p_h v'(H - w_H) + \lambda p_h + \mu(p_h - p_l) = 0 \qquad \text{(FOC1)}$$

$$\mu(p_h - p_l) = 0. \qquad \text{(FOC2)}$$

We first show that the PC holds as an equality by showing that $\lambda = 0$ is impossible. Suppose $\lambda = 0$. Then (FOC) gives

$$\begin{aligned}-p_h v'(H - w_H) + \mu(p_h - p_l) = 0 \\ -(1 - p_h) v'(L - w_L) - \mu(p_h - p_l) = 0.\end{aligned}$$

If $\mu > 0$, this says that $v'(L - w_L) < 0$, which is impossible. If $\mu = 0$, this says that $v'(L - w_L) = 0$, which is also impossible. Thus, $\lambda = 0$ is impossible and the PC holds as an equality.

Finally, if the peasant is risk neutral but the landlord is risk averse then if high effort produces a surplus over low effort, the landlord's profit-maximizing solution involves getting a fixed rent from the peasant, who becomes the residual claimant, bearing all the risk and taking all the profits and losses. First, we can rewrite FOC1 and FOC2 as

$$v'(H - w_H) - \lambda = \mu(p_h - p_l)/p_h \tag{7.9}$$
$$v'(L - w_L) - \lambda = -\mu(p_h - p_l)/(1 - p_h). \tag{7.10}$$

If $\mu > 0$, then $v'(H - w_H) > v'(L - w_L)$, so $H - w_H < L - w_L$, or $H - L < w_H - w_L$. But $\mu > 0$ implies that the ICC holds as an equality, so $(p_h - p_l)(w_H - w_L) = d_h - d_l$ and $d_h - d_l < (p_h - p_l)(H - L)$ from SO, implying $H - L > w_H - w_L$. This is a contradiction, and hence $\mu = 0$; that is, there is no optimum in which the ICC holds as an equality.

What, then, is an optimum? If $\mu = 0$, equations (7.9) and (7.10) imply that $H - w_h = L - w_L$, because v' is strictly increasing (because the landlord is risk averse). This means the landlord gets a fixed rent, as asserted.

7.6 Bob's Car Insurance

Bob wants to buy theft insurance for his car, which is worth \$1,200. If Bob is careful, the probability of theft is 5% and if he is careless, the probability of theft is 7.5%. Bob is risk averse (§2.4) with utility function $u(x) = \ln(x + 1)$, where x is in dollars and the disutility of being careful is $\epsilon > 0$, measured in utils. Suppose the car lasts one period and is either stolen or not stolen at the beginning of the period.

We first find the value of ϵ below which Bob will be careful. If he is careful, the value is $0.95\ln(1201) - \epsilon$, and if he is careless the value is $0.925\ln(1201)$. Being careful is worthwhile as long as $0.95\ln(1201) - \epsilon \geq 0.925\ln(1201)$, or $\epsilon \leq \ln(1201)/40 = 0.177$.

Suppose the insurance industry is competitive, so an insurer must offer Bob a "fair" policy, the expected payout of which equals the premium x. We can show that if Bob buys a policy with full coverage (that is, payout in case of theft equals \$1,200), his premium is $x = \$90$. To see this, note that Bob's his utility is $\ln(1201 - x)$ if he is careless, and $\ln(1201 - x) - \epsilon$ if he is careful, because he is fully compensated in case of theft. Thus, he will not be careful. The probability of theft is then 7.5%, and because the insurance company must give a fair lottery to Bob, $x = (0.075)1200 = 90$.

We can now show that Bob should buy insurance, whether or not he is careful without insurance. To see this, note that the expected value of the car plus insurance is $\ln(1201 - 90) \approx= 7.013$. This is greater that the payoff to being careless without insurance, which has value $0.925 \ln(1201) \approx 6.559$. If Bob is careful without insurance, his payoff is $0.95ln(1201) - \epsilon$, which is also less that the payoff with insurance.

This analysis assumes that Bob is fully insured. Suppose the company offered a fair policy at price $x$with deductible z, so that if the car is stolen, Bob receives $1200-x-z$. Show that, whether or not Bob is careful without insurance, the optimal value of z is zero; that is, Bob should fully insure. However, with no deductible, Bob will not be careful, and the insurance company will either assume the higher theft rate or require a positive deductible. Moreover, it is possible that even if Bob would be careful without insurance, there is no deductible with insurance that would lead Bob to be careful.

7.7 A Generic Principal-Agent Model

Principal-agent games are variations of the following scenario. The agent has a set $A = \{a_1, \ldots, a_n\}$ of available actions. The principal receives one of a set $S = \{s_1, \ldots, s_m\}$ of signals that depend on the action chosen by the agent. Let $p_{ij} = \mathrm{P}[s_j|a_i]$ be the conditional probability of signal s_j when the agent takes action a_i. Note that we must have

$$\sum_{j=1}^{m} p_{ij} = 1 \qquad \text{for } i = 1, \ldots, n.$$

The agent has utility function $u(w, a)$, where w is money, $a \in A$ is the action the agent performs, and the agent's reservation utility is u_o, which is the minimum expected utility that would induce the agent to work for the principal. The principal has payoff $\pi(a) - w$ for $a \in A$.[1]

To maximize his payoff, the principal must determine the payoff associated with getting the agent to perform each action $a_i \in A$, and then choose the largest of the n payoffs. Thus, for each $i = 1, \ldots, n$ the principal must find incentives w_{ij}, $j = 1, \ldots, m$ such that the principal agrees to pay the agent w_{ij} if he observes the signal s_j, and the incentive scheme $w_i = \{w_{i1}, \ldots, w_{im}\}$ induces the agent to choose a_i. The principal must

[1]We assume $\pi(a)$ has a stochastic component, so the principal cannot infer a from $\pi(a)$.

then choose the index i that maximizes his return

$$\pi(a_i) - \mathbf{E}_i w_i = \pi(a_i) - \sum_{j=1}^{m} p_{ij} w_{ij},$$

where $\mathbf{E}_i$ is expectation with respect to the probabilities that obtain when the agent chooses a_i.

Suppose the principal wants to induce the agent to choose action a_k. There are two constraints to the problem.

Participation Constraint. The expected utility to the agent must be enough to induce the agent to participate: $\mathbf{E}_k u(w_k, a_k) \geq u_o$, or

$$\sum_{j=1}^{m} p_{kj} u(w_{kj}, a_k) \geq u_o.$$

Incentive Compatibility Constraint. The chosen action a_k must maximize the payoff to the agent, among all actions $a_i \in A$, so $\mathbf{E}_k u(w_k, a_k) \geq \mathbf{E}_i u(w_k, a_i)$ for $i = 1, \ldots, n$, or

$$\sum_{j=1}^{m} p_{kj} u(w_{kj}, a_k) \geq \sum_{j=1}^{m} p_{ij} u(w_{kj}, a_i) \qquad \text{for } i = 1, \ldots, n.$$

Having determined the minimum expected cost $\mathbf{E}_k w_k$ of inducing the agent to choose action a_k, the principal chooses the index k to maximize $\pi(a_k) - \mathbf{E}_k w_k$.

Prove the following.

THEOREM 7.1 Fundamental Theorem of the Principal-Agent Model. *Solutions to the principal-agent problem have the following characteristics:*

1. *The agent is indifferent between participating and taking the reservation utility u_o.*
2. *If the agent is strictly risk averse and the principal's optimal action a_k is not the agent's most preferred action, then*

    a. *At least one of the incentive compatibility constraints holds as an equality.*

    b. *The principal's payoff is strictly lower than it would be if he could write an enforceable contract for the delivery of action a_k.*

3. *If the agent is risk neutral and if the return $\pi(a)$ is transferable (that is, it can be assigned to the agent), the contract that maximizes the principal's payoff is that which gives the principal a fixed payment and makes the agent the residual claimant on $\pi(a)$.*

The last finding says that individual agents (e.g., farmers, workers, managers) should always be residual claimants on their projects, as long as they are risk neutral and the return $\pi(a)$ to their actions is transferable. In practice, however, such agents are normally *not* residual claimants on their projects; managers may have salaries plus stock options, and farmers may have sharecropping contracts, but these contractual forms do not render the principals fixed claimants. The reason is that agents are risk averse, so that they would not want to be residual claimants and they are credit constrained, so they cannot credibly promise to make good on the project's losses in bad times. Moreover, $\pi(a)$ is often *not* transferable (e.g., in a large work team, it is difficult to isolate the contribution of a single employee).

To prove this theorem, we argue as follows. Suppose the principal wants to induce the agent to choose action a_k. If the participation constraint is not binding, we can reduce all the payments $\{w_{kj}\}$ by a small amount without violating either the participation or the incentive compatibility constraints. Moreover, if all of the incentive compatibility constraints are nonbinding, then the payoff system $\{w_{kj}\}$ is excessively risky, in the sense that the various $\{w_{kj}\}$ can be "compressed" around their expected value without violating the incentive compatibility constraint.

Formally, we form the Lagrangian

$$\mathcal{L} = \pi(a_k) - \mathbf{E}_k w_k + \lambda[\mathbf{E}_k u(w_k) - d(a_k) - u_o] \\ + \sum_{\substack{i=1\\ i\neq k}}^{n} \mu_i\{[\mathbf{E}_k u(w_k) - d(a_k)] - [\mathbf{E}_i u(w_k) - d(a_i)]\},$$

where $\mathbf{E}_i$ means take the expectation with respect to probabilities $\{p_{i1}, \ldots, p_{im}\}$, λ is the Lagrangian multiplier for the participation constraint, and μ_i is the Lagrangian multiplier for the ith incentive compatibility constraint. Writing the expectations out in full (in "real life" you would not do this, but it's valuable for pedagogical purposes), we get

$$\mathcal{L} = \pi(a_k) - \sum_{j=1}^{m} p_{kj} w_{kj} + \lambda\left[\sum_{j=1}^{m} p_{kj} u(w_{kj}, a_k) - u_o\right]$$

$$+\sum_{i=1}^{n}\mu_i\left[\sum_{j=1}^{m}p_{kj}u(w_{kj},a_k)-\sum_{j=1}^{m}p_{ij}u(w_{kj},a_i)\right].$$

The first-order conditions for the problem assert that at a maximum, $\partial\mathcal{L}/\partial w_{kj} = 0$ for $j = 1,\ldots,m$, and $\lambda,\mu_1,\ldots,\mu_n \geq 0$. Moreover, if $\lambda > 0$, then the participation constraint is binding (that is, holds as an equality), and if $\mu_i > 0$ for some $i = 1,\ldots,n$, then the incentive compatibility constraint holds for action a_i. In our case, we have

$$\frac{\partial\mathcal{L}}{w_{kj}} = -p_{kj} + \lambda p_{kj}u'(w_{kj},a_k)$$
$$+\sum_{i=1}^{n}\mu_i(p_{kj}-p_{ij})u'(w_{kj},a_k) = 0 \quad j = 1,\ldots,m.$$

Collecting terms, we have, for $j = 1,\ldots,m$,

$$\lambda p_{kj} + \sum_{i=1}^{n}\mu_i(p_{kj}-p_{ij}) = \frac{p_{kj}}{u'(w_{kj},a_k)}. \tag{7.11}$$

Now we sum this equation from $j = 1$ to m, noting that $\sum_j p_{ij} = 1$ for all i, getting

$$\lambda = \sum_{j=1}^{m}\frac{p_{kj}}{u'(w_{kj},a_k)} > 0,$$

which proves that the participation constraint is binding.

To see that at least one incentive compatibility constraint is binding, suppose all the μ_i are zero. Then equation (7.11) gives $\lambda = 1/u'(w_{kj},a_k)$ for all j. If the agent is risk averse, this implies all the w_{kj} are equal (because risk aversion implies strictly concave preference, which implies strictly monotonic marginal utility), which means a_k must be the agent's most preferred action.

To show that the principal's payoff would be higher if he could write an enforceable contract, note that with an enforceable contract, only the participation constraint would be relevant. To induce the agent to perform a_i, the principal would pay w_i^*, where $u(w_i^*,a_i) = u_o$. The principal will then choose action a_l such that $\pi(a_l) - w_l^*$ is a maximum.

Suppose the optimal action when no contract can be written is a_k and the wage structure is $w_k(s)$. The participation constraint is binding, so

$\mathbf{E}_k u(w_k) = d(a_k) + u_o$. Because a_k is not the agent's unconstrained preferred action (by assumption), $w_k(s)$ is not constant, and because the agent is strictly risk averse, we have

$$u(w_k^*, a_k) = u_o = \mathbf{E}u(w_k, a_k) < u(\mathbf{E}_k w_k, a_k),$$

so $w_k^* < \mathbf{E}_k w_k$. Because $\pi(a_l) - w_l^* \geq \pi(a_k) - w_k^* > \pi(a_k) - \mathbf{E}_k w_k$, we are done.

To prive the third assertion, suppose again that a_l is the principal's optimal choice when an enforceable contract can be written. Then a_l maximizes $\pi(a_i) - w_i^*$ where $u(w_i^*, a_i) = u_o$. If the agent is risk neutral, suppose the agent receives $\pi(a)$ and pays the principal the fixed amount $\pi(a_l) - w_l^*$. We can assume $u(w, a) = w + d(a)$ (why?). The agent then maximizes

$$\begin{aligned}\pi(a_i) - d(a_i) &= u(\pi(a_i)) - u(w_i^*) - u_o \\ &= \pi(a_i) - w_i^* - u_o,\end{aligned}$$

which of course occurs when $i = l$. This proves the theorem.

8

Signaling Games

> This above all: to thine own self be true, And it must follow, as the night the day, Thou canst not then be false to any man.
>
> Shakespeare

8.1 Signaling as a Coevolutionary Process

A Thompson's gazelle who spots a cheetah, instead of fleeing, will often "stott," which involves an 18-inch vertical jump, with legs stiff and white rump patch fully displayed to the predator. The only plausible explanation for this behavior (Alcock 1993) is that the gazelle is signaling the cheetah that it would be a waste of both their times and energies for the cheetah to chase the gazelle, because the gazelle is obviously very fit. Of course, if the cheetah could not understand this signal, it would be a waste of time and energy for the gazelle to emit it. Also, if the signal could be easily falsified, and the ability to stott had nothing to do with the probability of being caught, cheetahs would never have evolved to heed the signal in the first place.[1]

A *signal* is a special sort of physical interaction between two agents. Like other physical interactions, a signal changes the physical constitution of the agents involved. But unlike interactions among nonliving objects, or between a nonliving object and a living agent, a signal is the product of a *strategic dynamic* between sender and receiver, each of whom is pursuing distinct but interrelated objectives. Moreover, a signal is a specific *type* of strategic physical interaction, one in which the content of the interaction is determined by the sender, and it changes the receiver's behavior by altering the way the receiver evaluates alternative actions.

[1]For a review of evidence for costly signaling in birds and fish in the form of colorful displays that indicate health and vigor, see Olson and Owens 1998. On the more general topic of costly signaling, see Zahavi and Zahavi 1997 and section 8.4.

The most important fact about a signal is that it is generally the result of a *coevolutionary process between senders and receivers* in which both benefit from its use. For if a signal is costly to emit (and if its use has been stable over time), then the signal is most likely both *beneficial to the sender* and *worthy of belief for the receiver*; a sender is better off sending that signal rather than none, or some other, and a receiver is better off acting on it the way receivers traditionally have, rather than ignoring it or acting otherwise. The reason is obvious: if the receiver were *not* better off acting this way, a mutant who ignored (or acted otherwise on) the signal would be more fit than the current population of receivers, and would therefore increase its frequency in the population. Ultimately, so many receivers would ignore (or act otherwise on) the signal that, being costly to the sender, it would not be worth sending unless, of course, the "otherwise" were also beneficial to the sender.

Signaling systems are not always in equilibrium, and potentially beneficial mutations need not occur. Moreover, human beings are especially adept both at dissimulating (emitting "false" signals) and detecting such dissimulation (Cosmides and Tooby 1992). However, human beings are disposed to taking the signals around them at face value unless there are good reasons for doing otherwise (Gilbert 1991). The treatment of signals as emerging from a coevolutionary process and persisting as a Nash equilibrium of the appropriate game, is the starting point for a theory of signaling.

8.2 A Generic Signaling Game

In signaling games, player 1 has a "type" that is revealed to player 2 via a special "signal," to which player 2 responds by choosing an "action," the payoffs to the two players being a function of player 1's type and signal and player 2's action. Thus, the stage game that played so prominent a role in the general Bayesian game framework collapses, in the case of signaling games, to a pair of payoff functions.

Specifically, there are three players Sender, Receiver, and Nature. Nature begins by choosing from a set T of possible *types* or *states of affairs*, choosing $t \in T$ with probability $\rho(t)$. Sender observes t but Receiver does not. Sender then transmits a *signal* $s \in S$ to Receiver, who uses this signal to choose an *action* $a \in A$. The payoffs to the two players are $u(t,s,a)$ and $v(t,s,a)$, respectively. A pure strategy for Sender is thus a function $f: T \to S$, where $s = f(t)$ is the signal sent when Nature reveals type t, and a pure strategy for Receiver is a function $g : S \to A$, where $a = g(s)$ is the action taken when Receiver receives signal s. A mixed strategy for Sender is a probability distribution $P_S(s;t)$ over S for each $t \in T$, and a mixed strategy for Receiver is a probability distribution $p_R(a;s)$ over A for each signal s received. A Nash equilibrium for the game is thus a pair of probability distributions $(P_S(\cdot;t), p_R(\cdot, s))$ for each pair $\{(t,s)|t \in T, s \in S\}$ such that each agent uses a best response to the other, given the probability distribution $\rho(t)$ used by Nature to choose the type of Sender.

We say a signal $s \in S$ is *along the path of play*, given the strategy profile $(P_S(\cdot;t), p_R(\cdot;s))$, if there is a strictly positive probability that Sender will transmit s, that is, if

$$\sum_{t \in T} \rho(t) P_S(s;t) > 0.$$

If a signal is not along the path of play, we say it is *off the path of play*. If s is along the path of play, then a best response for Receiver maximizes Receiver's expected return, with a probability distribution over T given by

$$\mathrm{P}[t|s] = \frac{P_S(s;t)\rho(t)}{\sum_{t' \in T} P_S(s;t')\rho(t')}.$$

We thus require of P_S and p_R that

a. For every state $t \in T$, and all signals $s' \in S$ such that $P_S(s';t) > 0$, s' maximizes

$$\sum_{a \in A} u(t, s', a) p_R(a;s)$$

over all $s \in S$; that is, Sender chooses a best response to Receiver's pattern of reacting to S's signals;

b. For every signal $s \in S$ along the path of play, and all actions $a' \in A$ such that $p_R(a';s) > 0$, a' maximizes

$$\sum_{t \in T} v(t, s, a)\mathrm{P}[t|s]$$

over all $a \in A$; that is, Receiver chooses a best response to Sender's signal.

c. If a signal $s \in S$ is not along the path of play, we may choose $\mathrm{P}[t|s]$ arbitrarily such that (b) still holds. In other words, Receiver may respond arbitrarily to a signal that is never sent, provided this does not induce Sender to send the signal.

8.3 Sex and Piety: The Darwin-Fisher Model

In most species, females invest considerably more in raising their offspring than do males; for instance, they produce a few large eggs as opposed to the male's millions of small sperm. So, female fitness depends more on the *quality* of inseminations, whereas male fitness depends more on the *quantity* of inseminations (§6.26). Hence, in most species there is an *excess demand for copulations* on the part of males, for whom procreation is very cheap, and therefore there is a *nonclearing market for copulations*, with the males on the long side of the market (§9.13). In some species this imbalance leads to violent fights among males (dissipating the rent associated with achieving a copulation), with the winners securing the scarce copulations. But in many species, *female choice* plays a central role, and males succeed by being attractive rather than ferocious.

What criteria might females use to choose mates? We would expect females to seek mates whose appearance indicates they have genes that will enhance the survival value of their offspring. This is indeed broadly correct. But in many cases, with prominent examples among insects, fish, birds, and mammals, females appear to have *arbitrary prejudices* for dramatic, ornamental, and colorful displays even when such accoutrements clearly reduce male survival chances; for instance, the plumage of the bird of paradise, the elaborate structures and displays of the male bowerbird, and the stunning coloration of the male guppy. Darwin speculated that such characteristics improve the mating chances of males at the expense of the average fitness of the species. The great biologist R. A. Fisher (1915) offered the first genetic analysis of the process, suggesting that an arbitrary female preference for a trait would enhance the fitness of males with that trait, and hence the fitness of females who pass that trait to their male offspring, so the genetic predisposition for males to exhibit such a trait could become common in a species. Other analytical models of sexual selection, called *Fisher's runaway process* include Lande (1981), Kirkpatrick (1982), Pomiankowski (1987), and Bulmer (1989). We will follow Pomiankowski (1987), who showed that *as long as females incur no cost for being choosy, the Darwin-Fisher sexual selection process works, but even with a slight cost of being choosy, costly ornamentation cannot persist in equilibrium.*

We shall model runaway selection in a way that is not dependent on the genetics of the process, so it applies to cultural as well as genetic evolution. Consider a community in which there are an equal number of males and females, and there is a cultural trait that we will call *pious fasting*. Although both men and women can have this trait, only men act on it, leading to their death prior to mating with probability $u > 0$. However, both men and women pass the trait to their children through family socialization. Suppose a fraction t of the population have the pious-fasting trait.

Suppose there is another cultural trait, a *religious preference for pious fasting*, which we call being "choosy" for short. Again, both men and women can carry the choosy trait and pass it on to their children, but only women can act on it, by choosing mates who are pious fasters at rate $a > 1$ times that of otherwise equally desirable males. However, there may be a cost of exercising this preference, because with probability $k \geq 0$ a choosy woman may fail to mate. Suppose a fraction p of community members bears the religious preference for pious fasters.

We assume parents transmit their values to their offspring in proportion to their own values; for instance, if one parent has the pious-fasting trait and the other does not, then half their children will have the trait. Males who are pious fasters then exercise their beliefs, after which females choose their mates, and a new generation of young adults is raised (the older generation moves to Florida to retire).

Suppose there are n young adult males and an equal number of young adult females. Let x_{tp} be the fraction of young adults who are "choosy fasters," x_{-p} the fraction of "choosy nonfasters," x_{t-} the fraction of "nonchoosy fasters," and x_{--} the fraction of "nonchoosy nonfasters." Note that $t = x_{tp} + x_{t-}$ and $p = x_{tp} + x_{-p}$. If there is no correlation between the two traits, we would have $x_{tp} = tp$, $x_{t-} = t(1-p)$, and so on. But we cannot assume this, so we write $x_{tp} = tp + d$, where d (which biologists call *linkage disequilibrium*) can be either positive or negative. It is easy to check that we then have

$$\begin{aligned} x_{tp} &= tp + d \\ x_{t-} &= t(1-p) - d \\ x_{-p} &= (1-t)p - d \\ x_{--} &= (1-t)(1-p) + d. \end{aligned}$$

Although male and female young adults have equal fractions of each trait because their parents pass on traits equally to both pious fasting and mate choosing can lead to unequal frequencies in the "breeding pool" of parents in the next generation. By assumption, a fraction k of choosy females do not make it to the breeding pool, so if t^f is the fraction of pious-faster females in the breeding pool, then

$$t^f = \frac{t - kx_{tp}}{1 - kp},$$

where the denominator is the fraction of females in the breeding pool, and the numerator is the fraction of pious-faster females in the breeding pool. Similarly, if p^f is the fraction of choosy females in the breeding pool, then

$$p^f = \frac{p(1-k)}{1 - kp},$$

where the numerator is the fraction of choosy females in the breeding pool.

We now do the corresponding calculations for males. Let t^m be the fraction of pious-faster males, and p^m the fraction of choosy males in the breeding pool, after the losses associated with pious fasting are taken into account. We have

$$t^m = \frac{t(1-u)}{1 - ut},$$

where the denominator is the fraction of males, and the numerator is the fraction of pious-faster males in the breeding pool. Similarly,

$$p^m = \frac{p - ux_{tp}}{1 - ut},$$

where the numerator is the fraction of choosy males in the breeding pool.

By assumption, all $n^f = n(1 - kp)$ females in the breeding pool are equally fit. We normalize this fitness to 1. The fitnesses of pious and nonpious males in the breeding pool are, however, unequal. Suppose each female in the breeding pool mates once. There are then $n^f(1 - p^f)$ nonchoosy females, so they mate with $n^f(1 - p^f)(1 - t^m)$ nonpious males and $n^f(1 - p^f)t^m$ pious males. There are also $n^f p^f$ choosy females, who mate with $n^f p^f(1 - t^m)/(1 - t^m + at^m)$ nonpious males and $n^f p^f at^m/(1 - t^m + at^m)$ pious males (the numerators account for the $a:1$ preference for pious males, and the denominator is chosen so that the two terms add to $n^f p^f$). If we write

$$r_- = (1 - p^f) + \frac{p^f}{1 - t^m + at^m}$$

$$r_t = (1 - p^f) + \frac{ap^f}{1 - t^m + at^m},$$

then the total number of matings of nonpious males is $n^f(1-t^m)r_-$, and the total number of matings of pious males is $n^f t^m r_t$. The probability that a mated male is pious is therefore $t^m r_t$. Because the probability that a mated female is pious is t^f, and both parents contribute equally to the traits of their offspring, the fraction of pious traits in the next generation is $(t^m r_t + t^f)/2$. If we write $\beta_t = t^m r_t - t$ and $\beta_p = p^f - p$, then the change Δt in the frequency of the pious trait can be written as

$$\Delta t = \frac{t^m r_t + t^f}{2} - t = \frac{1}{2}\left(\beta_t + \frac{d\beta_p}{p(1-p)}\right). \tag{8.1}$$

What about the change in p across generations? The fraction of mated, choosy females is simply p^f, because all females in the breeding pool mate. The number n^m of males in the breeding pool is $n^m = n(1-ut)$, of which nx_{-p} are nonpious and choosy, whereas $n(1-u)x_{tp}$ are pious and choosy. Each nonpious male has $n^f r_-/n^m$ offspring, and each pious male has $n^f r_t/n^m$ offspring, so the total number of choosy male offspring per breeding female is just

$$p^{m'} = nx_{-p}r_-/n^m + n(1-u)x_{tp}r_t/n^m.$$

A little algebraic manipulation shows that this can be written more simply as

$$p^{m'} = p + \frac{d\beta_t}{t(1-t)}.$$

Then the change Δp in the frequency of the choosy trait can be written as

$$\Delta p = \frac{p^{m'} + p^f}{2} - p = \frac{1}{2}\left(\beta_p + \frac{d\beta_t}{t(1-t)}\right). \tag{8.2}$$

Let us first investigate (8.1) and (8.2) when choosy females are not less fit, so $k = 0$. In this case, $p^f = p$, so $\beta_p = 0$. Therefore, $\Delta t = \Delta p = 0$ exactly when $\beta_t = 0$. Solving this equation for t, we get

$$t = \frac{(a-1)p(1-u) - u}{u(a(1-u)-1)}. \tag{8.3}$$

This shows that there is a range of values of p for which an equilibrium frequency of t exists. Checking the Jacobian of the right-hand sides of (8.1) and (8.2) (see section 11.7 if you do not know what this means) we find that stability requires that the denominator of (8.3) be positive (do it as an exercise). Thus, the line of equilibria is upward sloping, and t goes from zero to one as p goes from $u/(a-1)(1-u)$ to $au/(a-1)$ (you can check that this defines an interval contained in $(0,1)$ for $0 < u < 1$ and $a(1-u) > 1$). This set of equilibria is shown in figure 8.1. This shows that the Darwin-Fisher sexual selection process is plausible, even though it lowers the average fitness of males in the community. In essence, the condition $a(1-u) > 1$ ensures that the benefit of sexual selection more than offsets the cost of the ornamental handicap.

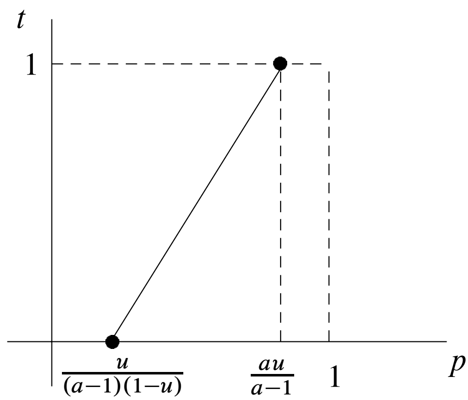


Figure 8.1. Equilibria in Darwin-Fisher sexual selection model when there is no selection against choosy females

Suppose, however, $k > 0$. If we then solve for $\Delta t = \Delta p = 0$ in (8.1) and (8.2), we easily derive the equation

$$d^2 = t(1-t)p(1-p).$$

But $t(1-t)p(1-p) = (x_{t-}+d)(x_{-p}+d)$, which implies $x_{t-} = x_{-p} = 0$. But then, nonchoosy females must mate only with nonpious males, which is impossible so long as there is a positive fraction of pious males. We conclude that *when choosiness is costly to females, sexual selection cannot exist*. Because in most cases we can expect some positive search cost to be involved in favoring one type of male over another, we conclude that sexual selection probably does not occur in equilibrium in nature. Of course, random mutations could lead to a disequilibrium situation in which females prefer certain male traits, leading to increased fitness of males with those traits. But when the fitness costs of such choices kick in, choosy females will decline until equilibrium is restored.

8.4 Biological Signals as Handicaps

Zahavi (1975), after close observation of avian behavior, proposed an alternative to the Darwin-Fisher sexual selection mechanism, a notion of costly signaling that he called the *handicap principle*. According to the handicap principle, a male who mounts an elaborate display is in fact signaling his good health and/or good genes, because an unhealthy or genetically unfit male lacks the resources to mount such a display. The idea was treated with skepticism for many years, because it proved difficult to model or empirically validate the process. This situation changed when Grafen (1990b) developed a simple analytical model of the handicap principle. Moreover, empirical evidence has grown in favor of the costly signaling approach to sexual selection, leading many to favor it over the Darwin-Fisher sexual selection model, especially in cases where female mate selection is costly.

Grafen's model is a special case of the generic signaling model presented in section 8.2. Suppose a male's type $t \in [t_{\min}, \infty)$ is a measure of male vigor (e.g., resistance to parasites). Females do best by accurately determining t, because an overestimate of t might lead a female to mate when she should not, and an underestimate might lead her to pass up a suitable mate. If a male of type t signals his type as $s = f(t)$, and a female uses this signal to estimate the male's fitness as $a = g(s)$, then in an equilibrium with truthful signaling we will have $a = t$. We suppose that the male's fitness is $u(t, s, a)$, with $u_t > 0$ (a male with higher t is more fit), $u_s < 0$ (it is costly to signal a higher level of fitness), and $u_a > 0$ (a male does better if a female thinks he's more fit). We assume the male's fitness function $u(t, s, g(s))$ is such that a more vigorous male will signal a higher fitness; that is, $ds/dt > 0$. Given $g(s)$, a male of type t will then choose s to maximize $U(s) = u(t, s, g(s))$, which has first-order condition

$$U_s(s) = u_s(t, s, g(s)) + u_a(t, s, g(s))\frac{dg}{ds} = 0. \tag{8.4}$$

If there is indeed truthful signaling, then this equation must hold for $t = g(s)$, giving us the differential equation

$$\frac{dg}{ds} = -\frac{u_s(g(s), s, g(s))}{u_a(g(s), s, g(s))}, \tag{8.5}$$

which, together with $g(s_{\min}) = t_{\min}$, uniquely determines $g(s)$. Because $u_s < 0$ and $u_a > 0$, we have $dg/ds > 0$, as expected.

Differentiating the first-order condition (8.4) totally with respect to t, we find

$$U_{ss}\frac{ds}{dt} + U_{st} = 0.$$

Because $U_{ss} < 0$ by the second-order condition for a maximum, and because $ds/dt > 0$, we must have $U_{st} > 0$. But we can write

$$\begin{aligned} U_{st} &= u_{st} + u_{at}g'(s) \\ &= \frac{u_{st}u_a(g(s), s, g(s)) - u_{at}u_s(g(s), s, g(s))}{u_a} > 0. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\left[\frac{u_s(t, s, g(s))}{u_a(t, s, g(s))}\right] = \frac{U_{st}}{u_a} > 0. \tag{8.6}$$

We can now rewrite (8.4) as

$$u_a(t, s, g(s))\left[\frac{u_s(t, s, g(s))}{u_a(t, s, g(s))} + g'(s)\right] = 0. \tag{8.7}$$

Because the fraction in this expression is increasing in t, and the expression is zero when $t = g(s)$, this shows $s = g^{-1}(t)$ is a local maximum, so the male maximizes fitness by truthfully reporting $s = g^{-1}(t)$, at least locally.

For an example of the handicap principle, suppose $u(t, s, a) = a^r t^s$, $0 < t < 1$, so (8.5) becomes $g'/g = -(1/r)\ln g$, which has solution $\ln g = ce^{-s/r}$. If we use $g(s_{\min}) = t_{\min}$ this gives

$$g(s) = t_{\min}^{e^{-\frac{s - s_{\min}}{r}}}$$

and

$$f(t) = s_{\min} - r\ln\frac{\ln t}{\ln t_{\min}}.$$

The reader will note an important element of unrealism in this model: it assumes that the cost of female signal processing and detection is zero, and hence signaling is perfectly truthful and reliable. If we allow for costly female choice, we would expect that signal detection would be imperfect, and there would be a positive level of dishonest signaling in equilibrium, and the physical process of signal development should involve an evolutionary dynamic intimately related to receiver neurophysiology (Dawkins and Guilford 1991; Guilford and Dawkins 1991, 1993). In contrast with the Darwin-Fisher model of sexual selection, we would not expect a small amount of costly female choice to undermine a signaling equilibrium, because there are direct fitness benefits to females in locating vigorous males.

8.5 The Shepherds Who Never Cry Wolf

Because we value truthfulness, one might have the impression that when both a truthful signaling and a nonsignaling equilibrium exist, the truthful signaling equilibrium should entail higher payoffs for at least some of the players. But that need not be the case. Here is a counterexample.

Two shepherds take their flocks each morning to adjoining pastures. Sometimes a wolf will attack one of the flocks, causing pandemonium among the threatened sheep. A wolf attack can be clearly heard by both shepherds, allowing a shepherd to come to the aid of his companion. But unless the wolf is hungry, the cost of giving aid exceeds the benefits, and only the shepherd guarding the threatened flock can see if the wolf is hungry.

There are three pure strategies for a threatened shepherd: never signal (N), signal if the wolf is hungry (H), and always signal (A). Similarly, there are three pure strategies for the shepherd who hears a wolf in the other pasture: never help (N), help if signaled (H), and always help (A).

We make the following assumptions. The payoff to each shepherd to a day's work when no wolf appears is 1. The cost of being attacked by a hungry wolf and a nonhungry wolf is a and $b < a$, respectively. The cost of coming to the aid of a threatened shepherd is d, and doing so prevents the loss to the threatened shepherd, so his payoff is still 1. Finally, it is common knowledge that the probability that a wolf is hungry is $p > 0$.

We assume the shepherds' discount rates are too high, or wolf visits too infrequent, to support a repeated-game cooperative equilibrium using trigger strategies, so the game is a one-shot. If the shepherds are self-regarding (that is, they care only about their own payoffs), of course neither will help the other, so we assume that they are brothers, and the total payoff to shepherd 1 (the threatened shepherd) is his payoff π_1 plus $k\pi_2$, where π_2 is the payoff of shepherd 2, and similarly, the total payoff to shepherd 2 (the potential helper) is $\pi_2 + k\pi_1$. If $ka > d > kb$, a shepherd prefers to aid his threatened brother when wolf is hungry (why?). So we assume this is the case. We also assume that $a - dk > c > b - dk$, which means that a threatened shepherd would want his brother to come to help only if the wolf is hungry (why?). So there ought to be a signaling equilibrium in this case. Note, however, that this signaling equilibrium will exist whether p is small or large, so for very large p, it might be worthwhile for a brother *always* to help, thus saving the cost c of signaling to his brother, and saving the cost kc to himself. This, in fact, is the case. Although this can be proved in general, you are asked in this problem to prove a special case.

Assume $k = 5/12$ (note that $k = 1/2$ for full brothers, but the probability that two brothers that *ostensibly* have the same father *in fact* have the same father is probably about 80% in human populations). Also assume $a = 3/4$, $b = 1/4$, $c = 19/48$, and $d = 1/4$. Finally, assume $p = 3/4$. After verifying that these inequalities hold, do the following:

a. Show that there is a signaling equilibrium and find the payoffs to the shepherds.
b. Show that there is pooling equilibrium in which a threatened shepherd never signals, and a shepherd always helps his threatened brother. Show that this equilibrium is Pareto-superior to the signaling equilibrium.
c. There is also a mixed-strategy Nash equilibrium (truthful signaling occurs, but not with certainty) in which the threatened shepherd sometimes signals, and the other shepherd sometimes helps without being asked. Find this equilibrium and its payoffs, and show that the payoffs are slightly better than the signaling equilibrium but not as high as the pooling equilibrium.

8.6 My Brother's Keeper

Consider the following elaboration on the theme of section 8.5. Suppose the threatened shepherd, whom we will call Sender, is either healthy, needy, or desperate, each of which is true with probability 1/3. His brother, whom we will call Donor, is either healthy or needy, each with probability 1/2. Suppose there are two signals that the threatened shepherd can give: a low-cost signal costing 0.1 and a high-cost signal costing 0.2. If he uses either one, we say he is "asking for help." We assume that the payoff for each brother is his own fitness plus 3/4 of his brother's fitness. Sender's fitness is 0.9 if healthy, 0.6 if needy and 0.3 if desperate, minus whatever he pays for signaling. Donor's fitness is 0.9 if healthy and 0.7 if needy. However, the has a resource that ensures his fitness is 1 if he uses it and the fitness of Sender is 1 (minus the signaling cost) if he transfers it to the Sender. The resource is perishable, so either he or his brother must use it in the current period.

a. Show that after eliminating "unreasonable" strategies (define carefully what you mean by "unreasonable"), there are six pure strategies for Sender, in each of which a healthy sender never signals: Never Ask (NN); Signal Low Only If Desperate (NL); Signal High Only If Desperate (NH); Signal Low If Desperate or Needy (LL); Signal Low If Needy, High If Desperate (LH); and Signal High If Needy or Desperate (HH). Similarly, there are ten strategies for Donor: Never Help (Never); Help If Healthy and Signal Is High (Healthy, High); Help If Healthy and Asked (Healthy, Asked); Help If Healthy (Healthy); Help If Signal Is High (High); Help If Healthy and Asked, or Needy and Signal Is High (Healthy,Asked/Needy,High); Help If Healthy or Signal Is High; Help If Asked (Asked); Help If Healthy or Asked (Healthy or Asked); and Help Unconditionally (Help).
b.* If you have a lot of time on your hands, or if you know a computer programming language, derive the 6×10 normal form matrix for the game.
c.* Show that there are seven pure-strategy equilibria. Among these there is one completely pooling equilibrium: (NN—Help). This, of course, affords the Sender the maximum possible payoff. However, the pooling equilibrium maximizes the sum of the payoffs to both players, so it will be preferred by both if they are equally likely to be Sender and Donor. This is asocial optimum even among the mixed strategy equilibria, but that is even harder to determine.
d. Show that the truthful signaling strategies (LH—Healty,Asked/Needy,High) form a Nash equilibrium, but that this equilibrium is strictly Pareto-inferior to the pooling (nonsignaling) equilibrium.

This model shows that there can be many signaling equilibria, but all may be inferior to complete altruism (NN—Help). This is doubtless because the coefficient of relatedness is so high (3/4 is the coefficient of relatedness between sisters in many bee species, where the queen mates with a single haploid male).

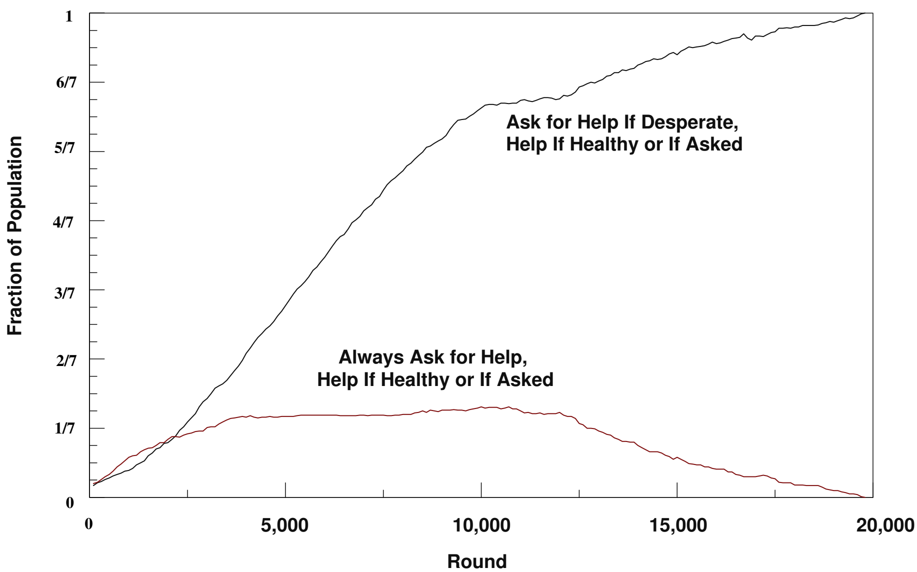


Figure 8.2. A signaling equilibrium in the brother's helper game

Simulating the model gives an entirely surprising result, as depicted in figure 8.2. For this simulation, I created 700 players, each randomly programmed to play one strategy as Sender and another as Donor. The players were randomly paired on each round, and one was randomly chosen to be Sender, the other Donor. After every 10 rounds, the strategies with the highest scores reproduced, and their offspring replaced those with the lowest scores. Figure 8.2 shows the outcome for the two strongest strategies. For the Donor, this involved using Help If Healthy or If Asked, and for Sender, either Signal Low If Desperate or Needy, or Signal Low If Desperate. After 20,000 rounds, the only remaining strategy (except for occasional mutations) is the latter, the other 59 strategies having disappeared. This is the signaling equilibrium that is best for the Donor but whose total fitness is inferior to the pooling equilibrium Never Ask, Always Help. Nor is this a fluke outcome: I ran the simulation 10 times with different seeds to the random number generator, and this equilibrium emerged every time. The implication is clear: *a signaling equilibrium can emerge from an evolutionary process even when it is inferior to a pooling equilibrium.*

8.7 Honest Signaling among Partial Altruists

In a certain fishing community, each fisher works alone on the open sea, earning a payoff that we will normalize to 1. A fisher occasionally encounters threatening weather. If the fisher does not escape the weather, his payoff is zero. If a threatened fisher has sufficient energy reserves, he can escape the bad weather, and his expected payoff is u, where $0 < u < 1$. We call such a fisher *secure*. However, with a certain probability p $(0 < p < 1)$ a threatened fisher does *not* have sufficient energy reserves. We say he is *in distress*.

If a threatened fisher sees another fisher on the horizon, he can send a signal to ask for help, at cost t, with $0 < t < 1$. If the fisher is in distress and a potential helper comes to his aid (we assume the potential helper is not threatened), the payoff to the distressed fisher is 1, but the cost to the helper is $c > 0$. Without the help, however, the distressed fisher succumbs to the bad weather and has payoff 0.

To complicate matters, a threatened fisher who is helped by another fisher but who is *not* distressed has payoff v, where $1 > v > u$. Thus, threatened fishers have an incentive to signal that they are in distress even when they are not. Moreover, fishers can tell when other fishers are threatened, but only the threatened fisher himself knows his own reserves, and hence whether he is in distress.

We assume that encounters of this type among fishers are one-shot affairs, because the probability of meeting the same distressed fisher again is very small. Clearly, unless there is an element of altruism, no fisher will help a threatened fisher. So let us suppose that in an encounter between fishers, the nonthreatened fisher receives a fraction $r > 0$ of the total payoff, including signaling costs, received by the threatened fisher (presumably because r is the degree of genetic or cultural relatedness between fishers). However, the helper bears the total cost c himself.

For example, if a fisher is in distress, signals for help, and receives help, the distressed fisher's payoff is $1-t$, and the helper's payoff is $r(1-t)-c$.

The nonthreatened fisher (fisher 1) who sees a threatened fisher (fisher 2) has three pure strategies: Never Help, Help If Asked and Always Help. fisher 2 also has three strategies: Never Ask, Ask When Distressed, Always Ask. We call the strategy pair {Help If Asked, Ask If Distressed} the *honest signaling* strategy pair. If this pair is Nash, we have an honest signaling equilibrium. This is called a *separating equilibrium* because agents truthfully reveal their situation by their actions. Any other equilibrium is called a *pooling equilibrium*, because agents' actions do not always reveal their situations.

The reasoning you are asked to perform below shows that when there are potential gains from helping distressed fishers (that is, $(1+r)(1-t) > c$), then if fishers are sufficiently altruistic, and signaling is sufficiently costly but not excessively costly, an honest signaling equilibrium can be sustained as a Nash equilibrium. The idea that signaling must be costly (but not too costly) to be believable was championed by Amotz Zahavi (1975), and modeled by Grafen (1990a), Maynard Smith (1991), Johnstone and Grafen (1992, 1993), and others in a notable series of papers. The general game-theoretic point is simple but extremely important: if a signal is not on balance truthful, it will not be heeded, so if it is costly to emit, it will not be emitted. Of course, there is much out-of-equilibrium behavior, so there is lots of room for duplicity in biology and economics.

a. Show that if

$$(1+r)\left[v-u+\frac{pt}{1-p}\right] < c < (1+r)(1-t), \tag{8.8}$$

then honest signaling is socially efficient (that is, maximizes the sum of the payoffs to the two fishers)? *Hint*: Set up the 3×3 normal form for the game, add up the entries in each box and compare. For the rest of the problem, assume that these conditions hold.

b. Show that there is always a pooling equilibrium in which fisher 2 uses Never Ask. Show that in this equilibrium, Fisher 1 Never Helps if

$$c > r[p+(1-p)(v-u)], \tag{8.9}$$

and Always Helps if the opposite inequality holds.

c. Show that if

$$v-u < \frac{c}{r} < 1$$

and

$$v-u < t < 1,$$

honest signaling is a Nash equilibrium.

d. Show that if t is sufficiently close to 1, honest signaling can be a Nash equilibrium even if it is not socially efficient.

e. Show that if honest signaling and {Never Ask, Never Help} are both Nash equilibria, then honest signaling has a higher total payoff than {Never Ask, Never Help}.
f. Show that if honest signaling and {Never Ask, Always Help} are both Nash equilibria, then honest signaling has a higher total payoff than {Never Ask, Always Help}.

8.8 Educational Signaling

Suppose there are two types of workers, high ability (h) and low ability (l), and the proportion of high-ability workers in the economy is $\alpha > 0$. Suppose workers invest in acquiring a level of schooling s that is both costly to obtain and productive. Specifically, suppose that a high-ability worker incurs a cost $c_h(s)$ of obtaining s years of schooling, whereas a low-ability workers incurs a cost of $c_l(s)$. We also assume schooling is more costly for low-ability workers than for high, so $c_h'(s) < c_l'(s)$ for all $s \geq 0$.

Schooling is also productive, so the marginal productivity of a high-ability worker with s years of schooling is $y_h(s)$, and the corresponding value for a low-ability worker is $y_l(s)$. We assume $y_h(0) = y_l(0) = 0$, and $y_h'(s) > y_l'(s) > 0$ for all $s \geq 0$, which means that high-ability workers have higher marginal products than low-ability workers do, and schooling increases the productivity of high-ability workers more than low. To simplify the diagrams, we assume y_h and y_l are linear functions of s.

Suppose employers cannot observe ability, but they do observe s, and if workers with different abilities obtain different amounts of schooling, they may offer a wage based on s. We assume the labor market is competitive, so all firms must offer a wage equal to the expected marginal product of labor.

A truthful signaling equilibrium in this case involves high- and low-ability workers choosing different amounts of schooling, so employers know their type by their schooling choices. They thus pay wages $y_h(s)$ to the high-ability workers and $y_l(s)$ to the low. If workers know this, high-ability workers will choose s to maximize $y_h(s) - c_h(s)$, and low-ability workers will choose s to maximize $y_l(s) - c_l(s)$. This is depicted in figure 8.3. Agents maximize their payoff by choosing the highest indifference curve that intersects their wage curve, which means tangency points between wage curves and indifference curves as illustrated. Moreover, neither type of agent would prefer to get the amount of schooling chosen by the other, because this would involve a lower level of utility; that is, the equilibrium point for each type lies below the indifference curve for the other type.

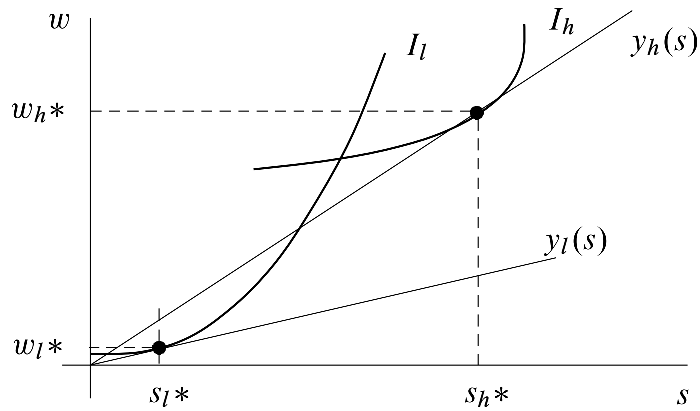


Figure 8.3. A truthful signaling equilibrium

a. Explain why there cannot be a truthful signaling equilibrium if the costs of schooling are the same for the two ability levels. Draw a diagram to illustrate your argument. *Hint*: Indifference curves for the same utility function cannot cross.
b. Modify figure 8.3 to illustrate the following assertion. If the optimum schooling level for the high-ability worker lies inside the optimal indifference curve for the low-ability worker, then the low-ability worker will mimic the high-ability worker and destroy the truthful signaling equilibrium.
c. However, high-ability workers may have a response to this: they may be able to increase their educational level to a point sufficiently high that it will no longer benefit the low-ability workers to imitate them. This is called an "educational rat race." Make a diagram illustrating this rat race, and another in which it is not worthwhile for high-ability workers to signal their quality.
d. Analyze the case of a pooling equilibrium, in which both high- and low-ability workers choose the same schooling level. Show that in this case employers do not use either the $y_h(s)$ or the $y_l(s)$ schedules, but rather set wages so that

$$w(s) = \alpha y_h(s) + (1 - \alpha) y_l(s) \tag{8.10}$$

for both types of workers. Show that in a pooling equilibrium, high-ability workers maximize their payoff subject to hitting the wage curve $w(s)$ and low-ability workers imitate their choice of educational level. Draw a diagram illustrating this result, and make sure the curves are drawn so neither high- nor low-ability workers have an incentive to switch unilaterally to the truthful signaling equilibrium.

This analysis does not exhaust the possibilities for a signaling equilibrium. There could also exist mixed strategy equilibria in which some low-ability workers imitate the high-ability workers and others do not.

There could also be strange Bayesian priors for the employers that would lead to strange pooling equilibria. For instance, if employers believe that a worker who does not choose $s = s_o$ for some given s_o is "crazy" and must be low ability. Then every worker may choose s_o to get the pooling wage, which is higher than the low ability wage. Such behavior by employers would be stupid, and they might be driven out of existence in a dynamic adjustment process.

8.9 Education as a Screening Device

Suppose a worker can be of high ability a_h with probability α, or low ability $a_l < a_h$ with probability $1 - \alpha$. Workers know their own ability, but employers do not. Workers can also choose to acquire high as opposed to low education, and this is observable by employers. Moreover, it costs c/a ($c > 0$) for a worker of ability a to acquire high education, so high education is more costly for the low-ability worker. We assume that workers are paid their expected marginal product, and the marginal product of a worker of ability a is just a, so high education does not improve worker productivity; education is at best a screening device, informing employers which workers are high ability. Suppose e_l is the event "worker chose low education," and e_h is the event "worker chose high education." Then, if w_l and w_h are the wage paid to low- and high-education workers, respectively, we have

$$w_k = \mathrm{P}[a_h|e_k]a_h + \mathrm{P}[a_l|e_k]a_l, \qquad k = l, h, \tag{8.11}$$

where $\mathrm{P}[a|e]$ is the conditional probability that the worker has ability a in the event e.

A Nash equilibrium for this game consists of a choice $e(a) \in \{e_l, e_h\}$ of education level for $a = a_h, a_l$, a set of probabilities $\mathrm{P}[a|e]$ for $a = a_h, a_l$, and $e = e_h, e_l$ that are consistent in the sense that if $\mathrm{P}[e] > 0$, then $\mathrm{P}[a|e]$ is given by Bayesian updating.

a. Show that there is a pooling equilibrium in which $e(a_h) = e(a_l) = e_l$, $w_h = w_l = \alpha a_h + (1-\alpha)a_l$, and $\mathrm{P}[a_l|e_l] = \mathrm{P}[a_l|e_h] = 1-\alpha$. In other words, employers disregard the education signal, and workers choose low education.
b. Show that there is some range of values for c such that there is a truthful signaling equilibrium in which $e(a_h) = e_h$, $e(a_l) = e_l$, $w_l = a_l$, $w_h = a_h$, $\mathrm{P}[a_l|e_l] = 1$, and $\mathrm{P}[a_l|e_h] = 0$. In other words, despite the fact that education does not increase worker productivity, workers can signal high ability by acquiring education, and employers reward high-ability workers with relatively high wages.
c. Suppose that with a small probability $\epsilon > 0$ a worker is given a free education, regardless of ability. Show that the pooling equilibrium does not have to specify arbitrarily the probabilities $\mathrm{P}[a_l|e_h]$ off the path of play, because $\mathrm{P}[e_h] = \epsilon > 0$, and because both ability types are equally likely to get a free education, we have $\mathrm{P}[a_l|e_h] = 1-\alpha$.
d. Show that, if c is sufficiently small, there are two pooling equilibria and no truthful signaling equilibrium. The first pooling equilibrium is as before. In the second pooling equilibrium, both ability types choose to be educated. Specifically, $e(a_h) = e(a_l) = e_h$, $w_l = a_l$, $w_h = \alpha a_h + (1-\alpha)a_l$, $\mathrm{P}[a_l|e_l] = 1$, and $\mathrm{P}[a_l|e_h] = 1-\alpha$. Note that this requires specifying the probabilities for e_l, which are off the path of play. The truthful signaling equilibrium is inefficient and inegalitarian, whereas the pooling equilibrium is inefficient but egalitarian. The pooling equilibrium is not very plausible, because it is more reasonable to assume that if a worker gets education, he is high ability.
e. Show that if we added a small exogenous probability $\epsilon > 0$ that a worker of either type is denied an education, all outcomes are along the path of play, and the posterior $\mathrm{P}[a_l|e_l] = 1-\alpha$ follows from the requirement of Bayesian updating.
f. Now suppose the educational level is a continuous variable $e \in [0,1]$. Workers then choose $e(a_h), e(a_l) \in [0,1]$, and employers face probabilities $\mathrm{P}[a_h|e], \mathrm{P}[a_l|e]$ for all education levels $e \in [0,1]$.
g. Show that for $e \in [0,1]$, there is a $\bar{e} > 0$ such that for any $e^* \in [0,\bar{e}]$, there is a pooling equilibrium where all workers choose educational level e^*. In this equilibrium, employers pay wages $w(e^*) = \alpha a_h +$

$(1-\alpha)a_l$ and $w(e \neq e^*) = a_l$. They have the conditional probabilities $P[a_l|e \neq e^*] = 1$ and $P[a_l|e = e^*] = 1-\alpha$.

h. Show that when $e \in [0, 1]$, if c is sufficiently large, there is a range of values of e^* such that there is a truthful signaling equilibrium where high-ability workers choose $e = e^*$, and low-ability workers choose $e = 0$. In this equilibrium, employers pay wages $w(e^*) = a_h$ and $w(e \neq e^*) = a_l$. They face the conditional probabilities $P[a_l|e \neq e^*] = 0$ and $P[a_l|e \neq e^*] = 1$.

8.10 Capital as a Signaling Device

Suppose there are many producers, each with a project to fund. There are two types of projects, each of which requires capital investment k. The "good" project returns 1 at the end of the period, and the "bad" project returns 1 with probability p $(0 < p < 1)$ at the end of the period, and otherwise returns 0. There are also many lenders. Whereas each producer knows the type of his own project, the lenders know only that the frequency of good projects in the economy is q $(0 < q < 1)$.

We assume the capital market is perfect and all agents are risk neutral (§2.4). Thus, each lender's reservation position is the risk-free interest rate $\rho > 0$, so a producer can always obtain financing for his project if he offers to pay an interest rate r that allows a lender to earn expected return ρ on his capital investment k.

We call a project with capital cost k *socially productive* if its expected return is greater than $k(1+\rho)$. This corresponds to the idea that although individual agents may be risk averse, the law of large numbers applies to creating a social aggregate, so a social surplus is created on all projects that return at least the risk-free interest rate.

a. Show that for any $p, q > 0$ there is a nonempty interval (k^g_{min}, k^g_{max}) of capital costs k such that no project is funded, despite the fact that a fraction q of the projects are socially productive.
b. Show that for any $p, q > 0$ there is a nonempty interval (k^b_{min}, k^b_{max}) of capital costs k such that all projects are funded, despite the fact that a fraction $1-q$ of the projects are not socially productive.

This is a sorry state of affairs, indeed! But is there not some way that an owner of a good project could signal this fact credibly? In a suitably religious society, perhaps the requirement that borrowers swear on a stack of Bibles that they have good projects might work. Or if producers have new projects available in each of many periods, we may have a "reputational equilibrium" in which producers with bad projects are not funded in future periods, and hence do not apply for loans in the current period. Or society might build debtors' prisons and torture the defaulters.

But suppose none of these is the case. Then equity comes to the rescue! Suppose each producer has an amount of capital $k^p > 0$. Clearly, if $k^p \geq k$, there will be no need for a credit market, and producers will invest in their projects precisely when they are socially productive (prove it!). More generally,

c. Show that for all $p, q, k > 0$ such that good projects are socially productive, and bad projects are socially unproductive, there is a wealth level $k^p_{min} > 0$ such that if all producers have wealth $k^p > k^p_{min}$; a producer's willingness to invest k^p in his project is a perfect indicator that the project is good. In this situation, exactly the good projects are funded, and the interest rate is the risk-free interest rate ρ.

The previous result says that if producers are sufficiently wealthy, there is a truthful signaling equilibrium, in which producers signal the quality of their projects by the amount of equity they are willing to put in them. But if there are lots of nonwealthy producers, many socially productive investments may go unfunded (Bardhan, Bowles and Gintis 2000).

9

Repeated Games

Tomorrow, and tomorrow, and tomorrow,
Creeps in this petty pace from day to day
To the last syllable of recorded time

Shakespeare

When a game $\mathcal{G}$ is repeated an indefinite number of times by the same players, many of the anomalies associated with finitely repeated games (§4.2) disappear. Nash equilibria of the repeated game arise that are not Nash equilibria of $\mathcal{G}$. The exact nature of these equilibria is the subject of the folk theorem (§9.10). We have encountered many games $\mathcal{G}$ in which most, or all Nash equilibria are Pareto-inefficient. Indeed, all the generic two-player games with two strategies, the prisoner's dilemma (§3.11), the battle of the sexes (§3.9), and the hawk-dove game (§3.10) are of this type.[1] The folk theorem asserts that if the signals of defection (that is, the signals that a player deviated from the behavior specified by the Nash equilibrium) are of sufficiently high quality, and if players have sufficiently long time horizons, the repeated game based on $\mathcal{G}$ can attain Pareto-efficiency, or at least approximate Pareto-efficiency as closely as desired.

The folk theorem requires that a defection on the part of a player carry a signal that is conveyed to other players. We say a signal is *public* if all players receive the same signal. We say the signal is *perfect* if it accurately reports whether or not the player in question defected. The first general folk theorem that does not rely on incredible threats was proved by Fudenberg and Maskin (1986) for the case of perfect public signals (§9.10).

Repeated game models provide elegant and compelling explanations of many on-going strategic interactions. The folk theorem, however, is highly overrated. The folk theorem achieves its magic through the unrealistic assumption that the defection signal can be made arbitrarily close to public

[1] We say a finite normal form game is *generic* if all the entries in the game matrix are distinct. More generically, we say that a property of a set of equations is generic if, when it holds for a particular set of parameters, it holds in a sufficiently small neighborhood of these parameters.

and arbitrarily accurate. For instance, as the number of players increases, the folk theorem continues to hold only if the signal becomes more accurate and closer to public. In fact, of course, as group size increases, the signal will, under plausible conditions, become both less public and noisier. A cogent analysis of cooperation in repeated games with self-regarding players under empirically plausible conditions remains to be developed.

9.1 Death and Discount Rates in Repeated Games

Suppose an agent plays a repeated game in which the payoff at the end of each period is π, the agent's discount rate is ρ, and the probability that the agent dies at the end of each period is $\sigma > 0$. We then can write the present value v as

$$v = \frac{\pi + (1-\sigma)v}{1+\rho},$$

because the agent receives π and, unless he dies, plays the lottery in which he wins v again, both at the end of the current period. Solving for v,we get

$$v = \frac{\pi}{\sigma + \rho}.$$

This result gives us some information as to what plausible discount rates are for humans in cases where we cannot arbitrage our gains and losses by banking them at the current interest rate. Given human life expectancy, this argument suggest a discount rate of about 2% to 3% per year.

9.2 Big Fish and Little Fish

Many species of fish are attacked by parasites that attach to their gills and inner mouth parts. Often such fish will form a symbiotic relationship with a smaller species of fish for whom the parasite is a food source. Mutual trust is involved, however, because the larger fish must avoid the temptation of eating the smaller fish, and the smaller fish must avoid the temptation of taking a chunk out of the larger fish, thereby obtaining a meal with much less work than picking around for the tiny parasites. This scenario, which is doubtless even more common and more important for humans than for fish, is explored in the following problem.

	C	D
C	5,5	-3,8
D	8,-3	0,0

Suppose Big Fish and Little Fish play the prisoner's dilemma shown in the diagram. Of course, in the one-shot game there is only one Nash equilibrium, which dictates that both parties defect. However, suppose the same players play the game at times $t = 0, 1, 2, \ldots$ This is then a new game, called a *repeated game*, in which the payoff to each is the sum of the payoffs over all periods, weighted by a *discount factor* δ, with $0 < \delta < 1$. Note that a discount factor δ relates to a *discount rate* ρ by the formula $\delta = 1/(1 + \rho)$. We call the game played in each period the *stage game* of a *repeated game* in which at each period the players can condition their moves on the complete previous history of the various stages. A strategy that dictates following one course of action until a certain condition is met, and then following a different strategy for the rest of the game is called a *trigger strategy*.

THEOREM 9.1 *The cooperative solution (5,5) can be achieved as a subgame perfect Nash equilibrium of the repeated game if δ is sufficiently close to unity, and each player uses the trigger strategy of cooperating as long as the other player cooperates, and defecting forever if the other player defects on one round.*

PROOF: We use the fact that for any discount factor δ with $0 < \delta < 1$,

$$1 + \delta + \delta^2 + \ldots = \frac{1}{1 - \delta}.$$

To see this, write

$$\begin{aligned} x &= 1 + \delta + \delta^2 + \ldots \\ &= 1 + \delta(1 + \delta + \delta^2 + \ldots) = 1 + \delta x, \end{aligned}$$

from which the result follows.

By the way, there is a faster way of arriving at the same result. Consider a repeated game that pays 1 now and in each future period, and the discount factor is δ. Let x be the value of the game to the player. The player receives 1 now, and then gets to play exactly the same game in the next period. Because the value of the game in the next period is x, its present value is δx. Thus $x = 1 + \delta x$, so $x = 1/(1 - \delta)$.

Now suppose both agents play the trigger strategy. Then, the payoff to each is $5/(1 - \delta)$. Suppose a player uses another strategy. This must involve cooperating for a number (possibly zero) of periods, then defecting forever; for once the player defects, his opponent will defect forever, the best response to which is to defect forever. Consider the game from the time t at which the first player defects. We can call this $t = 0$ without loss of generality. A fish that defects receives 8 immediately and nothing thereafter. Thus the cooperate strategy is Nash if and only if $5/(1-\delta) \geq 8$, or $\delta \geq 3/8$. When δ satisfies this inequality, the pair of trigger strategies is also subgame perfect, because the situation in which both parties defect forever is Nash subgame perfect. ■

There are lots of other subgame perfect Nash equilibria to this game. For instance,

THEOREM 9.2 *Consider the following trigger strategy for Little Fish: alternate $C, D, C, \ldots$ as long as Big Fish alternates $D, C, D, \ldots$. If Big Fish deviates from this pattern, defect forever. Suppose Big Fish plays the complementary strategy: alternate $D, C, D, \ldots$ as long as Little Fish alternates $C, D, C, \ldots$. If Little Fish deviates from this pattern, defect forever. These two strategies form a subgame perfect Nash equilibrium for δ sufficiently close to unity.*

PROOF: The payoffs are now $-3, 8, -3, 8, \ldots$ for Little Fish and $8, -3, 8, -3, \ldots$ for Big Fish. Let x be the payoffs to Little Fish. Little Fish gets -3 today, 8 in the next period, and then gets to play the game all over again starting two periods from today. Thus, $x = -3 + 8\delta + \delta^2 x$. Solving this, we get $x = (8\delta - 3)/(1 - \delta^2)$. The alternative is for Little Fish to defect at some point, the most advantageous time being when it is his turn to get -3. He then gets zero in that and all future periods. Thus, cooperating is Nash if and only if $x \geq 0$, which is equivalent to $8\delta - 3 \geq 0$, or $\delta \geq 3/8$. ■

9.3 Alice and Bob Cooperate

Alice and Bob play the game in the figure to the right an indefinite number of times. They use trigger strategies but do not discount the future. Show that if the probability p of continuing the game in each period is sufficiently large, then it is Nash for both Alice and Bob to cooperate (play C) in each period. What is the smallest value of p for which this is true?

Alice \ Bob	C	D
C	3,3	0,5
D	5,0	1,1

To answer this, let v be the present value of cooperating forever for Alice. Then $v = 3 + pv$, because cooperation pays 3, plus with probability p Alice gets the present value v again in the next period. Solving, we get $v = 3/(1-p)$. If Alice defects, she gets 5 now and then 1 forever, starting in the next period. The value of getting 1 forever is $v_1 = 1 + p \cdot v_1$, so $v_1 = p/(1-p)$. Thus Alice's total return to defecting is $5 + p/(1-p)$. Cooperating beats defecting for Alice if $3/(1-p) > 5 + p/(1-p)$. Solving, we find Alice should cooperate as long as $p > 50\%$.

9.4 The Strategy of an Oil Cartel

	Low	High
Low	184,168	104,176
High	208,88	128,96

Suppose there are two oil-producing countries, Iran and Iraq. Both can operate at either of two production levels: 2 or 4 million barrels a day. Depending on their decisions, the total output on the world market will be 4, 6, or 8 million barrels a day, and the price per barrel in these three cases is \$100, \$60, and \$40, respectively. Costs of production are \$8 per barrel for Iran and \$16 per barrel for Iraq. The normal form of the game is shown in the diagram. It is clear a prisoner's dilemma.

Suppose this game is repeated every day, and both countries agree to cooperate by producing the low output, each one threatening the other with a *trigger strategy:* "If you produce high output, even once, I will produce high output forever." Show that cooperation is now a Nash equilibrium if the discount rate is sufficiently low. What is the maximum discount rate that will sustain cooperation?

9.5 Reputational Equilibrium

Consider a firm that can produce a good at any quality level $q \in [0, 1]$. If consumers anticipate quality q_a, their demand x is given by

$$x = 4 + 6q_a - p.$$

Suppose the firm knows this demand curve, and takes q_a as given but can set the quality q supplied. The firm has no fixed costs, and the cost of producing one unit of the good of quality q is $2 + 6q^2$.

In each period $t = 1, 2, \ldots$ the firm chooses a quality level q and a price p. Consumers see the price but do not know the quality until they buy the good. Consumers follow a trigger strategy, in which they buy the good in each period in which $q \geq q_a$, but if $q < q_a$ in some period, they never buy from the firm again.

Suppose the firm uses discount factor $\delta = 0.9$. Define a *reputational equilibrium* as one in which quality q_a is supplied in each period. What are the conditions for a reputational equilibrium?

9.6 Tacit Collusion

Consider a duopoly operating over an infinite number of periods $t = 1, 2, \ldots$. Suppose the duopolists are price setters, so each pure strategy for firms 1 and 2 in period t takes the form of setting prices $p_1^t, p_2^t \geq 0$, respectively, conditioned on the history of prices in previous time periods, and a pure strategy for the whole game is a sequence of strategies, one for each period t. Suppose the profits in period t are given by $\pi_1(p_1^t, p_2^t)$ for firm 1 and $\pi_2(p_1^t, p_2^t)$ for firm 2. The payoffs to the firms for the whole game are then

$$\pi_1 = \sum_{t=1}^{\infty} \delta^t \pi_1(p_1^t, p_2^t), \qquad \pi_2 = \sum_{t=1}^{\infty} \delta^t \pi_2(p_1^t, p_2^t),$$

where δ is the common discount factor for the firms.

To specify the function $\pi_i(p_1^t, p_2^t)$, suppose the two firms have no fixed cost and constant marginal cost c, the firms face a downward-sloping demand curve, and the lowest price producer gets the whole market. Also, if the two producers have the same price, they share the market equally.

THEOREM 9.3 *There is a subgame perfect Nash equilibrium of this game in which* $p_1^t = p_2^t = c$ *for* $t = 1, 2, \ldots$.

Note that this is the "competitive" equilibrium in which profits are zero and price equals marginal cost. The existence of this Nash equilibrium is called Bertrand's paradox because it seems implausible (though hardly paradoxical!) that two firms in a duopoly actually behave in this manner.

THEOREM 9.4 *Suppose* $\delta > 50\%$, p^m *is the monopoly price (that is, the price that maximizes the profits of a monopolist) and* $c \leq p \leq p^m$. *Let* s *be a strategy profile which firm 1 sets* $p_1^t = p$, *firm 2 sets* $p_2^t = p$, $t = 1, 2, \ldots$, *and each firm responds to a deviation from this behavior on* *the part of the other firm by setting price equal to c forever. Then s is a subgame perfect Nash equilibrium.*

We call a Nash equilibrium of this type *tacit collusion*.

PROOF: Choose p satisfying the conditions of the theorem. Let $\pi(p)$ be total industry profits if price p is charged by both firms, so $\pi(p) \geq 0$. Suppose firm 2 follows the specified strategy. The payoff to firm 1 for following this strategy is

$$\pi^1(p) = \frac{\pi(p)}{2}\left(1 + \delta + \delta^2 + \ldots\right) = \frac{\pi(p)}{2(1-\delta)}. \tag{9.1}$$

The payoff to firm 1 for defecting on the first round by charging an amount $\epsilon > 0$ less than p is $\pi^1(p - \epsilon)$. Thus, $2(1 - \delta) < 1$ is sufficient for Nash. Clearly, the strategy is subgame perfect, because the Bertrand solution in which each firm charges marginal cost is Nash. ■

Intuition tells us that tacit collusion is more difficult to sustain when there are many firms. The following theorem, the proof of which we leave to the reader (just replace the "2" in the denominators of equation (9.1) by "n") shows that this is correct.

THEOREM 9.5 *Suppose there are $n > 1$ firms in the industry, but the conditions of supply and demand remain as before, the set of firms with the lowest price in a given time period sharing the market equally. Then, if $\delta > 1 - 1/n$ and $c \leq p \leq p^m$, the trigger strategies in which each firm sets a price equal to p in each period, and each firm responds to a deviation from this strategy on the part of another firm by setting price equal to c forever, form a subgame perfect Nash equilibrium.*

Another market condition that reduces the likelihood that tacit collusion can be sustained is incomplete knowledge on the part of the colluding firms. For instance, we have the following.

THEOREM 9.6 *Suppose there is an n-firm oligopoly, as described previously, but a firm that has been defected upon cannot implement the trigger strategy until $k > 1$ periods have passed. Then, tacit collusion can hold only if $\delta^{k+1} > 1 - 1/n$.*

This theorem shows that *tacit collusion is less likely to hold the more easily a firm can "hide" its defection*. This leads to the counterintuitive, but quite correct, conclusion that making contractual conditions public knowledge ("putting all your cards on the table") may *reduce* rather than *increase* the degree of competition in an industry.

PROOF: If it takes k periods after a defection to retaliate, the gain from defection is

$$\pi(p)(1+\delta+\delta^2+\ldots+\delta^k) = \frac{\pi(p)}{1-\delta}\left(1-\delta^{k+1}\right),$$

from which the result immediately follows, because the present value of cooperating forever is $\pi(p)/n(1-\delta)$. ■

9.7 The One-Stage Deviation Principle

Suppose s is a strategy profile for players in a repeated game, so s specifies what move each player makes at each stage of the game, depending on the prior history of moves in the game. We say s satisfies the *one-stage deviation principle* if no player can gain by deviating from s, either on or off the equilibrium path of the game tree, in a single stage, otherwise conforming to s. The following theorem is often very useful in analyzing repeated games, because it says that a strategy profile is subgame perfect if it satisfies the one stage deviation principle.

THEOREM 9.7 The One-Stage Deviation Principle. *A strategy profile s for a repeated game with positive discount rates, based on a finite stage game, is a subgame perfect Nash equilibrium if and only if it satisfies the one-stage deviation principle.*

For a formal proof of this theorem (it's not difficult) and some extensions see Fudenberg and Tirole (1991). Here is an informal proof. Obviously, subgame perfection implies the one-stage deviation principle. Suppose s is not subgame perfect. Then for some player i, there is an alternative strategy profile $\tilde{s}_i$ that offers i a higher payoff against s_{-i}, starting at one of i's information sets ν_i. If $\tilde{s}_i$ differs from s_i in only a finite number of places, we can assume $\tilde{s}_i$ has the fewest deviations from s_i among such alternatives for i. If $\tilde{s}_i$ has only one deviation, we are done. If there are more than one, then the final deviation must make i better off and again we are done. If $\tilde{s}_i$ has an infinite number of deviations, because the payoffs are bounded, and the discount rate is strictly positive, we can cut the game off at some period

t such that any possible gain to i after time t are as small as we want. This shows that there is also a finite deviation (cutting off $\tilde{s}_i$ after time t) that improves i's payoff starting at ν_i.

9.8 Tit for Tat

Consider a repeated prisoner's dilemma with two players. The payoffs are (r, r) for mutual cooperation, (p, p) for mutual defection, (t, s) when player 1 defects, and player 2 cooperates, and (s, t) when player 1 cooperates, and player 2 defects, where $t > r > p > s$ and $2r > s + t$. A tit for tat player cooperates on the first round of a repeated prisoner's dilemma, and thereafter does what the other player did on the previous round.

It is clear that tit for tat is not Nash when played against itself if and only if player 1 gains by defecting on the first move. Moreover, defecting on the first round, and going back to tit for tat when you partner plays tit for tat increases your payoff by $t + \delta s - r(1 + \delta)$, which is negative for $\delta_1 > (t - r)/(r - s)$. Note that $0 < \delta_1 < 1$.

After the first defect, the gain from defecting for k more rounds before returning to cooperate is $\delta^{k+1}(p - s(1 - \delta) - \delta r)$. If this is negative, then δ_1 is the minimum discount factor for which tit for tat is Nash against itself. If this is positive, however, then defecting forever increases payoff by $t - r - (r - p)\delta/(1 - \delta)$, which is negative for a discount factor greater than $\delta_2 = (t - r)/(t - p)$. Because $0 < \delta_2 < 1$, we find that there is always a discount factor less than unity, above which tit for tat is a Nash equilibrium.

However, tit for tat is not subgame perfect, because if a player defects on a particular round, tit for tat specifies that the two players will exchange cooperate and defect forever, which has a lower payoff than cooperating forever. Fortunately, we can revise tit for tat to make it subgame perfect. We define *contrite tit for tat* (Boyd 1989) as follows. A player is in either *good standing* or *bad standing*. In period 1 both players are in good standing. A player is in good standing in period $t > 1$ only if in period $t - 1$ (i) he cooperated and his partner was in good standing; (ii) he cooperated and he was in bad standing; or (iii) he defects and he was in good standing, while his partner was in bad standing. Otherwise the player is in bad standing in period $t > 1$. Contrite tit for tat says to cooperate unless you are in good standing and your partner is in bad standing. Using the one-stage deviation principle (§9.7), it is then easy to show that contrite tit for tat is subgame perfect.

9.9 I'd Rather Switch Than Fight

Consider a firm that produces a *quality good*, which is a good whose quality is costly to produce, can be verified only through consumer use, and cannot be specified contractually. In a single-period model, the firm would have no incentive to produce high quality. We develop a repeated game between firm and consumer, in which the consumer pays a price *greater* than the firm's marginal cost, using the threat of switching to another supplier (a trigger strategy) to induce a high level of quality on the part of the firm. The result is a nonclearing product market, with firms enjoying price greater than marginal cost. Thus, they are quantity constrained in equilibrium.

This model solves a major problem in our understanding of market economies: markets in quality goods do not clear, and the success of a firm hinges critically on its ability to sell a sufficient quantity of it product, something that is assured in a clearing market. Thus, the problem of competition in quality-goods markets is quite different from Walrasian general equilibrium models, in which the only problem is to produce at minimum cost.

Every Monday, families in Pleasant Valley wash their clothes. To ensure brightness, they all use bleach. Low-quality bleach can, with low but positive probability, ruin clothes, destroy the washing machine's bleach delivery gizmo, and irritate the skin. High-quality bleach is therefore deeply pleasing to Pleasant Valley families. However, high-quality bleach is also costly to produce. Why should firms supply high quality?

Because people have different clothes, washing machines, and susceptibility to skin irritation, buyers cannot depend on a supplier's reputation to ascertain quality. Moreover, a firm could fiendishly build up its reputation for delivering high-quality bleach and then, when it has a large customer base, supply low-quality bleach for one period, and then close up shop (this is called "milking your reputation"). Aggrieved families could of course sue the company if they have been hurt by low-quality bleach but such suits are hard to win and very costly to pursue. So no one does this.

If the quality q of bleach supplied by any particular company can be ascertained only after having purchased the product, and if there is no way to be compensated for being harmed by low-quality bleach, how can high quality be assured?

Suppose the cost to a firm of producing a gallon of the bleach of quality q is $b(q)$, where $b(0) > 0$ and $b'(q) > 0$ for $q \geq 0$. Each consumer is a customer of a particular supplier, and purchases exactly one gallon of bleach each Friday at price p from this supplier. If dissatisfied, the customer switches to another supplier at zero cost. Suppose the probability of being dissatisfied, and hence of switching, is given by the decreasing function $f(q)$. We assume an infinite time horizon with a fixed discount rate ρ.

a. Considering both costs $b(q)$ and revenue q as accruing at the end of the period, show that the value $v(q)$ to a firm from having a customer is

$$v(q) = \frac{p - b(q)}{f(q) + \rho}.$$

b. Suppose the price p is set by market competition, so it is exogenous to the firm. Show that the firm chooses quality q so that

$$p = b(q) + b'(q)g(q),$$

where $g(q) = -[f(q) + \rho]/f'(q)$, provided $q > 0$.

c. Show that quality is an increasing function of price.

Note that in this case firms are quantity constrained, because price is greater than marginal cost in a market (Nash) equilibrium, and that consumers are on the long side of the market.

This model raises an interesting question. What determines firm size? In the standard perfect competition model, firm size is determined by the condition that average costs be at a minimum. This is of course just silly, because a firm can always produce at any multiple of the "optimal firm size" simply by working the production process, whatever it might be, in parallel.[2] The monopolistic competition model, in which a firm has a downward-sloping demand curve, is better, but it does not apply to a case like ours, where firms are price takers, as in the perfect competition model, and firm size is determined by the dynamic process of movement of customers among firms. Here is one plausible model of such a process.

Suppose there are n firms in the bleach industry, all selling at the same price p. Suppose firm j has market share m_j^t in period t. Suppose for

[2]The standard treatment in microeconomic theory models *plant* size, not *firm* size. The important questions of vertical and horizontal integration, the real determinants of firm size, are virtually orthogonal to the question of plant size. Industrial economists have known this for a very long time. For a contemporary review of the literature on the subject, see Sutton (1997).

$j = 1, \ldots, n$, a fraction f_j of firm j's customers leave the firm in each period, and a fraction a_j of customers who have left firms are attracted to firm j. We say the bleach industry is *in equilibrium* if the market share of each firm is constant over time. We have the following.

THEOREM 9.8 *There is a unique asymptotically stable equilibrium in the bleach industry.*

PROOF: We normalize the number of customers in Pleasant Valley to one. Then, the number of customers leaving firm j is $m_j^t f_j$, so the total number of customers looking for new suppliers is $\sum_j m_j^t f_j$. A particular firm j attracts a fraction a_j of these. This assumes a firm can woo back a fraction a_j of its recently departed customers; the argument is the same if we assume the opposite. Thus, the net customer loss of firm j in period t is

$$f_j m_j^t - a_j \sum_{k=1}^{n} f_k m_k^t. \tag{9.2}$$

In equilibrium this quantity must be zero, and $m_k^t = m_k$ for all t and for $k = 1, \ldots, n$. This gives the equilibrium condition

$$m_j = \mu_j \sum_{k=1}^{n} f_k m_k, \tag{9.3}$$

where we have defined $\mu_k = a_k/f_k$. Note also that if we add up the n equations in (9.2), we get zero, so $\sum_k m_k^t = 1$ for all t, implying $\sum_k m_k = 1$. Summing (9.3), we arrive at the equilibrium conditions

$$m_j = \frac{\mu_j}{\sum_k \mu_k}.$$

Thus, there exists a unique industry equilibrium. To prove asymptotic stability, we define the $n \times n$ matrix $B = (b_{ij})$, where $b_{ij} = a_i f_j$ for $i \neq j$, and $b_{ii} = a_i f_i + (1 - f_i)$, $i, j = 1, \ldots, n$. Then, writing the column vector $m^t = (m_1^t, \ldots, m_n^t)$, we have $m^{t+1} = Bm^t$ and hence $m^t = B^t m^0$, where B^t is the tth power of B. The matrix B is a positive matrix, and it is easy to check that it has eigenvalue 1 with corresponding positive eigenvector $m = (m_1, \ldots, m_n)$. By Perron's theorem (see, for instance, Horn and Johnson, 1985, section 8.2), 1 is the unique maximal eigenvalue of B. Also $(1, 1, \ldots, 1)$ is a right eigenvector of B corresponding to the eigenvalue 1. It follows that B^t tends to the matrix whose columns are each m (see Horn and Johnson, 1985, theorem 8.2.8), which proves the theorem. ■

9.10 The Folk Theorem

The *folk theorem for repeated games* is so called because no one can discover who first thought of it; it is just part of the "folklore" of game theory. We shall first present a stripped-down analysis of the folk theorem with an example, and provide a somewhat more complete discussion in the next section.

Consider the stage game in section 9.2, reproduced in the diagram. There is of course one subgame perfect Nash equilibrium in which each player gets zero. Moreover, neither player can be forced to receive a negative payoff in the repeated game based on this stage game, because zero can be assured simply by playing D. Also, any point in the region OEABCF in figure 9.1 could be attained in the stage game, if the players could agree on a mixed strategy for each. To see this, note that if Big Fish uses C with probability α, and Little Fish uses C with probability β, then the expected payoff to the pair is $(8\beta - 3\alpha, 8\alpha - 3\beta)$, which traces out every point in the quadrilateral OEABCF for $\alpha, \beta \in [0, 1]$ (check it out!). Only the points in OABC are superior to the universal defect equilibrium (0,0), however.

	C	D
C	5,5	-3,8
D	8,-3	0,0

Consider the repeated game $\mathcal{R}$ based on the stage game $\mathcal{G}$ of section 9.2. The folk theorem says that under the appropriate conditions concerning the cooperate or defect signal available to players, any point in the region OABC can be sustained as the average per-period payoff of a subgame perfect Nash equilibrium of $\mathcal{R}$, provided the discount factors of the players are sufficiently near unity; that is, the players do not discount the future at a high rate.

More formally, consider any n-player game with finite strategy sets S_i for $i = 1, \ldots, n$, so the set of strategy profiles for the game is $S = \{(s_1, \ldots, s_n) | s_i \in S_i, i = 1, \ldots, n\}$. The payoff for player i is $\pi_i(s)$, where $s \in S$. For any $s \in S$, we write s_{-i} for the vector obtained by dropping the ith component of s, and for any $i = 1, \ldots, n$ we write $(s_i, s_{-i}) = s$. For a given player j, suppose the other players choose strategies m^j_{-j} such that j's best response m^j_j gives j the lowest possible payoff in the game. We call the resulting strategy profile m^j the *maximum punishment payoff* for j. Then, $\pi^*_j = \pi_j(m^j)$ is j's payoff when everyone else "gangs up on him." We call

$$\pi^* = (\pi^*_1, \ldots, \pi^*_n), \tag{9.4}$$

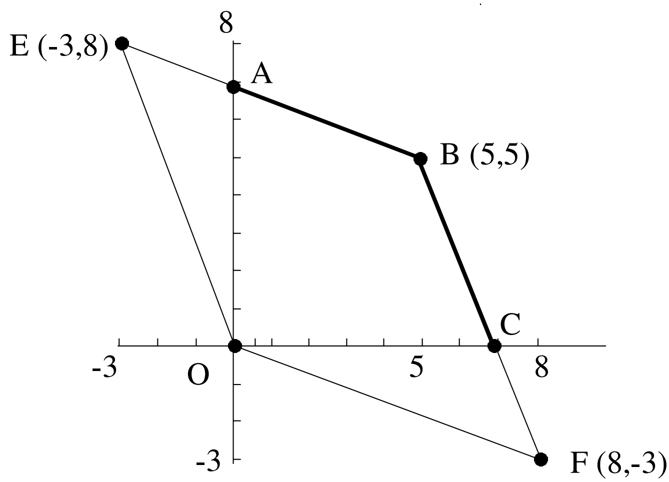


Figure 9.1. The folk theorem: any point in the region OABC can be sustained as the average per-period payoff the subgame perfect Nash equilibrium of the repeated game based on the stage game in section 9.2.

the *minimax point* of the game. Now define

$$\Pi = \{(\pi_1(s), \ldots, \pi_n(s)) | s \in S, \pi_i(s) \geq \pi_i^*, i = 1, \ldots, n\},$$

so Π is the set of strategy profiles in the stage game with payoffs at least as good as the maximum punishment payoff for each player.

This construction describes a stage game $\mathcal{G}$ for a repeated game $\mathcal{R}$ with discount factor δ, common to all the agents. If $\mathcal{G}$ is played in periods $t = 0, 1, 2, \ldots$, and if the sequence of strategy profiles used by the players is $s(1), s(2), \ldots$, then the payoff to player j is

$$\tilde{\pi}_j = \sum_{t=0}^{\infty} \delta^t \pi_j(s(t)).$$

Let us assume that information is *public* and *perfect*, so that when a player deviates from some agreed-upon action in some period, a signal to this effect is transmitted with probability one to the other players. If players can use mixed strategies, then any point in Π can be attained as payoffs to $\mathcal{R}$ by each player using the same mixed strategy in each period. However, it is not clear how a signal indicating deviation from a strictly mixed strategy should be interpreted. The simplest assumption guaranteeing the existence of such a signal is that there is a *public randomizing device* that can be seen by all players, and that players use to decide which pure strategy to use, given that they have agreed to use a particular mixed strategy. Suppose, for instance, the randomizing device is a circular disc with a pointer that can be spun by a flick of the finger. Then, a player could mark off a number of regions around the perimeter of the disc, the area of each being proportional to the probability of using each pure strategy in a given mixed strategy to be used by that player. In each period, each player flicks his pointer and chooses the appropriate pure strategy, this behavior is recorded accurately by the signaling device, and the result is transmitted to all players.

With these definitions behind us, we have the following, where for $\pi \in \Pi$, $\sigma_i(\pi) \in \Delta S_i$ is a mixed strategy for player i such that $\pi_i(\sigma_1, \ldots, \sigma_n) = \pi_i$:

THEOREM 9.9 Folk Theorem. *Suppose players have a public randomizing device, and the signal indicating cooperation or defection of each player is public and perfect. Then, for any $\pi = (\pi_1, \ldots, \pi_n) \in \Pi$, if the discount factor is sufficiently close to unity, there is a Nash equilibrium of the repeated game such that π_j is j's payoff for $j = 1, \ldots, n$ in each period. The equilibrium is effected by each player i using $\sigma_i(\pi)$ as long as no player has been signaled as having defected, and playing the minimax strategy m_i^j in all future periods after player j is first detected defecting.*

The idea behind this theorem is straightforward. For any such $\pi \in \Pi$, each player j uses the strategy $\sigma_j(\pi)$ that gives payoffs π in each period, provided the other players do likewise. If one player deviates, however, all other players play the strategies that impose the maximum punishment payoff on j forever. Because $\pi_j \geq \pi_j^*$, player j cannot gain from deviating from $\sigma_j(\pi)$, so the profile of strategies is a Nash equilibrium.

Of course, unless the strategy profile $(m_1^j, \ldots, m_n^j)$ is a Nash equilibrium for each $j = 1, \ldots, n$, the threat to minimax even once, let alone forever, is not a credible threat. However, we do have the following:

THEOREM 9.10 The folk theorem with Subgame Perfection. *Suppose $y = (y_1, \ldots, y_n)$ is the vector of payoffs in a Nash equilibrium of the underlying one-shot game, and $\pi \in \Pi$ with $\pi_i \geq y_i$ for $i = 1, \ldots, n$. Then, if δ is* *sufficiently close to unity, there is a subgame perfect Nash equilibrium of the repeated game such that π_j is j's payoff for $j = 1, \ldots, n$ in each period.*

To see this, note that for any such $\pi \in \Pi$, each player j uses the strategy s_j that gives payoffs π in each period, provided the other players do likewise. If one player deviates, however, all players play the strategies that give payoff vector y forever. Fudenberg and Maskin (1986) show that subgame perfection is possible even for the more general case where $\pi \in \Pi$.

9.11 The Folk Theorem and the Nature of Signaling

Repeated game theory is a key analytical tool in modeling the coordination of behavior in teams and other groups. Indeed, economists often claim that the folk theorem and its many variants have solved the problem of achieving social efficiency with self-regarding agents (that is, agents who care only about their personal payoffs). Unfortunately, such claims are exaggerated. Folk theorems generally only assert what occurs when the discount factor is near unity, so individuals are arbitrarily far-sighted, when the signals indicating cooperation and defection are arbitrarily close to zero and when the signals are sufficiently "public" in nature. In the real world, none of these assumptions is realistic, and their realism decreases with group size. Consider, for instance, the issue of the quality of signals of player behavior.

We say a signal is *imperfect* if it sometimes misreports whether or not the player in question defected. A public imperfect signal reports the same information to all players but it may be incorrect. The folk theorem was extended to imperfect public signals by Fudenberg, Levine and Maskin (1994), as was analyzed in section 9.10.

If different players receive different signals, or some receive no signal at all, we say the signal is *private*. The case of private signals has proved much more daunting than that of public signals. However, folk theorems for private but near-public signals (that is, where there is an arbitrarily small deviation ϵ from public signals) have been developed by several game theorists, including Sekiguchi (1997), Piccione (2002), Ely and Välimäki (2002), Bhaskar and Obara (2002), Hörner and Olszewski (2006), and Mailath and Morris (2006).

The problem is that in the real world, private signals are generally not near-public. For instance, in a large work team, each member will generally work directly with only a few other workers, so the signal indicating cooperation or defection of a worker will be sent only to a small subset of team members. Of course, if all individuals were obligate truth-tellers, their information could be pooled, and the public information assumption would be plausible. But, there is no reason for self-regarding individuals to be truthful, so we would need a mechanism to render truth-telling incentive compatible for self-regarding agents. Although this is possible in some case, in general such mechanisms are not known. In fact, in human societies, gossip is probably the major mechanism for turning private into public information, and the veracity of communication in gossip depends on individuals having an ethic of truth-telling and a strong regard for personal reputation that would be tarnished were lying detected. Ethics and self-esteem are, however, prime examples of other-regarding, not self-regarding, behavior (Gintis 2009).

It is also the case that, given the plethora of equilibria available in the repeated game case, there is no reason to believe a set of individuals would or could spontaneously play, or learn to play, a folk theorem equilibrium. One could argue that specific cultural institutions can arise that direct individuals to play their part in a particular equilibrium (Binmore 2005; Bicchieri 2006) but it can be shown that implementing the Nash equilibrium is infeasible unless individuals have an other-regarding predisposition to conform to social norms even when it is costly to do so (Gintis 2009).

9.12 The Folk Theorem Fails in Large Groups

Suppose the acts involved in cooperating in section 9.2 are in fact complex and demanding, so there is some probability $\epsilon > 0$ that a player will make a mistake, playing D instead of C. Trigger strategies are devastating in such a situation, because with probability 1 eventually one player will make a mistake, and both will defect forever after. If it were possible to distinguish mistakes from intentional defection, there would be no difficulty in sustaining cooperation. Suppose, however, that there is not. There may nevertheless be a *trembling hand* cooperative equilibrium of the following form: if either player defects, both defect for k rounds, and then both return to cooperation (no matter what happened in the defection rounds). Given ϵ, of course k should be chosen to be the *smallest* integer such that it pays to cooperate rather than defect.

Does such a k exist? Let v be the value of the game when both players use the following "trembling hand" strategy. There are two "phases" to the game. In the "cooperate" phase, (try to) play C, and in the "punishment" phase, play D. If the game is in the cooperate phase and either agent plays D (on purpose or by mistake), the game moves to the punishment phase. If the punishment phase has lasted for k rounds, the game moves to the cooperate phase. The game starts in the cooperate phase.

It is clear that there is no gain from playing C in the punishment phase. Can there be a gain from playing D in the cooperate phase?

Here is where the one-stage deviation principle becomes useful. If the above strategies do not form a subgame perfect Nash equilibrium, then playing D in the cooperate phase and then returning to the trembling hand strategy has a higher payoff than cooperating. The payoff to playing D in the cooperate phase and then returning to the trembling hand strategy is just $8(1-\epsilon)+\delta^{k+1}v$, because your partner also plays D with probability ϵ, in which case you get nothing but still must wait k periods to resume cooperating. Thus, cooperating is Nash when $8(1-\epsilon)+\delta^{k+1}v$ is less than v, or

$$v(1-\delta^{k+1}) > 8(1-\epsilon). \tag{9.5}$$

But what is v? Well, v must satisfy the equation

$$v = (1-\epsilon)\epsilon(-3+\delta^{k+1}v)+(1-\epsilon)^2(5+\delta v)+\epsilon[(1-\epsilon)8+\delta^{k+1}v]. \tag{9.6}$$

To see this, note that if you both cooperate at the start of the game, with probability $(1-\epsilon)\epsilon$, you play C and your partner plays D, in which case you get -3 now and after $k+1$ periods, you're back into the cooperate phase, the present value of which is $\delta^{k+1}v$. This is the first term in (9.6). With probability $(1-\epsilon)^2$ you both play C, so you get 5 and v again in the next period. This is the second term. With probability ϵ you play D, in which case you get 8 if your partner plays C, zero if your partner plays D, and in either case, you get v after $k+1$ periods. This is the final term in (9.6).

Solving (9.6) for v, we get

$$v = \frac{5(1-\epsilon)}{1-\delta(1-\epsilon)^2-\delta^k\epsilon(2-\epsilon)}.$$

Suppose, for instance, $\epsilon = 15\%$ and $\delta = 95\%$. It is easy to check by hand calculation that $v = 85$ if there is no punishment phase ($k = 0$)

but the payoff to defecting is $87.55 > v$. For one round of punishment ($k = 1$), $v = 67.27$, but the value of defecting is $8(1-\epsilon) + \delta^2 v = 67.51$, so punishment is still not sufficiently severe. For two rounds of punishment ($k = 2$), $v = 56.14$, and $8(1-\epsilon) + \delta^3 v = 54.93$, so two rounds of punishment are needed to sustain cooperation.

Suppose, now, that the group is of size $n > 2$, so the probability at least one member accidentally errs is now $1 - (1-\epsilon)^n$. We can show that as $\epsilon \to 1/n$, cooperation becomes impossible. I will not be rigorous but it is easy to see that the argument can be made as rigorous as desired. We make the conservative assumption that the group is sufficiently large that we can ignore the cost to each player of another player defecting. Moreover, let us assume that n is so large, and ϵn sufficiently close to one that we can use the approximation $(1 + 1/n)^n \approx 1/e$, where $e \approx 2.718$ is the base of the natural logarithms (in fact, $n \geq 4$ is usually good enough). Finally, suppose it takes k rounds of no cooperation to make cooperating a best response. We have the recursion equation

$$v = (5 + \delta v)(1-\epsilon)^n + (8 + \delta^k v) + (1 - \epsilon - (1-\epsilon)^n)(5 + \delta^k v).$$

The first term on the right-hand side is the payoff when there are no errors. The second term is the payoff when the agent errs. The third term is the payoff in all other cases. With our approximations, this reduces to

$$v = \frac{e(5 + 3\epsilon)}{e(1-\delta^k) - \delta(1-\delta)},$$

Now, the payoff to defecting once and returning to cooperation is $v_d = 8 + \delta^k v$, so cooperating is a best response if and only if $8 + \delta^k v < v$, which reduces to

$$8\delta - 3e(1-\epsilon) + (3e(1-\epsilon) - 8)\delta^k > 0,$$

which is false for small ϵ, for all integers $k \geq 1$, because $3e \approx 8.15 > 8 > 8\delta$.

9.13 Contingent Renewal Markets Do Not Clear

In many exchanges, including those between (a) employer and employee, (b) lender and borrower, and (c) firm and customer, the agent on one side of the exchange gives money (employer, lender, customer), whereas the agent on the other side of the exchange gives a promise (employee, borrower, firm). The employee promises to work hard, the borrower promises to repay the loan, and the firm promises to provide high-quality products. Rarely, however, is this promise subject to a contract that can be enforced at reasonably low cost.

Let us call the player who gives money the *principal*, and the player who gives promises the *agent*. In the absence of an enforceable contract, why do agents keep their promises? Of course, some agents are just honest but there are doubtless enough dishonest people that exchange would break down if enforcement were based purely on the integrity of the agents.

Perhaps the threat of suing is sufficient to secure agent compliance. But generally such threats are not credible. Taking an employee to court for not working hard enough is rare. A lender can sue a borrower for nonpayment, but if the borrower has no wealth, there is not much to collect. A customer can sue a firm for faulty goods but this is done only in cases where a product has caused extensive personal injury. So why, then, do agents generally keep their promises?

The answer is that if agents do not keep their promises, principals dump them: employers fire workers who shirk, lenders refuse future loans to borrowers who have defaulted, and customers switch to new suppliers when dissatisfied. All three actions represent *trigger strategies*: the exchange between principal and agent is renewed indefinitely (perhaps with some exogenous probability of dissolution), the principal using the threat of nonrenewal to secure compliance. We call this *contingent renewal* exchange.

A *contingent renewal market* is a market in which exchanges between buyers and sellers are regulated by contingent renewal relationships. Because the principal (employer, lender, consumer) in such markets uses a trigger strategy (the threat of nonrenewal) to elicit performance from the agent (worker, borrower, firm), the loss of the relationship must be costly to the agent. But if the price is set in such markets to equate supply and demand, the cost to an agent of being cut off by the principal is zero, because the agent will secure another position in the next period at the prevailing price. Hence, if the principal uses a trigger strategy, there must be a positive probability that there is an excess supply of agents. It follows that *in a Nash equilibrium of a contingent renewal market, there is an excess supply of agents*.

This conclusion nicely explains some of the most pervasive facts about market economies. Consider, for instance, labor markets. In the neoclassical model, the wage rate adjusts to equate the supply of and the demand for labor. The general condition of labor markets, however, is excess supply. Often this takes the form of explicit unemployment, which neoclassical economists explain using complex models involving search costs, friction, adaptive expectations, exotic intertemporal elasticities and the like. Using Occam's razor (always opt for the simplest explanation first), a contingent renewal labor market does the job. There simply cannot be full employment in such models (Gintis 1976; Shapiro and Stiglitz 1984; Bowles 1985; Bowles and Gintis 1993). Excess supply in labor markets takes the form not only of unemployment but of "underemployment": workers hold one position but are capable and willing to fill a "better" position, even at the going wage or a bit below, but they cannot secure such a position.

Another example is credit markets. In the neoclassical model, the interest rate adjusts to equate the supply of and the demand for loans. The general condition of credit markets, however, is excess demand. Why does the interest rate not rise to cut off this excess demand? There are two basic reasons (Stiglitz and Weiss 1981; Stiglitz 1987). First, an increase in the interest rate will drive borrowers who have low-risk, low-expected-return projects out of the market, and increase the expected riskiness of the remaining pool of borrowers. Second, an interest rate increase will induce borrowers to increase the riskiness of their investment projects, thus lowering the lender's expected return.

Because risksharing, requiring the borrower to put up a fraction of the equity in a project, is the most widely used and effective means of endogenous contract enforcement in credit markets, it follows that *lending is directed predominantly toward wealthy agents*. This basic fact of life, which seems so perverse from the neoclassical standpoint (loans should be from the wealthy to the nonwealthy), is perfectly comprehensible from the standpoint of models in which contract enforcement is endogenous, even without contingent renewal. Contingent renewal (making available a line of credit, contingent on performance) adds the dimension that a certain subset of nonwealthy borrowers with good projects can get loans, facing the threat of falling into the pool of unemployed "credit seekers" should their credit line be terminated.

A third example is consumer goods markets. In the neoclassical model, price adjusts until supply and demand are equal. This implies that firms can sell as much as they want, subject to the market price, and choose how much to produce according to cost considerations. Everyday observation tells a different story: firms try to create sales, and generally they can produce with ease whatever is needed to satisfy the demand they have generated. Of course, there are models of monopolistic competition in which firms have differentiated products and downward-sloping demand curves, but these notions do not capture the critical point that there is a strategic interaction between buyer and seller, and the price is determined in part by this interaction.

In our game-theoretic model, equilibrium price in markets with quality goods must exceed marginal cost, not because of the price elasticity of demand, as in monopolistic competition, but because a high price ensures that it is costly to lose a client, thus reassuring the buyer that the seller has an incentive to produce a high-quality good, under the threat of the buyer finding another supplier (Klein and Leffler 1981; Gintis 1989).

9.14 Short-Side Power in Contingent Renewal Markets

We say a principal P has power over an agent A if P can impose, or credibly threaten to impose, sanctions on A but A has no such capacity vis-à-vis P (Bowles and Gintis 1992). This definition is doubtless incomplete and unnuanced but conforms to standard notions in analytical political theory (Simon 1953; Dahl 1957; Harsanyi 1962). In the neoclassical model there is no power, because all markets clear and contracts are costlessly enforced. In contingent renewal markets, however, principals have power over agents because they can impose costs on agents by terminating them. Because agents are in excess supply, without collusion agents can exercise no parallel threat over their principals. It follows that employers have power over employees, lenders have power over borrowers, and consumers have power over the firms from which they buy. We may call this *short-side power* because it always lies with the transactor on the short side of the market; that is, the side for which the quantity of desired transactions is the lesser.

Contingent renewal markets do not clear, and in equilibrium they allocate power to agents located on the short side of the market.

9.15 Money Confers Power in Contingent Renewal Markets

If we review the cast of characters in our various contingent renewal markets, we find a strong regularity: the principal gives money to the agent, and the principal is on the short side of the market. For instance, the employer, the lender, and the consumer hand over money to the worker, the borrower, and the supplying firm, and the latter are all short-siders. The reason for this is that the money side of contracts is relatively easy to enforce. This important regularity is implicit in most repeated game principal-agent models, where it is assumed that the principal can make credible promises (to wit, the incentive scheme), whereas agent cannot.

The application of the notion that "money talks" is particularly dramatic in the case of consumer goods markets. In neoclassical theory, consumer sovereignty means that free markets (under the appropriate conditions) lead to efficient allocations. What the term *really* means in people's lives is that because firms are on the long side of the market (they are quantity constrained), consumers can tell producers how to behave. People are then truly sovereign. Probably nowhere in the daily lives of ordinary people do they feel more power and gain more respect, than when acting as consumers, constantly pandered to by obsequious suppliers interested in staying in their good graces, and benefiting from the difference between price and marginal cost.

9.16 The Economy Is Controlled by the Wealthy

It is neither a criticism nor profound to observe that the wealthy control the market economy. This cannot be explained in neoclassical economics, in which the wealthy have great *purchasing power* but this does not translate into any sort of economic power over others. As Paul Samuelson (1957:894) has noted, "in a perfectly competitive market it really does not matter who hires whom; so let labor hire capital." The result, expressed long ago by Joseph Schumpeter (1934), is a decentralization of power to consumers: "The people who direct business firms only execute what is prescribed for them by wants." These views taken together imply the competitive economy as an arena of "solved political problems" (Lerner 1972). However, it is not correct in the case of labor, capital and quality goods, as we illustrate below.

9.17 Contingent Renewal Labor Markets

In this section we develop a repeated game between employer and employee in which the employer pays the employee a wage higher than the expected value of his next best alternative, using the threat of termination (a trigger strategy) to induce a high level of effort, in a situation where it is infeasible to write and enforce a contract for labor effort. When all employers behave in this manner, we have a nonclearing market in equilibrium.

Suppose an employer's income per period is $q(e)$, an increasing, concave function of the effort e of an employee. The employee's payoff per period $u = u(w,e)$ is an increasing function of the wage w and a decreasing function of effort e. Effort is known to the employee but is only imperfectly observable by the employer. In each period, the employer pays the employee w, the employee chooses effort e, and the employer observes a signal that registers the employee as "shirking" with probability $f(e)$, where $f'(e) < 0$. If the employee is caught shirking, he is dismissed and receives a fallback with present value z. Presumably z depends on the value of leisure, the extent of unemployment insurance, the cost of job search, the startup costs in another job, and the present value of the new job. The employer chooses w to maximize profits. The tradeoff the employer faces is that a higher wage costs more but it increases the cost of dismissal to the employee. The profit-maximizing wage equates the marginal cost to the marginal benefit.

The employee chooses $e = e(w)$ to maximize the discounted present value v of having the job, where the flow of utility per period is $u(w,e)$. Given discount rate ρ and fallback z, the employee's payoff from the repeated game is

$$v = \frac{u(w,e) + [1 - f(e)]v + f(e)z}{1+\rho},$$

where the first term in the numerator is the current period utility, which we assume for convenience to accrue at the end of the period, and the others measure the expected present value obtainable at the end of the period, the weights being the probability of retaining or losing the position. Simplifying, we get

$$v = \frac{u(w,e) - \rho z}{\rho + f(e)} + z.$$

The term ρz in the numerator is the forgone flow of utility from the fallback, so the numerator is the net flow of utility from the relationship, whereas $f(e)$ in the denominator is added to the discount rate ρ, reflecting the fact that future returns must be discounted by the probability of their accrual as well as by the rate of time preference.

The employee varies e to maximize v, giving the first-order condition

$$\frac{\partial u}{\partial e} - \frac{\partial f}{\partial e}(v - z) = 0, \tag{9.7}$$

which says that the employee increases effort to the point where the marginal disutility of effort is equal to the marginal reduction in the expected loss occasioned by dismissal. Solving (9.7) for e gives us the employee's best response $e(w)$ to the employer's wage offer w.

We assume that the employer can hire any real number n of workers, all of whom have the effort function $e(w)$, so the employer solves

$$\max_{w,n} \; \pi = q(ne(w)) - wn.$$

The first-order conditions on n and w give $q'e = w$ and $q'ne' = n$, which together imply

$$\frac{\partial e}{\partial w} = \frac{e}{w}. \tag{9.8}$$

This is the famous *Solow condition* (Solow 1979).

The best-response function and part of the employer's choice of an optimal enforcement strategy (w^*) are shown in figure 9.2, which plots effort against salary. The iso-v function v^* is one of a family of loci of effort levels and salaries that yield identical present values to the employee. Their slope, $-(\partial v/\partial w)/(\partial v/\partial e)$, is the marginal rate of substitution between wage and effort in the employee's objective function. Preferred iso-v loci lie to the right.

By the employee's first-order conditions (9.7), the iso-v loci are vertical where they intersect the best-response function (because $\partial v/\partial e = 0$). The negative slope of the iso-v functions below $e(w)$ results from the fact that in this region the contribution of an increase in effort, via $(\partial f/\partial e)(v - z)$, to the probability of keeping the job outweigh the effort-disutility effects. Above $e(w)$, the effort-disutility effects predominate. Because v rises along $e(w)$, the employee is unambiguously better off at a higher wage. One of

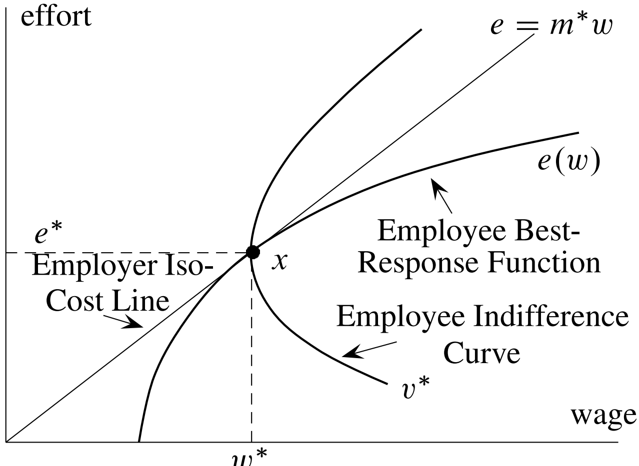


Figure 9.2. The employee's best-response function

the employer's iso-cost loci is labeled $e = m^*w$, where m^* is the profit-maximizing effort per dollar. The employer's first-order condition identifies the equilibrium wage w^* as the tangency between the employer's iso-cost function, $e = m^*w$ and the employee's effort function, with slope e', or point x in the figure.

It should be clear that the contingent renewal equilibrium at x is not first-best, because if the parties could write a contract for effort, any point in the lens-shaped region below the employee's indifference curve v^* and above the employer's iso-cost line $e = m^*w$ makes both parties strictly better off than at x. Note that if we populated the whole economy with firms like this, we would in general have $v > z$ in market equilibrium, because if $v = z$, (9.7) shows that $\partial u/\partial e = 0$, which is impossible so long as effort is a disutility. This is one instance of the general principle enunciated previously, that *contingent renewal markets do not clear in (Nash) equilibrium*, and the agent whose promise is contractible (usually the agent paying money) is on the long side of the market.

Perhaps an example would help visualize this situation. Suppose the utility function is given by

$$u(w, e) = w - \frac{1}{1-e}$$

and the shirking signal is given by

$$f(e) = 1 - e.$$

You can check that $e(w)$ is then given by

$$e(w) = 1 - a - \sqrt{a^2 + \rho a},$$

where $a = 1/(w - \rho z)$. The reader can check that this function indeed has the proper shape: it is increasing and concave, is zero when $w = 2 + \rho(1 + z)$, and approaches unity with increasing w.

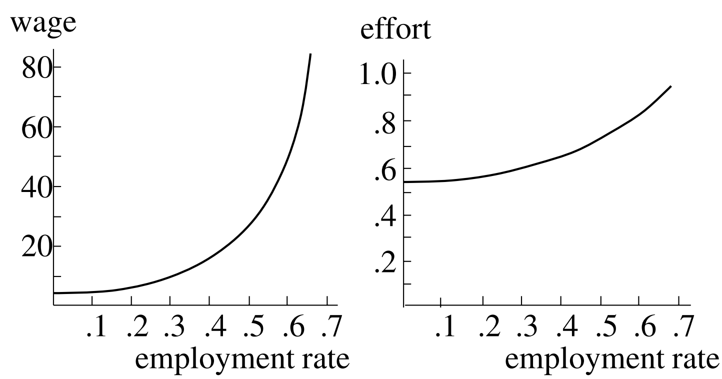


Figure 9.3. Wage and effort as functions of the employment rate in a contingent renewal labor market

The solution for the employer's optimum w, given by the Solow condition (9.8), is very complicated, so I will approximate the solution. Suppose $\rho = 0.05$ and the employment rate is $q \in [0, 1]$. An employee dismissed at the end of the current period therefore has a probability q of finding a job right away (we assume all firms are alike), and so regains the present value v. With probability $1 - q$, however, the ex-employee remains unemployed for one period and tries again afterward. Therefore we have

$$z = qv + (1 - q)z/(1 + \rho),$$

assuming the flow of utility from being unemployed (in particular, there is no unemployment insurance) is zero. Solving, we have

$$z = \frac{(1 + \rho)q}{q + \rho} v.$$

For a given unemployment rate q, we can now find the equilibrium values of w, e, v, and z, and hence the employer's unit labor cost e/w. Running this through Mathematica, the equilibrium values of w and e as the employment rate q goes from zero to 0.67 are depicted in figure 9.3.

Note that although effort increases only moderately as the unemployment rate drops from 100% to 33%, the wage rate increases exponentially as the unemployment rate approaches 33%. I could not find a solution for $q > 0.67$. The actual unemployment rate can be fixed by specifying the firm's production function and imposing a zero profit condition. However this is accomplished, there will be positive unemployment in equilibrium.

10

Evolutionarily Stable Strategies

There is but a step between the sublime and the ridiculous.

Leo Tolstoy

In 1973 the biologist John Maynard Smith and the mathematician G. R. Price wrote an article in *Nature* showing how game theory applies to the behavior of animals (Maynard Smith and Price 1973). Maynard Smith went on to write a book on the subject (Maynard Smith 1982), which has become a classic. The idea of applying game theory to animals, and not just the higher primates, but fish, dung beetles, fireflies, and pond scum as well, seemed strange at the time, because game theory had always been the preserve of hyperrationality. Animals hardly fit the bill. Maynard Smith made three critical shifts from traditional game theory. The first is in the concept of a strategy, the second in the concept of equilibrium, and a third in the nature of agent interactions.

Strategy. In classical game theory, *players* have strategy sets from which they choose particular strategies. In biology, *species* have strategy sets (genotypic variants), of which *individuals* inherit one or another variant, perhaps mutated, that they then play in their strategic interactions. This extends nicely to the treatment of culture in human society, in which case we say that *society* has the strategy set (the set of alternative cultural forms) and *individuals* inherit or choose among them.

Equilibrium. In place of the Nash equilibrium, Maynard Smith and Price used the *evolutionarily stable strategy* (ESS) concept. A strategy is an ESS if a whole population using that strategy cannot be invaded by a small group with a mutant genotype. Similarly, a cultural form is an ESS if, upon being adopted by all members of a society (firm, family, etc.), no small group of individuals using an alternative cultural form can invade. We thus move from explaining the actions of individuals to modeling the diffusion of forms of behavior ("strategies") in society.

Player interactions. In place of the one-shot and repeated games of classical game theory, Maynard Smith introduced the notion of the *repeated, random pairing of agents* who play strategies based on their genome but not on the previous history of play.

The ESS concept is particularly useful because it says something about the dynamic properties of a system without being committed to any particular dynamic model. As we shall see, however, an evolutionary system with a symmetrical two-player stage game can be dynamically stable without being an ESS (§12.9).

10.1 Evolutionarily Stable Strategies: Definition

Consider a two-player normal form game in which both players have the set $S = \{s_1, \ldots, s_n\}$ of pure strategies, and the payoffs to an agent playing $s_i \in S$ and another agent playing $s_j \in S$ are π^1_{ij} for the first and $\pi^2_{ij} = \pi^1_{ji}$ for the second. We call such a game *symmetric in payoffs*. In addition, we assume the agents cannot condition their play on whether they are "player 1" or "player 2." We call such a game *symmetric in strategies*. If a game is symmetric in both payoffs and strategies, we simply call the game *symmetric*. We call $A = (\pi^1_{ij})$ the *matrix* of the symmetric game. Note that A represents only the payoffs for the row player, because the payoffs to the column player are just the transpose of A.

Let $\mathcal{G}$ be a symmetric game with matrix A (we'll call it the *stage game*) and large population of agents. In each period $t = 1, 2, \ldots$, agents are randomly paired and they play the stage game $\mathcal{G}$ once. Each agent is of type i for some $s_i \in S$, meaning that the agent uses strategy s_i in the stage game. If the proportion of agents of type j is p_j at a particular time, we say the *state* of the population is $\sigma = p_1 s_1 + \ldots + p_n s_n$. Note that we must have $p_1, \ldots, p_n \geq 0$ and $\sum_i p_i = 1$. The payoff at that time to a player of type i when the state of the population is σ is defined by

$$\pi_{i\sigma} = \sum_{j=1}^{n} \pi_{ij} p_j, \tag{10.1}$$

which is the player's expected payoff before being assigned a particular partner. These conditions define a new game, called the *evolutionary game* corresponding to the stage game $\mathcal{G}$.

Suppose the state of the population is σ, and some small subpopulation plays a "mutant" strategy $\tau = q_1 s_1 + \ldots + q_n s_n$, in the sense that q_i is the frequency of pure strategy s_i in this subpopulation. We say the mutant is of "type τ," and has payoff

$$\pi_{\tau\sigma} = \sum_{i,j=1}^{n} q_i \pi_{ij} p_j ,$$

when a random member of its population meets a random member of the population σ.

Suppose the state of the population is $\sigma = p_1 s_1 + \ldots + p_n s_n$. The expected payoff to a randomly chosen member of the population is thus just $\pi_{\sigma\sigma}$. If we replace a fraction $\epsilon > 0$ of the population with a "mutant" of type τ, the new state of the population is

$$\mu = (1-\epsilon)\sigma + \epsilon\tau,$$

so the payoff to a randomly chosen nonmutant is

$$\pi_{\sigma\mu} = (1-\epsilon)\pi_{\sigma\sigma} + \epsilon\pi_{\sigma\tau},$$

and the expected payoff to a mutant is

$$\pi_{\tau\mu} = (1-\epsilon)\pi_{\tau\sigma} + \epsilon\pi_{\tau\tau}.$$

We say the mutant type can *invade* the population if $\sigma \neq \mu$ and for all sufficiently small $\epsilon > 0$,

$$\pi_{\tau\mu} \geq \pi_{\sigma\mu},$$

which says that, on average, a mutant does at least as well against the new population as does a nonmutant. We say σ is an *evolutionarily stable strategy* (ESS) if it cannot be invaded by any mutant type, in a sense defined precisely below.

We assume that mutants can employ mixed strategies in applying the ESS criterion, because as we shall see later (§12.7), with this assumption evolutionarily stable strategies have powerful dynamic properties. A Nash equilibrium in an evolutionary game can consist of a *monomorphic* population of agents, each playing the same mixed strategy, or a *polymorphic* population, a fraction of the population playing each of the underlying pure strategies in proportion to its contribution to the mixed Nash strategy. The two interpretations are interchangeable under many conditions, and we shall not commit ourselves exclusively to either interpretation. Because the stage game is a one-shot, it is rarely plausible to hold that an individual will play a strictly mixed strategy. Thus, in general, the heterogeneous population interpretation is superior. The heterogeneous mutant τ must then possess some internal mechanism for maintaining the constant frequency distribution $q_1, \ldots, q_n$ from period to period. We relax this assumption when we treat evolutionary games as dynamical systems in chapter 12.

10.2 Properties of Evolutionarily Stable Strategies

Prove the following properties of evolutionarily stable strategies:

a. Strategy $\sigma \in \Delta S$ is an ESS if, for every mutant type $\tau \in \Delta S$, there is an $\epsilon_\tau > 0$ such that for all $\epsilon \in (0, \epsilon_\tau)$ and defining $\mu = (1-\epsilon)\sigma + \epsilon\tau$, we have

$$\pi_{\sigma\mu} > \pi_{\tau\mu}. \tag{10.2}$$

b. We say that $\sigma \in \Delta S$ has a *uniform invasion barrier* if there is some $\epsilon_o \in (0, 1)$ such that (10.2) holds for all $\tau \neq \sigma$ and all $\epsilon \in (0, \epsilon_o)$. Strategy σ is an ESS if and only if it has a uniform invasion barrier.

c. Strategy $\sigma \in \Delta S$ is an ESS if and only if, for any mutant type $\tau \in \Delta S$, we have

$$\pi_{\sigma\sigma} \geq \pi_{\tau\sigma},$$

and if $\pi_{\sigma\sigma} = \pi_{\tau\sigma}$, then

$$\pi_{\sigma\tau} > \pi_{\tau\tau}.$$

This says that $\sigma \in \Delta S$ is an ESS if and only if a mutant cannot do better against an incumbent than an incumbent can do against another incumbent, and if a mutant does as well as an incumbent against another incumbent, then an incumbent must do better against a mutant than a mutant does against another mutant. Note here that we are assuming mutants can use mixed strategies.

d. An evolutionarily stable strategy is a Nash equilibrium that is isolated in the set of symmetric Nash equilibria (that is, it is a strictly positive distance from any other symmetric Nash equilibrium).

e. Every strict Nash equilibrium in an evolutionary game is an ESS.

10.3 Characterizing Evolutionarily Stable Strategies

THEOREM 10.1 *Suppose symmetric two-player game $\mathcal{G}$ has two pure strategies. Then, if $\pi_{11} \neq \pi_{21}$ and $\pi_{12} \neq \pi_{22}$, $\mathcal{G}$ has an evolutionarily stable strategy.*

PROOF: Suppose $\pi_{11} > \pi_{21}$. Then, pure strategy 1 is a strict Nash equilibrium, so it is an evolutionarily stable strategy. The same is true if $\pi_{22} > \pi_{12}$. So suppose $\pi_{11} < \pi_{21}$ and $\pi_{22} < \pi_{12}$. Then, we can show that the game has a unique completely mixed symmetric equilibrium p, where each player uses strategy 1 with probability $\alpha_p \in (0, 1)$. The payoff to strategy 1 against the mixed strategy $(\alpha_p, 1-\alpha_p)$ is then $\alpha_p\pi_{11} + (1-\alpha_p)\pi_{12}$, and the payoff to strategy 2 against this mixed strategy is $\alpha_p\pi_{21} + (1-\alpha_p)\pi_{22}$. Because these must be equal, we find that $\alpha_p = (\pi_{22} - \pi_{12})/\Delta$, where $\Delta = \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12} < 0$. Note that under our assumptions, $0 < \alpha_p < 1$, so there is a unique completely mixed Nash equilibrium $(\alpha_p, 1-\alpha_p)$.

Now let α_q be the probability a mutant player uses pure strategy 1. Because each pure strategy is a best response to α_p, α_q must also be a best response to α_p, so clearly, $\pi_{qp} = \pi_{pp}$. To show that p is an ESS, we must show that $\pi_{pq} > \pi_{qq}$. We have

$$\pi_{pq} = \alpha_p[\pi_{11}\alpha_q + \pi_{12}(1-\alpha_q)] + (1-\alpha_p)[\pi_{21}\alpha_q + \pi_{22}(1-\alpha_q)]$$

and

$$\pi_{qq} = \alpha_q[\pi_{11}\alpha_q + \pi_{12}(1-\alpha_q)] + (1-\alpha_q)[\pi_{21}\alpha_q + \pi_{22}(1-\alpha_q)].$$

Subtracting and simplifying, we get

$$\pi_{pq} - \pi_{qq} = -(\alpha_p - \alpha_q)^2\Delta > 0,$$

which proves we have an ESS. ■

THEOREM 10.2 *Using the same notation, the stage game has a strictly mixed Nash equilibrium if and only if $\pi_{11} > \pi_{21}$ and $\pi_{22} > \pi_{12}$, or $\pi_{11} < \pi_{21}$ and $\pi_{22} < \pi_{12}$. The equilibrium is an ESS only if the second set of inequalities holds.*

PROOF: It is easy to check that if there is a mixed strategy equilibrium, the frequency α of pure strategy 1 must satisfy

$$\alpha = \frac{\pi_{22} - \pi_{12}}{\Delta}, \qquad \text{where } \Delta = \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12}.$$

Suppose $\Delta > 0$. Then $0 < \alpha < 1$ if and only if $0 < \pi_{22} - \pi_{12} < \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12}$, which is true if and only if $\pi_{11} > \pi_{21}$ and $\pi_{22} > \pi_{12}$. If $\Delta < 0$, a similar argument shows that $0 < \alpha < 1$ if and only if the other pair of inequalities holds.

Suppose there is a "mutant" that uses pure strategy 1 with probability β. Thus, in general,

$$\begin{aligned}\pi_{\gamma\delta} &= \gamma\delta\pi_{11} + \gamma(1-\delta)\pi_{12} + (1-\gamma)\delta\pi_{21} + (1-\gamma)(1-\delta)\pi_{22} \\ &= \gamma\delta\Delta + \delta(\pi_{21} - \pi_{22}) + \gamma(\pi_{12} - \pi_{22}) + \pi_{22}.\end{aligned}$$

It follows that

$$\pi_{\alpha\alpha} - \pi_{\beta\alpha} = (\alpha - \beta)[\alpha\Delta - (\pi_{22} - a_{12})] = 0,$$

so the equilibrium is an ESS if and only if $\pi_{\alpha\beta} > \pi_{\beta\beta}$. But

$$\begin{aligned}\pi_{\alpha\beta} - \pi_{\beta\beta} &= \alpha\beta\Delta + \beta(a_{21} - a_{22}) + \alpha(a_{12} - a_{22}) + a_{22} \\ &\qquad - \beta^2\Delta - \beta(a_{21} - a_{22}) - \beta(a_{12} - a_{22}) - a_{22} \\ &= \beta(\alpha - \beta)\Delta + (\alpha - \beta)(a_{12} - a_{22}) \\ &= (\alpha - \beta)(\beta\Delta + a_{12} - a_{22}) \\ &= (\alpha - \beta)(\beta\Delta - \alpha\Delta) \\ &= -(\alpha - \beta)^2\Delta.\end{aligned}$$

Thus, the equilibrium is an ESS if and only if $\Delta < 0$, which is equivalent to $a_{11} < a_{21}$ and $a_{22} < a_{12}$. This proves the assertion. ■

THEOREM 10.3 *Suppose* $\sigma = \alpha_1 s_1 + \ldots + \alpha_n s_n \in \Delta S$ *is an ESS, where* s_i *is a pure strategy and* $\alpha_i > 0$ *for* $i = 1, \ldots, n$. *Suppose* $\tau = \beta_1 s_1 + \ldots + \beta_n s_n \in \Delta S$ *is also an ESS. Then,* $\beta_i = \alpha_i$ *for* $i = 1, \ldots n$. *In other words, the support of an ESS cannot strictly contain the support of another ESS.*

THEOREM 10.4 *If* $\sigma \in \Delta S$ *is weakly dominated, then* σ *is not an ESS.*

THEOREM 10.5 *An evolutionary game whose stage game has a finite number of pure strategies can have only a finite number of evolutionarily stable strategies.*

PROOF: Suppose there are an infinite number of distinct evolutionarily stable strategies. Then there must be two, say σ and τ, that use exactly the same pure strategies. Now τ is a best response to σ, so σ must do better against τ than τ does against itself. But σ does equally well against τ as τ does against τ. Thus, σ is not an ESS and similarly for τ. ■

By the *distance* between two strategies $\sigma = \sum_i p_i s_i$ and $\tau = \sum_i q_i s_i$, we mean the distance in $\mathbf{R}^n$ between the points $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$. The following is proved in Hofbauer and Sigmund (1998). Note that the theorem implies that an evolutionarily stable strategy σ is an *isolated Nash equilibrium*, in the sense that there is an $\epsilon > 0$ such that no strategy $\tau \neq \sigma$ within distance ϵ of σ is a Nash equilibrium.

THEOREM 10.6 *Strategy* $\sigma \in \Delta S$ *is an ESS if and only if there is some* $\epsilon > 0$ *such that* $\pi_{\sigma\tau} > \pi_{\tau\tau}$ *for all* $\tau \in \Delta S$ *within distance* ϵ *of* σ.

PROOF: Suppose σ is an ESS, so for any $\tau \neq \sigma$, there is an $\tilde{\epsilon}(\tau)$ such that

$$\pi_{\tau,(1-\epsilon)\sigma+\epsilon\tau} < \pi_{\sigma,(1-\epsilon)\sigma+\epsilon\tau} \quad \text{for all } \epsilon \in (0, \tilde{\epsilon}(\tau)). \tag{10.3}$$

In fact, we can choose $\tilde{\epsilon}(\tau)$ as follows. If (10.3) holds for all $\epsilon \in (0,1)$, then let $\tilde{\epsilon}(\tau) = 1$. Otherwise, let $\tilde{\epsilon}$ be the smallest $\epsilon > 0$ such that (10.3) is violated and define

$$\tilde{\epsilon}(\tau) = \frac{\pi_{\sigma\sigma} - \pi_{\tau\sigma}}{\pi_{\tau\tau} - \pi_{\tau\sigma} - \pi_{\sigma\tau} + \pi_{\sigma\sigma}}.$$

It is easy to check that $\tilde{\epsilon}(\tau) \in (0, 1]$ and (10.3) are satisfied. Let $T \subset S$ be the set of strategies such that if $\tau \in T$, then there is at least one pure strategy used in σ that is not used in τ. Clearly, T is closed and bounded, $\sigma \notin T$, $\tilde{\epsilon}(\tau)$ is continuous, and $\tilde{\epsilon}(\tau) > 0$ for all $\tau \in T$. Hence, $\tilde{\epsilon}(\tau)$ has a strictly positive minimum ϵ^* such that (10.3) holds for all $\tau \in T$ and all $\epsilon \in (0, \epsilon^*)$.

If τ is a mixed strategy and s is a pure strategy, we define $s(\tau)$ to be the weight of s in τ (that is, the probability that s will be played using τ). Now consider the neighborhood of s consisting of all strategies τ such that $|1-s(\tau)| < \epsilon^*$ for all pure strategies s. If $\tau \neq s$, then $\epsilon^* > 1-s(\tau) = \epsilon > 0$

for some pure strategy s. Then $\tau = (1-\epsilon)s + \epsilon r$, where $r \in T$. But then (10.3) gives $\pi_{r\tau} < \pi_{s\tau}$. If we multiply both sides of this inequality by ϵ and add $(1-\epsilon)\pi_{s\tau}$ to both sides, we get $\pi_{\tau\tau} < \pi_{s\tau}$, as required. The other direction is similar, which proves the assertion. ■

THEOREM 10.7 *If $\sigma \in \Delta S$ is a completely mixed evolutionarily stable strategy (that is, it uses all pure strategies with positive probability), then it is the unique Nash equilibrium of the game and $\pi_{\sigma\tau} > \pi_{\tau\tau}$ for all $\tau \in \Delta S$, $\tau \neq \sigma$.*

PROOF: If σ is completely mixed, then for any $tau \in S$, $\pi_{\sigma\sigma} = \pi_{\tau\sigma}$, because any pure strategy has the same payoff against σ as σ does against σ. Therefore, any mixed strategy has the same payoff against σ as σ has against σ. For similar reasons, $\pi_{\sigma\tau} = \pi_{\sigma\sigma}$. Thus, σ is an ESS and if τ is any other strategy, we must have $\pi_{\sigma\tau} > \pi_{\tau\tau}$. ■

10.4 A Symmetric Coordination Game

Consider a two-player pure coordination game in which both players win $a > 0$ if they both choose Up, and they win $b > 0$ if they both choose Down, but they get nothing otherwise. Show that this game has a mixed-strategy equilibrium with a lower payoff than either of the pure-strategy equilibria. Show that this game is symmetric, and the mixed-strategy equilibrium is not an ESS. Show that there are, however, two ESSs. This example shows that sometimes adding the ESS requirement eliminates implausible and inefficient equilibria.

10.5 A Dynamic Battle of the Sexes

The battle of the sexes (§3.9) is not symmetric, and hence the concept of an evolutionarily stable strategy does not apply. However, there is an obvious way to recast battle of the sexes so that it becomes symmetric. Suppose when two players meet, one is randomly assigned to be player 1, and the other player 2. A pure strategy for a player can be written as "xy," which means "if I am Alfredo, I play x, and if I am Violetta, I play y." Here x stands for Opera and y stands for Gambling. There are thus four pure strategies, OO, OG, GO, GG. This game is symmetric, and the normal form matrix (only the payoff to player 1 is shown) is

	OO	OG	GO	GG
OO	3/2,3/2	1,1/2	1/2,1	0,0
OG	1/2,1	0,0	1,2	1/2,1/2
GO	1,1/2	2,1	0,0	1,1/2
GG	0,0	1,1/2	1/2,1	3/2,3/2

Let $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\delta = 1 - \alpha - \beta - \gamma \geq 0$ be the fraction of players who use strategy OO, OG, GO, and GG, respectively (or, equivalently, let $(\alpha, \beta, \gamma, \delta)$ be the mixed strategy of each player). Show that there are two pure-strategy Nash equilibria, OO and GG, and for each $\alpha \in [0, 1]$, there is a mixed-strategy Nash equilibrium αOO $+ (1/3 - \alpha)$OG $+ (2/3 - \alpha)$GO $+ \alpha$GG. Show that the payoffs to these equilibria are 3/2 for the pure-strategy equilibria and 2/3 for each of the mixed-strategy equilibria. It is easy to show that the first two equilibria are ESSs, and the others are not–they can be invaded by either OO or GG.

10.6 Symmetrical Throwing Fingers

Similarly, although throwing fingers (§3.8) is not a symmetric game, and hence the concept of an evolutionarily stable strategy does not apply, there is an obvious way to recast throwing fingers so that it becomes symmetric. Suppose when two players meet, one is randomly assigned to be player 1, and the other player 2. A pure strategy for a player can be written as "xy," which means "if I am player 1, I show x fingers, and if I am player 2, I show y fingers." There are thus four pure strategies, 11, 12, 21, and 22. Show that this game is symmetric, and derive the normal form matrix (only the payoff to player 1 is shown)

	11	12	21	22
11	0,0	−1,1	1,−1	0,0
12	1,−1	0,0	0,0	−1,1
21	−1,1	0,0	0,0	1,−1
22	0,0	1,−1	−1,1	0,0

Let $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\delta = 1 - \alpha - \beta - \gamma \geq 0$ be the fraction of players who use strategy 11, 12, 21, and 22, respectively (or, equivalently, let $(\alpha, \beta, \gamma, \delta)$ be the mixed strategy of each player). Show that a Nash equilibrium is characterized by $\alpha = 1/2-\gamma$, $\beta = \gamma$ (which implies $\delta = \alpha$). It is easy to show that any such Nash equilibrium can be invaded by any distinct strategy $(\alpha', \beta', \gamma', \delta')$ with $\alpha' = 1/2 - \gamma'$, $\beta' = \gamma'$, so there is no evolutionarily stable strategy for throwing fingers.

10.7 Hawks, Doves, and Bourgeois

THEOREM 10.8 *The mixed-strategy equilibrium in the hawk-dove game (§3.10) is an ESS.*

PROOF: The payoff to H is $\alpha(v - w)/2 + (1 - \alpha)v = v - \alpha(v + w)/2$, and the payoff to D is $(1 - \alpha)(v/2) = v/2 - \alpha(v/2)$. These are equated when $\alpha = v/w$, which is < 1 if $w > v$. To show that this mixed-strategy equilibrium is an ESS, note that $\pi_{11} = (v-w)/2$, $\pi_{21} = 0$, $\pi_{22} = v/2$, and $\pi_{12} = v$. Thus $\pi_{11} = (v - w)/2 < 0 = \pi_{21}$ and $\pi_{22} = v/2 < v = \pi_{12}$, so the equilibrium is an ESS. ■

Note that in the hawk-dove-bourgeois game (§6.41), the bourgeois strategy is a strict Nash equilibrium, and hence is an ESS.

10.8 Trust in Networks II

We now show that the completely mixed Nash equilibrium found in trust in networks (§6.23) is not an ESS and can be invaded by trusters. In case you think this means this equilibrium is dynamically unstable, think again! See Trust in Networks III (§12.10).

For specificity, we take $p = 0.8$. You can check that the equilibrium has inspect share $\alpha^* \approx 0.71$ trust share $\beta^* \approx 0.19$, and defect share $\gamma^* \approx 0.10$. The payoff to the equilibrium strategy s is $\pi_{ss} \approx 0.57$. The payoff to trust against the equilibrium strategy is of course $\pi_{ts} = \pi_{ss} \approx 0.57$, but the payoff to trust against itself is $\pi_{tt} = 1$, so trust can invade.

10.9 Cooperative Fishing

In a certain fishing village, two fisherman gain from having the nets put out in the evening. However, the fishermen benefit equally whether or not they share the costs of putting out the nets. Suppose the expected catch is ν, the cost of putting out the nets to each is c_1 if each fisherman does it alone, and the cost to each is $c_2 < c_1$ if they do it together. We assume $\nu/2 > c_1$, so it is worthwhile for a fisherman to put out the nets even if he has to do it alone. But because $c_2 < c_1$, he prefers help. On the other hand, by free-riding on the first fisherman's effort (that is, by not helping), the other fisherman gets $\nu/2$ anyway.

	Put Out	Free Ride
Put Out	$\frac{\nu}{2} - c_2, \frac{\nu}{2} - c_2$	$\frac{\nu}{2} - c_1, \frac{\nu}{2}$
Free Ride	$\frac{\nu}{2}, \frac{\nu}{2} - c_1$	0,0

Figure 10.1. Cooperative fishing

Figure 10.1 shows the normal form game, where each fisherman has the available strategies put out (P) and free ride (F)? We can show there is a unique mixed-strategy equilibrium and that this strategy is an ESS. It is easy to see there are no pure-strategy symmetric equilibria, because $\nu/2 > c_1$. There are two pure-strategy asymmetric equilibria, FP and PF. Consider a mixed-strategy equilibrium where a fraction α of the population plays P. The payoff to P is then

$$\alpha\left(\frac{\nu}{2} - c_2\right) + (1-\alpha)\left(\frac{\nu}{2} - c_1\right) = \frac{\nu}{2} - [\alpha c_2 + (1-\alpha)c_1].$$

The payoff to F is simply $\alpha\nu/2$. Equating the two payoffs, we get

$$\alpha = \frac{\frac{\nu}{2} - c_1}{\frac{\nu}{2} + c_2 - c_1}.$$

Note that we have $0 < \alpha < 1$, so this is a strictly mixed Nash equilibrium. Is this mixed strategy, which we will call M, an evolutionarily stable strategy? We have $\pi_{11} = v/2 - c_2$, $\pi_{21} = v/2$, $\pi_{22} = 0$, and $\pi_{12} = v/2 - c_1$. Thus $\pi_{11} = v/2 - c_2 < v/2 = \pi_{21}$ and $\pi_{22} = 0 < v/2 - c_1 = \pi_{12}$, so the equilibrium is an ESS.

10.10 Evolutionarily Stable Strategies Are Not Unbeatable

It is easy to show that x is an ESS in the game shown in the diagram. We shall see later that it is also asymptotically stable in the replicator dynamic (§12.9). Nevertheless, it is not an *unbeatable strategy*, in the sense of always having the highest payoff when invaded by multiple mutants.

	x	y	z
x	1,1	1,1	0,0
y	1,1	0,0	1,0
z	0,0	0,1	0,0

a. Show that x is an ESS.
b. Show that if a fraction ϵ_y of y-players and a fraction $\epsilon_z > \epsilon_y$ of z-players simultaneously invade, then y has a higher payoff than x.
c. Is the average payoff to the invaders higher than the payoff to x?

To complicate the picture, some game theorists have *defined* the ESS concept as "unbeatability" in the preceding sense. In a famous article, Boyd and Lorberbaum (1987) showed that "no pure strategy is an ESS in the repeated prisoner's dilemma game," and Farrell and Ware (1989) extended this by showing that no mixed strategy using a finite number of pure strategies is an ESS. Finally, Lorberbaum (1994) extended this to all nondeterministic strategies, and Bendor and Swistak (1995) showed that, for a sufficiently low discount rate, no pure strategy is an ESS in any nontrivial repeated game. In all cases, however, the ESS criterion is interpreted as "unbeatability" in the preceding sense. But unbeatability is not a very important concept, because it has no interesting dynamic properties. Be sure you understand how invasion by a pair of pure mutant strategies is not the same as being invaded by a single mixed strategy, and also be able to explain the intuition behind the preceding example.

10.11 A Nash Equilibrium That Is Not an EES

Suppose agents consume each other's products but not their own. An agent can produce one or two units per period at cost 1, and then he meets another consumer. They can agree to exchange either one or two units. The utility of consumption is 2 per unit consumed. The first strategy is thus "exchange equal for equal, but at most one unit of the good," and strategy two is "exchange equal for equal, but at most two units of the good." The payoff

	1	2
1	1,1	1,1
2	1,1	2,2

matrix is shown in the diagram. Show that one of the Nash equilibria consists of evolutionarily stable strategies, but the other does not. What does this say about the ESS criterion and the elimination of weakly dominated strategies?

10.12 Rock, Paper, and Scissors Has No ESS

A Nash equilibrium that is not an ESS may nevertheless be quite important. Consider, for instance, rock, paper, and scissors (§6.25). Show that the unique, completely mixed Nash equilibrium to this game is not an ESS. We will see later (§12.14) that under a replicator dynamic, rock, paper, and scissors traces out closed orbits around the equilibrium (1/3, 1/3, 1/3), as suggested in the example of the lizard *Uta stansburiana* (§6.25).

10.13 Invasion of the Pure-Strategy Mutants

	L	C	R
U	1,1	1,1	1,1
M	1,1	0,0	a,a
D	1,1	a,a	0,0

It is possible that a Nash equilibrium be immune to invasion by any *pure* strategy mutant but not by an appropriate *mixed*-strategy mutant. This is the case with respect to the game in the diagram if one assumes $a > 2$. Here the first strategy is an ESS if only pure-strategy mutants are allowed, but not if mixed strategy mutants are allowed. Show that a mixed strategy using pure strategies 2 and 3 each with probability 1/2 can invade a Nash equilibrium consisting of strategy 1 alone.

Is there an evolutionarily stable strategy using pure strategies 2 and 3? Because mutants are normally considered to be rare, it is often plausible to restrict consideration to single mutant types or to mixed strategies that include only pure strategies used in the Nash equilibrium, plus at most one mutant.

10.14 Multiple Evolutionarily Stable Strategies

Using the matrix in the diagram, show that there are two evolutionarily stable strategies, one using the first two rows and columns, and the second using the second and third strategies. Show that there is also a completely mixed Nash equilibrium that is stable against invasion by pure strategies but is not an ESS.

	L	C	R
U	5,5	7,8	2,1
M	8,7	6,6	5,8
D	1,2	8,5	4,4

Prove the latter either by finding a mixed strategy that does invade. *Hint*: Try one of the evolutionarily stable strategies or use a previously proved theorem. If you want to cheat, look up a paper by Haigh (1975), which develops a simple algorithm for determining whether a Nash equilibrium is an ESS.

10.15 Evolutionarily Stable Strategies in Finite Populations

Consider a population in which agents are randomly paired in each period and each pair plays a 2×2 game. Let $r_{\mu\nu}$ be the payoff to playing μ when your partner plays ν. Let $r(\mu)$ and $r(\nu)$ be the expected payoffs to an μ-type and a ν-type agent, respectively.

Suppose there are n agents, m of which play the "mutant" strategy μ, the rest playing the "incumbent" strategy ν. Then we have

$$
\begin{aligned}
r(\mu) &= \left(1 - \frac{m-1}{n}\right) r_{\mu\nu} + \frac{m-1}{n} r_{\mu\mu} \\
r(\nu) &= \left(1 - \frac{m}{n}\right) r_{\nu\nu} + \frac{m}{n} r_{\nu\mu}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
r(\nu) - r(\mu) = &\left(1 - \frac{m}{n}\right)(r_{\nu\nu} - r_{\mu\nu}) \\
&+ \frac{m}{n}(r_{\nu\mu} - r_{\mu\mu}) + \frac{1}{n}(r_{\mu\mu} - r_{\mu\nu}). \qquad (10.4)
\end{aligned}
$$

We say a strategy ν is *noninvadable* if there is an $\epsilon > 0$ such that for all feasible mutants $\mu \neq \nu$ and all positive m such that $m/n < \epsilon$, $r(\nu) > r(\mu)$. When this condition fails, we say ν is *invadable*. We say a strategy ν is *Nash* if ν is a best reply to itself or, equivalently, if there is a Nash equilibrium in which only ν is played. We say a strategy ν is *evolutionarily stable* if there is a population size n such that ν is noninvadable for all populations of size n or greater.

While it is obviously possible for a Nash strategy to be invadable, it is also possible for a non-Nash strategy to be noninvadable, even by a Nash strategy. To see this, let $r_{\mu\nu} = 0$, $r_{\nu\mu} = n$, $r_{\mu\mu} = n + 1$, and $r_{\nu\nu} = -1$. Then ν is not Nash, because $r_{\mu\nu} > r_{\nu\nu}$, μ is Nash because $r_{\mu\mu} > r_{\mu\nu}$ and $r_{\mu\mu} > r_{\nu\mu}$. But, $r(\nu) - r(\mu) = 1/n > 0$ for any m.

THEOREM 10.9 *Strategy ν is evolutionarily stable if and only if ν is a Nash strategy, and for any μ that is a best reply to ν, ν is a better reply to μ than μ is to itself, or if μ is as good a reply to itself as ν is to μ, then μ is a better reply to itself than μ is to ν.*

PROOF: Suppose ν is evolutionarily stable but is not Nash. Then there is some μ such that $r_{\nu\nu} < r_{\mu\nu}$. Let $m = 1$. Then for sufficiently large n we have $r(\nu) < r(\mu)$ in

$$r(\nu) - r(\mu) = \left(1 - \frac{m}{n}\right)(r_{\nu\nu} - r_{\mu\nu}) + \frac{m}{n}(r_{\nu\mu} - r_{\mu\mu}) + \frac{1}{n}(r_{\mu\mu} - r_{\mu\nu}). \tag{10.5}$$

Hence, ν must be Nash. Now suppose ν is evolutionarily stable and $r_{\nu\nu} = r_{\mu\nu}$ but $r_{\nu\mu} < r_{\mu\mu}$. Equation (10.5) becomes

$$r(\nu) - r(\mu) = \frac{1}{n}\left\{m[r_{\nu\mu} - r_{\mu\mu}] + [r_{\mu\mu} - r_{\mu\nu}]\right\}.$$

Given $\epsilon > 0$, choose $\overline{m}$ so that the term in brackets is negative, and then choose n so that $\overline{m}/n < \epsilon$. Then $r(\nu) < r(\mu)$ for all positive $m \le \overline{m}$, which is a contradiction. So suppose in addition to $r_{\nu\nu} = r_{\mu\nu}$ and $r_{\nu\mu} = r_{\mu\mu}$, we have $r_{\mu\mu} < r_{\mu\nu}$. Then clearly $r(\nu) - r(\mu) = [r_{\mu\mu} - r_{\mu\nu}]/n < 0$, again a contradiction. This proves that the stated conditions are necessary. We can reverse the argument to prove the conditions are sufficient as well. ■

The forgoing conditions are not those of Maynard Smith, which state that ν is evolutionarily stable if and only if ν is Nash and for any μ that is a best reply to n, ν is a better reply to μ than μ is to itself or, equivalently, for any mutant μ, either $r_{\nu\nu} > r_{\mu\nu}$, or $r_{\nu\nu} = r_{\mu\nu}$ and $r_{\nu\mu} > r_{\mu\mu}$. However, we can derive Maynard Smith's conditions by letting $m, n \to \infty$ in (10.4) in such a manner that $m/n = \epsilon$, but the limit argument cannot be used to conclude that $r(\nu) > r(\mu)$ in the "large finite" case.

To see this, note that in the limit we have

$$r(\nu) - r(\mu) = (1-\epsilon)[r_{\nu\nu} - r_{\mu\nu}] + \epsilon[r_{\nu\mu} - r_{\mu\mu}].$$

The conclusion follows immediately from this equation. The limit argument cannot be used to conclude that $r(\nu) > r(\mu)$ in the "large finite" case if $r_{\nu\nu} = r_{\mu\nu}$ and $r_{\nu\mu} = r_{\mu\mu}$.

Let $p = m/n, a = r_{\nu\nu} - r_{\mu\nu}, b = r_{\nu\mu} - r_{\mu\mu} - r_{\nu\nu} + r_{\mu\nu}$, and $c = r_{\mu\mu} - r_{\mu\nu}$. Then (10.4) becomes

$$r(\nu) - r(\mu) = \frac{1}{n}(na + mb + c). \tag{10.6}$$

Suppose ν can be invaded by mutant strategy μ, and the system follows any dynamic in which a strategy with a higher payoff increases in frequency. Then, if $n(a+b) > c$, μ will invade until μ is the largest integer less than $-(na+c)/b$. Otherwise μ will invade until ν is extinct.

In the case of partial invasion in the preceding example, we say μ is *quasi-evolutionarily stable*. Note that μ is quasi-evolutionarily stable with respect to ν for very large n if and only if μ and ν are part of a completely mixed Nash equilibrium (assuming there are no other feasible pure strategies).

10.16 Evolutionarily Stable Strategies in Asymmetric Games

Many situations can be modeled as evolutionary games, except for the fact that the two players are not interchangeable. For instance, in one-card two-round poker with bluffing (§6.21), the player going first has a different set of strategies from the player going second. Yet, despite the lack of symmetry, we simulated the game quite nicely as an evolutionary game. Analogous situations include interactions between predator and prey, boss and worker, male and female, incumbent and intruder, among a host of others.

This is not simply a technicality; it makes no sense to say that a "mutant meets its own type" when the game is asymmetric, so the ESS criterion has no meaning. The obvious way around this problem is to define a homogeneous set of players who in each period are paired randomly, one of the pair being randomly assigned to be player 1, and the other to be player 2. We may call this the "symmetric version" of the asymmetric evolutionary game. However, *an evolutionarily stable strategy in the symmetric version*

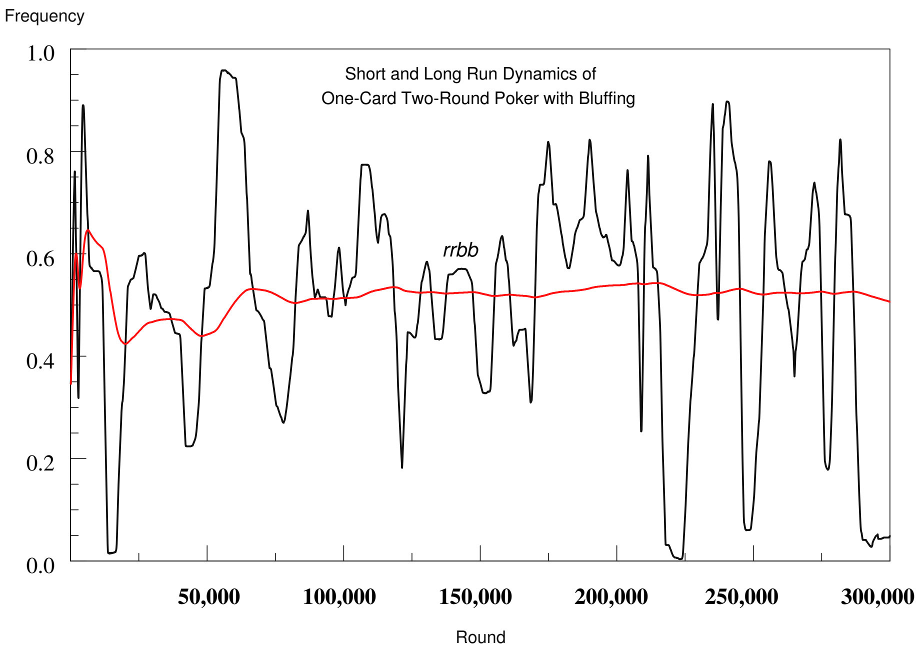


Figure 10.2. An agent-based model of one-card two-round poker with bluffing

of an asymmetric evolutionary game must be a strict Nash equilibrium; that is, each type in the asymmetric game must use exactly *one* pure strategy (Selten 1980). To see this, suppose there is a Nash equilibrium $u = (\sigma_1, t_2)$ of the symmetric version, where a player uses strictly mixed strategy σ_1 when assigned to be player 1 and uses t_2 (pure or mixed) when assigned to be player 2. Consider a mutant that uses $v = (s_1, t_2)$, where s_1 is a pure strategy that appears with positive weight in σ_1. Then v does as well against u as u does against itself, and v does as well against v as u does against v. All this is true because the payoff to s_1 against t_2 in the asymmetric game is equal to the payoff to σ_1 against t_2 by the fundamental theorem (§3.6).

It follows that *mixed-strategy Nash equilibria in asymmetric evolutionary games are never evolutionarily stable in the symmetric version of the game*. As we shall see later, this situation reflects the fact that mixed-strategy Nash equilibria in asymmetric evolutionary games with a replicator dynamic are never asymptotically stable (§12.17). Some game theorists consider this a weakness of evolutionary game theory (Mailath 1998), but in fact it reflects a deep and important regularity of social interaction. In asymmetric evolutionary games, the frequency of different types of behavior goes through periodic cycles through time.

For a dramatic example of this important insight, we return to our model of one-card two-round poker with bluffing (figure 6.6). In this agent-based model, I lowered the mutation rate to 1% and ran the model for 300,000 periods. The results are shown in figure 10.2 for one of the player 1 types, $rrbb$ (bluff all the way). Note that the *average frequency* of each strategy settles down to the theoretically predicted equilibrium value, but the *period-to-period frequencies* fluctuate wildly in the medium run. Strategy $rrbb$, which has the equilibrium frequency of about 50%, sometimes goes for thousands of periods with frequency under 5% or over 90%.

11

Dynamical Systems

History is Spirit at war with itself.
Georg Wilhelm Freidrich Hegel

We have studied Nash equilibria of games, but *do* games reach Nash equilibrium and, if so, by what process? If there are several Nash equilibria, to which one does the game go? Indeed, what are Nash equilibria equilibria of? To answer these questions we will study the behavior of *dynamical systems* that are generally not in equilibrium but which, under appropriate conditions, approach a state of equilibrium over time, or orbit equilibria the way planets orbit the sun, or have some other love-hate relationship with equilibria (e.g., strange attractors).

We can apply several analytical tools in treating strategic interactions as dynamical systems, including difference equations, stochastic processes (such as Markov chains and diffusion processes), statistical mechanics, and differential equations. The differential equation approach is the most basic and has the quickest payoff, so that is what we will develop in this chapter.

11.1 Dynamical Systems: Definition

Suppose $x = (x_1, \ldots, x_n)$ is a point in n-dimensional space $\mathbf{R}^n$ that traces out a curve through time. We can describe this as

$$x = x(t) = (x_1(t), \ldots, x_n(t)) \qquad \text{for } -\infty < t < \infty.$$

Often we do not know $x(t)$ directly, but we do know the forces determining its rate and direction of change in some region of $\mathbf{R}^n$. We thus have

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \mathbf{x} \in \mathbf{R}^n, \tag{11.1}$$

where the "dot" indicates the derivative with respect to t, so $\dot{\mathbf{x}} = dx/dt$. We always assume f has continuous partial derivatives. If we write these vector equations out in full, we get

$$\frac{dx_1}{dt} = f^1(x_1, \ldots, x_n),$$
$$\frac{dx_2}{dt} = f^2(x_1, \ldots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = f^n(x_1, \ldots, x_n),$$

We call this a set of *first-order ordinary differential equations* in n unknowns. It is "first-order" because no derivative higher than the first appears. It is "ordinary" as opposed to "partial" because we want to solve for a function of the single variable t, as opposed to solving for a function of several variables.

We call $\mathbf{x}(t)$ a *dynamical system* if it satisfies such a set of ordinary differential equations, in the sense that $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ for t in some (possibly infinite) interval. A *fixed point*, also called a *critical point*, or a *stationary point*, is a point $\mathbf{x}^* \in \mathbf{R}^n$ for which $f(\mathbf{x}^*) = 0$.

11.2 Population Growth

Suppose the rate of growth of fish in a lake is r. Then the number y of fish in the lake is governed by the equation

$$\dot{y} = ry.$$

We can solve this equation by "separation of variables," bringing all the expressions involving t on the right, and all the expressions involving y on the left. This is not possible for just any differential equation, of course, but it is possible in this case. This gives

$$\frac{dy}{y} = r\,dt.$$

Now we integrate both sides, getting $\ln y = rt + a$, where a is a constant of integration. Taking the antilogarithm of both sides, we get

$$y = be^{rt},$$

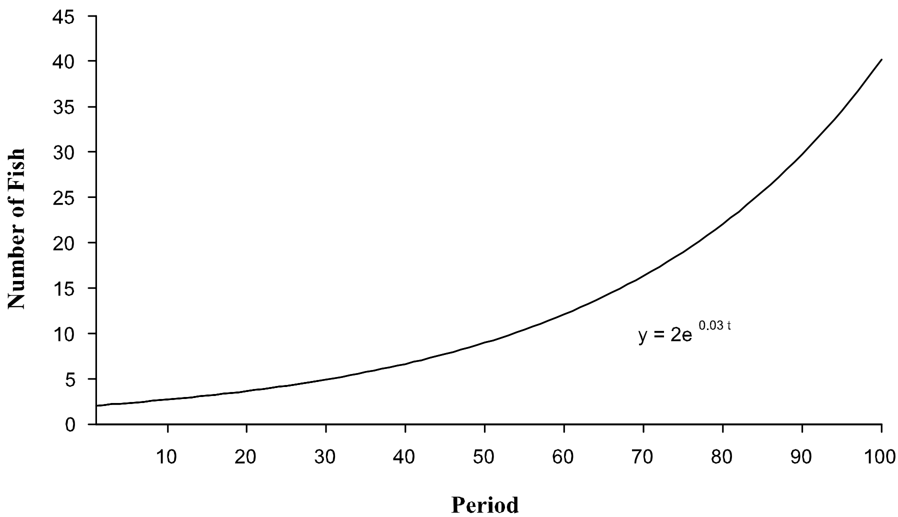


Figure 11.1. The exponential growth of fish in lake. The initial population is $y_o = 2$, and the rate of growth is $r = 3.0\%$.

where $b = e^a$ is another constant of integration.

We determine the constant of integration by noting that if the number of the fish in the lake at time $t = 0$ is y_o, then we must have $b = y_o$. This gives the final solution

$$y = y_o e^{rt}. \tag{11.2}$$

This function is graphed in figure 11.1.

11.3 Population Growth with Limited Carrying Capacity

Equation (11.2) predicts that the fish population can grow without bounds. More realistically, suppose that the more fish, the lower the rate of growth of fish. Let η be the "carrying capacity" of the lake—the number of fish such that the rate of growth of the fish population is zero. The simplest expression for the growth rate of the fish population, given that the growth rate is r when y is near zero, is then $r(1 - y/\eta)$. Our differential equation then becomes

$$\dot{y} = r\left(1 - \frac{y}{\eta}\right)y \qquad \eta, r > 0. \tag{11.3}$$

Note that the dynamical system given by this equation has two fixed points: $y^* = 0$, where the fish population is zero, and $y^* = \eta$, where the population is just equal to the carrying capacity.

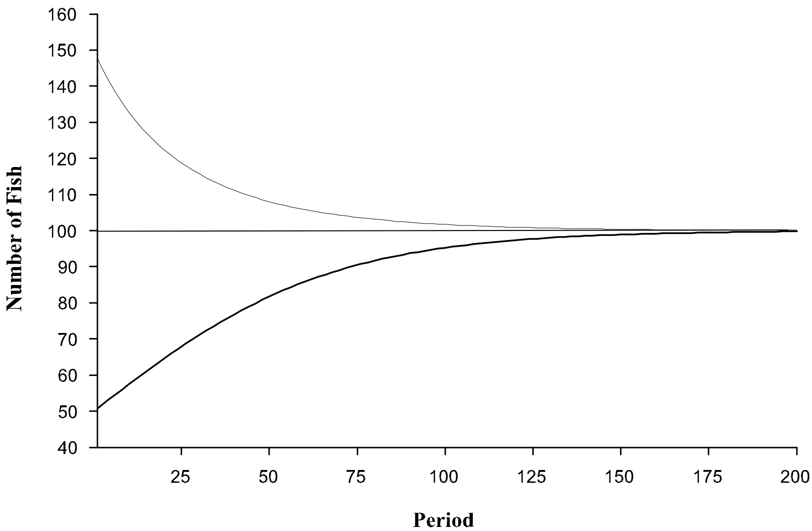


Figure 11.2. Population growth with limited carrying capacity

To solve the equation, we separate variables, getting

$$\frac{dy}{y(\eta - y)} = \frac{r}{\eta}dt.$$

We now integrate both sides, getting

$$\int \frac{dy}{y(\eta - y)} = \frac{r}{\eta}t + a, \tag{11.4}$$

where a is a constant of integration. We use the method of partial fractions to write

$$\frac{1}{y(\eta - y)} = \frac{1}{\eta}\left[\frac{1}{\eta - y} + \frac{1}{y}\right].$$

Thus, we have

$$\int \frac{dy}{y(\eta - y)} = \frac{1}{\eta}\left[\int \frac{dy}{\eta - y} + \int \frac{dy}{y}\right]$$
$$= \frac{1}{\eta}\ln\frac{y}{\eta - y}.$$

Substituting into (11.4), we get

$$\ln\frac{y}{\eta - y} = rt + a\eta.$$

Taking antilogarithms of both sides, this becomes

$$\frac{y}{\eta - y} = be^{rt},$$

where $b = e^{a\eta}$ is another constant of integration. If the number of fish in the lake at time $t = 0$ is y_o, then we must have $b = y_o/(\eta - y_o)$, which can be either positive or negative, depending on whether the initial fish population is larger or smaller than the stationary population size η.

Now we can solve this equation for y, getting

$$y = \frac{\eta}{Ae^{-rt} + 1},$$

where $A = (\eta - y_o)/y_o$. Note that this equation predicts a smooth movement from disequilibrium to stationarity as $t \to \infty$. A picture of the process is given in figure 11.2.

11.4 The Lotka-Volterra Predator-Prey Model

Foxes eat rabbits. Suppose we normalize the rabbit population at a point in time to a fraction x of its maximum, given the carrying capacity of its habitat when foxes are absent, and suppose the fox population at a point in time is a fraction y of its maximum, given the carrying capacity of its habitat when there is an unlimited supply of rabbits. Suppose foxes are born at the rate $\delta_1 x$ but die at the rate $\gamma_1(1-x)$. We then have $\dot{y}/y = \delta_1 x - \gamma_1(1-x)$, which we can write as

$$\dot{y} = \delta y(x - \gamma), \qquad \delta > 0, 1 > \gamma > 0, \tag{11.5}$$

where we have written $\delta = \delta_1 + \gamma_1$ and $\gamma = \gamma_1/(\delta_1 + \gamma_1)$. Equation (11.5) expresses the rate of growth $\dot{y}/y$ as a function of the frequency of rabbits.

Suppose the natural rate of growth of rabbits is $g > 0$, but predation reduces the rate of growth by μy, so

$$\dot{x} = x(g - \mu y). \tag{11.6}$$

Now, (11.5) and (11.6) form a pair of differential equations in two unknowns (x and y), the solution to which is a dynamical system known as the *Lotka-Volterra predator-prey model.*

How do we solve this equation? There is no solution in closed form (e.g., using polynomials, trigonometric functions, logarithms, and exponentials). We can, however, discover the properties of such equations without solving them explicitly.

We begin such an analysis with a *phase diagram* of the differential equations. The phase diagram for the Lotka-Volterra model is depicted in figure 11.3.

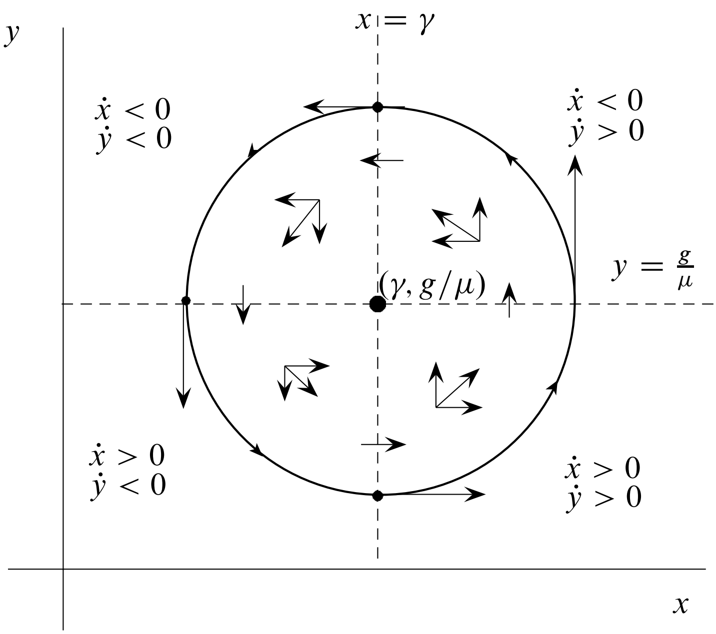


Figure 11.3. Phase diagram of Lotka-Volterra system

The horizontal dotted line represents the condition $dx/dt = 0$, and the vertical dotted line represents the condition $dy/dt = 0$. The fixed point is at $(\gamma, g/\mu)$, where the two intersect. The little arrows show in which direction the flow of the dynamical system moves for that particular point (x, y). The arrows point northward when $dy/dt > 0$ and southward when $dy/dt < 0$, and they point eastward when $dx/dt > 0$ and westward when $dx/dt < 0$. The arrows are vertical where $dx/dt = 0$ because motion is purely north-south instantaneously at such a point, and are horizontal where $dy/dt = 0$, because motion is purely east-west at such a point. In each of the four quadrants marked off by the dotted lines, the direction of the flow is qualitatively similar. Thus to the northeast of the fixed point, the flow is northwest; to the northwest, the flow is southwest; to the southwest, the flow is southeast; and to the southeast of the fixed point, the flow is to the northeast. So it is clear that the flow circles counterclockwise about the fixed point. However, we cannot tell a priori whether the flow circles into the fixed point, circles outward to infinity, or forms closed circuits about the fixed point.

To show that the Lotka-Volterra has closed orbits (§11.5), we find a function that is constant on any trajectory of the dynamical system and show that this function is monotonic (strictly increasing or decreasing) along a ray starting from the fixed point and pointing northeast.[1]

Suppose we have such a function f and consider a path starting at a point $\mathbf{x}$ on the ray, making one complete revolution around the fixed point and hitting the ray again, say at $\mathbf{y}$. Because f is constant on the path, $f(\mathbf{x}) = f(\mathbf{y})$. But because f is monotonic on the ray, we must have $\mathbf{x} = \mathbf{y}$, so the path is a closed orbit (§11.5). First, we eliminate t from the equations for $\dot{x}$ and $\dot{y}$ by dividing the first by the second, getting

$$\frac{dy}{dx} = \frac{\delta y(x-\gamma)}{x(g-\mu y)}.$$

Now we separate variables, pulling all the x's to the right, and all the y's to the left:

$$\frac{g-\mu y}{y}dy = \frac{\delta(x-\gamma)}{x}dx.$$

[1]We say a function $f(x)$ is (1) *increasing* if $x > y$ implies $f(x) \geq f(y)$; (2) *strictly increasing* if $x > y$ implies $f(x) > f(y)$; (3) *decreasing* if $x > y$ implies $f(x) \leq f(y)$; and (4) *strictly decreasing* if $x > y$ implies $f(x) < f(y)$.

Now we integrate both sides, getting

$$g \ln y - \mu y = \delta x - \delta\gamma \ln x + C,$$

where C is an arbitrary constant of integration. Bringing all the variables over to the left and taking the antilogarithm, we get

$$y^g x^{\delta\gamma} e^{-(\mu y + \delta x)} = e^C. \tag{11.7}$$

So now we have an expression that is constant along any trajectory of the Lotka-Volterra dynamical system.

Now, consider a ray (x, y) that starts at the fixed point $(\gamma, g/\mu)$ and moves to the northeast in a direction heading away from the origin. We can write this as $x = \gamma s$, $y = (g/\mu)s$, where s is a parameter measuring the distance from the fixed point. Note that when $s = 1$, (x, y) is at the fixed point. Substituting in (11.7), we get

$$\left(\frac{g}{\mu}\right)^g \gamma^{\delta\gamma} s^{g+\delta\gamma} e^{-(g+\delta\gamma)s} = e^C.$$

This looks forbidding, but it's really not. We pull the first two terms on the left over to the right, and then take the $(g + \delta\gamma)$-th root of both sides. The right-hand side is a complicated constant, which we can abbreviate by D, and the left is just se^{-s}, so we have

$$se^{-s} = D. \tag{11.8}$$

If we can show that the left-hand side is strictly decreasing for $s > 1$, we are done, because then any $s > 1$ that satisfies (11.8) must be unique. We take the derivative of the left-hand side, getting

$$e^{-s} - se^{-s} = (1 - s)e^{-s},$$

which is negative for $s > 1$. This shows that the dynamical system moves in closed orbits (§11.5) around the fixed point.

It follows from this analysis that if the system begins out of equilibrium, both the fraction of rabbits and foxes will go through constant-amplitude oscillations around their equilibrium values forever. We shall later characterize this as an *asymmetric evolutionary game* (§12.17) for which this oscillatory behavior is quite typical.

11.5 Dynamical Systems Theory

With these examples under our belt, we can address the basic theory of dynamical systems (a.k.a. differential equations).[2]

Suppose a dynamical system is at a point $\mathbf{x}_o$ at time t_o. We call the locus of points through which the system passes as $t \to \infty$ the *forward trajectory* of the system through $\mathbf{x}_o$, or the *trajectory* of the system starting at $\mathbf{x}_o$. The *backward trajectory* of the system through $\mathbf{x}_o$ is the locus of points through which the system passes as $t \to -\infty$. The forward and backward trajectories are together called the *trajectory* through $\mathbf{x}_o$.

Clearly if a dynamical system is at a fixed point $\mathbf{x}^*$, it will stay there forever, so the trajectory starting at $\mathbf{x}^*$ is simply $\mathbf{x}^*$ itself. However, if we perturb the system a little from $\mathbf{x}^*$ by choosing a new initial point $\mathbf{x}_o$ at time $t = 0$, there are several things that can happen. We begin with a couple of definitions. If $\mathbf{x} \in \mathbf{R}^n$, and $r > 0$, we define a *ball of radius* r around $\mathbf{x}$, which we write $B_r(\mathbf{x})$, as the set of points $\mathbf{y} \in \mathbf{R}^n$ whose distance from $\mathbf{x}$ is less than r. We define a *neighborhood* of $\mathbf{x}$ to be any subset of $\mathbf{R}^n$ that contains some ball around $\mathbf{x}$. Finally, we say a set in $\mathbf{R}^n$ is an *open set* if it is a neighborhood of each of its points. Note that a set is open if and only if it contains a ball of some positive radius around each of its points.

We define an ϵ*-perturbation* of the dynamical system at a fixed point $\mathbf{x}^*$ to be a trajectory of the system starting at some $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, where $\epsilon > 0$ and $\mathbf{x}_o \neq \mathbf{x}^*$. We say a trajectory $\mathbf{x}(t)$ *approaches* $\mathbf{x}^*$ if $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. We say a trajectory $\mathbf{x}(t)$ ϵ*-escapes* $\mathbf{x}^*$ if there is some t_o such that $\mathbf{x}(t) \notin B_\epsilon(\mathbf{x}^*)$ for $t > t_o$; that is, after some point in time, the trajectory never gets closer than ϵ to $\mathbf{x}^*$.

If there is some $\epsilon > 0$ such that for any $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, the trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$, we say the fixed point at $\mathbf{x}^*$ is *asymptotically stable*. The set of points $\mathbf{x}_o \in \mathbf{R}^n$ such that a trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$ is called the *basin of attraction* of the fixed point $\mathbf{x}^*$. If every point where the differential equation is defined is in the basin of attraction of $\mathbf{x}^*$, we say the fixed point is *globally stable*.

If $\mathbf{x}^*$ is not asymptotically stable, but for any ball $B_\epsilon(\mathbf{x}^*)$ there is another ball $B_\delta(\mathbf{x}^*)$ such that for any point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$, the trajectory starting at $\mathbf{x}_o$

[2] There are many excellent texts on differential equations. Some of my favorites are Perko 1991, Hirsch and Smale 1974, Epstein 1997, and Hofbauer and Sigmund 1998. The last of these is a beautiful summary of evolutionary dynamics.

never leaves $B_\epsilon(\mathbf{x}^*)$, we say the fixed point at $\mathbf{x}^*$ is *neutrally stable*. Neutral stability means that a sufficiently small perturbation about the fixed point never leads the system too far away from the fixed point. A special case is when any trajectory through $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$ is a *closed orbit*; that is, the trajectory starting at $\mathbf{x}_o$ eventually returns to $\mathbf{x}_o$.

If $\mathbf{x}^*$ is neither asymptotically stable nor neutrally stable, we say $\mathbf{x}^*$ is *unstable*. Thus, $\mathbf{x}^*$ is unstable if there is an $\epsilon > 0$ such that for any ball $B_\delta(\mathbf{x}^*)$, there is a point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$ such that the trajectory starting at $\mathbf{x}_o$ ϵ-escapes $\mathbf{x}^*$.

11.6 Existence and Uniqueness

THEOREM 11.1 Existence, Uniqueness, and Continuous Dependence on Initial Conditions. *Suppose that f in equation (11.1) has continuous derivatives on an open set D containing a point x_o. Then there is some interval $I = [-t_o, t_o]$ and a unique trajectory $\mathbf{x}(t)$ satisfying (11.1) defined on I with $\mathbf{x}(0) = x_o$. Moreover, $\mathbf{x}(t)$ depends smoothly upon x_o in the following sense: there is some $\delta > 0$, and a unique function $\mathbf{x}(t, \mathbf{y})$ that satisfies (11.1) on an interval $[-t_1, t_1]$ with $\mathbf{x}(0, \mathbf{y}) = \mathbf{y}$, for all $\mathbf{y} \in B_\delta(\mathbf{x}_o)$. Moreover, $\mathbf{x}(t, \mathbf{y})$ has continuous partial derivatives, and continuous second partial derivatives with respect to t.*

This theorem says that if $f(\mathbf{x})$ is suitably well behaved, the dynamical system (11.1) has a unique, twice-differentiable trajectory through each point $\mathbf{x}_o$, and the trajectory varies differentiably as we vary $\mathbf{x}_o$. In particular, two trajectories can never cross.

THEOREM 11.2 Continuous Dependence on Parameters. *Let $\mu \in \mathbf{R}^k$ be a set of k parameters, and suppose $f(\mathbf{x}, \mu)$ has continuous partial derivatives in a neighborhood of $(\mathbf{x}_o, \mu_o) \in \mathbf{R}^{n+k}$. Then there is a $t_1 > 0$, a $\delta > 0$, an $\epsilon > 0$, and a unique function $\mathbf{x}(t, \mathbf{y}, \mu)$ that satisfies*

$$\dot{\mathbf{x}} = f(\mathbf{x}(t, \mathbf{y}, \mu), \mu) \tag{11.9}$$

with $\mathbf{x}(0, \mathbf{y}, \mu) = \mathbf{y}$, for $t \in [-t_1, t_1]$, $\mathbf{y} \in B_\delta(\mathbf{x}_o)$, and $\mu \in B_\epsilon(\mu_o)$. Moreover, $\mathbf{x}(t, \mathbf{y}, \mu)$ has continuous partial derivatives.

This theorem says that if $f(\mathbf{x}, \mu)$ is suitably well behaved, the trajectories of the dynamical system (11.9) vary differentiably as we vary the parameters μ.

11.7 The Linearization Theorem

Given a dynamical system (11.1), we define the *Jacobian* of f at a point $\mathbf{x} \in \mathbf{R}^n$ to be the $n \times n$ matrix $Df(\mathbf{x}) = (a_{ij})$ where

$$a_{ij} = \frac{\partial f^i}{\partial x_j}(\mathbf{x}) \qquad \text{for } i, j = 1, \ldots, n.$$

Suppose $\mathbf{x}^*$ is a fixed point of the dynamical system (11.1), and let $A = Df(\mathbf{x}^*)$ be the Jacobian of the system at $\mathbf{x}^*$. We define the *linearization* of the original dynamic system (11.1) at $\mathbf{x}^*$ to be the linear dynamical system

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\end{aligned}$$

or, more succinctly,

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad \mathbf{x} \in \mathbf{R}^n. \tag{11.10}$$

Note that the fixed point of linearization has been moved from $\mathbf{x}^*$ to 0 (we could keep the fixed point at $\mathbf{x}^*$ by defining the linearization as $\dot{\mathbf{x}} = A(\mathbf{x} - \mathbf{x}^*)$, but this needlessly complicates the notation).

We define the eigenvalues of the matrix A in (11.10) to be the set of (possibly complex) numbers λ that satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{11.11}$$

for some vector $\mathbf{x} \neq 0$. This equation can be rewritten as $(A - \lambda I)x = 0$, which holds for $x \neq 0$ only if the determinant of $A - \lambda I$ is zero. This determinant is given by

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

Because this is a polynomial of degree n, we know from linear algebra—you can refer to Hirsch and Smale (1974) for details—there are exactly n (possibly complex) eigenvalues, if we account properly for their "multiplicity." At any rate, we shall only deal in this book with dynamical systems in one or two dimensions, where the calculation of the eigenvalues is very simple.

We call the dynamical system (11.1) *hyperbolic* at a fixed point $\mathbf{x}^*$ if every eigenvalue of the Jacobian matrix $Df(x^*)$ has nonzero real part. We then have the following.

THEOREM 11.3 Linearization Theorem. *Suppose the dynamical system (11.1) is hyperbolic at fixed point* $\mathbf{x}^*$*. Then* $\mathbf{x}^*$ *is asymptotically stable if its linearization (11.10) is asymptotically stable. Also, if* $\mathbf{x}^*$ *is asymptotically stable, then no eigenvalue of the Jacobian matrix* $Df(x^*)$ *has strictly positive real part.*

When no eigenvalue has a strictly positive real part at $\mathbf{x}^*$, but one or more eigenvalues have a zero real part, $\mathbf{x}^*$ may be either stable or unstable.

11.8 Dynamical Systems in One Dimension

If $n = 1$, equation (11.1) becomes

$$\dot{x} = f(x), \qquad x \in \mathbf{R}. \tag{11.12}$$

Suppose $f(x)$ has the shape shown in figure 11.4. We call a diagram like the one in figure 11.4 a *phase diagram*—a depiction of the state space of the dynamic system with little arrows showing the direction of movement of the system at representative points in the state space.

It is obvious that fixed points x_a and x_c are stable, whereas fixed point x_b is unstable. To see that this agrees with the linearization theorem 11.3, note that the Jacobian at a point x is just the one-dimensional matrix $(f'(x))$, and the eigenvalue of this matrix is just $f'(x)$. Thus, the system has a fixed point at x^* if $f(x^*) = 0$, and this fixed point is hyperbolic if $f'(x^*) \neq 0$. Note that in figure 11.4 all three fixed points are hyperbolic, because $f'(x) < 0$ at x_a and x_c, and $f'(x) > 0$ at x_b. The linearization of (11.12) at fixed point x^* is $\dot{x} = f'(x^*)x$, which has solution

$$x(t) = x(0)e^{f'(x^*)t},$$

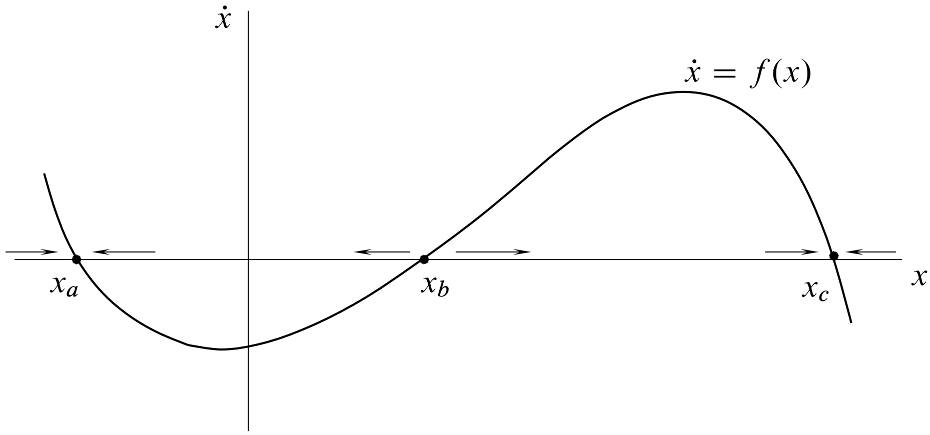


Figure 11.4. Stable and unstable fixed points

which is obviously stable when $f'(x^*) < 0$ and unstable when $f'(x^*) > 0$. Applying the linearization theorem, we find that the fixed points at x_a and x_c are stable, whereas fixed point x_b is unstable.

We can also apply the linearization theorem to the population growth with limited carrying capacity dynamical system (11.3). This system has two fixed points, $y = 0$ and $y = \eta$. The Jacobian at a point y is just

$$r\left(1 - \frac{2y}{\eta}\right),$$

which has the value $r > 0$ at $y = 0$ and the value $-r < 0$ at $y = \eta$. The linearization of the dynamical system at $y = 0$ is thus

$$\dot{y} = ry,$$

which has the solution $y = ae^{rt}$. This explodes to infinity, so $y = 0$ is unstable.

The linearization of the dynamical system at $y = \eta$ is

$$\dot{y} = -ry$$

with solution $y = ae^{-rt}$. This converges to zero so $y = \eta$ is an asymptotically stable fixed point.

We conclude that in this model, the fixed point $y = \eta$ is globally stable, and is approached exponentially from any $y > 0$.

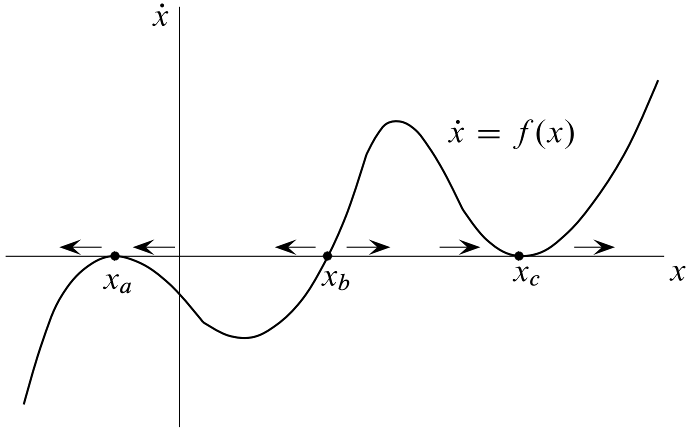


Figure 11.5. The phase diagram of nonhyperbolic fixed points in a one-dimensional system. Note that x_a is unstable to the left and locally stable to the right, B is unstable, and C is locally stable to the left, unstable to the right.

If (11.12) is not hyperbolic at a fixed point x^* (that is, $f'(x^*) = 0$), then we cannot apply the linearization theorem. We illustrate this in the phase diagram shown in figure 11.5. Here the fixed point at x_b is unstable, just as before. But at fixed points x_a and x_c, the Jacobians are zero (that is, $f'(x_a) = f'(x_c) = 0$), so the fixed points are not hyperbolic. Note that linearization has the solution $x(t) = 0$, which of course tells us nothing about the dynamical system. In fact, we can easily see that the system approaches the fixed point from the right of x_a but ϵ-escapes the fixed point to the left of x_a for small ϵ. At x_c the system approaches the fixed point from the left of x_c but ϵ-escapes the fixed point from right of x_c for small ϵ.

11.9 Dynamical Systems in Two Dimensions

We can write the equations for a dynamical system in two dimensions as

$$\begin{aligned}\dot{x} &= f(x, y),\\ \dot{y} &= g(x, y).\end{aligned} \tag{11.13}$$

Suppose this has a fixed point at a point (x^*, y^*). We can write the Jacobian of the system at this point as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}.$$

The linearization of the dynamical system about (x^*, y^*) can then be written as

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{11.14}$$

where $\mathbf{x}$ is the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Let α = trace of $A/2 = (a_{11} + a_{22})/2$, let $\beta = \det(A) = a_{11}a_{22} - a_{21}a_{12}$, and let $\gamma = \alpha^2 - \beta$, the *discriminant* of A. It is easy to check that the eigenvalues of (11.14) are $\lambda_1 = \alpha + \sqrt{\gamma}$ and $\lambda_2 = \alpha - \sqrt{\gamma}$. Note that if $\beta = 0$, the two equations in (11.14) are multiples of each other, so the system is indeterminate. Thus, we assume that $\beta \neq 0$, which implies that (11.14) has the unique critical point $(0, 0)$. We have the following.

THEOREM 11.4 *If $\gamma > 0$, the dynamical system (11.14) is governed by the equations*

$$x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t} \tag{11.15}$$

$$y(t) = ce^{\lambda_1 t} + de^{\lambda_2 t} \tag{11.16}$$

for constants a, b, c, and d that depend on the initial conditions. It follows that the dynamical system is hyperbolic with distinct eigenvalues $\lambda_1 = \alpha + \sqrt{\gamma}$ and $\lambda_2 = \alpha - \sqrt{\gamma}$.

a. *If $\lambda_1, \lambda_2 < 0$, which occurs when $\alpha < 0$ and $\beta > 0$, the fixed point at (0,0) is globally stable. This is called a* stable node.
b. *If $\lambda_1, \lambda_2 > 0$, which occurs when $\alpha, \beta > 0$, the fixed point at (0,0) is unstable and every trajectory starting at a nonfixed point (x_o, y_o) approaches ∞ as $t \to \infty$. This is called an* unstable node .
c. *If $\lambda_1 > 0$ and $\lambda_2 < 0$, which occurs when $\beta < 0$, the system is unstable, but if (x_o, y_o) lies on the straight line*

$$(a_{11} - \lambda_2)x + a_{12}y = 0,$$

the system converges to the fixed point as $t \to \infty$. This line is called the stable manifold *of the system. Also, if (x_o, y_o) lies on the straight line*

$$(a_{11} - \lambda_1)x + a_{12}y = 0,$$

the system converges to the fixed point as $t \to -\infty$. *This line is called the* unstable manifold *of the system. The fixed point is called a* saddle point.

d. *If the system starts at* $(x(0), y(0)) = (x_o, y_o)$, *then*

$$a = \frac{(a_{11} - \lambda_2)x_o + a_{12}y_o}{2\sqrt{\gamma}}, b = -\frac{(a_{11} - \lambda_1)x_o + a_{12}y_o}{2\sqrt{\gamma}} \tag{11.17}$$
$$c = \frac{a_{21}x_o + (a_{22} - \lambda_2)y_o}{2\sqrt{\gamma}}, d = -\frac{a_{21}x_o + (a_{22} - \lambda_1)y_o}{2\sqrt{\gamma}}.$$

The proof of this theorem, which is left to the reader, is simple. The main point is to show that the answer satisfies (11.14). By theorem 11.1, there are no other solutions. The constants a, b, c, and d in (11.17) are solutions to the four equations

$$x(0) = x_o = a + b$$
$$y(0) = y_o = c + d$$
$$\dot{x}(0) = a_{11}x_o + a_{12}y_o = a\lambda_1 + b\lambda_2$$
$$\dot{y}(0) = a_{21}x_o + a_{22}y_o = c\lambda_1 + d\lambda_2,$$

which follow directly from (11.14), (11.15), and (11.16).

The phase diagram for a stable node in the case $\gamma > 0$ is elementary: the trajectories all converge to the fixed point; in the case of an unstable node, the trajectories all move away from the fixed point. But the case of a saddle point is more interesting and is depicted in figure 11.6.

THEOREM 11.5 *If* $\gamma < 0$, *the dynamical system (11.14) is satisfied by the equations*

$$x(t) = e^{\alpha t}[a \cos \omega t + b \sin \omega t] \tag{11.18}$$
$$y(t) = e^{\alpha t}[c \cos \omega t + d \sin \omega t], \tag{11.19}$$

where $\omega = \sqrt{-\gamma}$. *The system is hyperbolic if and only if* $\alpha \neq 0$, *and its eigenvalues are* $\lambda = \alpha \pm \omega\sqrt{-1}$, *so trajectories circle around the fixed point with period* $2\pi/\omega$.

a. *If* $\alpha < 0$, *the fixed point is globally stable. This is called a* stable focus.
b. *If* $\alpha > 0$, *the fixed point is unstable. This is called an* unstable focus.

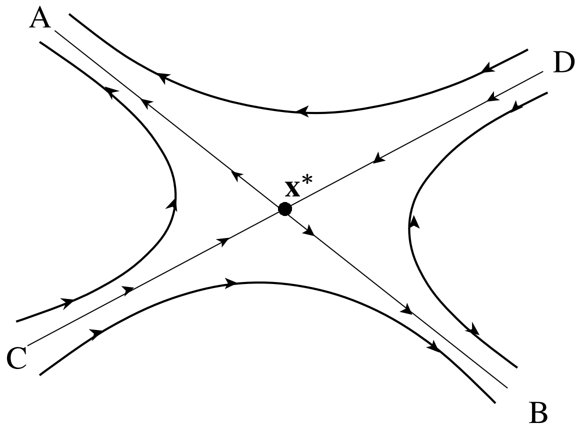


Figure 11.6. The stable and unstable manifolds of a saddle point. Note that AB is the unstable manifold and CD is the stable manifold of the saddle point.

c. *If* $\alpha = 0$*, the fixed point is neutrally stable, and all trajectories are closed orbits. This is called a* center.

d. *If the system starts at* $(x(0), y(0)) = (x_o, y_o)$*, then*

$$a = x_o, \quad b = \frac{(a_{11} - a_{22})x_o + 2a_{12}y_o}{2\omega} \tag{11.20}$$
$$c = y_o, \quad d = \frac{2a_{21}x_o + (a_{22} - a_{11})y_o}{2\omega}.$$

Note that the coefficients in (11.20) are derived from the solutions to (11.18) and (11.19) for $t = 0$. To understand why the trajectories circle the critical point $(0, 0)$, note that from elementary trigonometry, we have

$$\begin{aligned} a\cos\omega t + b\sin\omega t &= a_o\left(a'\cos\omega t + b'\sin\omega t\right) \\ &= a_o\left(\cos\theta\cos\omega t + \sin\theta\sin\omega t\right) \\ &= a_o\cos(\omega t - \theta) \end{aligned}$$

where $a_o = \sqrt{a^2 + b^2}$, $a' = a/a_o$, $b' = b/a_o$, and $\theta = \arccos a'$. A similar equation holds for both $\dot{x}$ and $\dot{y}$, so the trajectory of (11.20) is an ellipse.

THEOREM 11.6 *If* $\gamma = 0$*, the dynamical system (11.14) satisfies the equations*

$$x(t) = e^{\alpha t}(at + b) \tag{11.21}$$
$$y(t) = e^{\alpha t}(ct + d), \tag{11.22}$$

and if the system starts at $(x(0), y(0)) = (x_o, y_o)$*, we have*

$$a = (a_{11} - \alpha)x_o + a_{12}y_o \quad b = x_o \tag{11.23}$$

$$c = a_{21}x_o + (a_{22} - \alpha)y_o \quad d = y_o. \tag{11.24}$$

The system has the single eigenvalue α*, and it is hyperbolic if and only if* $\alpha \neq 0$.

a. *If* $\alpha > 0$*, the origin is an unstable node.*
b. *If* $\alpha < 0$*, the origin is an stable node.*

figure 11.7 summarizes the behavior of the linear two-dimensional system of differential equations. Note that we have not said what happens when $\beta = \det(A) = 0$. This is called a *degenerate critical point* and is not of much interest. It means one of the differential equations is a multiple of the other, so there is really only one equation.

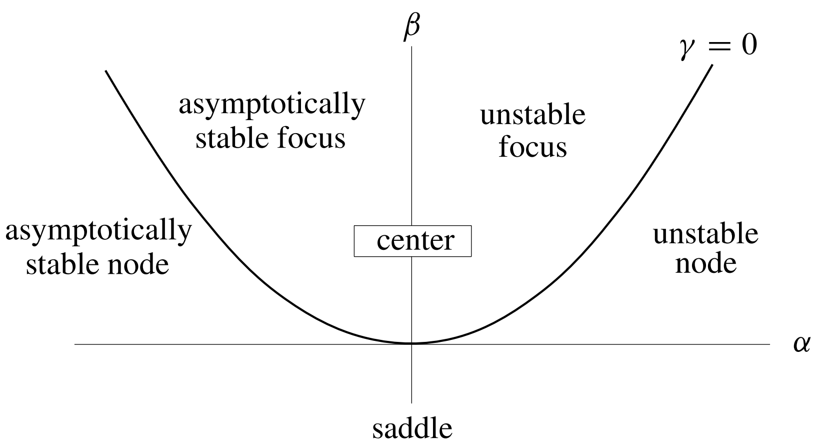


Figure 11.7. A summary of types of fixed points in a two-dimensional dynamical system. The discriminant γ is negative within the parabola $\gamma = 0$ and positive without.

11.10 Exercises in Two-Dimensional Linear Systems

For each of the following differential equations, draw a phase diagram, write out the general solution, and use the results of section 11.9 to determine analytically the nature of the fixed point. Determine the path through the point $(x(0), y(0)) = (1, 1)$.

a. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda, \mu > 0$.
b. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda > 0$, $\mu < 0$.
c. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda, \mu < 0$.
d. $\dot{x} = \lambda x + y$, $\dot{y} = \lambda y$ for $\lambda > 0$.
e. $\dot{x} = \lambda x + y$, $\dot{y} = \lambda y$ for $\lambda < 0$.
f. $\dot{x} = ax - by$, $\dot{y} = bx + ay$ for $a, b > 0$.
g. $\dot{x} = -x - y$, $\dot{y} = x - y$.
h. $\dot{x} = 3x - 2y$, $\dot{y} = x + y$.
i. $\dot{x} = 3x + y$, $\dot{y} = -x + y$.
j. $\dot{x} = y$, $\dot{y} = -x + 2y$.

For instance, the phase diagram for problem f is shown in figure 11.8. It is clear that this is a focus, but we cannot tell whether it is stable or unstable. Indeed, this should depend on the the parameters a and b. The matrix of the dynamical system is given by

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

so we have $\alpha = a$, $\beta = a^2 + b^2$, $\gamma = -b^2$, and the eigenvalues are $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. Because $a > 0$, this is an unstable focus.

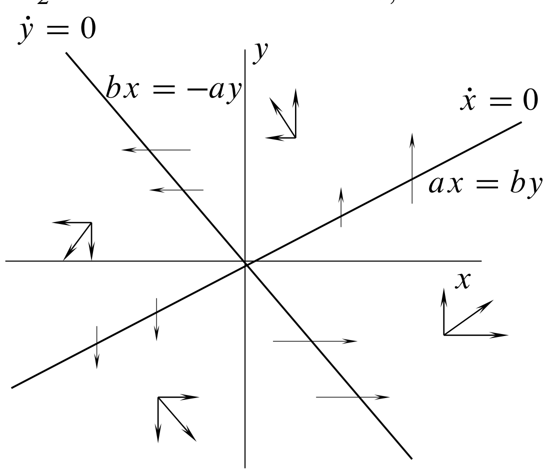


Figure 11.8. Phase Diagram for Problem f

11.11 Lotka-Volterra with Limited Carrying Capacity

The general Lotka-Volterra model (§11.4) has the form

$$\begin{aligned} \dot{x} &= x(a - by), \\ \dot{y} &= y(-c + dx). \end{aligned} \tag{11.25}$$

This model has the absurd property that if the predator is absent, the prey grows at the constant rate a forever. Suppose we add a limited carrying capacity term (§11.3), so the first equation in (11.25) becomes

$$\dot{x} = x(a - by - \epsilon x),$$

where $\epsilon > 0$, corresponding to capacity a/ϵ. We will show that the fixed point of this system is a stable focus for small ϵ and a stable node for large ϵ (but still satisfying $\epsilon < ad/c$).

To see this, note that interior equilibrium satisfies $x^* = c/d$ and $y^* = (ad - c\epsilon)/bd$. The Jacobian, evaluated at (x^*, y^*) is

$$J = \begin{bmatrix} -c\epsilon/d & -bc/d \\ y^* & 0 \end{bmatrix}.$$

The eigenvalues of the Jacobian at the equilibrium are

$$\frac{-c\epsilon \pm \sqrt{c}\sqrt{c\epsilon^2 + 4cd\epsilon - 4ad^2}}{2d}.$$

When ϵ is small, the term under the square root sign is negative, so both eigenvalues have negative real parts. The equilibrium in this case is a stable focus. If ϵ is large, but $\epsilon > ad/c$, it is easy to show that both eigenvalues are negative, so the equilibrium is a stable node.

11.12 Take No Prisoners

Two firms share a nonmarketable, nonexcludable resource R that lowers production costs but is subject to overcrowding and depletion. Suppose that when firm 1 has size x and firm 2 has size y, the profits of the two firms are given by

$$\begin{aligned} \pi^x(x, y) &= \gamma_x(R - x - y) - \alpha_x, \\ \pi^y(x, y) &= \gamma_y(R - x - y) - \alpha_y, \end{aligned}$$

where $\gamma_x, g_y > 0$. Suppose also that the firms' growth rates are equal to their profit rates, so

$$\dot{x} = x(\gamma_x(R - x - y) - \alpha_x),$$
$$\dot{y} = y(\gamma_y(R - x - y) - \alpha_y).$$

We assume that $\gamma_x R > \alpha_x$, $\gamma_y R > \alpha_y$, and $\gamma_x/\alpha_x > \gamma_y/\alpha_y$.

a. Show that if $y = 0$ the model has an unstable fixed point at $x = 0$ and an asymptotically stable fixed point at $x = x^* = R - a_x/g_x$.
b. Show that if $x = 0$ the model has an unstable equilibrium at $y = 0$ and an asymptotically stable fixed point at $y = y^* = R - a_y/g_y$.
c. Show that the complete model has three fixed points, (0,0), $(x^*, 0)$, and $(0, y^*)$, of which only the second is asymptotically stable.

We conclude that both firms cannot coexist in equilibrium.

11.13 The Hartman-Grobman Theorem

The linearization theorem 11.3 tells us that we can determine whether a hyperbolic fixed point of a dynamical system is asymptotically stable or unstable by looking at its linearization. This is a fairly weak statement, because we have discovered a lot more about the nature of equilibria than just stability. We have, for instance, distinguished nodes, foci, and saddles, and in the latter case, we have found that there are always stable and unstable manifolds. It turns out that in the hyperbolic case, each of these properties of the linearization of a dynamical system is also possessed by the system itself. This is the famous *Hartman-Grobman theorem*. To state the theorem, however, we need a new definition.

Suppose the dynamical system defined by $\dot{\mathbf{x}} = f(\mathbf{x})$ has a fixed point at $\mathbf{x}^*$, and the dynamical system defined by $\dot{\mathbf{y}} = f(\mathbf{y})$ has a fixed point at $\mathbf{y}^*$. We say that the two systems are *topologically equivalent* at these fixed points if there are balls $B_\epsilon(\mathbf{x}^*)$ and $B_\delta(\mathbf{y}^*)$ around the two fixed points and a continuous one-to-one mapping $\phi : B_\epsilon(\mathbf{x}^*) \to B_\delta(\mathbf{y}^*)$ with a continuous inverse that takes trajectories of the dynamical system lying in $B_\epsilon(\mathbf{x}^*)$ into trajectories of the dynamical system lying in $B_\delta(\mathbf{y}^*)$ and preserves the direction of time.

Intuitively, two dynamical systems are topologically equivalent at $\mathbf{x}^*$ and $\mathbf{y}^*$ if we can perform the following operation. Draw the phase diagram in the neighborhood of $\mathbf{x}^*$ on a rubber sheet, including trajectories and arrows indicating the direction of time. Now stretch the rubber sheet without tearing or folding until it looks just like the phase diagram for the second dynamical system in a neighborhood of $\mathbf{y}^*$. If this is possible, then the systems are topologically equivalent.

THEOREM 11.7 Hartman-Grobman. *If* $\mathbf{x}^*$ *is a hyperbolic fixed point of the dynamical system given by* $\dot{\mathbf{x}} = f(\mathbf{x})$*, then this fixed point is topologically equivalent to the fixed point at the origin of the linearization of the system* $\dot{\mathbf{x}} = A\mathbf{x}$*, where* $A = Df(\mathbf{x}^*)$ *is the Jacobian matrix of the system evaluated at* $\mathbf{x}^*$.

This means that we can determine the qualitative behavior of a dynamical system in a neighborhood of a hyperbolic fixed point by looking at its linearization, which is of course much easier to analyze. Indeed, we have fully characterized such equilibria for one- and two-dimensional systems. Higher-dimensional linear systems are harder to analyze, but they too can be completely characterized and are essentially combinations of one- and two-dimensional systems, placed at angles to each other in higher dimensions.

11.14 Features of Two-Dimensional Dynamical Systems

Two-dimensional dynamical systems have lots of nice properties not shared by higher-dimensional systems. This appears to be due to the famous *Jordan curve theorem*, which says that any continuous, non-self-intersecting, closed curve in the plane divides the plane into two connected pieces—an "inside" and an "outside." Trajectories of a dynamical system are of course continuous and non-self-intersecting, though not generally closed.

Let $\mathbf{x}(t)$ be a trajectory of the dynamical system (11.1). We say a point $\mathbf{y} \in \mathbf{R}^n$ is an ω-limit point of the trajectory if there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \mathbf{x}(t_n) = \mathbf{y}.$$

For instance, if $\mathbf{x}^*$ is an asymptotically stable fixed point, then $\mathbf{x}^*$ is the ω-limit of every trajectory starting at a point in the basin of attraction of $\mathbf{x}^*$. In some cases, a trajectory can actually have lots of ω-limit points. For instance, suppose the fixed point $\mathbf{x}^*$ is an unstable spiral, but there is a closed orbit at some distance from $\mathbf{x}^*$. Then a trajectory starting at a point near $\mathbf{x}^*$ can spiral out, getting closer and closer to the closed orbit. Each point on the closed orbit is thus an ω-limit of the trajectory. If a trajectory is bounded (that is, is contained in some ball), then it must have at least one ω-limit point.

THEOREM 11.8 Poincaré-Bendixson. *Suppose* $\mathbf{x}(t)$ *is a bounded trajectory of (11.13), and* Ω *is the set of* ω*-limit points of the trajectory. Then if* Ω *contains no fixed points of (11.13),* Ω *is a periodic orbit of (11.13).*

The following theorem is also often useful.

THEOREM 11.9 *Suppose equation (11.13) has a closed orbit* Γ *and let* U *be the interior region bounded by* Γ. *Then* U *contains a fixed point of (11.13).*

12

Evolutionary Dynamics

> Through the animal and vegetable kingdoms, nature has scattered the seeds of life abroad with the most profuse and liberal hand; but has been comparatively sparing in the room and nourishment necessary to rear them.
>
> T. R. Malthus

> Fifteen months after I had begun my systematic enquiry, I happened to read for amusement "Malthus on Population" . . . it at once struck me that . . . favorable variations would tend to be preserved, and unfavorable ones to be destroyed. Here, then, I had at last got a theory by which to work.
>
> Charles Darwin

Our study of evolutionary dynamics is built around the replicator equations. We begin by defining the replicator dynamic, deriving it in several distinct ways, and exploring its major characteristics (§12.1–§12.6). The next several sections make good on our promise to justify Nash equilibrium in terms of dynamical systems, as we exhibit the relationship between dynamic stability of evolutionary models, on the one hand, and dominated strategies (§12.7), Nash equilibria (§12.8), evolutionarily stable strategies (§12.9), and connected sets of Nash equilibria, on the other. Many of the results we obtain remain valid in more general settings (e.g., when the dynamic has an aggregate tendency toward favoring more fit strategies, but not necessarily as strongly as the replicator dynamic).

We next turn to asymmetric evolutionary games (§12.17), which have the surprising property, a property that is extremely important from the point of view of understanding real-world evolutionary dynamics, that strictly mixed Nash equilibria are never asymptotically stable but are often neutrally stable, leading to generically stable orbits (the Lotka-Volterra model has orbits, but it is not generic, as small changes in the coefficients lead to the equilibrium being either a stable or an unstable focus). In asymmetric games, the limit points of dynamic processes are generally Nash equilibria, but virtually nothing stronger than this can be asserted, including the elimination of weakly dominated strategies (Samuelson and Zhang 1992).

12.1 The Origins of Evolutionary Dynamics

The central actor in an evolutionary system is the *replicator*, which is an entity having some means of making approximately accurate copies of itself. The replicator can be a gene, an organism (defining "accurate copy" appropriately in the case of sexual reproduction), a strategy in a game, a belief, a technique, a convention, or a more general institutional or cultural form. A *replicator system* is a set of replicators in a particular environmental setting with a structured pattern of interaction among agents. An *evolutionary dynamic* of a replicator system is a process of change over time in the frequency distribution of the replicators (and in the nature of the environment and the structure of interaction, though we will not discuss these here), in which strategies with higher payoffs reproduce faster in some appropriate sense.

In addition to having an evolutionary dynamic, evolutionary systems may generate novelty if random errors ("mutations" or "perturbations") occur in the replication process, allowing new replicators to emerge and diffuse into the population if they are relatively well adapted to the replicator system.

The stunning variety of life forms that surround us, as well as the beliefs, practices, techniques, and behavioral forms that constitute human culture, are the product of evolutionary dynamics.

Evolutionary dynamics can be applied to a variety of systems, but we consider here only two-player *evolutionary games*, which consist of a *stage game* played by pairs of agents in a large population, each "wired" to play some pure strategy. We assume the game is symmetric (§10.1), so the players cannot condition their actions on whether they are player 1 or player 2. In each time period, agents are paired, they play the stage game, and the results determine their rate of replication. We generally assume there is random pairing, in which case the payoff to an agent of type i playing against the population is given by equation (10.1); note that this assumes that the population is very large, so we treat the probability of an agent meeting his own type as equal to fraction p_i of the population that uses the ith pure strategy. More generally, we could assume spatial, kinship, or other patterns of assortation, in which case the probability of type i meeting type j depends on factors other than the relative frequency p_j of type j in the population.

There are various plausible ways to specify an evolutionary dynamic. See Friedman (1991) and Hofbauer and Sigmund (1998) for details. Here we discuss only *replicator dynamics*, which are quite representative of evolutionary dynamics in general. Our first task is to present a few of the ways a replicator dynamic can arise.

12.2 Strategies as Replicators

Consider an evolutionary game where each player follows one of n pure strategies s_i for $i = 1, \ldots, n$. The play is repeated in periods $t = 1, 2, \ldots$. Let p_i^t be the fraction of players playing s_i in period t, and suppose the payoff to s_i is $\pi_i^t = \pi_i(p^t)$, where $p^t = (p_1^t, \ldots, p_n^t)$. We look at a given time t, and number the strategies so that $\pi_1^t \leq \pi_2^t \leq \ldots \leq \pi_n^t$.

Suppose in every time period dt, each agent with probability $\alpha dt > 0$ learns the payoff to another randomly chosen other agent and changes to the other's strategy if he perceives that the other's payoff is higher. However, information concerning the difference in the expected payoffs of the two strategies is imperfect, so the larger the difference in the payoffs, the more likely the agent is to perceive it, and change. Specifically, we assume the probability p_{ij}^t that an agent using s_i will shift to s_j is given by

$$p_{ij}^t = \begin{cases} \beta(\pi_j^t - \pi_i^t) & \text{for } \pi_j^t > \pi_i^t \\ 0 & \text{for } \pi_j^t \leq \pi_i^t \end{cases}$$

where β is sufficiently small that $p_{ij} \leq 1$ holds for all i, j. The expected fraction $\mathbf{E}p_i^{t+dt}$ of the population using s_i in period $t + dt$ is then given by

$$\begin{aligned} \mathbf{E}p_i^{t+dt} &= p_i^t - \alpha dt\ p_i^t \sum_{j=i+1}^{n} p_j^t \beta(\pi_j^t - \pi_i^t) + \sum_{j=1}^{i} \alpha dt\ p_j^t p_i^t \beta(\pi_i^t - \pi_j^t) \\ &= p_i^t + \alpha dt\ p_i^t \sum_{j=1}^{n} p_j^t \beta(\pi_i^t - \pi_j^t) \\ &= p_i^t + \alpha dt\ p_i^t \beta(\pi_i^t - \bar{\pi}^t), \end{aligned}$$

where $\bar{\pi}^t = \pi_1^t p_1^t + \ldots + \pi_n^t p_n^t$ is the average return for the whole population. If the population is large, we can replace $\mathbf{E}p_i^{t+dt}$ by p_i^{t+dt}. Subtracting p_i^t from both sides, dividing by dt, and taking the limit as $dt \to 0$, we get

$$\dot{p}_i^t = \alpha\beta p_i^t(\pi_i^t - \bar{\pi}^t), \qquad \text{for } i = 1, \ldots, n, \tag{12.1}$$

which is called the *replicator dynamic*. Because the constant factor $\alpha\beta$ merely changes the rate of adjustment to stationarity but leaves the stability properties and trajectories of the dynamical system unchanged, we often simply assume $\alpha\beta = 1$ (§12.5).

Several points are worth making concerning the replicator dynamic. First, *under the replicator dynamic, the frequency of a strategy increases exactly when it has above-average payoff.* In particular, this means that the replicator dynamic is not a best-reply dynamic; that is, agents do not adopt a best reply to the overall frequency distribution of strategies in the previous period. Rather, the agents in a replicator system have limited and localized knowledge concerning the system as a whole. Some game theorists call such agents "boundedly rational," but this term is very misleading, because the real issue is the distribution of information, not the degree of rationality.

Second, if we add up all the equations, we get $\sum_i \dot{p}_i^t = 0$, so if $\sum_i p_i^t = 1$ at one point in time, this remains true forever. Moreover, although a particular replicator can become extinct at $t \to \infty$, a replicator that is not represented in the population at one point in time will never be represented in the population at any later point in time. So, replicator dynamics deal poorly with innovation. A more general system adds a term to the replicator equation expressing the spontaneous emergence of replicators through mutation.

Third, our derivation assumes that there are no "mistakes;" that is, players never switch from a better to a worse strategy. We might suspect that small probabilities of small errors would make little difference, but I do not know under what conditions this intuition is valid.

Note that taking expected values allows us to *average* over the possible behaviors of an agent, so that even if there is a positive probability that a player will switch from better to worse, on average the player will not. The replicator dynamic compels a dynamical system *always* to increase the frequency of a strategy with above average payoff. If we do *not* take expected values, this property fails. For instance, if there is a probability $p > 0$, no matter how small, that a player will go from better to worse, and if there are n players, then there is a probability $p^n > 0$ that *all* players will switch from better to worse. We would have a "stochastic dynamic" in which movement over time probably, but not necessarily, increases the frequency of successful strategies. If there is a single stable equilibrium, this might not cause much of a problem, but if there are several, such rare accumulations of error will inevitably displace the dynamical system from the basin of attraction of one equilibrium to that of another (see chapter 13).

It follows that the replicator dynamic, by abstracting from stochastic influences on the change in frequency of strategies, is an idealized version of how systems of strategic interaction develop over time, and is accurate only if the number of players is very large in some appropriate sense, compared to the time interval of interest. To model the stochastic dynamic, we use stochastic processes, which are Markov chains and their continuous limits, diffusion processes. We provide an introduction to such dynamics in chapter 13.

It is satisfying that as the rate of error becomes small, the deviation of the stochastic dynamic from the replicator dynamic becomes arbitrarily small with arbitrarily high probability (Freidlin and Wentzell 1984). But the devil is in the details. For instance, as long as the probability of error is positive, under quite plausible conditions a stochastic system with a replicator dynamic will make regular transitions from one asymptotically stable equilibrium to another, and superior mutant strategies may be driven to extinction with high probability; see chapter 13, as well as Foster and Young (1990) and Samuelson (1997) for examples and references.

12.3 A Dynamic Hawk-Dove Game

There is a desert that can support n raptors. Raptors are born in the evening and are mature by morning. There are always at least n raptors alive each morning. They hunt all day for food, and at the end of the day, the n raptors that remain search for nesting sites (all raptors are female and reproduce by cloning). There are two types of nesting sites: good and bad. On a bad nesting site, a raptor produces an average of u offspring per night, and on a good nesting site, she produces an average of $u + 2$ offspring per night. However, there are only $n/2$ good nesting sites, so the raptors pair off and vie for the good sites.

	Hawk	Dove
Hawk	$u-1$, $u-1$	$u+2$, u
Dove	u, $u+2$	$u+1$, $u+1$

There are two variants of raptor: *hawk raptors* and *dove raptors*. When a dove raptor meets another dove raptor, they do a little dance and with equal probability one of them gets the good site. When a dove raptor meets a hawk raptor, the hawk raptor takes the site without a fight. But when two hawk raptors meet, they fight to the point that the expected number of offspring for each is one less than if they had settled for a bad nesting site. Thus the payoff to the two "strategies" hawk and dove are as shown in the diagram.

Let p be the fraction of hawk raptors in the population of n raptors. We Assume n is sufficiently large that we can consider p to be a continuous variable, and we also assume that the number of days in the year is sufficiently large that we can treat a single day as an infinitesimal dt of time. We can then show that *there is a unique equilibrium frequency p^* of hawk raptors and the system is governed by a replicator dynamic.*

In time period dt, a single dove raptor expects to give birth to

$$f_d(p)dt = (u + 1 - p)dt$$

little dove raptors overnight, and there are $n(1-p)$ dove raptors nesting in the evening, so the number of dove raptors in the morning is

$$n(1-p)(1 + (u + 1 - p)dt) = n(1-p)(1 + f_d(p)dt).$$

Similarly, the number of hawk raptors in the evening is np and a single hawk raptor expects to give birth to

$$f_h(p)dt = (u + 2(1-p) - p)dt$$

little hawk raptors overnight, so there are

$$np(1 + (u + 2(1-p) - p)dt) = np(1 + f_h(p)dt)$$

hawk raptors in the morning. Let

$$f(p) = (1-p)f_d(p) + pf_h(p),$$

so $f(p)dt$ is the total number of raptors born overnight and $n(1 + f(p)dt)$ is the total raptor population in the morning. We then have

$$p(t + dt) = \frac{np(t)(1 + f_h(p)dt)}{n(1 + f(p)dt)} = p(t)\frac{1 + f_h(p)dt}{1 + f(p)dt},$$

which implies

$$\frac{p(t+dt)-p(t)}{dt} = p(t)\left\{\frac{f_h(p)-f(p)}{1+f(p)dt}\right\}.$$

If we now let $dt \to 0$, we get

$$\frac{dp}{dt} = p(t)[f_h(p)-f(p)]. \tag{12.2}$$

This is of course a replicator dynamic, this time derived by assuming that agents reproduce genetically but are selected by their success in playing a game. Note that $p(t)$ is constant (that is, the population is in equilibrium) when $f_h(p) = f(p)$, which means $f_h(p) = f_d(p) = f(p)$.

If we substitute values in equation (12.2), we get

$$\frac{dp}{dt} = p(1-p)(1-2p). \tag{12.3}$$

This equation has three fixed points: $p = 0, 1, 1/2$. From our discussion of one-dimensional dynamics (§11.8), we know that a fixed point is asymptotically stable if the derivative of the right-hand side is negative, and is unstable if the derivative of the right-hand side is positive. It is easy to check that the derivative of $p(1-p)(1-2p)$ is positive for $p = 0, 1$ and negative for $p = 1/2$. Thus, a population of all dove raptors or all hawk raptors is stationary, but the introduction of even one raptor of the other type will drive the population toward the heterogeneous asymptotically stable equilibrium.

12.4 Sexual Reproduction and the Replicator Dynamic

Suppose the *fitness* (that is, the expected number of offspring) of members of a certain population depends on a single *genetic locus*, at which there are two genes (such creatures, which includes most of the "higher" plants and animals, are called *diploid*). Suppose there are n alternative types of genes (called *alleles*) at this genetic locus, which we label $g_1, \ldots, g_n$. An individual whose gene pair is (g_i, g_j), whom we term an "ij-type," then has fitness w_{ij}, which we interpret as being the expected number of offspring surviving to sexual maturity. We assume $w_{ij} = w_{ji}$ for all i,j.

Suppose sexually mature individuals are randomly paired off once in each time period, and for each pair (g_i, g_j) of genes, g_i taken from the first and g_j taken from the second member of the pair, a number of offspring of type ij are born, of which w_{ij} reach sexual maturity. The parents then die.

THEOREM 12.1 *For each* $i = 1, \ldots, n$ *let* $p_i(t)$ *be the frequency of* g_i *in the population. Then, fitness of a* g_i *allele is given by* $w_i(t) = \sum_{j=1}^n w_{ij} p_j(t)$*, the average fitness in the population is given by* $w(t) = \sum_{i=1}^n p_i w_i(t)$*, and the following replicator equations hold:*

$$\dot{p}_i = p_i[w_i(t) - w(t)] \qquad \textit{for } i = 1, \ldots, n. \tag{12.4}$$

PROOF: For any $i = 1, \ldots, n$, let y_i be the number of alleles of type g_i, and let y be the total number of alleles, so $y = \sum_{j=1}^n y_j$ and $p_i = y_i/y$. Because p_j is the probability that a g_i allele will meet a g_j allele, the expected number of g_i genes in the offspring of a g_i gene is just $\sum_j w_{ij} p_j$, and so the total number of g_i alleles in the next generation is $y_i \sum_j w_{ij} p_j$. This gives the differential equation

$$\dot{y}_i = y_i \sum_{j=1}^n w_{ij} p_j.$$

Differentiating the identity $\ln p_i = \ln y_i - \ln y$ with respect to time t, we get

$$\frac{\dot{p}_i}{p_i} = \frac{\dot{y}_i}{y_i} - \sum_{j=1}^n \frac{\dot{y}_j}{y} = \sum_{j=1}^n w_{ij} p_j - \sum_{j=1}^n \frac{\dot{y}_j}{y_j} p_j = \sum_{j=1}^n w_{ij} p_j - \sum_{j,k=1}^n w_{jk} p_j p_k,$$

which is the replicator dynamic.

The following important theorem was discovered by the famous biologist R. A. Fisher.

THEOREM 12.2 Fundamental Theorem of Natural Selection. *The average fitness* $w(t)$ *of a population increases along any trajectory of the replicator dynamic (12.4), and satisfies the equation*

$$\dot{w} = 2 \sum_{i=1}^n p_i (w_i - w)^2.$$

Note that the right-hand side of this equation is twice the fitness variance.

PROOF: Let W be the $n \times n$ matrix (w_{ij}) and let $p(t) = (p_1(t), \ldots, p_n(t))$ be the column vector of allele frequencies. The fitness of allele i is then

$$w_i = \sum_{j=1}^n w_{ij} p_j,$$

and the average fitness is

$$w = \sum_{i=1}^{n} p_i w_i = \sum_{i,j=1}^{n} p_i w_{ij} p_j.$$

Then,

$$\begin{aligned}\dot{w} &= 2\sum_{i,j=1}^{n} p_j w_{ji} \dot{p}_i = 2\sum_{i,j=1}^{n} p_j w_{ji} p_i (w_i - w) \\ &= 2\sum_{i=1}^{n} p_i (w_i - w) w_i = 2\sum_{i=1}^{n} p_i (w_i - w)^2,\end{aligned}$$

where the last equation follows from $\sum_{i=1}^{n} p_i (w_i - w) w = 0.$ ■

The above model can be extended in a straightforward manner to a situation in which the parents live more than one generation, and the fundamental theorem can be extended to include many genetic loci, provided they do not interact. However, it is a bad mistake to think that the fundamental theorem actually holds in the real world (this is often referred to as the *Panglossian fallacy*, named after Voltaire's Dr. Pangloss, who in *Candide* declared that "all is for the best in this, the best of all possible worlds"). Genes *do* interact, so that the fitness of an allele depends not just on the allele, but on the other alleles in the individual's genetic endowment. Such genes, called *epistatic genes*, are actually quite common. Moreover, the fitness of populations may be *interdependent* in ways that reduce fitness over time (see, for instance, section 11.4, which describes the Lotka-Volterra predator-prey model). Finally, stochastic effects ignored in replicator dynamics can lead to the elimination of very fit genes and even populations.

12.5 Properties of the Replicator System

Given the replicator equation (12.1), show the following:

a. For $1 \leq i < j \leq n$, show that

$$\frac{d}{dt}\left(\frac{p_i}{p_j}\right) = \left(\frac{p_i}{p_j}\right)(\pi_i - \pi_j).$$

b. Suppose that there is an $n \times n$ matrix $A = (a_{ij})$ such that for each $i = 1, \ldots, n$, $\pi_i = \sum_j a_{ij} p_j$; that is, a_{ij} is the payoff to player i when paired with player j in the stage game. Show that adding a constant to a column of A does not change the replicator equation and hence does not change the dynamic properties of the system. Note that this allows us to set the diagonal of A to consist of zeros, or set the last row of A to consist of zeros, in analyzing the dynamics of the system.
c. How does the column operation described in the previous question affect the Nash equilibria of the stage game? How does it affect the payoffs?

A more general form of (12.1) is

$$\dot{p}_i^t = a(p,t)p_i^t(\pi_i^t - \bar{\pi}^t) \qquad \text{for } i = 1, \ldots, n, \tag{12.5}$$

where $p = (p_i, \ldots, p_n)$, π_i^t and $\bar{\pi}^t$ are defined as in (12.1) and $a(p,t) > 0$ for all p, t. We will show that for any trajectory $p(t)$ of (12.5) there is an increasing function $b(t) > 0$ such that $q(t) = p(b(t))$ is a trajectory of the original replicator equation (12.1). Thus, multiplying the replicator equations by a positive function preserves trajectories and the direction of time, altering only the time scale.

12.6 The Replicator Dynamic in Two Dimensions

Suppose there are two types of agents. When an agent of type i meets an agent of type j, his payoff is α_{ij}, $i, j = 1, 2$. Let p be the fraction of type 1 agents in the system.

a. Use section 12.5 to show that we can assume $\alpha_{21} = \alpha_{22} = 0$, and then explain why the replicator dynamic for the system can be written

$$\dot{p} = p(1-p)(a + bp), \tag{12.6}$$

where $a = \alpha_{12}$ and $b = \alpha_{11} - \alpha_{12}$.
b. Show that in addition to the fixed point $p = 0$ and $p = 1$, there is an interior fixed point p^* of this dynamical system (that is, a p^* such that $0 < p^* < 1$) if and only if $0 < -a < b$ or $0 < a < -b$.
c. Suppose p^* is an interior fixed point of (12.6). Find the Jacobian of the system and show that p^* is an asymptotically stable equilibrium if and only if $b < 0$, so $0 < a < -b$. Show in this case that both of the other fixed points of (12.6) are unstable.

d. If p^* is an unstable interior fixed point of (12.6), show that the fixed points $p = 0$ and $p = 1$ are both asymptotically stable equilibria.
e. Show that if $z = p/(1-p)$, then the replicator equation becomes

$$\dot{z} = (1-p)z(\alpha_{11}z + \alpha_{12}), \tag{12.7}$$

and this has an interior asymptotically stable equilibrium $z^* = -\alpha_{12}/\alpha_{11}$ if and only if $\alpha_{11} < 0 < \alpha_{12}$.
f. Now use section 12.5 to show that this has the same trajectories as the simpler differential equation

$$\dot{z} = z(\alpha_{11}z + \alpha_{12}). \tag{12.8}$$

Show that the general solution to (12.8) is given by $z(t) = \alpha_{12}/(ce^{-\alpha_{12}t} - \alpha_{11})$, where $c = \alpha_{12}/z(0) + \alpha_{11}$. In this case we can verify directly that there is an interior asymptotically stable equilibrium if and only if $\alpha_{11} < 0 < \alpha_{12}$.

12.7 Dominated Strategies and the Replicator Dynamic

All Nash equilibria of a game survive the iterated elimination of strongly dominated strategies, but not of weakly dominated strategies (see chapter 3). Not surprisingly, strongly dominated strategies do not survive in a replicator dynamic. Suppose there are n pure strategies in the stage game of an evolutionary game in which $p_i(t)$ is the fraction of the population playing strategy i at time t. Recall that a strategy is *completely mixed* if $p_i(t) > 0$ for all i. We have the following theorem.

THEOREM 12.3 *Let $p(t) = (p_1(t), \ldots, p_n(t))$ be a completely mixed trajectory of the replicator dynamic (12.1) and suppose strategy i is recursively strongly dominated (§4.1). Then, strategy i does not survive the replicator dynamic; that is,* $\lim_{t\to\infty} p_i(t) = 0$.

To see this, first suppose i is strongly dominated by p^{o}. We write $\pi(p, q)$ for the payoff to strategy p against strategy q. Then, $\pi(p^{\mathrm{o}}, p) > \pi_i(p)$ for all mixed strategies p. Because the set of mixed strategies is closed and bounded, $\epsilon = \min_p(\pi(p^{\mathrm{o}}, p) - \pi_i(p))$ is strictly positive. Let

$$f(p) = \ln(p_i) - \sum_{j=1}^{n} p_j^{\mathrm{o}} \ln(p_j).$$

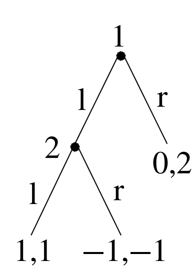
It is easy to check that $df(p(t))/dt \leq -\epsilon$, so $p_i(t) \to 0$.

It seems likely that this proof can be extended to the case of iterated domination. For instance, suppose strategy j is not strongly dominated until strongly dominated strategy i has been eliminated. By the preceding argument, when t is sufficiently large, i is used with vanishingly small probability, so now j is strongly dominated, and hence we can apply the preceding argument to j. And so on. The theorem is proved for the case of strategies that are iteratively strongly dominated by pure strategies in Samuelson and Zhang (1992). The case of strategies strongly dominated by mixed strategies is also treated in Samuelson and Zhang (1992), but a stronger condition on the dynamic, which they call *aggregate monotonic* , is needed to ensure elimination.

What about weakly dominated strategies? If the pure strategies against which a weakly dominated strategy does poorly are *themselves* driven out of existence by a replicator dynamic, then the weakly dominated strategy may persist in the long run. However, we do have the following two theorems.

THEOREM 12.4 *Let* $p(t) = (p_1(t), \ldots, p_n(t))$ *be a completely mixed trajectory of the replicator dynamic (12.1), and suppose* $p(t)$ *converges to a limit* p^* *as* $t \to \infty$*. Then* p^* *cannot assign unitary probability to a weakly dominated strategy.*

THEOREM 12.5 *Let* $p(t) = (p_1(t), \ldots, p_n(t))$ *be a completely mixed trajectory of the replicator dynamic (12.1), and let* α_{ij} *be the payoff of pure strategy* s_i *against* s_j*. Suppose pure strategy* s_i *is weakly dominated by* s_k*, so* $\pi(s_k, s_j) > \pi(s_i, s_j)$ *for some pure strategy* s_j*. Suppose* $\lim_{t\to\infty} p_j(t) > 0$*. Then,* $\lim_{t\to\infty} p_i(t) = 0$.



It is worthwhile thinking about the implications of this theorem for the persistence of a non-subgame-perfect Nash equilibrium under a replicator dynamic. Consider the little game in the diagram. Clearly, there are two Nash equilibria. The first is (l,l), where each player gets 1. But if player 2 is greedy, he can threaten to play r, the best response to which on the part of player 1 is r. Thus, (r,r) is a second Nash equilibrium. This equilibrium is, however, not subgame perfect, because player 2's threat of playing r is not credible.

a. Construct the normal form game and show that strategy r for player 2 is weakly dominated by strategy l.
b. Write the replicator equations for this system and find the fixed points of the replicator dynamic. Show that the replicator equations are

$$\dot{\alpha} = \alpha(1 - \alpha)(2\beta - 1) \tag{12.9}$$
$$\dot{\beta} = -2\beta(1 - \beta)(1 - \alpha). \tag{12.10}$$

Note that the underlying game is not symmetric in this case.
c. Find the Jacobian of the dynamical system at each of the fixed points, draw a phase diagram for the system, and show that any trajectory that does not start at a fixed point tends to the subgame perfect equilibrium as $t \to \infty$. Compare your results with the phase diagram in figure 12.1.
d. If you are inquisitive, study some other non-subgame-perfect Nash equilibria of various games and try to generalize as to (1) the relationship between non-subgame perfection and weakly dominated strategies, and (2) the conditions under which a non-subgame perfect Nash equilibrium can persist in a replicator dynamic.

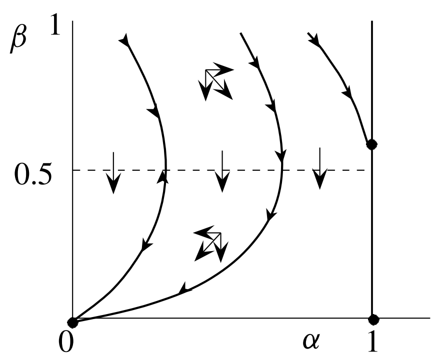


Figure 12.1. Phase diagram for dynamics of equations (12.9) and (12.10)

12.8 Equilibrium and Stability with a Replicator Dynamic

Consider an evolutionary game with n pure strategies and stage game payoff π_{ij} to an i-player who meets a j-player. If $p = (p_1, \ldots, p_n)$ is the frequency of each type in the population, the expected payoff to an i-player is then $\pi_i(p) = \sum_{j=1}^{n} p_j \pi_{ij}$, and the average payoff in the game is $\bar{\pi}(p) = \sum_{i=1}^{n} p_i \pi_i(p)$. The replicator dynamic for this game is then given by

$$\dot{p}_i = p_i(\pi_i(p) - \bar{\pi}(p)). \tag{12.11}$$

We are now at the point of motivating the importance of the Nash equilibrium as the fundamental equilibrium concept of game theory. We have

THEOREM 12.6 *The following hold, provided an evolutionary game satisfies the replicator dynamic (12.11).*

a. *If p^* is a Nash equilibrium of the stage game, p^* is a fixed point of the replicator dynamic.*
b. *If p^* is not a Nash equilibrium of the stage game, then p^* is not an evolutionary equilibrium .*
c. *If p^* is an asymptotically stable equilibrium of the replicator dynamic, then it is an isolated Nash equilibrium of the stage game (that is, it is a strictly positive distance from any other Nash equilibrium).*

The first of these assertions follows directly from the fundamental theorem of mixed-strategy Nash equilibrium (§3.6). To prove the second assertion, assume p^* is *not* isolated. Then, there is an i and an $\epsilon > 0$ such that $\pi_i(p^*) - \bar{\pi}(p^*) > \epsilon$ in a ball around p^*. But then the replicator dynamic implies p_i grows exponentially along a trajectory starting at any point in this ball, which means p^* is not asymptotically stable. The third part is left to the reader.

In general, the converse of these assertions is false. Clearly, there are fixed points of the replicator dynamic that are not Nash equilibria of the evolutionary game, because if an i-player does not exist in the population at one point in time, it can never appear in the future under a replicator dynamic. Therefore, for any i, the state $p_i = 1$, $p_j = 0$ for $j \neq i$ is a fixed point under the replicator dynamic.

Also, a Nash equilibrium need not be an asymptotically stable equilibrium of the replicator dynamic. Consider, for instance, the two-player pure coordination game that pays each player one if they both choose L or R, but zero otherwise. There is a Nash equilibrium in which each chooses L with probability 1/2. If p is the fraction of L-choosers in the population, then the payoff to an L-player is $\pi_L(p) = p$ and the payoff to an R-player is $\pi_R(p) = 1 - p$. The average payoff is then $\bar{\pi}(p) = p^2 + (1-p)^2$, so $\pi_L(p) - \bar{\pi}(p) = p - p^2 - (1-p)^2$. The Jacobian is then $3 - 4p$, which is positive at $p^* = 1/2$, so the fixed point is unstable. This is of course intuitively clear, because if there is a slight preponderance of one type of player, then all players gain from shifting to that type.

12.9 Evolutionary Stability and Asymptotically Stability

Consider the replicator dynamic (12.11) for the evolutionary game described in section 12.8. We have the following theorem.

THEOREM 12.7 *If p^* is an evolutionarily stable strategy of the stage game, then p^* is an asymptotically stable equilibrium of the replicator dynamic (12.11). Moreover, if p^* uses all strategies with positive probability, then p^* is a globally stable fixed point of the replicator dynamic.*

This theorem, which is due to Taylor and Jonker (1978), is proved nicely in Hofbauer and Sigmund (1998:70–71).

The fact that a point is an asymptotically stable equilibrium in a symmetric game does *not* imply that the point is an ESS, however. The diagram represents the stage game of an evolutionary game that has a locally stable fixed point that is not and evolutionarily stable strategy. Show the following:

	s_1	s_2	s_3
s_1	2,2	1,5	5,1
s_2	5,1	1,1	0,4
s_3	1,5	4,0	3,3

a. The game has a unique Nash equilibrium, in which the three strategies are used in proportions $(15/35, 11/35, 9/35)$.
b. This Nash equilibrium is not evolutionarily stable, because it can be invaded by the third strategy.
c. The eigenvalues of the Jacobian of the replicator dynamic equations are $3(-3 \pm 2i\sqrt{39})/35$, so the fixed point is a stable focus.

12.10 Trust in Networks III

In trust in networks (§6.23), we found a completely mixed Nash equilibrium, which in section 10.8 we found to be evolutionarily unstable, because in could be invaded by trusters. We now show that this equilibrium is in fact globally stable under the replicator dynamic. We illustrate this dynamic in figure 12.2. Note that south of the equilibrium, the fraction of trusters increases, but eventually the path turns back on itself and the fraction of trusters again increases. This is another example of an evolutionary equilibrium that is not an evolutionary stable strategy: near the equilibrium, the dynamic path moves away from the equilibrium before veering back toward it.

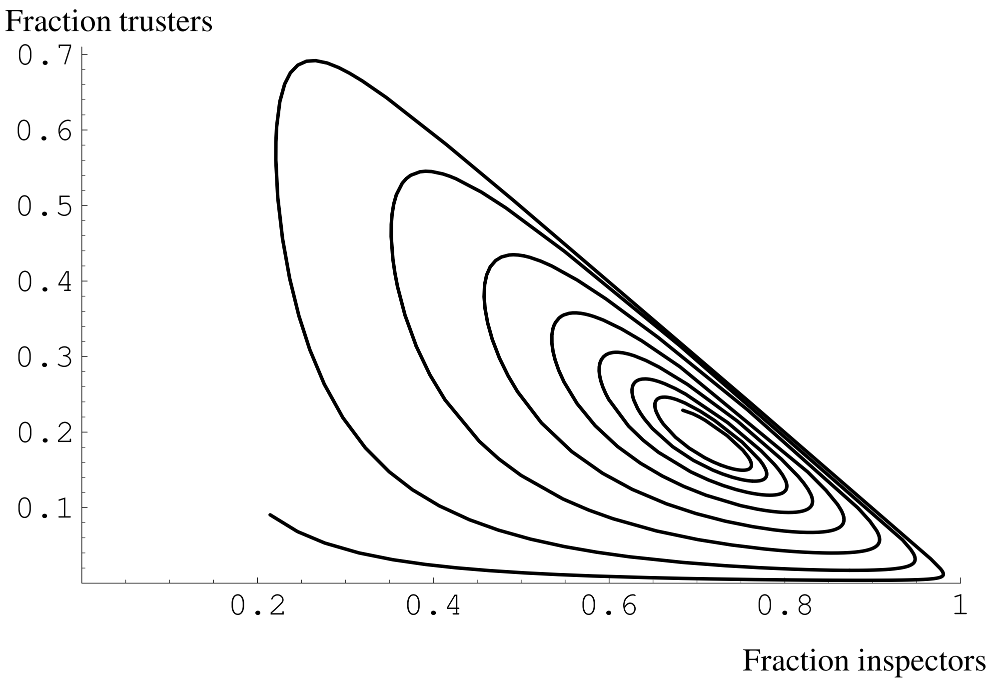


Figure 12.2. A typical path of the trust in networks dynamical system

12.11 Characterizing 2×2 Normal Form Games II

Suppose a normal form game is generic in the sense that no two payoffs for the same player are equal. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are the payoff matrices for Alice and Bob, so the payoff to Alice's strategy s_i against Bob's strategy t_j is a_{ij} for Alice and b_{ij} for Bob. We say two generic 2×2 games with payoff matrices (A, B) and (C, D) are *equivalent* if, for all $i, j = 1, 2$

$$a_{ij} > a_{kl} \equiv c_{ij} > c_{kl}$$

and

$$b_{ij} > b_{kl} \equiv d_{ij} > d_{kl}.$$

In particular, if a constant is added to the payoffs to all the pure strategies of one player when played against a given pure strategy of the other player, the resulting game is equivalent to the original.

Show that equivalent 2×2 generic games have the same number of pure Nash equilibria and the same number of strictly mixed Nash equilibria. Show also that every generic 2×2 game is equivalent to either the prisoner's dilemma (§3.11), the battle of the sexes (§3.9), or the hawk-dove (§3.10). Note that this list does not include throwing fingers (§3.8), which is not generic. *Hint*: Refer to the figure in section 3.7. First order the strategies so the highest payoff for player 1 is a_1. Second, add constants so that $c_1 = d_1 = b_2 = d_2 = 0$. Because the game is generic, $a_1 > 0$, and either $a_2 > 0$ (case I) or $a_2 < 0$ (case II). Third, explain why only the signs of c_2 and b_1, rather than their magnitudes, remain to be analyzed. If either is positive in case I, the game has a unique equilibrium found by the iterated elimination of dominated strategies, and is equivalent to the prisoner's dilemma. The same is true in case II if either $b_1 > 0$ or $c_2 < 0$. The only remaining case I situation is $b_1, c_1 < 0$, which is equivalent to the battle of the sexes, with two pure- and one mixed-strategy equilibrium. The only remaining case II is $b_1 < 0, c_2 > 0$, which is equivalent to hawk-dove, and there is a unique mixed-strategy equilibrium.

Now show that if two 2×2 normal form games are equivalent, then their corresponding Nash equilibria have the same stability properties. It follows that there are really only three types of generic, 2×2 two-player games: (a) a single, stable, pure-strategy equilibrium; (b) a coordination game, with two stable pure-strategy Nash equilibrium, separated by an unstable mixed-strategy equilibrium; and (c) a hawk-dove game, which has a unique, stable Nash equilibrium in mixed strategies. Show that a 2×2 two-player ESS is asymptotically stable in the replicator dynamic.

12.12 Invasion of the Pure-Strategy Nash Mutants II

In section 10.13 we exhibited a Nash equilibrium that cannot be invaded by any pure strategy mutant but can be invaded by an appropriate mixed-strategy mutant. We can show that this Nash equilibrium is unstable under the replicator dynamic. This is why we insisted that the ESS concept be defined in terms of mixed- rather than pure-strategy mutants; an ESS is an asymptotically stable equilibrium only if the concept is so defined.

To show instability, let γ, α, and β be the fraction of agents using strategies 1, 2, and 3, respectively. It is straightforward to check that the replicator equation governing strategies 2 and 3 are given by

$$\dot{\alpha} = \alpha(-\alpha + \beta(a-1) + (\alpha+\beta)^2 - 2\alpha\beta a)$$
$$\dot{\beta} = \beta(-\beta + \alpha(a-1) + (\alpha+\beta)^2 - 2\alpha\beta a).$$

The Jacobian of the equations is the zero matrix at the fixed point $\gamma = 1$, that is, where $\alpha = \beta = 0$. Thus, linearization does not help us. However, we can easily calculate that when $\alpha = \beta, \gamma > 0$, $\dot{\alpha} + \dot{\beta}$ is strictly positive when $\alpha = \beta$ and $a > 2$, so $\dot{\gamma} < 0$ arbitrarily close to the equilibrium $\gamma = 1$. This proves that the Nash equilibrium using strategy 1 is unstable. Note that the Nash equilibrium using strategies 2 and 3 with probability 1/2 is an ESS, and is evolutionarily stable under the replicator dynamic.

Use the same method to check that the two evolutionarily stable equilibria in section 10.14 are asymptotically stable equilibria, and the completely mixed Nash equilibrium, which you showed was resistant to invasion by pure but not mixed strategies, is a saddle point under the replicator dynamic.

12.13 A Generalization of Rock, Paper, and Scissors

The game in the diagram, where we assume $-1 < \alpha < 1$ and $\alpha \neq 0$, is a generalization of Rock, Paper, and Scissors in which agents receive a nonzero payoff α if they meet their own type. We can show that the game has a unique Nash equilibrium in which each player chooses each strategy with probability 1/3, but this equilibrium is not evolutionarily stable for $\alpha > 0$. Moreover, we can show that the equilibrium is a hyperbolic fixed point under the replicator dynamic and is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$.

	R	*S*	*P*
R	α,α	1,−1	−1,1
S	−1,1	α,α	1,−1
P	1,−1	−1,1	α,α

12.14 *Uta stansburiana* in Motion

Determine the dynamic behavior of the male lizard population in section 6.25 under a replicator dynamic.

12.15 The Dynamics of Rock, Paper, and Scissors

	α	β	γ
α	0,0	r,s	s,r
β	s,r	0,0	r,s
γ	r,s	s,r	0,0

Consider the rock-paper-scissors type game in the diagram, where r and s are nonzero. Suppose α, β, and $\gamma = 1 - \alpha - \beta$ are the fraction of the population playing the three strategies, and suppose in each period members are randomly paired and they play the game. What is the replicator dynamic for the game? How does the behavior of the system depend on r and s? Prove the following:

Show that when $r, s < 0$, this system has three stable pure strategy equilibria, as well as three unstable Nash equilibria using two pure strategies. Then show that rock, paper, and scissors has the mixed-strategy Nash equilibrium $(\alpha, \beta) = (1/3, 1/3)$ with the following dynamic properties:

a. The fixed point cannot be a saddle point.
b. The fixed point is an asymptotically stable equilibrium if $r + s > 0$ and unstable if $r + s < 0$.
c. The fixed point is a focus if $r \neq s$, and a node if $r = s$.
d. If $r + s = 0$, as in the traditional Rock, Paper and Scissors game, the fixed point of the linearized system is a center, so the system is not hyperbolic. Thus, we cannot determine the dynamic for this case from the Hartman-Grobman theorem. However, we can show that the fixed point is a center, so all trajectories of the system are periodic orbits.

12.16 The Lotka-Volterra Model and Biodiversity

Suppose two species interact in a fixed environment. If u and v represent the number of individuals of species A and B respectively, the system follows the differential equations

$$\dot{u} = u\left[a\frac{u}{u+v} + b\frac{v}{u+v} - k(u+v)\right]$$
$$\dot{v} = v\left[c\frac{u}{u+v} + d\frac{v}{u+v} - k(u+v)\right],$$

where $k > 0$ and $(d - b)(a - c) > 0$. We interpret these equations as follows: the growth rate of each species is frequency dependent, but all share an ecological niche and hence are subject to overcrowding, the intensity of which is measured by k. For instance, suppose individuals meet at random. Then, an A meets another A with probability $u/(u+v)$, and they may reproduce at rate a, although an A meets a B with probability $v/(u+v)$, in which case the A eats the B ($b > 0$) or vice versa ($b < 0$). Show the following.

a. Let $w = u + v$, the size of the total population, and $p = u/w$, the fraction of species A in the population. The stationary fraction p^* of species A is given by

$$p^* = \frac{d-b}{a-c+d-b},$$

which is strictly positive and independent of the crowding factor k.

b. If we think of w as the whole population, then the payoff π_A to species A, the payoff π_B to species B, and the mean payoff $\bar{\pi}$ to a member of the population, *abstracting from the overcrowding factor k*, are given by

$$\begin{aligned} \pi_A &= ap + b(1-p), \\ \pi_B &= cp + d(1-p), \\ \bar{\pi} &= p\pi_A + (1-p)\pi_B. \end{aligned}$$

Show that p satisfies the replicator dynamic

$$\dot{p} = p(\pi_A - \bar{\pi}),$$

even taking into account the overcrowding factor. This equation indicates that the frequency of species A in the population is independent of the crowding factor in the dynamic interaction between the two species. Moreover, the stability conditions for p are also independent of k, so we conclude: *If neither species can become extinct when the crowding factor k is low, the same is true no matter how large the crowding factor k.*

c. We can generalize this result as follows. Suppose there are n species, and let u_i be the number of individuals in species i, for $i = 1, \ldots, n$. Define $w = \sum_j u_j$ and for $i = 1, \ldots, n$ let $p_i = u_i/w$, the relative frequency of species i. Suppose the system satisfies the equations

$$\dot{u}_i = u_i \left[a_{i1}p_1 + \ldots + a_{in}p_n - ku\right]$$

for $k > 0$. We assume the $\{a_{ij}\}$ are such that there is a positive stationary frequency for each species. Show that the system satisfies the differential equations

$$\frac{\dot{p}_i}{p_i} = \sum_{j=1}^{n} a_{ij} p_j - \sum_{j,k=1}^{n} a_{jk} p_j p_k$$

for $i = 1, \ldots, n$. Show that this represents a replicator dynamic if the payoffs to the various species abstract from the crowding factor k. Once again we find that the frequency of each species is independent of the crowding factor, and if the ecology is sustainable with low crowding factor (that is, no species goes extinct), then it remains so with high crowding factor.

This result is surprising, perhaps. How do we account for it? It is easy to see that the *absolute* number of individuals in each species in equilibrium is proportional to $1/k$. Thus, when k is large, the justification for using a replicator dynamic is no longer valid: with considerable probability, the stochastic elements abstracted from in the replicator dynamic act to reduce some p_i to zero, after which it cannot ever recover unless the ecological system is repopulated from the outside. For an example of dynamics of this type, see Durrett and Levin (1994).

12.17 Asymmetric Evolutionary Games

Consider two populations of interacting agents. In each time period, agents from one population (row players) are randomly paired with agents from the other population (column players). The paired agents then play a game in which row players have pure strategies $S = \{s_1, \ldots, s_n\}$ and column players have pure strategies $T = \{t_1, \ldots, t_m\}$. Agents are "wired" to play one of the pure strategies available to them, and the payoffs to an i-type (that is, a row player wired to play s_i) playing a j-type (that is, a column player wired to play t_j) are α_{ij} for the i-type and β_{ij} for the j-type. We call the resulting game an *asymmetric evolutionary game*.

Suppose the frequency composition of strategies among column players is $q = (q_1, \ldots, q_m)$, where q_j is the fraction of j-types among column players. Then, the payoff to an i-type row player is

$$\alpha_i(q) = \sum_{j=1}^{m} q_j \alpha_{ij}.$$

Similarly if the frequency composition of strategies among row players is $p = (p_1, \ldots, p_n)$, where p_i is the fraction of i-types among row players, then the payoff to a j-type column player is

$$\beta_j(p) = \sum_{i=1}^{n} p_i \beta_{ij}.$$

We say $s_i \in S$ is a *best response* to $q \in Q$ if $\alpha_i(q) \geq \alpha_k(q)$ for all $s_k \in S$, and we say $t_j \in T$ is a *best response* to $p \in P$ if $\beta_j(p) \geq \beta_k(p)$ for all $t_k \in T$. A *Nash equilibrium* in an asymmetric evolutionary game is a frequency composition $p^* \in P$ of row players and $q^* \in Q$ of column players such that for all $s_i \in S$, if $p_i^* > 0$, then s_i is a best response to q^*, and for all $s_j \in T$, if $q_j^* > 0$, then t_j is a best response to p^*.

Note that there is a natural correspondence between the mixed-strategy Nash equilibria of a two-player normal form game as defined in section 3.4 and the Nash equilibria of an asymmetric evolutionary game. Thus, if we take an arbitrary two-player game in which row players and column players are distinguished and place the game in an evolutionary setting, we get an asymmetric evolutionary game. Hence, the dynamics of asymmetric evolutionary games more or less represent the dynamics of two-player games in general.[1]

A replicator dynamic for an asymmetric evolutionary game is given by the $n + m - 2$ equations

$$\begin{aligned} \dot{p}_i &= p_i(\alpha_i(q) - \alpha(p, q)) \\ \dot{q}_j &= q_j(\beta_j(p) - \beta(p, q)), \end{aligned} \tag{12.12}$$

[1]I say more or *less* because in fact the assumption of random pairings of agents is not at all characteristic of how agents are paired in most strategic interaction settings. More common is some form of *assortative interaction*, in which agents with particular characteristics have a higher than chance probability of interacting. Assortative interactions, for instance, are a more favorable setting for the emergence of altruism than panmictic interactions.

where $\alpha(p,q) = \sum_i p_i\alpha_i(q)$, $\beta(p,q) = \sum_j q_j\beta_j(p_i)$, $i = 1,\dots,n-1$, and $j = 1,\dots,m-1$. Note that although the *static* game pits the row player against the column player, the *evolutionary* dynamic pits row players against themselves and column players against themselves. This aspect of an evolutionary dynamic is often misunderstood. We see the conflict between a predator and its prey, or between a pathogen and its host, and we interpret the "survival of the fittest" as the winner in this game. But, in fact, in an evolutionary sense predators fight among themselves for the privilege of having their offspring occupy the predator niche in the next period and improve their chances by catching more prey. Meanwhile the prey are vying among themselves for the privilege of having their offspring occupy the prey niche, and they improve their chances by evading predators for an above-average period of time.

What nice properties does this dynamic have? Theorem 12.3 continues to hold: only strategies that are not recursively strongly dominated survive the replicator dynamic. A version of theorem 12.6 also holds in this case: a Nash equilibrium of the evolutionary game is a fixed point under the replicator dynamic, a limit point of a trajectory under the replicator dynamic is a Nash equilibrium, and an asymptotically stable equilibrium of the replicator dynamic is a Nash equilibrium. Even theorem 12.7 continues to hold: an evolutionarily stable strategy is an asymptotically stable equilibrium under the replicator dynamic. However, as we have seen in section 10.16, *an evolutionarily stable strategy of an asymmetric evolutionary game must be a strict Nash equilibrium*; that is, both row and column players must be monomorphic in equilibrium, there being only one type of player on each side. So, in all but trivial cases, evolutionary stability does *not* obtain in asymmetric evolutionary games. Because evolutionary stability is closely related to being an asymptotically stable equilibrium under the replicator dynamic the following theorem (Hofbauer and Sigmund 1998) is not surprising.

THEOREM 12.8 *A strictly mixed-strategy Nash equilibrium of asymmetric evolutionary games is not an asymptotically stable equilibrium under the replicator dynamic.*

Actually, this situation applies to a much larger class of evolutionary dynamics than the replicator dynamic. See Samuelson and Zhang (1992) for details.

For a simple example of theorem 12.8, consider the case where $n = m = 2$; that is, row and column players each have two pure strategies. We have the following theorem.

THEOREM 12.9 *In the asymmetric evolutionary game in which each player has two pure strategies, a mixed-strategy Nash equilibrium* (p^*, q^*) *is either unstable or an evolutionary focal point. In the latter case, all trajectories are closed orbits around the fixed point, and the time average of the frequencies* $(p(t), q(t))$ *around an orbit is* (p^*, q^*)*:*

$$\begin{aligned} \frac{1}{T}\int_0^T q(t)dt &= \frac{\alpha}{\gamma} = q^* \\ \frac{1}{T}\int_0^T p(t)dt &= \frac{\beta}{\delta} = p^*. \end{aligned} \tag{12.13}$$

When the time average of a dynamical system equals its equilibrium position, we say the system is *ergodic*.

PROOF: Check out the following fact. If a constant is added to each entry in a column of the matrix $A = \{\alpha_{ij}\}$, or to each row of the matrix $B = \{\beta_{ij}\}$, the replicator equations (12.12) remain unchanged. We can therefore assume $\alpha_{11} = \alpha_{22} = \beta_{11} = \beta_{22} = 0$. Writing $p = p_1$ and $q = q_1$, the replicator equations then become

$$\begin{aligned} \dot{p} &= p(1-p)(\alpha - \gamma q) \\ \dot{q} &= q(1-q)(\beta - \delta p), \end{aligned}$$

where $\alpha = \alpha_{12}$, $\beta = \beta_{12}$, $\gamma = \alpha_{12} + \alpha_{21}$, and $\delta = \beta_{12} + \beta_{21}$. A mixed-strategy equilibrium then occurs when $0 < \alpha/\gamma, \beta/\delta < 1$, and is given by $p^* = \beta/\delta$, $q^* = \alpha/\gamma$. The Jacobian at the fixed point is

$$J(p^*, q^*) = \begin{bmatrix} 0 & -\gamma p^*(1-p^*) \\ -\delta q^*(1-q^*) & 0 \end{bmatrix}.$$

Note that if α and β (or equivalently γ and δ) have the same sign, this is a saddle point (theorem 11.4) and hence unstable. You can check that in this case at least one of the monomorphic fixed points is asymptotically stable. Because the mixed-strategy equilibrium is hyperbolic, the fixed point is also a saddle under the replicator dynamic, by the Hartman-Grobman theorem (theorem 11.7). In case this argument whizzed by you, make a phase diagram to get a feel for this very common situation.

If α and β have opposite signs, the linearized system is neutrally stable, so the mixed-strategy equilibrium is not hyperbolic. Although we cannot apply the Hartman-Grobman theorem, a sketch of the phase diagram shows that trajectories spiral around the fixed point. We can then determine that trajectories are closed orbits by exhibiting a function that is constant on trajectories. To see this, we divide the second replicator equation by the first, getting

$$\frac{dq}{dp} = \frac{(\beta - \delta p)q(1-q)}{(\alpha - \gamma q)p(1-p)}.$$

Separating variables, we get

$$\frac{\alpha - \gamma q}{q(1-q)} dq = \frac{\beta - \delta p}{p(1-p)} dp.$$

Integrating both sides and simplifying, we get

$$\alpha \ln(q) - (\alpha - \gamma)\ln(1-q) - \beta \ln(p) + (\beta - \delta)\ln(1-p) = C,$$

for some constant C. Suppose $\alpha > \gamma$. Then, this function is monotonic in the q direction, so the spirals must in fact be closed orbits. If $\alpha \leq \gamma$, then we must have $\beta > \delta$, so the function is monotonic in the p direction, so again the spirals are closed orbits.

To check the ergodic property of the system in the case of neutral stability, consider a trajectory $(p(t), q(t))$ starting at a point $(p(0), q(0)) = (p_o, q_o)$. We integrate both sides of the equation $\dot{p}/p(1-p) = \alpha - \gamma q$ with respect to time, getting

$$\ln(p(t)) + \ln(1-p(t)) = A + \alpha t - \gamma \int_0^t q(\tau)d\tau,$$

where the constant of integration A satisfies $A = \ln(p_o) + \ln(1-p_o)$. If the period of the trajectory is T, so $p(T) = p_o$ and $q(T) = q_o$, then letting $t = T$ in the previous expression gives

$$\frac{1}{T}\int_0^T q(t)dt = \frac{\alpha}{\gamma} = q^*.$$

A similar argument justifies the second equation in (12.13) as well. This proves the theorem. ■

12.18 Asymmetric Evolutionary Games II

To gain some feeling for the argument in section 12.17, check out the dynamic properties of the asymmetric evolutionary game versions of the following games. *Hint*: In most cases the results follow easily from performing the row and column manipulations that leave zeros on the diagonals of the two payoff matrices.

a. Section 12.17 (draw the phase diagram).
b. The mixed-strategy equilibrium of the Big John and Little John game (§3.1).
c. The mixed-strategy equilibrium of the battle of the sexes (§3.9).

12.19 The Evolution of Trust and Honesty

	H	D
I	3,2	2,1
T	4,3	1,4

Consider an asymmetric evolutionary game with buyers and sellers. Each seller can be either honest (H) or dishonest (D), and each buyer can either inspect (I) or trust (T). Let p be the fraction of buyers who inspect and let q be the fraction of sellers who are honest. Suppose the payoff matrix for an encounter between a buyer and a seller is given as in the figure in the diagram. The payoff to inspect is then $3q + 2(1-q) = q + 2$, the payoff to trust is $4q + (1-q) = 3q + 1$, the payoff to be honest is $2p + 3(1-p) = -p + 3$, and the payoff to be dishonest is $p + 4(1-p) = -3p + 4$.

Suppose we have a replicator dynamic, such that the fraction of inspectors grows at a rate equal to its fitness minus the average fitness of buyers. Buyer average fitness is $p(q+2) + (1-p)(3q+1) = 3q + 1 - p(2q-1)$, so the inspector growth rate is $q + 2 - [3q + 1 - p(2q-1)] = (1-p)(1-2q)$, and we have the replicator equation

$$\dot{p} = p(1-p)(1-2q). \tag{12.14}$$

Similarly, the fraction of honest sellers grows at a rate equal to its fitness minus the average fitness among sellers, giving

$$\dot{q} = q(1-q)(2p-1). \tag{12.15}$$

a. Show that these two coupled differential equations have five fixed points, (0,0), (0,1), (1,0), (1,1), and (1/2,1/2).

b. Show that the first four fixed points are unstable.
c. Show that the equilibrium at (1/2,1/2) is not hyperbolic: its linearization is a center. It follows that we cannot use the Hartman-Grobman theorem to ascertain the type of fixed point.
d. Draw a phase diagram and show that the trajectories are spirals moving counterclockwise around the fixed point.

How might we prove that the fixed point is a center? Suppose we could find a function $f(p,q)$ that is constant on trajectories of the system. If we could then show that f is strictly increasing along an appropriate ray from the fixed point to the northeast, we would be done, because only closed orbits are then possible. This is precisely what we did in section 11.4 to show that the trajectories of the Lotka-Volterra equations are orbits around the fixed point. See also sections 12.14 and 12.17.

Eliminating t from (12.14) and (12.15), we get

$$\frac{dq}{dp} = \frac{(q-q^2)(2p-1)}{(p-p^2)(1-2q)}.$$

Separating the variables, this becomes

$$\frac{1-2p}{p-p^2}dp = -\frac{1-2q}{q-q^2}dq.$$

Integrating both sides and combining terms, we get $\ln(p-p^2)(q-q^2) = C$ for some constant C. We simplify by taking the antilogarithm of both sides, getting $(p-p^2)(q-q^2) = e^C$. Thus, $f(p,q) = p(1-p)q(1-q)$ is constant on trajectories of the dynamical system. Consider a ray from the origin through the fixed point. We may parametrize this by $p = q = s$, which hits the fixed point when $s = 1/2$. Then, $f(p(s),q(s)) = s^2(1-s)^2$, so $df(p(s),q(s))/ds = 2s(1-s)(1-2s)$. This is strictly positive for $1/2 < s < 1$. If the trajectory were not a center, it would hit this ray more than once, and $f(p(s),q(s))$ would have a larger value the second time than the first, which is impossible. This proves that (1/2,1/2) is a center.

13

Markov Economies and Stochastic Dynamical Systems

God does not play dice
with the Universe.
Albert Einstein

Time-discrete stochastic processes are powerful tools for characterizing some dynamical systems. The prerequisites include an understanding of Markov chains (§13.1). Time-discrete systems behave quite differently from dynamical systems based on systems of ordinary differential equations. This chapter presents a Markov model of adaptive learning that illustrates the concept of stochastic stability, as developed in Young (1998). After developing some of the theoretical results, we provide an agent-based model.

13.1 Markov Chains

A *finite Markov chain* is a dynamical system that in each time period $t = 0, 1, \ldots$ can be any one of n *states*, such that if the system is in state i in one time period, there is a probability p_{ij} that the system will be in state j in the next time period. Thus, for each i, we must have $\sum_j p_{ij} = 1$, because the system must go somewhere in each period. We call the $n \times n$ matrix $P = \{p_{ij}\}$ the *transition probability matrix* of the Markov chain, and each p_{ij} is called a *transition probability*. A *denumerable Markov chain* has an infinite number of states $t = 1, 2, \ldots$, and is otherwise the same. If we do not care whether the finite or denumerable case obtains, we speak simply of a *Markov chain*.

Many games can be viewed as Markov chains. Here are some examples:

a. Suppose two gamblers have wealth k_1 and k_2 dollars, respectively, and in each period they play a game in which each has an equal chance of winning one dollar. The game continues until one player has no more wealth. Here the state of the system is the wealth w of player 1,

$p_{w,w+1} = p_{w,w-1} = 1/2$ for $0 < w < k_1 + k_2$, and all other transition probabilities are zero.

b. Suppose n agents play a game in which they are randomly paired in each period, and the stage game is a prisoner's dilemma. Players can remember the last k moves of their various partners. Players are also given one of r strategies, which determine their next move, depending on their current histories. When a player dies, which occurs with a certain probability, it is replaced by a new player who is a clone of a successful player. We can consider this a Markov chain in which the state of the system is the history, strategy, and score of each player, and the transition probabilities are just the probabilities of moving from one such state to another, given the players' strategies (§13.8).

c. Suppose n agents play a game in which they are randomly paired in each period to trade. Each agent has an inventory of goods to trade and a strategy indicating which goods the agent is willing to trade for which other goods. After trading, agents consume some of their inventory and produce more goods for their inventory, according to some consumption and production strategy. When an agent dies, it is replaced by a new agent with the same strategy and an empty inventory. If there is a maximum-size inventory and all goods are indivisible, we can consider this a finite Markov chain in which the state of the system is the strategy and inventory of each player and the transition probabilities are determined accordingly.

d. In a population of beetles, females have k offspring in each period with probability f_k, and beetles live for n periods. The state of the system is the fraction of males and females of each age. This is a denumerable Markov chain, where the transition probabilities are calculated from the birth and death rates of the beetles.

We are interested in the long-run behavior of Markov chains. In particular, we are interested in the behavior of systems that we expect will attain a long-run equilibrium of some type independent from its initial conditions. If such an equilibrium exists, we say the Markov chain is *ergodic*. In an ergodic system, history does not matter: every initial condition leads to the same long-run behavior. Nonergodic systems are history dependent. It is intuitively reasonable that the repeated prisoner's dilemma and the trading model described previously are ergotic. The gambler model is not ergodic, because the system could end up with either player bankrupt.[1] What is your intuition concerning the beetle population, if there is a positive probability that a female has no offspring in a breeding season?

It turns out that there is a very simple and powerful theorem that tells us exactly when a Markov chain is ergodic and provides a simple characterization of the long-run behavior of the system. To develop the machinery needed to express and understand this theorem, we will define a few terms. Let $p_{ij}^{(m)}$ be the probability of being in state j in m periods if the chain is currently in state i. Thus, if we start in state i at period 1, the probability of being in state j at period 2 is just $p_{ij}^{(1)} = p_{ij}$. To be in state j in period 3 starting from state i in period 1, the system must move from state i to some state k in period 2, and then from k to j in period 3. This happens with probability $p_{ik}p_{kj}$. Adding up over all k, the probability of being in state j in period 3 is

$$p_{ij}^{(2)} = \sum_k p_{ik}p_{kj}.$$

Using matrix notation, this means the matrix of two-period transitions is given by

$$P^{(2)} = \{p_{ij}^{(2)} | i, j = 1, 2, \ldots\} = P^2.$$

Generalizing, we see that the k-period transition matrix is simply P^k. What we are looking for, then, is the limit of P^k as $k \to \infty$. Let us call this limit (supposing it exists) $P^* = \{p_{ij}^*\}$. Now P^* must have two properties. First, because the long-run behavior of the system cannot depend on where it started, $p_{ij}^* = p_{i'j}^*$ for any two states i and i'. This means that all the rows of P^* must be the same. Let us denote the (common value of the) rows by $u = \{u_1, \ldots, u_n\}$, so u_j is the probability that the Markov chain will be in state j in the long run. The second fact is that

$$PP^* = P \lim_{k\to\infty} P^k = \lim_{k\to\infty} P^{k+1} = P^*.$$

This means u must satisfy

$$u_j = \lim_{m\to\infty} p_{ij}^{(m)} \quad \text{for } i = 1, \ldots, n \tag{13.1}$$

[1]Specifically, you can show that the probability that player 1 wins is $k_1/(k_1 + k_2)$, and if player 1 has wealth w at some point in the game, the probability he will win is $w/(k_1 + k_2)$.

$$u_j = \sum_i u_i p_{ij} \tag{13.2}$$

$$\sum_k u_k = 1, \tag{13.3}$$

for $j = 1, \ldots, n$. Note that (13.2) can be written in matrix notation as $u = uP$, so u is a *left eigenvector* of P. The first equation says that u_j is the limit probability of being in state j starting from any state, the second says that the probability of being in state j is the probability of moving from some state i to state j, which is $u_i p_{ij}$, summed over all states i, and the final equation says u is a probability distribution over the states of the Markov chain. The *recursion equations* (13.2) and (13.3) are often sufficient to determine u, which we call the *invariant distribution* or *stationary distribution* of the Markov chain.

In the case where the Markov chain is finite, the preceding description of the stationary distribution is a result of the *Frobenius-Perron* theorem (Horn and Johnson 1985), which says that P has a maximum eigenvalue of unity, and the associated left eigenvector, which is the stationary distribution $(u_1, \ldots, u_n)$ for P, exists and has nonnegative entries. Moreover, if P^k is strictly positive for some k (in which case we say P is irreducible), then the stationary distribution has strictly positive entries.

In case a Markov chain is not ergodic, it is informative to know the whole matrix $P^* = (p^*_{ij})$, because p_{ij} tell you the probability of being absorbed by state j, starting from state i. The Frobenius-Perron theorem is useful here also, because it tells us that all the eigenvalues of P are either unity or strictly less than unity in absolute value. Thus, if $D = (d_{ij})$ is the $n \times n$ diagonal matrix with the eigenvalues of P along the diagonal, then $D^* = \lim_{k\to\infty} D^k$ is the diagonal matrix with zeros everywhere except unity where $d_{ii} = 1$. But, if M is the matrix of left eigenvectors of P, then $MPM^{-1} = D$, which follows from the definitions, implies $P^* = M^{-1}D^*M$. This equation allows us to calculate P^* rather easily.

A few examples are useful to get a feel for the recursion equations. Consider first the n-state Markov chain called the *random walk on a circle*, in which there are n states, and from any state $t = 2, \ldots, n-1$ the system moves with equal probability to the previous or the next state, from state n it moves with equal probability to state 1 or state $n-1$, and from state 1 it moves with equal probability to state 2 and to state n. In the long run, it is intuitively clear that the system will be all states with equal probability

$1/n$. To derive this from the recursion equations, note that the probability transition matrix for the problem is given by

$$P = \begin{bmatrix} 0 & 1/2 & 0 & \dots & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & \dots & 0 & 1/2 & 0 \end{bmatrix}.$$

The recursion equations for this system are given by

$$\begin{aligned} u_1 &= \frac{1}{2}u_n + \frac{1}{2}u_2 \\ u_i &= \frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1} \quad i = 2, \dots, n-1 \\ u_n &= \frac{1}{2}u_1 + \frac{1}{2}u_{n-1} \end{aligned}$$

$$\sum_{i=1}^{n} u_i = 1.$$

Clearly, this set of equations has solution $u_i = 1/n$ for $i = 1, \dots, n$. Prove that this solution is unique by showing that if some u_i is the largest of the $\{u_k\}$, then its neighbors are equally large.

Consider next a closely related n-state Markov chain called the *random walk on the line with reflecting barriers*, in which from any state $2, \dots, n-1$ the system moves with equal probability to the previous or the next state, but from state 1 it moves to state 2 with probability 1, and from state n it moves to state $n-1$ with probability 1. Intuition in this case is a bit more complicated, because states 1 and n behave differently from the other states. The probability transition matrix for the problem is given by

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

The recursion equations for this system are given by

$$\begin{aligned} u_1 &= u_2/2 \\ u_i &= u_{i-1}/2 + u_{i+1}/2 \quad i = 2, \ldots, n-1 \\ u_n &= u_{n-1}/2 \\ \sum_{i=1}^{n} u_i &= 1. \end{aligned}$$

It is easy to check directly that $u_i = 1/(n-1)$ for $i = 2, \ldots, n-1$, and $u_1 = u_n = 1/2(n-1)$. In fact, there is a general method for solving difference equations of this type, as described in section 13.6.

We can use the same methods to find other characteristics of a Markov chain. Consider, for instance, the finite random walk, between points $-w$ and w, starting at k, with $0 < k < w$. We assume the end points are absorbing, so we may think of this as a gambler's wealth, where he is equally likely to win, lose, or draw in each period, until he is bankrupt or has reached wealth w. The recursion equations for the mean time to absorption into state $-w$ or w are then given by

$$\begin{aligned} m_{-w} &= 0 \\ m_w &= 0 \\ m_n &= m_n/3 + m_{n-1}/3 + m_{n+1}/3 + 1 \qquad -w < n < w. \end{aligned}$$

We can rewrite the recursion equation as

$$m_{n+1} = 2m_n - m_{n-1} - 3.$$

We can solve this, using the techniques of section 13.6. The associated characteristic equation is $x^2 = 2x - 1$, with double root $x = 1$, so $m_n = a + nb$. To deal with the inhomogeneous part (−3), we try adding a quadratic term, so $m_n = a + bn + cn^2$. We then have

$$a + b(n+1) + c(n^2 + 2n + 1) = 2(a + bn + cn^2) - (a + b(n-1) + c(n-1)^2) - 3$$

which simplifies to $c = 2/3$. To solve for a and b, we use the boundary conditions $m_{-w} = m_w = 0$, getting

$$m_n = \frac{3}{2}(w^2 - n^2).$$

We can use similar equations to calculate the probability p_n of being absorbed at $-w$ if one starts at n. In this case, we have

$$\begin{aligned} p_{-w} &= 1 \\ p_w &= 0 \\ p_n &= p_n/3 + p_{n-1}/3 + p_{n+1}/3 \qquad 0 < n < w. \end{aligned}$$

We now have $p_i = a + bi$ for constants a and b. Now, $p_{-w} = 1$ means $a - bw = 1$, and $p_w = 0$ means $a + bw = 0$, so

$$p_i = \frac{1}{2}\left(1 - \frac{i}{w}\right).$$

Note that the random walk is "fair" in the sense that the expecting payoff if you start with wealth i is equal to $w(1 - p_i) - wp_i = i$.

For an example of a denumerable Markov chain, suppose an animal is in state $d_k = k + 1$ if it has a $k + 1$-day supply of food. The animal forages for food only when $k = 0$, and then he finds a $k + 1$-day supply of food with probability f_k, for $k = 0, 1, \ldots$. This means that the animal surely finds enough food to subsist for at least one day. This is a Markov chain with $p_{0k} = f_k$ for all k, and $p_{k,k-1} = 1$ for $k \geq 1$, all other transition probabilities being zero. The recursion equations in this case are

$$u_i = u_\text{o} f_i + u_{i+1}$$

for $i \geq 0$. If we let $r_k = f_k + f_{k+1} + \ldots$ for $k \geq 0$ (so r_k is the probability of finding at least a $k + 1$ days' supply of food when foraging), it is easy to see that $u_k = r_k u_\text{o}$ satisfies the recursion equations; that is,

$$r_i u_\text{o} = u_\text{o} f_i + r_{i+1} u_\text{o}.$$

The requirement that $\sum_i u_i = 1$ becomes $u_\text{o} = 1/\mu$, where $\mu = \sum_{k=0}^{\infty} r_k$. To see that μ is the expected value of the random variable d, note that

$$\begin{aligned} \mathbf{E}d &= 1f_\text{o} + 2f_1 + 3f_2 + 4f_3 + 5f_4 + \ldots \\ &= r_\text{o} + f_1 + 2f_2 + 3f_3 + 4f_4 \ldots \\ &= r_\text{o} + r_1 + f_2 + 2f_3 + 3f_4 + \ldots \\ &= r_\text{o} + r_1 + r_2 + f_3 + 2f_4 + \ldots \\ &= r_\text{o} + r_1 + r_2 + r_3 + f_4 + \ldots, \end{aligned}$$

and so on.[2]

We conclude that if this expected value does not exist, then no stationary distribution exists. Otherwise, the stationary distribution is given by

$$u_i = r_i/\mu \qquad \text{for } i = 0, 1, \ldots.$$

Note that $\mu = 1/u_o$ is the expected number of periods between visits to state 0, because μ is the expected value of d. We can also show that $1/u_k = \mu/r_k$ is the expected number of periods μ_k between visits to state k, for any $k \geq 0$. Indeed, the fact that $u_k = 1/\mu_k$, where u_k is the probability of being in state k in the long run and μ_k is the expected number of periods between visits to state k, is a general feature of Markov chains with stationary distributions. It is called the *renewal equation*.

Let us prove that $\mu_k = \mu/r_k$ for $k = 2$ in the preceding model, leaving the general case to the reader. From state 2 the Markov chain moves to state 0 in two periods, then requires some number j of periods before it moves to some state $k \geq 2$, and then in $k - 2$ transitions moves to state 2. Thus, if we let v be the expected value of j and we let w represent the expected value of k, we have $\mu_k = 2 + v + w - 2 = v + w$. Now v satisfies the recursion equation

$$v = f_o(1 + v) + f_1(2 + v) + r_2(1),$$

because after a single move the system remains in state 0 with probability f_o and the expected number of periods before hitting $k > 1$ is $1+v$ (the first term), or it moves to state 1 with probability f_1 and the expected number of periods before hitting $k > 1$ is $2 + v$ (the second term), or hits $k > 1$ immediately with probability r_2 (the final term). Solving, we find that $v = (1 + f_1)/r_2$. To find w, note that the probability of being in state k conditional on $k \geq 2$ is f_k/r_2. Thus $v + w = \mu/r_2$ follows from

$$\begin{aligned} w &= \frac{1}{r_2}(2f_2 + 3f_3 + \ldots) \\ &= \frac{1}{r_2}(\mu - 1 - f_1). \end{aligned}$$

[2]More generally, noting that $r_k = \mathrm{P}[d \geq k]$, suppose x is a random variable on $[0, \infty)$ with density $f(x)$ and distribution $F(x)$. If x has finite expected value, then using integration by parts, we have $\int_0^\infty [1-F(x)]dx = \int_0^\infty \int_x^\infty f(y)dydx = xf(x)|_0^\infty + \int_0^\infty xf(x)dx = \mathbf{E}[x]$.

13.2 The Ergodic Theorem for Markov Chains

When are equations (13.1)-(13.3) true, and what exactly do they say? To answer this, we will need a few more concepts. Throughout, we let M be a finite or denumerable Markov chain with transition probabilities $\{p_{ij}\}$. We say a state j can be *reached* from a state i if $p_{ij}^{(m)} > 0$ for some positive integer m. We say a pair of states i and j *communicates* if each is reached from the other. We say a Markov chain is *irreducible* if every pair of states communicates.

If M is irreducible, and if a stationary distribution u exists, then all the u_i in (13.1) are *strictly positive*. To see this, suppose some $u_j = 0$. Then by (13.2), if $p_{ij} > 0$, then $p_i = 0$. Thus, any state that reaches j in one period must also have weight zero in u. But a state i' that reaches j in two periods must pass through a state i that reaches j in one period, and because $u_i = 0$, we also must have $u_{i'} = 0$. Extending this argument, we say that any state i that reaches j must have $u_i = 0$, and because M is irreducible, all the $u_i = 0$, which violates (13.3).

Let q_i be the probability that, starting from state i, the system returns to state i in some future period. If $q_i < 1$, then it is clear that with probability one, state i can occur only a finite number of times. Thus, in the long run we must have $u_i = 0$, which is impossible for a stationary distribution. Thus in order for a stationary distribution to exist, we must have $q_i = 1$. We say a state i is *persistent* or *recurrent* if $q_i = 1$. Otherwise, we say state i is *transient*. If all the states of M are recurrent, we say that M is recurrent.

Let μ_i the be the expected number of states before the Markov chain returns to state i. Clearly, if i is transient, then $\mu_i = \infty$, but even if i is persistent, there is no guarantee that $\mu_i < \infty$. We call μ_i the *mean recurrence time* of state i. If the mean recurrence time of state i is μ_i, M is in state i on average one period out of every μ_i, so we should have $u_i = 1/\mu_i$. In fact, this can be shown to be true whenever the Markov chain has a stationary distribution. This is called the *renewal theorem* for Markov chains. We treat the renewal theorem as part of the ergodic theorem. Thus, if M is irreducible, it can have a stationary distribution only if μ_i is finite, so $u_i = 1/\mu_i > 0$. We say a recurrent state i in a Markov chain is *null* if $\mu_i = \infty$, and otherwise we call the state *non-null*. An irreducible Markov chain cannot have a stationary distribution unless all its recurrent states are non-null.

We say state i in a Markov chain is *periodic* if there is some integer $k > 1$ such that $p_{ii}^{(k)} > 0$ and $p_{ii}^{(m)} > 0$ implies m is a multiple of k. Otherwise, we say M is *aperiodic*. It is clear that if M has a non-null, recurrent, periodic state i, then M cannot have a stationary distribution, because we must have $u_i = \lim_{k\to\infty} p_{ii}^{(k)} > 0$, which is impossible unless $p_{ii}^{(k)}$ is bounded away from zero for sufficiently large k.

An irreducible, non-null recurrent, aperiodic Markov chain is called *ergodic*. We have shown that if an irreducible Markov chain is not ergodic, it cannot have a stationary distribution. Conversely, we have the following *ergodic theorem* for Markov chains, the proof of which can be found in Feller (1950).

THEOREM 13.1 *An ergodic Markov chain M has a unique stationary distribution, and the recursion equations (13.1)-(13.3) hold with all $u_i > 0$. Moreover $u_j = 1/\mu_j$ for each state j, where μ_j is the mean recurrence time for state j.*

We say a subset M' of states of M is *isolated* if no state in M' reaches a state not in M'. Clearly an isolated set of states is a Markov chain. We say M' is an *irreducible set* if M' is isolated and all pairs of states in M' communicate. Clearly, an irreducible set is an irreducible Markov chain. Suppose a Markov chain M consists of an irreducible set M' plus a set A of states, each of which reaches M'. Then, if u' is a stationary distribution of M', there is a stationary distribution u for M such that $u_i = u'_i$ for $i \in M'$ and $u_i = 0$ for $i \in A$. We can summarize this by saying that a Markov chain that consists of an irreducible set of states plus a set of transient states has a unique stationary distribution in which the frequency of the transient states is zero and the frequency of recurrent states is strictly positive. We call such a Markov chain *nearly irreducible*, with transient states A and an absorbing set of states M'.

More generally, the states of a Markov chain M can be uniquely partitioned into subsets $A, M_1, M_2 \ldots$ such that for each i, M_i is nearly irreducible and each state in A reaches M_i for some i. The states in A are thus transient, and if each M_i is non-null and aperiodic, it has a unique stationary distribution. However, M does not have a stationary distribution unless it is nearly irreducible.

13.3 The Infinite Random Walk

The random walk on the line starts at zero and then, with equal probability in each succeeding period, does not move, or moves up or down one unit. It is intuitively clear that in the long run, when the system has "forgotten" its starting point, is equally likely to be in any state. Because there are an infinite number of states, the probability of being in any particular state in the long run is thus zero. Clearly this Markov chain is irreducible and aperiodic. It can be shown to be recurrent, so by the ergodic theorem, it must be null-recurrent. This means that even though the Markov random walk returns to any state with probability one, its mean recurrence time is infinite.

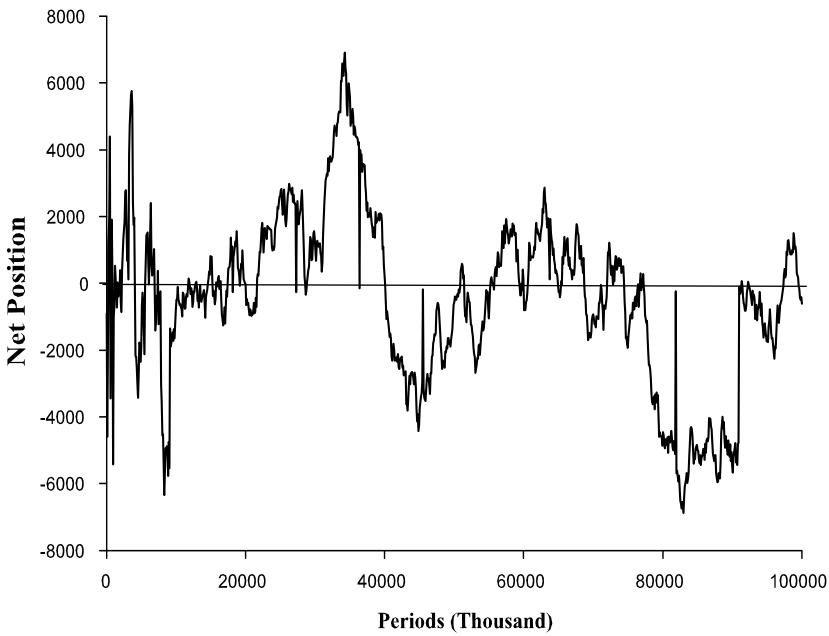


Figure 13.1. The random walk on the line

Perhaps the fact that the recurrence time for the random walk is infinite explains why individuals tend to see statistical patters in random data that are not really there. Figure 13.1 plots the random walk for 100 million periods. The result looks biased in favor of forward from about period 20 million to 50 million, backward 75 million, forward 90 million, and forward thereafter. Of course the maximum deviation from the mean (zero) is less than 2% of the total number of periods.

13.4 The Sisyphean Markov Chain

As an exercise, consider the following *Sisyphean Markov chain*, in which Albert has a piano on his back and must climb up an infinite number of steps $k = 1, 2, \ldots$. At step k, with probability b_k, he stumbles and falls all the way back to the first step, and with probability $1 - b_k$ he proceeds to the next step. This gives the probability transition matrix

$$P = \begin{bmatrix} b_1 & 1-b_1 & 0 & 0 & 0 & \ldots \\ b_2 & 0 & 1-b_2 & 0 & 0 & \ldots \\ b_3 & 0 & 0 & 1-b_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The recursion equations for this system are

$$u_1 = \sum u_i b_i$$
$$u_{k+1} = u_k(1 - b_k) \quad \text{for } k \geq 1,$$

which are satisfied only if

$$u_1(b_1 + (1 - b_1)b_2 + (1 - b_1)(1 - b_2)b_3 + \ldots) = u_1,$$

so either

$$b_1 + (1 - b_1)b_2 + (1 - b_1)(1 - b_2)b_3 + \ldots = 1, \tag{13.4}$$

or $u_1 = \infty$ (note that $u_1 \neq 0$). If $b_i = \alpha$ for some $\alpha \in [0, 1]$ and all $i = 1, 2, \ldots$, it is easy to see that (13.4) is true (let the left-hand side equal $x < \infty$, subtract b_1 from both sides, and divide by $1 - b_1$; now the left-hand side is just x again; solve for x).

Now, because $\sum_i u_i = 1$, u_1, which must satisfy

$$u_1[1 + (1 - b_1) + (1 - b_1)(1 - b_2) + \ldots] = 1.$$

This implies that the Markov chain is ergodic if $b_k = \alpha$ for $\alpha \in (0, 1)$ and indeed $u_i = 1/\alpha$ for $i = 1, \ldots$. The Markov chain is not ergodic if $b_k = 1/k$, however, because the mean time between passages to state 1 is infinite ($b_1 + b_2 + \ldots = \infty$).

13.5 Andrei Andreyevich's Two-Urn Problem

After Andrei Andreyevich Markov discovered the chains that bear his name, he proved the ergodic theorem for finite chains. Then he looked around for an interesting problem to solve. Here is what he came up with—this problem had been solved before, but not rigorously.

Suppose there are two urns, one black and one white, each containing m balls. Of the $2m$ balls, r are red and the others are blue. At each time period $t = 1, \ldots$ two balls are drawn randomly, one from each urn, and each ball is placed in the other urn. Let state i represent the event that there are $i \in [0, \ldots, r]$ red balls in the black urn. What is the probability u_i of state i in the long run?

Let $P = \{p_{ij}\}$ be the $(r+1) \times (r+1)$ probability transition matrix. To move from i to $i-1$, a red ball must be drawn from the black urn, and a blue ball must be drawn from the white urn. This means $p_{i,i-1} = i(m-r+i)/m^2$. To remain in state i, either both balls drawn are red or both are blue, $p_{i,i} = (i(r-i)+(m-i)(m-r+i))/m^2$. To move from i to $i+1$, a blue ball must be drawn from the black urn, and a red ball must be drawn from the white urn. This means $p_{i,i+1} = (m-i)(r-i)/m^2$. All other transition probabilities are zero.

The recursion equations in this case are given by

$$u_i = u_{i-1} p_{i-1,i} + u_i p_{ii} + u_{i+1} p_{i+1,i} \tag{13.5}$$

for $i = 0, \ldots, r+1$, where we set $u_{-1} = u_{r+2} = 0$. I do not know how Andrei solved these equations, but I suspect he guessed at the answer and then showed that it works. At any rate, that is what I shall do. Our intuition concerning the ergodic theorem suggests that in the long run the probability distribution of red balls in the black urn are the same as if m balls were randomly picked from a pile of $2m$ balls (of which r are red) and put in the black urn. If we write the number of combinations of n things taken r at a time as $\binom{n}{r} = n!/r!(n-r)!$, then u should satisfy

$$u_i = \binom{m}{i}\binom{m}{r-i} \Big/ \binom{2m}{r}.$$

The denominator in this expression is the number of ways the r red balls can be allocated to the $2m$ possible positions in the two urns, and the numerator is the number of ways this can be done when i red balls are in the black urn. You can check that u now satisfies the recursion equations.

13.6 Solving Linear Recursion Equations

In analyzing the stationary distribution of a Markov chain, we commonly encounter an equation of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \ldots + a_k u_{n-k}, \tag{13.6}$$

along with some boundary conditions, including $u_i \geq 0$ for all i and $\sum_i u_i = 1$. Note that this recursion equation is *linear* in the sense that if $u_n = g_i(n)$ for $i = 1, \ldots m$ are m solutions, then so are all the weighted sums of the form $u_n = \sum_{j=1}^{m} b_j g(j)$ for arbitrary weights $b_1, \ldots, b_m$.

A general approach to solving such equations is presented by Elaydi (1999) in the general context of difference equations. We here present a short introduction to the subject, especially suited to analyzing Markov chains. First, form the associated k-degree *characteristic equation*

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n, \tag{13.7}$$

The general solution to (13.6) is the weighted sum, with arbitrary coefficients, of solutions $g(n)$ of the following form. First, suppose r is a non-repeated root of (13.7). Then $g(n) = r^n$ is a solution to (13.6). If the root r of (13.7) is repeated m times, then $g(n) = n^j r^n$ are independent solutions for $j = 1, \ldots, m$. Now, choose the weights of the various terms to satisfy the system's boundary conditions.

For instance, the stationary distribution for the reflecting boundaries model with n states, given by (13.4), satisfies

$$u_k = 2u_{k-1} - u_{k-2},$$

for $k = 2, \ldots n-1$, with boundary conditions $u_1 = u_2/2 = u_{n-1}/2 = u_n$ and $u_1 + \ldots + u_n = 1$. The characteristic equation is $x^2 = 2x - 1$, which has the double root $x = 1$. Thus the general form of the solution is $u_k = a \cdot 1^k + bk \cdot 1^k = a + bk$. The symmetry condition then implies that $b = 0$, and the condition $\sum_i u_i = 1$ implies $a = 1/(n-1)$.

Sometimes the recursion equations have an *inhomogeneous* part, as the $g(i)$ in

$$u_i = u_{i-1} p_{i-1,i} + u_i p_{ii} + u_{i+1} p_{i+1,i} + g(i) \tag{13.8}$$

There is no general rule for finding the solution to the inhomogeneous part, but generally trying low-degree polynomials works.

13.7 Good Vibrations

	l	r
l	5,5	0,0
r	0,0	3,3

Consider the pure coordination game in the diagram. We can check using the techniques of chapter 6 that there are two pure-strategy equilibria, ll and rr, as well as a mixed strategy equilibrium. If we represent the out-of-equilibrium dynamics of the game using a replicator process (see chapter 12), the pure strategy equilibria will be stable and the mixed strategy equilibrium unstable. But the concept of stability that is used, although at first glance compelling and intuitive, may be unrealistic in some cases. The idea is that if we start at the equilibrium ll, and we subject the system to a small disequilibrium shock, the system will move back into equilibrium. But in the real world, dynamical systems may be *constantly* subject to shocks, and if the shocks come frequently enough, the system will not have time to move back close to equilibrium before the next shock comes.

The evolutionary models considered in chapters 10 and 12 are certainly subject to continual random "shocks," because agents are paired randomly, play mixed strategies with stochastic outcomes, and update their strategies by sampling the population. We avoided considering the stochastic nature of these processes by implicitly assuming that random variables can be replaced by their expected values, and mutations occur infrequently compared with the time to restore equilibrium. But these assumptions need not be appropriate.

We may move to stochastic differentical equations, where we add a random error term to the right-hand side of an equation such as (11.1). This approach is very powerful, but uses sophisticated mathematical techniques, including stochastic processes and partial differential equations.[3] Moreover, applications have been confined mainly to financial economics. Applying the approach to game theory is very difficult, because stochastic differential equations with more than one independent variable virtually never have a closed-form solution. Consider the following alternative approach, based on the work of H. Peyton Young (1998) and others. We start by modeling adaptive learning with and without errors.

[3]For relatively accessible expositions, see Dixit 1993 and Karlin and Taylor 1981.

13.8 Adaptive Learning

How does an agent decide what strategy to follow in a game? We have described three distinct methods so far in our study of game theory. The first is to determine the expected behavior of the other players and choose a best response ("rational expectations"). The second is to inherit a strategy (e.g., from one's parents) and blindly play it. The third is to mimic another player by switching to the other player's strategy, if it seems to be doing better than one's own. But there is a fourth, and very commonly followed, modus operandi: follow the history of how other players have played against you in the past, and choose a strategy for the future that is a best response to the past play of others. We call this *adaptive learning*, or *adaptive expectations*.

To formalize this, consider an evolutionary game in which each player has limited memory, remembering only $h = \{h_1, h_2, \ldots, h_m\}$, the last m moves of the players with whom he has been paired. If the player chooses the next move as a best response to h, we say the player follows adaptive learning.

Suppose, for instance, two agents play the coordination game in section 13.7, but the payoffs to ll and rr are both $5, 5$. Let $m = 2$, so the players look at the last two actions chosen by their opponents. The best response to ll is thus l, the best response to rr is r, and the best response to rl or lr is any combination of l and r. We take this combination to be: play l with probability 1/2 and r with probability 1/2. There are 16 distinct "states" of the game, which we label $abcd$, where each of the letters can be l or r, b is the previous move by player 1, a is player 1's move previous to this, d is the previous move by player 2, and c is player 2's move previous to this. For instance, $llrl$ means player 1 moved l on the previous two rounds, whereas player 2 moved first r and then l.

We can reduce the number of states to 10 by recognizing that because we do not care about the order in which the players are counted, a state $abcd$ and a state $cdab$ are equivalent. Eliminating redundant states, and ordering the remaining states alphabetically, the states become $llll$, $lllr$, $llrl$, $llrr$, $lrlr$, $lrrl$, $lrrr$, $rlrl$, $rlrr$, and $rrrr$. Given any state, we can now compute the probability of a transition to any other state on the next play of the game. For instance, $llll$ (and similarly $rrrr$) is an *absorbing* state in the sense that, once it is entered, it stays there forever. The state $lllr$ goes to states $llrl$ and $lrrl$, each with probability 1/2. The state $llrl$ goes either to $llll$ where it stays forever, or to $lllr$, each with probability

1/2. The state $lrlr$ goes to $rlrl$ and $rrrr$ each with probability 1/4, and to $rlrr$ with probability 1/2. And so on.

We can summarize the transitions from state to state in a 10×10 matrix $M = (m_{ij})$, where $m_{abcd,efgi}$ is the probability of moving from state $abcd$ to state $efgi$. We call M a *probability transition matrix*, and the dynamic process of moving from state to state is a *Markov chain* (§13.1). Because matrices are easier to describe and manipulate if their rows and columns are numbered, we will assign numbers to the various states, as follows: $llll = 1, lllr = 2, \ldots rrrr = 10$. This gives us the following probability transition matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Also, if we represent the 10 states by the 10 10-dimensional row vectors $\{v_1, \ldots, v_{10}\}$, where $v_1 = (1,0,\ldots,0)$, $v_2 = (0,1,0,\ldots,0)$, and so on, then it is easy to see that, if we are in state v_i in one period, the probability distribution of states in the next period is just $v_i M$, meaning the product of v_i, which is a 1×10 row vector, and M, which is a 10×10 matrix, so the product is another 1×10 row vector. It is also easy to see that the sum of the entries in $v_i M$ is unity and that each entry represents the probability that the corresponding state will be entered in the next period.

If the system starts in state i at $t = 0$, $v_i M$ is the probability distribution of the state it is in at $t = 1$. The probability distribution of the state the system at $t = 2$ can be written as

$$v_i M = p_1 v_1 + \ldots + p_{10} v_{10}.$$

Then, with probability p_j the system has probability distribution $v_j M$ in the second period, so the probability distribution of states in the second period is

$$p_1 v_1 M + \ldots + p_{10} v_{10} M = v_i M^2.$$

Similar reasoning shows that the probability distribution of states after k periods is simply $v_i M^k$. Thus, just as M is the probability transition matrix for one period, so is M^k the probability transition matrix for k periods. To find out the long-run behavior of the system, we therefore want to calculate

$$M^* = \lim_{k\to\infty} M^k.$$

I let Mathematica, the computer algebra software package, calculate M^k for larger and larger k until the entries in the matrix stopped changing or became vanishingly small, and I came up with the following matrix:

$$M^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 5/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5/6 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, no matter where you start, you end up in one of the absorbing states, which is a Pareto-optimal Nash equilibrium. We call pure-strategy Nash equilibria in which all players choose the same strategy *conventions* (Young 1998). We conclude that *adaptive learning leads with probability 1 to a convention.*

13.9 The Steady State of a Markov Chain

There is a simpler way to compute M^* in the previous case. The computation also gives a better intuitive feel for the steady-state solution to the adaptive learning dynamical system generated by a pure coordination game. We know that whatever state we start the system in, we will end up in either state $llll$ or state $rrrr$. For state $abcd$, let P[$abcd$] be the probability that we end up in $llll$ starting from $abcd$. Clearly, P[$llll$] $= 1$ and P[$rrrr$] $= 0$. Moreover, P[$lllr$] $=$ P[$llrl$]$/2 +$ P[$lrrl$]$/2$, because $lllr$ moves to either $llrl$ or to $lrrl$ with equal probability. Generalizing, you can check that, if we define

$$v = (\mathrm{P}[llll], \mathrm{P}[lllr], \ldots, \mathrm{P}[rrrr])',$$

the column vector of probabilities of being absorbed in state $llll$, then we have

$$Mv = v.$$

If we solve this equation for v, subject to $v[1] = 1$, we get

$$v = (1, 2/3, 5/6, 1/2, 1/3, 1/2, 1/6, 2/3, 1/3, 0)',$$

which then must be the first column of M^*. The rest of the columns are zero, except for the last, which must have entries so the rows each sum up to unity. By the way, I would not try to solve the equation $Mv = v$ by hand unless you're a masochist. I let Mathematica do it (v is a *left eigenvector* of M, so Mathematica has a special routine for finding v easily).

13.10 Adaptive Learning II

Now consider the pure coordination game illustrated in section 13.7, where the ll convention Pareto-dominates the rr convention. How does adaptive learning work in such an environment? We again assume each player finds a best response to the history of the other player's previous two moves. The best response to ll and rr are still l and r, respectively, but now the best response to rl or lr is also l. Now, for instance, $lllr$ and $lrlr$ both go to $llll$ with probability 1. The probability transition matrix now becomes as shown.

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

To calculate

$$M^* = \lim_{k\to\infty} M^k$$

is relatively simple, because in this case $M^k = M^4$ for $k \geq 4$. Thus, we have

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, if you start in state $rrrr$, you stay there; otherwise, after four steps you arrive at $llll$ and remain there forever. We conclude that *with adaptive learning, if the system starts in a nonconventional state, it always ends up in the Pareto-efficient conventional state*.

13.11 Adaptive Learning with Errors

We now investigate the effect on a dynamic adaptive learning system when players are subject to error. Consider the pure coordination game illustrated in section 13.7, but where the payoffs to ll and rr are equal. Suppose each player finds a best response to the history of the other player's previous two moves with probability $1-\epsilon$, but chooses incorrectly with probability $\epsilon > 0$. The probability transition matrix now becomes

$$M = \begin{pmatrix} a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & c & d & 0 & 0 & 0 \\ c & 1/2 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 1/2 & c \\ 1/4 & 1/2 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & d & 0 & c & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 2b & a \end{pmatrix},$$

where $a = (1-\epsilon)^2$, $b = \epsilon(1-\epsilon)$, $c = (1-\epsilon)/2$, $d = \epsilon/2$, and $e = \epsilon^2$. Note that now *there are no absorbing states*. To see what happens in the long run, suppose $\epsilon = 0.01$, so errors occur 1% of the time. Using Mathematica to calculate M^*, we find *all the rows are the same*, and each row has the entries

$$(0.442\ 0.018\ 0.018\ 0.001\ 0.0002\ 0.035\ 0.018\ 0.0002\ 0.018\ 0.442)$$

In other words, you spend about 88.4% of the time in one of the conventional states, and about 11.6% of the time in the other states.

It should be intuitively obvious how the system behaves. If the system is in a conventional state, say $llll$, it remains there in the next period with probability $(1-\epsilon)^2 = 98\%$. If one player makes an error, the state moves to $lllr$. If there are no more errors for a while, we know it will return to $llll$ eventually. Thus, it requires multiple errors to "kick" the system to a new convention. For instance, $llll \rightarrow lllr \rightarrow lrrr \rightarrow rrrr$ can occur with just two errors: $llll \rightarrow lllr$ with one error, $lllr \rightarrow lrrr$ with one error, and $lrrr \rightarrow rrrr$ with no errors, but probability 1/2. We thus expect convention flips about every 200 plays of the game.

To test our "informed intuition," I ran 1000 repetitions of this stochastic dynamical system using Mathematica. Figure 13.2 reports on the result.

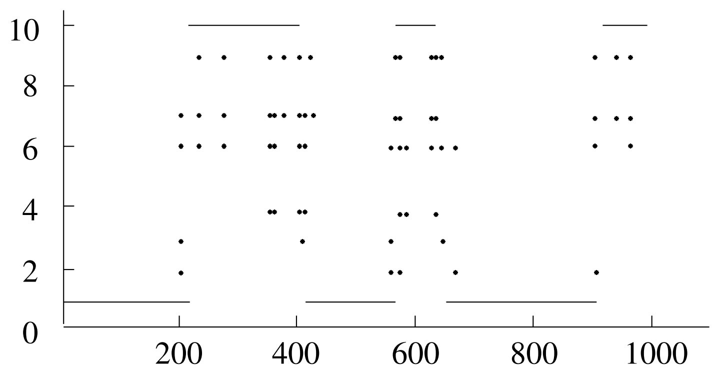


Figure 13.2. An agent-based model adaptive learning with errors.

13.12 Stochastic Stability

We define a state in a stochastic dynamical system to be *stochastically stable* if the long-run probability of being in that state does not become zero or vanishingly small as the rate of error ϵ goes to zero. Clearly, in the previous example $llll$ and $rrrr$ are both stochastically stable and no other state is. Consider the game in section 13.7. It would be nice if the Pareto-dominant equilibrium ll were stochastically stable, and no other state were stochastically stable. We shall see that is the case. Now the probability transition matrix becomes

$$M = \begin{pmatrix} a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 2b & a & 0 & e & 0 & 0 & 0 & 0 & 0 \\ a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b \\ 0 & 0 & a & b & 0 & b & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b \\ a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 2b & a \end{pmatrix},$$

where $a = (1-\epsilon)^2$, $b = \epsilon(1-\epsilon)$, and $e = \epsilon^2$. Again there are no absorbing states. If $\epsilon = 0.01$, we calculate M^*, again we find *all the rows are the same*, and each row has the entries

$$(0.9605 \quad 0.0198 \quad 0.0198 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0).$$

In other words, the system spends 96% of the time in the Pareto-dominant conventional states and virtually all of the remaining time in “nearby states.” It is clear (though it should be formally proved) that ll is the only stochastically stable state.

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Table of Symbols

$\{a, b, x\}$	Set with members a, b and x
$\{x \mid p(x)\}$	The set of x for which $p(x)$ is true
$p \wedge q$	p and q
$p \vee q$	p or q
iff	if and only if
$p \Rightarrow q$	p implies q
$p \Leftrightarrow q$	p if and only if q
(a, b)	Ordered pair: $(a, b) = (c, d)$ iff $a = c \wedge b = d$
$a \in A$	a is a member of the set A
$a \notin A$	a is not a member of the set A
$A \times B$	$\{(a, b) \mid a \in A \wedge b \in B\}$
$\mathbf{R}$	The real numbers
$\mathbf{R}^n$	The n-dimensional real vector space
$(x_1, \ldots, x_n) \in \mathbf{R}^n$	An n-dimensional vector
$f{:}A \to B$	A function $b = f(a)$, where $a \in A$ and $b \in B$
$f(\cdot)$	A function f where we suppress its argument
$f^{-1}(y)$	The inverse of function $y = f(x)$
$\sum_{x=a}^{b} f(x)$	$f(a) + \ldots + f(b)$
$S_1 \times \ldots \times S_n$	$\{(s_1, \ldots s_n) \mid s_i \in S_i, i = 1, \ldots n\}$
ΔS	Set of probability distributions (lotteries) over S
$[a, b]$	$\{x \in \mathbf{R} \mid a \le x \le b\}$
$[a, b)$	$\{x \in \mathbf{R} \mid a \le x < b\}$
$(a, b]$	$\{x \in \mathbf{R} \mid a < x \le b\}$
(a, b)	$\{x \in \mathbf{R} \mid a < x < b\}$
$A \cup B$	$\{x \mid x \in A \vee x \in B\}$
$A \cap B$	$\{x \mid x \in A \wedge x \in B\}$
$A - B$	$\{x \mid x \in A \wedge x \notin B\}$
A^c	$\{x \mid x \notin A\}$
$\cup_\alpha A_\alpha$	$\{x \mid x \in A_\alpha$ for some $\alpha\}$
$\cap_\alpha A_\alpha$	$\{x \mid x \in A_\alpha$ for all $\alpha\}$
$A \subset B$	$A \neq B \wedge (x \in A \Rightarrow x \in B)$
$A \subseteq B$	$x \in A \Rightarrow x \in B$

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Answers

Probability Theory: Answers

1.7 Craps

Roller wins immediately with any one of (16,25,34,43,52,61,56,65), which have probability 8/36. Let $p(4)$ be the probability of rolling a 4. Because there are three ways to roll a 4 (13, 22, 31) out of 36 possible rolls, $p(4) = 3/36$. If the Roller first rolls 4, let $q(4)$ be the probability he wins. The ways of rolling a 2, 7, or 12 are (11, 61, 52, 43, 34, 25, 16, 66), so the probability of "crapping out" is 8/36. This gives us the equation

$$q(4) = 3/36 + (1 - 3/36 - 8/36)q(4)$$

because if you do not crap out or roll a 4, the probability of rolling a 4 before crapping out is still $q(4)$. We can solve this for $q(4)$, getting $q(4) = 3/11$. Thus, the probability Roller wins by first rolling a 4 is $p(4)q(4) = 9/396$. We have $p(5) = 4/36$ and $q(5) = 4/36+(1-4/36-8/36)q(5)$, so $q(5) = 4/12$, and the probability of winning by first throwing a 5 is $p(5)q(5) = 16/432$. Similarly, $p(6) = 5/36$ and $q(6) = 5/13$, so the probability of winning by first throwing a 6 is 25/468. Also, $p(8) = p(6) = 5/36$, so the probability of winning by first throwing an 8 is also 25/468. Again, $p(9) = p(5)$, so the probability of winning by first throwing a 9 is 16/432. Finally $p(10) = p(4)$, so the probability of winning by first throwing a 10 is 9/396. Thus, the probability of winning is

$$8/36+9/396+16/432+25/468+24/468+16/432+9/396 = 6895/15444,$$

or about 44.645%. So, you can see why the casino likes Craps. But, why does Roller like Craps?

1.8 A Marksman Contest

If Alice plays Bonnie twice, she wins the contest with probability $pq+(1-p)qp = pq(2-p)$, but if she plays Carole twice, she wins $qp+(1-q)pq = pq(2-q)$, which is larger.

1.9 Sampling

A die has six possible outcomes. Throwing six dice is like sampling one die six times with replacement. Thus, there are $6^6 = 46656$ ordered configurations of the 6 dice. There are 6 outcomes in which all the faces are the same. Thus, the probability is $6/46656 = 0.0001286$. A more straightforward solution is to note that the second through sixth die must match the first, which happens with probability $(1/6)^5$.

1.10 Aces Up

There are 52 ways to choose the first card, and 51 ways to choose the second card, so there are 52×51 different ways to choose two cards from the deck. There are 4 ways to choose the first ace, and 3 ways to choose the second, so there are 4×3 ways to choose a pair of aces. Thus, the probability of choosing a pair of aces is $12/(52 \times 51) = 1/221 \approx 0.0045248 \approx 0.45248\%$.

1.11 Permutations

Let's first solve the problem for a particular n, say $n = 3$. We can write the $n! = 6$ permutations as follows:

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{array}$$

Each row has exactly $2 = (n-1)!$ matches and there are $3 = n$ rows, so the total number of matches is $6 = n \times (n-1)! = n!$. Thus the average number of matches is $6/6 = n!/n! = 1$. You can generalize this to show that the average number of matches for any n is 1.

1.13 Mechanical Defects

This is sampling 2 times without replacement from a set of 7 objects. There are $7!/(7-2)! = 7 \times 6 = 42$ such samples. How many of these are two nondefective machines? How many samples of two are there from a population of 5 (the number of nondefectives)? The answer is $5!/(5-2)! = 5 \times 4 = 20$. Thus, the probability is $20/42 = 0.4762$.

1.14 Mass Defection

The number of ways of selecting 10 items from a batch of 100 items equals the number of combinations of 100 things taken 10 at a time, which is 100!/10!90!. If the batch is accepted, all of the 10 items must have been chosen from the 90 nondefective items. The number of such combinations of ten items is 90!/10!80!. Thus, the probability of accepting the batch is

$$\begin{aligned}\frac{(90!/10!80!)}{(100!/10!90!)} &= \frac{90!90!}{80!100!} \\ &= \frac{90 \times 89 \times \ldots \times 81}{100 \times 99 \times \ldots 91},\end{aligned}$$

which is approximately 33.04%.

1.15 House Rules

Here is an equivalent game: you ante \$1,000 and choose a number. The house rolls the three dice, and pays you \$2,000 for one match, \$3,000 for two matches, and \$4,000 for three matches. The probability of one match is

$$\binom{3}{1}\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{75}{216},$$

the probability of two matches is

$$\binom{3}{2}\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{15}{216},$$

and the probability of three matches is 1/216. The expected payoff is thus

$$2000\frac{75}{216} + 3000\frac{15}{216} + 4000\frac{1}{216} = \frac{19900}{216} = 921.3.$$

Thus, you can expect to lose \$78.70 every time you play.

1.17 A Guessing Game

Suppose the first guess is k. This is correct with probability $1/n$, high with probability $(k-1)/n$, and low with probability $(n-k)/n$. Thus, the expected number of guesses given that the first guess is k is given by

$$f(n|k) = \frac{1}{n} + \frac{(k-1)[1+f(k-1)]}{n} + \frac{(n-k)[1+f(n-k)]}{n},$$

where $f(0) = 0$. But we also know that

$$f(n) = f(n|1)/n + ... + f(n|n)/n.$$

Thus, we have

$$\begin{aligned} f(n) &= \frac{1}{n} + \sum_{k=1}^{n}(k-1)[1+f(k-1)]/n^2 + \sum_{k=1}^{n}(n-k)[1+f(n-k)]/n^2 \\ &= \frac{1}{n} + \sum_{k=1}^{n}[n-1+(k-1)f(k-1)+(n-k)f(n-k)]/n^2 \\ &= 1 + \frac{2}{n^2}\sum_{k=1}^{n-1}kf(k). \end{aligned}$$

Let us solve this recursive equation. Note that

$$\begin{aligned} f(n) &= 1 + \frac{2}{n^2}[f(1)+2f(2)+...+(n-1)f(n-1)] \\ &= 1 + \frac{2(n-1)}{n^2}f(n-1) \\ &\quad + \frac{(n-1)^2}{n^2}\frac{2}{(n-1)^2} \\ &\quad \times[f(1)+2f(2)+...+(n-2)f(n-2)] \\ &= 1 + \frac{2(n-1)}{n^2}f(n-1) + \frac{(n-1)^2}{n^2}[f(n-1)-1]. \end{aligned}$$

Collecting terms and rearranging a bit, we have

$$\frac{nf(n)-3}{n+1} = \frac{(n-1)f(n-1)-3}{n} + \frac{2}{n}.$$

If we write $g(n) = [nf(n)-3]/(n+1)$, the last equation becomes

$$g(n) = g(n-1) + \frac{2}{n},$$

with $g(1) = [f(1) - 3]/2 = -1$. Thus,

$$g(n) = -3 + 2\sum_{k=1}^{n} k^{-1}.$$

Finally,

$$f(n) = \frac{n+1}{n}\left[-3 + 2\sum_{k=1}^{n} k^{-1}\right] + \frac{3}{n}.$$

We can approximate $f(n)$ for large n by noting that

$$\sum_{k=1}^{n} k^{-1} = \frac{3}{2} \approx \frac{3}{2} + \int_{3}^{n} \frac{dk}{k} = \frac{3}{2} + \ln\left(\frac{n}{3}\right).$$

Thus,

$$f(n) \approx \frac{n+1}{n} \ln\left(\frac{n}{3}\right) + \frac{3}{n} \approx \ln\left(\frac{n}{3}\right) \approx \ln(n).$$

for large n.

1.18 North Island, South Island

Let P_s be the probability of finding the treasure if Bob is on South Island. Then we have

$$P_n = q_n + r_n P_s + (1 - q_n - r_n) P_n$$

and

$$P_s = e_s P_s + (1 - e_s - r_s) P_n.$$

Now, solve these two equations for P_n.

1.21 Extrasensory Perception

Suppose Alice's first draw, a_1, is less than the other player's draw, b. Then the probability Alice's next draw, a_2, is higher than a_1 is given by:

$$\mathrm{P}[a_2 > a_1 | b > a_1] = \frac{\mathrm{P}[a_2 > a_1 \wedge b > a_1]}{\mathrm{P}[b > a_1]}.$$

The numerator in this expression is equal to the probability that a_1 is the lowest of three draws, which is 1/3, and the denominator is equal to the probability that a_1 is the lowest of two draws, which is 1/2. Thus, Alice beats herself on the second draw with probability 2/3, and the overall probability she wins is $(1/2)+(1/2)(2/3)=5/6$.

1.22 Les Cinq Tiroirs

We depict the whole event space as a rectangle with six pieces. Piece A, which consists of 20% of the space, represents the event "the object is not in any drawer."

A	D1	16%
	D2	16%
	D3	16%
	D4	16%
20%	D5	16%

The other five events, D1, D2, D3, D4, and D5, represent the event where the object is in one of the drawers. Because these are equally likely, each such event represents (1-0.2)/5 = 16% of the space.

The probability of D1, which we may write $\mathrm{P}[D1]$ is, of course 16%. The probability of D2 given not D1 is $\mathrm{P}[D2|D1^c]$. We can evaluate this by

$$\mathrm{P}[D2|D1^c]=\frac{\mathrm{P}[D2\wedge D1^c]}{\mathrm{P}[D1^c]}=\frac{\mathrm{P}[D2]}{1-0.16}=0.16/0.84\approx 19\%.$$

The probability of D3 given not D1 or D2 is $\mathrm{P}[D3|D1^c\wedge D2^c]$. We can evaluate this by

$$\mathrm{P}[D3|D1^c\wedge D2^c]=\frac{\mathrm{P}[D3\wedge D1^c\wedge D2^c]}{\mathrm{P}[D1^c\wedge D2^c]}=\frac{\mathrm{P}[D3]}{1-0.16-0.16}$$
$$=0.16/0.68\approx 23.5\%.$$

You can check that the probability of finding the object in the fourth drawer, given that it was not in any previous drawer, is $0.16/0.52 = 30.77\%$, and the probability that it is in the fifth drawer given that it is neither of the first four is $0.16/0.36 = 44.44\%$. So the probability of finding the object rises from drawer 1 to drawer 5.

What about the probability of not finding the object? Let N be the event "the object is in none of the drawers." the $\mathrm{P}[N]=0.2$. What is $\mathrm{P}[N|D1^c]$, the probability it is none of the drawers if it is not in the first drawer. Well, by definition of conditional probability,

$$\mathrm{P}[N|D1^c] = \frac{\mathrm{P}[N \wedge D1^c]}{\mathrm{P}[D1^c]} = \frac{\mathrm{P}[N]}{\mathrm{P}[D1^c]} = 0.2/0.84 = 23.81\%.$$

The probability the object is in none of the drawers if it is found not to be in either of the first two is, similarly (do the reasoning!) $0.2/0.68 = 29.41\%$. It is easy now to do the rest of the problem (do it!).

1.23 Drug Testing

We have $\mathrm{P}[A] = 1/20$ and $\mathrm{P}[\mathrm{Pos}|A] = \mathrm{P}[\mathrm{Neg}|A^c] = 19/20$. Thus,

$$\begin{aligned}\mathrm{P}[A|\mathrm{Pos}] &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + \mathrm{P}[\mathrm{Pos}|A^c]\,\mathrm{P}[A^c]} \\ &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + (1 - \mathrm{P}[\mathrm{Neg}|A^c])\mathrm{P}[A^c]} \\ &= \frac{(19/20)(1/20)}{(19/20)(1/20) + (19/20)(1/20)} = 1/2.\end{aligned}$$

We can answer the problem without using Bayes' rule just by counting. Suppose we test 10,000 people (the number does not matter). Then $10,000 \times 0.05 = 500$ use drugs (on average), of whom $500 \times 0.95 = 475$ test positive (on average). But 9,500 do not use drugs (again, on average), and $9,500 \times (1 - 0.95) = 475$ also test positive (on average). Thus of the 950 ($= 475 + 475$) who test positive, exactly 50% use drugs (on average).

1.25 Urns

For any $k = 0, \ldots, n$, let p_k^s be the probability that you are drawing from the kth urn, given then you have drawn s red balls from the urn. Let R^s be the event "drew s red balls from the urn," and let U_k be the event "you are drawing from urn k." Then, we have

$$p_k^s = \frac{P[R^s|U_k]P[U_k]}{\sum_{i=0}^n P[R^s|U_i]P[U_i]} = \frac{P[R^s|U_k]}{\sum_{i=0}^n P[R^s|U_i]} = \frac{k^s}{\sum_{i=0}^n i^s}.$$

Let R be the event "the next ball drawn is red." Then, the probability the next ball will be red, given that we have already drawn s red balls, which we can write $P[R|R^s]$, is given by

$$P[R|R^s] = \sum_{i=0}^{n} P[R|U_i]p_i^s = \sum_{i=0}^{n} P[R|U_i]\frac{i^s}{\sum_{j=0}^{n} j^s}$$
$$= \sum_{i=0}^{n} \frac{i}{n+1}\frac{i^s}{\sum_{j=0}^{n} j^s} = \frac{\sum_{j=0}^{n} j^{s+1}}{(n+1)\sum_{j=0}^{n} j^s}.$$

These expressions have closed form evaluations, but we can approximate them more easily by integrals. Thus,

$$\sum_{i=0}^{n} i^s \approx \int_0^n x^s dx = n^{s+1}/(s+1).$$

Thus,

$$P[R|R^s] \approx \frac{n^{s+2}}{s+2}\frac{1}{n+1}\frac{s+1}{n^{s+1}} = \frac{n}{n+1}\frac{s+1}{s+2}.$$

1.26 The Monty Hall Game

Let p be the event that the contestant chooses the winning door, say door A, so $\mathrm{P}[p] = 1/3$. Let q be the event that Monty Hall chooses a door, say door B, from among the other two doors, and door B has no prize behind it. From Bayes' rule, we have

$$\mathrm{P}[p|q] = \frac{\mathrm{P}[q|p]\mathrm{P}[p]}{\mathrm{P}[q]}.$$

But $\mathrm{P}[q|p] = 1$, because Monty Hall cannot choose door A, so if p holds, than q must also hold. Thus we have

$$\mathrm{P}[p|q] = \frac{1}{3\mathrm{P}[q]}.$$

If Monty Hall chose a door that he *knew* has no prize behind it, then $\mathrm{P}[q] = 1$, so $\mathrm{P}[p|q] = 1/3$. The probability that the prize is behind door C is then $1 - 1/3 = 2/3$, so the contestant doubles the probability of winning the prize by shifting from door A to door C. However, if Monty Hall chooses *randomly* between doors B and C, then $P[q] = 2/3$, so $P[p|q] = 1/2$. The probability that the prize is behind door C is then $1 - 1/2 = 1/2$, so the contestant cannot gain from shifting.

It is instructive to generalize this to n doors. The contestant chooses a door, say A, and the event q is now that Monty Hall opens all the other doors but one, and none has a prize behind it. Does the contestant gain from switching?

We now have $P[p] = 1/n$ and $P[q|p] = 1$. Thus $P[p|q] = 1/nP[q]$. If Monty Hall always chooses a door with no prize behind it, then $P[q] = 1$, so $P[p|q] = 1/n$, and the probability that the prize is behind the remaining door is then $1 - 1/n = (n-1)/n$. Thus, for $n \geq 3$, the contestant gains by switching. However, if Monty Hall chose randomly, then $P[q] = (n-2)/n$. This is because the probability that the prize is behind one of the two doors he did not choose is just $2/n$. In this case, then, $P[p|q] = 1/(n-2)$, so the probability the prize is behind the other unopened door is $(n-1)/(n-2) > 1/(n-2)$, so the contestant gains (a lot!) from shifting.

1.27 The Logic of Murder and Abuse

First, from Bayes' rule,

$$P[C|A] = \frac{P[A|C]\,P[C]}{P[A]}.$$

This is the probability that a man murders his wife if he has abused her. But from (d) above, $P[A|C] = 9/10$; from (c) $P[C] = 1/4000$; from (a), $P[A] = 1/20$; so we find $P[C|A] = 4.50\%$.

"I object!" says the chief prosecutor. "The defense ignores the fact that Nicole was *murdered*. What we *really* must know is $P[C|AB]$, the probability a *murdered* woman who was abused by her husband was murdered by him." "But," splutters the astounded judge, "how could you calculate such a complex probability?" A computer projector is brought into the court, and the chief prosecutor reveals the following calculation, the astute jurors taking mental notes. "We have," says the prosecutor,

$$P[C|AB] = \frac{P[ABC]}{P[AB]} = \frac{P[AC]}{P[ABC] + P[ABC^c]} =$$

$$\frac{\mathrm{P}[A|C]\,\mathrm{P}[C]}{\mathrm{P}[AC]+\mathrm{P}[A|BC^c]\,\mathrm{P}[BC^c]}=\frac{\mathrm{P}[A|C]\,\mathrm{P}[C]}{\mathrm{P}[A|C]\,\mathrm{P}[C]+\mathrm{P}[A](\mathrm{P}[B]-\mathrm{P}[C])},$$

where $\mathrm{P}[A|BC^c] = \mathrm{P}[A]$ by (e). From (b), $\mathrm{P}[B] = 1/200$, so $\mathrm{P}[C|B] = \mathrm{P}[C]/\mathrm{P}[B] = 1/2$, so we have $\mathrm{P}[C|AB] = 18/19 = 94.74\%$.

1.29 The Greens and the Blacks

Let A be the event "A bridge hand contains at least two aces." Let B be the event "A bridge hand contains at least one ace." Let C be the event "A bridge hand contains the ace of spades."

Then $\mathrm{P}[A|B]$ is the probability that a hand contains two aces if it contains one ace and hence is the first probability sought. Also $\mathrm{P}[A|C]$ is the probability a hand contains two aces if it contains the ace of spades, which is the second probability sought. By Bayes' rule,

$$\mathrm{P}[A|B]=\frac{\mathrm{P}[AB]}{\mathrm{P}[B]}=\frac{\mathrm{P}[A]}{\mathrm{P}[B]}\quad\text{and}\quad\mathrm{P}[A|C]=\frac{\mathrm{P}[AC]}{\mathrm{P}[C]}.$$

Clearly, $\mathrm{P}[C] = 0.25$, because all four hands are equally likely to get the ace of spades.

To calculate $\mathrm{P}[B]$, note that the total number of hands with no aces is the number of ways to take 13 objects from 48 (the 52 cards minus the four aces), which is $\binom{48}{13}$.

The probability of a hand having at least one ace is then

$$\mathrm{P}[B]=\frac{\binom{52}{13}-\binom{48}{13}}{\binom{52}{13}}=1-\frac{39\times 38\times 37\times 36}{52\times 51\times 50\times 49}=0.6962.$$

The probability of at least two aces is the probability of at least one ace minus the probability of exactly one ace. We know the former, so let's calculate the latter.

The number of hands with exactly one ace is four times $\binom{48}{12}$, because you can choose the ace in one of four ways, and then choose any combination of 12 cards from the 48 non-aces. But

$$\frac{4\times\binom{48}{12}}{\binom{52}{13}}=\frac{39\times 38\times 37}{51\times 50\times 49}\approx 0.4388,$$

which is the probability of having exactly one ace. The probability of at least two aces is thus

$$\mathrm{P}[A] = .6962 - .4388 = .2574$$

(to four decimal places).

Now $\mathrm{P}[AC]$ is the probability of two aces including the ace of spades. The number of ways to get the ace of spades plus one other ace is calculated as follows: take the ace of spades out of the deck, and form hands of twelve cards. The number of ways of getting no aces from the remaining cards is $\binom{48}{12}$, so the number of hands with one other ace is $\binom{51}{12} - \binom{48}{12}$. The probability of two aces including the ace of spades is thus

$$\frac{\binom{51}{12} - \binom{48}{12}}{\binom{52}{13}} = .1402.$$

Thus, $\mathrm{P}[AC] = .1402$. We now have

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]} = \frac{.1402}{.25} = .5608 > \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{.2574}{.6962} = .3697.$$

1.30 The Brain and Kidney Problem

Let A be the event "the jar contains two brains," and let B be the event "the mad scientist pulls out a brain." Then $\mathrm{P}[A] = \mathrm{P}[A^c] = 1/2$, $\mathrm{P}[B|A] = 1$, and $\mathrm{P}[B|A^c] = 1/2$. Then from Bayes' rule, the probability that the remaining blob is a brain is $\mathrm{P}[A|B]$, which is given by

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]\mathrm{P}[A]}{\mathrm{P}[B|A]\mathrm{P}[A] + \mathrm{P}[B|A^c]\mathrm{P}[A^c]} = \frac{1/2}{1/2 + (1/2)(1/2)} = 2/3.$$

1.31 The Value of Eyewitness Testimony

Let G be the event "Cab that hit Alice was green," let B be the event "cab that hit Alice was blue," let WB be the event "witness records seeing blue cab," and finally, let WG be the event "witness records seeing green cab." We have $\mathrm{P}[G] = 85/100 = 17/20$, $\mathrm{P}[B] = 15/100 = 3/20$, $\mathrm{P}[WG|G] = \mathrm{P}[WB|B] = 4/5$, $\mathrm{P}[WB|G] = \mathrm{P}[WG|B] = 1/5$. Then Bayes' rule yields

$$\mathrm{P}[B|WB] = \frac{\mathrm{P}[WB|B]\mathrm{P}[B]}{\mathrm{P}[WB|B]\mathrm{P}[B] + \mathrm{P}[WB|G]\mathrm{P}[G]},$$

which evaluates to 12/29.

1.32 When Weakness Is Strength

Suppose a player is picked randomly to shoot in each round. It remains true in this case that Alice and Bob will shoot at each other until only one of them remains. However, clearly Carole now prefers to have a one-on-one against Bob rather than against Alice, so Carole will shoot at Alice if given the chance. Now

$$\pi_a(ab) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{5}\pi_a(ab),$$

so $\pi_a(ab) = 5/9$ and $\pi_b(ab) = 4/9$. Similar reasoning gives $\pi_a(ac) = 2/3$ and $\pi_c(ac) = 1/3$. Finally,

$$\pi_b(bc) = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{5}\pi_b(bc)\right) + \frac{1}{2} \times \frac{1}{2}\pi_b(bc),$$

from which we conclude $\pi_b(bc) = 8/13$ and $\pi_c(bc) = 5/13$. Now clearly $\pi_a[a] = \pi_a(ac) = 2/3$, $\pi_b[a] = 0$, and $\pi_c[a] = 1/3$. Similarly, it is easy to check that

$$\begin{aligned}
\pi_b[b] &= (4/5)\pi_b(bc) + (1/5)\pi_b \\
\pi_a[b] &= (1/5)\pi_a \\
\pi_c[b] &= (4/5)pi_c(bc) + (1/5)\pi_c \\
\pi_c[c] &= (1/2)\pi_c(bc) + (1/2)\pi_c \\
\pi_b[c] &= (1/2)\pi_b(bc) + (1/2)\pi_b \\
\pi_a[c] &= (1/2)\pi_a.
\end{aligned}$$

Moving to the final calculations, we have

$$\pi_b = \frac{1}{3}\left[0 + \frac{4}{5}\pi_b(bc) + \frac{1}{5}\pi_b + \frac{1}{2}\pi_b(bc) + \frac{1}{2}\pi_b\right].$$

We can solve this for π_b, getting $\pi_b = 24/69$. The similar equation for marksman Alice is

$$\pi_a = \frac{1}{3}\left[\frac{2}{3} + \frac{1}{5}\pi_a + \frac{1}{2}\pi_a\right],$$

which gives $\pi_a = 20/69$. Finally,

$$\pi_c = \frac{1}{3}\left[\frac{1}{3} + \frac{4}{5}\pi_c(bc) + \frac{1}{5}\pi_c + \frac{1}{2}\pi_c(bc) + \frac{1}{2}\pi_c\right],$$

which gives $\pi_c = 25/69$. Clearly, $\pi_c > \pi_b > \pi_a$, so the meek inherit the earth.

1.33 From Uniform to Exponential

Let p_k be the probability that $n = k$. Then, Alice wins \$$k$ with probability p_k, so her average winnings are

$$\begin{aligned} W &= 2p_2 + 3p_3 + 4p_4 + \dots \\ &= 2 + p_3 + 2p_4 + \dots \\ &= 2 + (p_3 + p_4 + \dots) + (p_4 + p_5 + \dots) + \dots \\ &= 2 + \mathrm{P}[n > 2] + \mathrm{P}[n > 3] + \dots \\ &= 2 + 1/2! + 1/3! + \dots = e, \end{aligned}$$

where $e \approx 2.71$ is the base of the natural logarithms.

1.34 Laplace's Law of Succession

Suppose there are n balls in the urn, and assume the number of white balls is uniformly distributed between 0 and n. Let A_k be the event "there are k white balls," and let B_{rm} be the event "of m balls chosen with replacement, r are white." Then $\mathrm{P}[A_k] = 1/(n+1)$, and by Bayes' rule we have

$$\mathrm{P}[A_k|B_{rm}] = \frac{\mathrm{P}[B_{rm}|A_k]\mathrm{P}[A_k]}{\mathrm{P}[B_{rm}]}.$$

Now it is easy to check that

$$\mathrm{P}[B_{rm}|A_k] = \binom{m}{r}\left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{m-r}$$

and

$$\mathrm{P}[B_{rm}] = \sum_{k=0}^{n} \mathrm{P}[A_k]\mathrm{P}[B_{rm}|A_k]. \tag{A1}$$

The probability of choosing a white ball on the next draw is then

$$\begin{aligned} \sum_{k=0}^{n}\left(\frac{k}{n}\right)\mathrm{P}[A_k|B_{rm}] &= \sum_{k=0}^{n}\frac{k\mathrm{P}[B_{rm}|A_k]}{n(n+1)\mathrm{P}[B_{rm}]} \\ &= \frac{1}{(n+1)\mathrm{P}[B_{rm}]}\binom{m}{r}\sum_{k=0}^{n}\left(\frac{k}{n}\right)^{r+1}\left(1-\frac{k}{n}\right)^{m-r}. \end{aligned}$$

To approximate this expression, note that if n is large, equation (A1) is a Riemann sum representing the integral

$$\mathrm{P}[B_{rm}] \approx \frac{1}{n+1}\binom{m}{r}\int_0^1 x^r(1-x)^{m-r} = \frac{1}{(n+1)(m+1)}, \tag{A2}$$

where the integral is evaluated by integration by parts r times. Replacing m by $m+1$ and r by $r+1$ in the preceding expression, we see that equation (AA2) is approximately

$$\frac{1}{(n+1)\mathrm{P}[B_{rm}]}\frac{m!}{r!(m-r)!}\frac{(r+1)!(m-r)!}{(m+2)(m+1)!} = \frac{r+1}{m+2}.$$

Eliminating Dominated Strategies: Answers

4.3 Exercises in Eliminating Dominated Strategies

(c) $N_2 < J_2$, $C_1 < N_1$, $J_2 < C_2$, $N_1 < J_1$.
(d) $C > D$, $e > a$, $B > E$, $c > b$, $B > A$, $c > d$, $B > C$,$c > e$.

4.6 Second-Price Auction

Suppose first you win, and let v_s be the second-highest bid. If you had bid more than v_i, you still would have won, and your gain would still be the same, namely $v_i - v_s \geq 0$. If you had bid lower than v_i, there are three subcases: you could have bid more than, equal to, or less than v_s. If you had bid more than v_s, you would have had the same payoff, $v_i - v_s$. If you had bid equal to v_s, you could have lost the auction in the playoff among the equally high bidders, and if you had bid less than v_s, you certainly would have lost the auction. Hence, nothing beats bidding v_i in case you win.

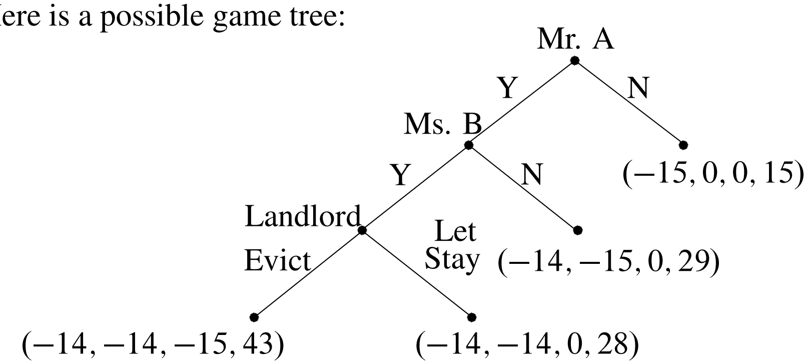
But suppose you bid v_i and lost. Let v_h be the highest bid and v_s be the second-highest bid. Because you lost, your payoff is zero, so if you had bid less than v_i, you would still have lost, so you could not improve your payoff this way. If had you bid more than v_i, it would not matter unless you had bid enough to win the auction, in which case your gain would have been $v_s - v_i$. Because $v_i \neq v_h$, we must have $v_i \leq v_s$, as v_s is the second-highest offer. Thus, you could not have made a positive gain by bidding higher than v_i.

Hence, bidding v_i is a best response to any set of bids by the other players.

a. Because "truth telling" is a dominant strategy, it remains a best response no matter what the other players do.
b. Yes, it could matter. For instance, suppose you are bidder 1 and all other bidders $i = 2, \ldots, n$ follow the strategy of bidding zero first, and bidding \$1 more than the highest bid, provided the highest bid is less that v_i. Then, if you bid an amount greater than the largest v_i for the other players, you win and pay zero. If you bid your value v_1, by contrast, and some $v_i > v_1 + 1$, you will not win the auction.
c. If every player uses truth telling except you, you can bid a very small amount lower that the highest value v_i, ensuring that the winner of the lottery has very small payoff.

4.8 The Eviction Notice

Here is a possible game tree:



4.9 Hagar's Battles

Each side should deploy its troops to the most valuable battlefields. To see this, suppose player 1 does not. Let x_j be the highest value battlefield unoccupied by player 1, and let x_i be the lowest value battlefield occupied by player 1. What does player 1 gain by switching a soldier from x_i to x_j? If both are occupied by player 2, there is no change. If neither is occupied by player 2, player 1 gains $a_j - a_i > 0$. If player 2 occupies x_j but not x_i, player 1 loses a_i by switching, and player 2 loses a_j, so player 1 gains $a_j - a_i > 0$. Similarly if player 2 occupies x_i but not x_j.

Another explanation: Suppose you occupy a_i but not a_j, where $a_j > a_i$. The figure below shows that the gain from switching from a_i to a_j is positive in all contingencies.

Enemy Occupies

	a_i not a_j	a_j not a_i	a_i and a_j	neither
loss	lose i	lose i	lose i	lose i
gain	gain j	gain j	gain j	gain j
net gain	$j - i$	$j - i$	$j - i$	$j - i$

4.10 Military Strategy

First we can eliminate all country I strategies that do not arrive at A. This leaves six strategies, which we can label fcb, feb, fed, hed, heb, and hgd.

We can also eliminate all country A strategies that stay at A at any time, or that hit h or f. This leaves the six strategies bcb,beb,bed,ded,deb,dgd. The payoff matrix is:

	bcb	beb	bed	ded	deb	dgd
fcb	−1	−1	1	1	−1	1
feb	−1	−1	−1	−1	−1	1
fed	1	−1	−1	−1	−1	−1
hed	1	−1	−1	−1	−1	−1
heb	−1	−1	−1	−1	−1	1
hgd	1	1	−1	−1	1	−1

Now feb is weakly dominated by fcb, as is heb. Moreover, we see that fed and hed are weakly dominated by hgd. Thus there are two remaining strategies for country I, "south" (hgd) and "north" (fcb).

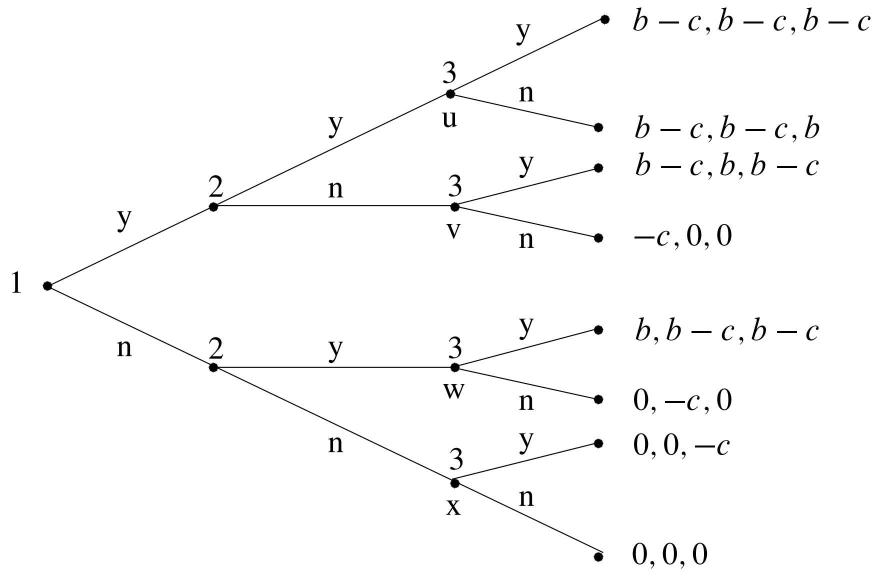
Also bcb is dominated by beb and dgd is dominated by ded, so we may drop them. Moreover, beb and deb are the same "patrol north," whereas bed and ded are the same "patrol south." This gives us the reduced game:

	patrol north	patrol south
attack north	−1,1	1,−1
attack south	1,−1	−1,1

So this complicated game is just the heads-tails game, which we will finish solving when we do mixed-strategy equilibria!

4.12 Strategic Voting

We can solve this by pruning the game tree. We find that player 1 chooses no, and players 2 and 3 choose yes, with payoff $(b, b-c, b-c)$. It is best to go first. The game tree is the following:



Note that this does not give a full specification of the strategies, because player 2 has 4 strategies and player 3 has 16 strategies. The preceding description says only what players 2 and 3 do "along the game path," that is, as the game is actually played.

To describe the Nash equilibrium in full, let us write player 3's strategies as "uvwx," where u, v, w, and x are each either y (yes) or n (no) and indicate the choice at the corresponding node in the game tree, starting from the top. Then the third player's choice is nyyn. Similarly, player 2's choice is ny, and the first player's is, of course, n.

If player 3 chooses nnnn, player 2 chooses yn, and player 1 chooses y, we have another Nash equilibrium (check it out!), in which player 3 now gets b and the other two get $b-c$. The equilibrium is strange because it means that player 3 should make suboptimal choices at nodes v and w; he says he will choose "no" but in fact he will choose "yes" at these nodes, because this gives him a higher payoff. The strategy nnnn is called an *incredible threat*, because it involves player 3 threatening to do something that he in fact will not do when it comes time to do it. But if the others believe him, he will never have to carry out his threat! We say such a Nash equilibrium violates subgame perfection.

Pure-Strategy Nash Equilibria: Answers

5.4 The Tobacco Market

a. Let's not use numbers until we need to. We can write $p = a - bq$, where $q = q_1 + q_2 + q_3$, and q_i is the amount sent to market by farmer i. Farmer 1 maximizes $pq_1 = (a - bq)q_1$. If there is an interior solution, the first-order condition on q_1 must satisfy

$$a - b(q_2 + q_3) - 2bq_1 = 0.$$

If all farmers ship the same amount of tobacco, then $q_2 = q_3 = q_1$, so this equation becomes $4bq_1 = a$, which gives $q_1 = q_2 = q_3 = a/4b$, and $q = 3a/4b$, so $p = a/4$. The revenue of each farmer is $pq = a^2/16b$. In our case $a = 10$ and $b = 1/100000$, so the price is \$2.50 per pound, and each farmer ships 250,000 pounds and discards the rest. The price support does not matter, because $p >$ \$0.25. Each farmer has profit \$625,000.

If the second and third farmers send their whole crop to market, then $q_2 + q_3 = 1,200,000$. In this case even if farmer 1 shipped nothing, the market price would be $10 - 1,200,000/100,000 = -2 < 0.25$, so the price support would kick in. Farmer 1 should then also ship all his tobacco at \$0.25 per pound, and each farmer has profit \$150,000.

b. You can check that there are no other Nash equilibria. If one farmer sends all his crop to market, the other two would each send 400,000/3 pounds to market. But then the first farmer would gain by sending less to market.

5.5 The Klingons and the Snarks

Suppose the Klingons choose a common rate r of consumption. Then each eats 500 snarks, and each has payoff

$$u = 2000 + 50r - r^2.$$

Setting the derivative u' to zero, we get $r = 25$, so each has utility $u = 2000 + 50(25) - 25^2 = 2625$.

Now suppose they choose their rates separately. Then

$$u_1 = \frac{4000r_1}{r_1 + r_2} + 50r_1 - r_1^2.$$

Setting the derivative of this to zero, we get the first-order condition

$$\frac{\partial u_1}{\partial r_1} = \frac{4000r_2}{(r_1 + r_2)^2} + 50 - 2r_1 = 0,$$

and a symmetrical condition holds for the second Klingon:

$$\frac{\partial u_2}{\partial r_2} = \frac{4000r_1}{(r_1 + r_2)^2} + 50 - 2r_2 = 0.$$

These two imply

$$\frac{r_2}{r_1} = \frac{r_1 - 25}{r_2 - 25},$$

which has solutions $r_1 = r_2$ and $r_1 + r_2 = 25$. The latter, however, cannot satisfy the first-order conditions. Setting $r_1 = r_2$, we get

$$\frac{4000}{4r_1} + 50 - 2r_1 = 0,$$

or $1000/r_1 + 50 - 2r_1 = 0$. This is a quadratic that is easy to solve. Multiply by r_1, getting $2r_1^2 - 50r_1 - 1000 = 0$, with solution $r = (50 + \sqrt(2500 + 8000))/4 = 38.12$. So the Klingons eat about 50% faster than they would if they cooperated! Their utility is now $u = 2000 + 50r_1 - r_1^2 = 2452.87$, lower than if they cooperated.

5.6 Chess: The Trivial Pastime

We will have to prove something more general. Let's call a game *Chessian* if it is a finite game of perfect information in which players take turns, and the outcome is either (win,lose), (lose,win), or (draw,draw), where win is preferred to draw, and draw is preferred to lose. Let us call a game *certain* if it has a solution in pure strategies. If a Chessian game is certain, then clearly either one player has a winning strategy or both players can force a draw. Suppose there were a Chessian game that is not certain. Then there must be a *smallest* Chessian game that is not certain (that is, one with fewest nodes). Suppose this has k nodes. Clearly, $k > 1$, because it is obvious that a Chessian game with one node is certain. Take any node all of whose child nodes are terminal nodes (why must this exist?). Call this node A. Suppose Red (player 1) chooses at A (the argument is similar if Black chooses). If one of the terminal nodes from A is (win,lose), label A (lose,win); if all of the terminal nodes from A are (lose,win), label A (win,lose); otherwise label A (draw,draw). Now erase the branches from A, along with their terminal nodes. Now we have a new, smaller, Chessian game, which is certain, by our induction assumption. It is easy to see that if Red has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Similarly, if Black has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Finally, if both players can force a draw in the smaller game, their respective strategies must force a draw in the larger game.

5.7 No-Draw, High-Low Poker

You can check that 0.75RR + 0.25SR, SR and 0.33RR + 0.67RS, SR are additional Nash equilibria.

Mixed-Strategy Nash Equilibria: Answers

6.4 Tennis Strategy

$$
\begin{aligned}
\sigma &= \alpha b_r + (1-\alpha) f_r, \quad \tau = \beta b_s + (1-\beta) f_s \\
\pi_{b_s} &= \alpha \pi_1(b_s, b_r) + (1-\alpha)\pi_1(b_s, f_r) \\
&\quad \text{where } \pi_1(b_s, b_r) = \text{the server's payoff to } b_s, b_r \\
&= .4\alpha + .7(1-\alpha) = .7 - .3\alpha \\
\pi_{f_s} &= \alpha \pi_1(f_s, b_r) + (1-\alpha)\pi_1(f_s, f_r) \\
&= .8\alpha + .1(1-\alpha) = .1 + .7\alpha \\
.7 - .3\alpha &= .1 + .7\alpha \Rightarrow \boxed{\alpha = 3/5} \\
\pi_{b_r} &= \beta \pi_2(b_s, b_r) + (1-\beta)\pi_2(f_s, b_r) \\
&= .6\beta + .2(1-\beta) = .2 + .4\beta \\
\pi_{f_r} &= \beta \pi_2(b_s, f_r) + (1-\beta)\pi_2(f_s, f_r) \\
&= .3\beta + .9(1-\beta) = .9 - .6\beta \\
.2 + .4\beta &= .9 - .6\beta \Rightarrow \boxed{\beta = 7/10}
\end{aligned}
$$

Payoffs to Players:

$$
\pi_1 = .4 \cdot \frac{3}{5} + .7 \cdot \frac{2}{5} = .52, \quad \pi_2 = .6 \cdot \frac{7}{10} + .2 \cdot \frac{3}{10} = .48.
$$

6.8 Robin Hood and Little John

The payoff matrix is:

	G	W
G	$-\delta - \tau_{lj}/2$ $-\delta - \tau_r/2$	0 $-\tau_r$
W	$-\tau_{lj}$ 0	$-\epsilon - \tau_{lj}/2$ $-\epsilon - \tau_r/2$

The pure Nash equilibria are:

$$
\begin{aligned}
GG&: \quad \tau_r, \tau_{lj} \geq 2\delta \\
WG&: \quad 2\delta \geq \tau_{lj} \\
GW&: \quad 2\delta \geq \tau_r.
\end{aligned}
$$

For the mixed-strategy equilibrium, we have

$$\alpha_{lj} = \frac{\epsilon + \tau_{lj}/2}{\epsilon + \delta}, \qquad \alpha_r = \frac{\epsilon + \tau_r/2}{\epsilon + \delta}$$

for $2\delta > \tau_r, \tau_{lj}$.

Suppose $\tau_r > \tau_{lj}$. Then, the socially optimal δ is any δ satisfying $\tau_r > 2\delta > \tau_{lj}$, because in this case it never pays to fight. The cost of crossing the bridge is τ_{lj} (or $\tau_r + 2\tau_{lj}$), including the crossing time itself. Of course, this makes Robin Hood wait all the time. He might prefer to lower or raise the costs of fighting. Will he? The payoff to the game to the players when $\tau_r > 2\delta > \tau_{lj}$ is $(-\tau_{lj}, 0)$.

Suppose Robin Hood can shift to lower-cost confrontation: we lower δ so $\tau_r > \tau_{lj} > 2\delta$. Then, GG is dominant, and the gain to the two players is $(-\delta - \tau_{lj}/2, -\delta - \tau_r/2)$, which is better for Robin Hood if and only if $-\tau_{lj} < -\delta - \tau_{lj}/2$, or $2\delta < \tau_{lj}$, *which is true*! Therefore, *Robin Hood gains if he can shift to a lower-cost form of fighting*.

Suppose Robin Hood can shift to a higher-cost warfare. We raise δ so $2\delta > \tau_r > \tau_{lj}$. Now the mixed-strategy solution obtains, and the payoff to Robin Hood is $(-\delta - \tau_{lj}/2)(\in +\tau_{lj}/2)/(\in +\delta)$, which it is easy to see is always less than $-\tau_{lj}$. Thus, *Robin Hood never wants to shift to a higher-cost form of fighting*, even though he would win some of the time.

6.9 The Motorist's Dilemma

The normal form matrix for the game is:

	G	W	C
G	$-\delta - \tau/2, -\delta - \tau/2$	$0, -\tau$	$0, -\tau$
W	$-\tau, 0$	$-\epsilon - \tau/2, -\epsilon - \tau/2$	$-\tau, 0$
C	$-\tau, 0$	$0, -\tau$	$-\delta - \tau/2, -\delta - \tau/2$

Write $\sigma = \tau/2\delta < 1$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ represent Bob and Alice's mixed strategies, where (u_1, u_2) means play G with probability u_1, play W with probability u_2, and play C with probability $1 - u_1 - u_2$. Similarly for (v_1, v_2). Let $\delta = \{(x, y) | 0 \leq x, y, x + y \leq 1\}$, so δ is the strategy space for both players.

It is easy to check that the payoff to the pair of mixed strategies (u, v) for Bob is

$$\begin{aligned} f_1(u,v) = -(2\delta v_1 + (\delta+\tau/2)(v_2-1))u_1 - ((\delta-\tau/2)(v_1-1) \\ + (\delta+\epsilon)v_2)u_2 + (\delta-\tau/2)v_1 \\ + (\delta+\tau/2)(v_2-1), \qquad \text{(A3)} \end{aligned}$$

and the payoff $f_2(u, v)$ to Alice is, by symmetry, $f_2(u, v) = f_1(v, u)$. The players reaction sets are given by

$$R_1 = \{(u,v) \in \delta \times \delta | f_1(u,v) = \max_{\mu} f_1(\mu, v)\}$$
$$R_2 = \{(u,v) \in \delta \times \delta | f_2(u,v) = \max_{\mu} f_2(\mu, v)\},$$

and the set of Nash equilibria is $R_1 \cap R_2$.

If the coefficients of u_1 and u_2 are negative in equation (A3), then (0,0) is the only best response for Bob.

6.11 Frankie and Johnny

Let π be the payoff to Johnny, and write $\overline{x} = (x_f + x_j)/2$. If $x_f < x_j$, then $y < \overline{x}$ implies $\pi = x_f$, and otherwise $\pi = x_j$. If $x_f > x_j$, then $y < \overline{x}$ implies $\pi = x_j$, and otherwise $\pi = x_f$. Since $\Pr\{y < \overline{x}\} = F(\overline{x})$, we have $\pi = x_f F(\overline{x}) + x_j(1-F(\overline{x}))$ for $x_f \leq x_j$, and $\pi = x_j F(\overline{x}) + x_f(1-F(\overline{x}))$ for $x_f > x_j$.

First, suppose $x_f < x_j$. The first-order conditions on x_f and x_j are then $\pi_{x_f} = F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, and $\pi_{x_j} = 1 - F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, from which it follows that $F(\overline{x}) = 1/2$. Substituting into the first-order conditions gives $x_f = \overline{x} - 1/2f(\overline{x})$, $x_j = \overline{x} + 1/2f(\overline{x})$. Since π should be a minimum for Frankie, the second order condition must satisfy $\pi_{x_f x_f} = f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 > 0$. Since π should be a maximum for Johnny, the second order condition must satisfy $\pi_{x_j x_j} = -f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 < 0$.

For instance, if y is drawn from a uniform distribution then $\overline{x} = 1/2$ and $f(\overline{x}) = 1$, so $x_f = 0$ and $x_j = 1$. For another example, suppose $f(x)$ is quadratic, symmetric about $x = 1/2$, and $f(0) = f(1) = 0$. Then it is easy to check that $f(x) = 6x(1-x)$. In this case $\overline{x} = 1/2$ and $f(\overline{x}) = 3/2$, so $x_f = 1/6$ and $x_j = 5/6$.

6.13 Cheater-Inspector

Let α be the probability of trusting. If there is a mixed-strategy equilibrium in the n-round game, the payoff to cheating in the first period is $\alpha n + (1-\alpha)(-an) = \alpha n(1+a) - an$, and the payoff to being honest is $g_{n-1} + b(1-\alpha)$. Equating these, we find

$$\alpha = \frac{g_{n-1} + b + an}{n(1+a)+b},$$

assuming $g_{n-1} < n$ (which is true for $n = 0$, and which we will show is true for larger n by induction). The payoff of the n-round game is then

$$g_n = g_{n-1} + b\frac{n - g_{n-1}}{n(1+a)+b}.$$

It is easy to check that $g_1 = b/(1+a+b)$ and $g_2 = 2b/(1+a+b)$, which suggests that

$$g_n = \frac{nb}{1+a+b}.$$

This can be checked directly by assuming it to be true for g_{n-1} and proving it true for g_n. This is called “proof by induction”: prove it for $n = 1$, then show that it is true for some integer n, it is true for $n+1$. Then it is true for all integers n

$$\begin{aligned}
g_n &= g_{n-1} + b\frac{n - g_{n-1}}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + b\frac{n - \frac{b(n-1)}{1+a+b}}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b}\frac{n+na+nb-b(n-1)}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b} \\
&= \frac{bn}{1+a+b}.
\end{aligned}$$

6.16 Big John and Little John Revisited

Let σ be the mixed strategy for Big John, who climbs with probability α, and let τ be the strategy for Little John, who climbs with probability β. Let π_{c_i} and π_{w_i} be the payoffs to climbing and waiting, respectively, for player i. Then we have

$$
\begin{aligned}
\sigma &= \alpha c_1 + (1-\alpha) w_1, \quad \tau = \beta c_2 + (1-\beta) w_2 \\
\pi_{c_1} &= \beta \pi_1(c_1, c_2) + (1-\beta)\pi_1(c_1, w_2) \\
&\quad \text{where } \pi_1(c_1, c_2) = \text{Big John's payoff to } c_1, c_2 \\
&= 5\beta + 4(1-\beta) = 4 + \beta \\
\pi_{w_1} &= \beta \pi_1(w_1, c_2) + (1-\beta)\pi_1(w_1, w_2) \\
&= 9\beta + 0(1-\beta) = 9\beta \\
4 + \beta &= 9\beta \Rightarrow \boxed{\beta = 1/2} \\
\pi_{c_2} &= \alpha \pi_2(c_1, c_2) + (1-\alpha)\pi_2(w_1, c_2) \\
&= 3\alpha + (1-\alpha) = 1 + 2\alpha \\
\pi_{w_2} &= \alpha \pi_2(c_1, w_2) + (1-\alpha)\pi_2(w_1, w_2) \\
&= 4\alpha + 0(1-\alpha) = 4\alpha \\
1 + 2\alpha &= 4\alpha \Rightarrow \boxed{\alpha = 1/2}.
\end{aligned}
$$

Payoffs to Players:

$$\pi_1 = 5 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = \frac{9}{2}, \quad \pi_2 = 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 2.$$

Note: Show two other ways to find payoffs

6.21 One-Card, Two-Round Poker with Bluffing

The reduced normal form is as follows:

	ss	sf	f
rrbb	0,0	4,−4	2,−2
rrbf	1,−1	0,0	2,−2
rrf	2,−2	1,−1	0,0
rfbb	−5,5	0,0	2,−2
rfbf	−4,4	4,−4	2,−2
rff	−3,3	−3,3	0,0
fbb	−4,4	1,−1	0,0
fbf	−3,3	−3,3	0,0
ff	−2,2	−2,2	−2,2

The last six strategies for player 1 are weakly dominated by rrbb. Eliminating these strategies gives the following reduced normal form.

	ss	sf	f
rrbb	0,0	4,−4	2,−2
rrbf	1,−1	0,0	2,−2
rrf	2,−2	1,−1	0,0

If 2 uses α ss + β sf + $(1-\alpha-\beta)$ f, the payoffs to 1's strategies are:

$$\begin{aligned}
\text{rrbb:}\quad & 4\beta + 2(1-\alpha-\beta) = -2\alpha + 2\beta + 2\\
\text{rrbf:}\quad & \alpha + 2(1-\alpha-\beta) = -\alpha - 2\beta + 2\\
\text{rrf:}\quad & 2\alpha + \beta
\end{aligned}$$

If rrbb and rrbf are used, we have $\beta = \alpha/4$; if rrbb and rrf are used, we have $4\alpha = \beta + 2$. If rrbf and rrf are used we have $\alpha + \beta = 2/3$. Thus, if all three are used, we have $\alpha = 8/15, \beta = 2/15$, and $1 - \alpha - \beta = 1/3$. The payoff is 18/15 = 6/5.

If 1 uses γ rrbb + δ rrbf + $(1-\gamma-\delta)$ rrf, the payoffs to 2's strategies are

$$\begin{aligned}
\text{ss:}\quad & -\delta - 2(1-\gamma-\delta) = 2\gamma + \delta - 2\\
\text{sf:}\quad & -4\gamma - (1-\gamma-\delta) = -3\gamma + \delta - 1\\
\text{f:}\quad & -2\gamma - 2\delta
\end{aligned}$$

Thus, if ss and sf are used, $\gamma = 1/5$. If ss and f are both used, $4\gamma + 3\delta = 2$, so if all are used, $3\delta = 2 - 4/5 = 6/5$, and $\delta = 2/5$. Then $1 - \gamma - \delta = 2/5$. The payoff is $4/5 - 2 = -1/5 - 1 = -6/5$, so it all works out.

There is a Nash equilibrium

$$\frac{8}{15}\text{ss} + \frac{2}{15}\text{sf} + \frac{1}{3}\text{f}, \quad \frac{1}{5}\text{rrbb} + \frac{2}{5}\text{rrbf} + \frac{2}{5}\text{rrf},$$

with a payoff of 6/5 to player 1.

Note that we have arrived at this solution by eliminating weakly dominated strategies. Have we eliminated any Nash equilibria this way?

6.23 Trust in Networks

Let α and β be the fraction of inspectors and trusters, respectively, and write $\gamma = 1-\alpha-\beta$. Then we have

$$\begin{aligned}\pi_I &= \alpha\pi_{II} + \beta\pi_{IT} + \gamma\pi_{ID}\\ \pi_T &= \alpha\pi_{TI} + \beta\pi_{TT} + \gamma\pi_{TD}\\ \pi_D &= \alpha\pi_{DI} + \beta\pi_{DT} + \gamma\pi_{DD}\end{aligned}$$

where $\pi_{II} = p^2, \pi_{IT} = p, \pi_{ID} = -2(1-p), \pi_{TI} = p, \pi_{TT} = 1, \pi_{TD} = -2, \pi_{DI} = 2(1-p), \pi_{DT} = 2$, and $\pi_{DD} = -1$. Solving simultaneously for the completely mixed Nash equilibrium, we find $\pi^*(p) = 4(1-2p)^2/(1+p)(3p-1)$, which has derivative $8(1-7p+10p^2)/(1-3p)^2(1+p)^2$, which is positive for $p \in (0.5, 1]$.

6.27 A Mating Game

Let $\alpha = \alpha_H + \alpha_E$, and $\beta = \beta_H + \beta_E$. You can check that the payoffs for males are (a) $\pi^m_{FF} = 1$; (b) $\pi^m_{FR} = 3(2-\alpha)/4$; (c) $\pi^m_{RF} = 3\alpha/4$; (d) $\pi^m_{RR} = 1$. The payoffs for females are (a) $\pi^f_{FF} = 1-\beta/4$; (b) $\pi^f_{FR} = 2-\beta$; (c) $\pi^f_{RF} = \beta/2$; (d) $\pi^f_{RR} = 1-\beta/4$. Also, $\alpha, \beta = 2$ for FF, $\alpha, \beta = 1$ for FR and RF, and $\alpha, \beta = 0$ for RR. Now you can form the 4×4 normal form matrix, and the rest is straightforward.

6.28 Coordination Failure

Suppose player 1 uses the three pure strategies with probabilities α,β, and $\gamma = 1-\alpha-\beta$, respectively, and 2 uses the pure strategies with probabilities

a, b, and $c = 1 - a - b$. We can assume without loss of generality that $\alpha \geq 1/3$ and $\beta \geq \gamma$. The payoffs to a, b, and c are

$$\begin{aligned}\pi_a &= 50\beta + 40(1-\alpha-\beta) = 40 - 40\alpha + 10\beta,\\ \pi_b &= 40\alpha + 50(1-\alpha-\beta) = 50 - 10\alpha - 50\beta,\\ \pi_c &= 50\alpha + 40\beta.\end{aligned}$$

We have $\alpha + 2\beta \geq 1$, so $\beta \geq (1-\alpha)/2$. Then,

$$\begin{aligned}\pi_c - \pi_a &= 50\alpha + 40\beta - [40 - 40\alpha + 10\beta] = 90\alpha + 30\beta - 40\\ &> 90\alpha + 30(1-\alpha)/2 - 40 = 15 + 75\alpha - 40 = 75\alpha - 25 > 0.\end{aligned}$$

Thus, c is better than a. Also,

$$\begin{aligned}\pi_c - \pi_b &= 50\alpha + 40\beta - [50 - 10\alpha - 50\beta] = 60\alpha + 90\beta - 50\\ &> 60\alpha + 90(1-\alpha)/2 - 50 = 45 + 15\alpha - 50 = 15\alpha - 5 > 0,\end{aligned}$$

so c is better than b. Thus, player 2 will use c, and his payoff is $50\alpha + 40\beta > 50\alpha + 20(1-\alpha) = 20 + 30\alpha > 30$. The payoff to 1 is then $40\alpha + 50\beta > 40\alpha + 25(1-\alpha) = 25 + 15\alpha > 30$. Thus, both are better off than with the 30 payoff of the Nash equilibrium.

6.29 Colonel Blotto Game

The payoff matrix, giving Colonel Blotto's return (the enemy's payoff is the negative of this) is as follows:

		Enemy Strategies			
		(3,0)	(0,3)	(2,1)	(1,2)
Colonel Blotto Strategies	(4,0)	4	0	2	1
	(0,4)	0	4	1	2
	(3,1)	1	−1	3	0
	(1,3)	−1	1	0	3
	(2,2)	−2	−2	2	2

Suppose the enemy uses all strategies. By symmetry, 1 and 2 must be used equally, and 3 and 4 must be used equally. Let p be the probability of using (3,0), and q be the probability of using (2,1). The expected return to Colonel Blotto is then

$$\begin{array}{rl} 4p + \quad 2q + q = & 4p + 3q \\ 4p + q + 2q = & 4p + 3q \\ p - p + 3q \quad = & 3q \\ -p + p \quad + 3q = & 3q \\ -2p - 2p + 2q + 2q = & -4p + 4q. \end{array}$$

Colonel Blotto cannot use all strategies in a mixed strategy, because there is no p that makes all entries in this vector equal. Suppose we drop Colonel Blotto's (3,1) and (1,3) strategies and choose p to solve $4p + 3q = -4p + 4q$ and $2p + 2q = 1$. Thus, $p = 1/18$ and $q = 4/9$. There are other Nash equilibria.

6.30 Number Guessing Game

Clearly, the game is determined in the first two rounds. Let us write my strategies as (g h l), for "first guess g, if high guess h and if low guess l." If a high guess is impossible, we write (1 x l), and if a low guess is impossible, we write (3 h x). For instance, (1x3) means "first choose 1, and if this is low, then choose 3." Then, we have the following payoff matrix for Bob (the payoff to Alice is minus the payoff to Bob):

Bob \ Alice	(102)	(103)	(213)	(310)	(320)
1	1	1	2	2	3
2	2	3	1	3	2
3	3	2	2	1	1

First show that Bob will use a completely mixed strategy. It is obvious that no Nash equilibrium uses only a single pure strategy of Alice, and you can show that no Nash equilibrium uses one of the 10 pairs of pure strategies for Alice. This leaves the 10 triples of strategies for Alice. It is arduous to check all of these, but the procedures described in section 6.44. It turns out that there is only one Nash equilibrium, in which we drop (1x2) and (32x). Then, equating the costs of the other three, we find that Bob uses the mixed strategy (0.4,0.2,0.4) against Alice's mixed strategy (0.2,0.6,0.2). The payoff is 1.8 to player 1. It is easy to check that Alice's excluded strategies are more costly than this for Alice.

6.31 Target Selection

Suppose Attacker uses mixed strategy $x = (x_1, \ldots, x_n)$ and Defender uses strategy $y = (y_1, \ldots, y_n)$, and these form a Nash equilibrium. If $x_j = 0$, then the best response of Defender must set $y_j = 0$. Suppose $x_i > 0$ for some $i > j$. Then, by switching x_i and x_j, Attacker gains $a_j - pa_i y_i \geq a_j - a_i > 0$.

6.32 A Reconnaissance Game

The normal form matrix is as follows:

	counter full defend	counter half defend	no counter full defend	no counter half defend
reconnoiter, full attack	$a_{11} - c + d$	$a_{12} - c + d$	$a_{11} - c$	$a_{12} - c$
reconnoiter, half attack	$a_{21} - c + d$	$a_{22} - c + d$	$a_{21} - c$	$a_{22} - c$
no reconnoiter, full attack	$a_{11} + d$	$a_{12} + d$	a_{11}	a_{12}
no reconnoiter, half attack	$a_{21} + d$	$a_{22} + d$	a_{21}	a_{22}

With the given payoffs and costs, the entries in the normal form game become

46,–46	22,–22	39,–39	27,–27
10,–10	34,–34	39,–39	27,–27
55,–55	31,–31	48,–48	24,–24
19,–19	43,–43	12,–12	36,–36

Suppose Defender does not counter and full defends with probability p. Then, Attacker faces

$$39p + 27(1-p) = 12p + 27$$
$$39p + 27(1-p) = 12p + 27$$
$$48p + 24(1-p) = 24p + 24$$
$$12p + 36(1-p) = -24p + 36.$$

Check the third and fourth. We have $-24p+36 = 24p+24$, so $p = 1/4$. Suppose attacker does not reconnoiter and full attacks with probability q. Then, $-48q - 12(1-q) = -24q - 36(1-q)$, so $q = 1/2$. You must check that no other strategy has a higher payoff, and you will find this to be true. The payoffs are $(30, -30)$. If you are ambitious, you can check that there are many other Nash equilibria, all of which involve (0,0,1/4,3/4) for Defender. How do you interpret this fact?

6.33 Attack on Hidden Object

We have

	P	F
PP	$2\gamma - \gamma^2$	$\beta\gamma$
PF	γ	γ
FP	$\beta(1-\alpha(1-\beta))$	β
FF	β^2	$2\beta - \beta^2$

Note that the second row is strongly dominated by the third, and the third row is weakly dominated by the fourth row. Moreover, it is clear that if $\beta^2 > 2\gamma - \gamma^2$, then the first row is strictly dominated by the fourth, so there is a unique Nash equilibrium in which Bob plays FF and Alice plays P. The condition for this is

$$\alpha > \frac{\beta + \sqrt{1-\beta^2} - 1}{\beta}.$$

If $\beta^2 < 2\gamma - \gamma^2 < \beta(1-\alpha(1-\beta))$, then PP is strictly dominated by FP, and (FP,P) is a pure-strategy equilibrium. Finally, if $2\gamma - \gamma^2 > b(1-\alpha(1-\beta))$ you can check that there is a strictly mixed Nash equilibrium including PP, FP for Bob, and F and P for Alice.

Principal-Agent Models: Answers

7.1 Gift Exchange

Choosing w and N to maximize profits gives the first-order conditions

$$\pi_w(w,N) = [f'(eN)e' - 1]N = 0 \tag{A4}$$
$$\pi_N(w,N) = f'(eN)e - w = 0. \tag{A5}$$

Solving these equations gives the Solow condition.

The second partials are

$$\pi_{ww} = [f''Ne'^2 + f'e'']N < 0, \qquad \pi_{NN} = f''e^2 < 0,$$
$$\pi_{wN} = f''Nee' + f'e' - 1 = f''Nee' < 0.$$

It is easy to check that the second-order conditions are satisfied: $\pi_{ww} < 0$, $\pi_{NN} < 0$, and $\pi_{ww}\pi_{NN} - \pi_{wN}^2 > 0$.

To show that $dw/dz > 1$, differentiate the first-order conditions (AA5) totally with respect to w and N:

$$\begin{aligned}\pi_{ww}\frac{dw}{dz} + \pi_{wN}\frac{dN}{dz} + \pi_{wz} &= 0\\ \pi_{Nw}\frac{dw}{dz} + \pi_{NN}\frac{dN}{dz} + \pi_{Nz} &= 0.\end{aligned} \tag{A6}$$

Solving these two equations in the two unknowns dw/dz and dN/dz, we find

$$\frac{dw}{dz} = -\frac{\pi_{NN}\pi_{wz} - \pi_{Nw}\pi_{Nz}}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2}. \tag{A7}$$

But we also calculate directly that

$$\pi_{wz} = -[f''Ne'^2 + f'e'']N = -\pi_{ww},$$
$$\pi_{Nz} = -f'e' - f''Nee' = -f'e' - \pi_{Nw}.$$

Substituting these values in (A7), we get

$$\frac{dw}{dz} = 1 - \frac{\pi_{Nw}f'e'}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2},$$

and the fraction in this expression is negative (the denominator is positive by the second-order conditions, while $\pi_{Nw} < 0$ and $f', e' > 0$).

Because $dw/dz > 1$, it follows from the chain rule that

$$\frac{de}{dz} = e'\left[\frac{dw}{dz} - 1\right] > 0,$$

$$\frac{dN}{dz} = \frac{-\pi_{wz} - \pi_{ww}\frac{dw}{dz}}{\pi_{wN}}$$
$$= \frac{\pi_{ww}}{\pi_{wN}}\left(1 - \frac{dw}{dz}\right) < 0 \qquad \text{[by (AA6), (AA7)]},$$

$$\frac{d\pi}{dz} = \frac{\partial\pi}{\partial w}\frac{dw}{dz} + \frac{\partial\pi}{\partial N}\frac{dN}{dz} + \frac{\partial\pi}{\partial z} = \frac{\partial\pi}{\partial z} = -f'e'N < 0.$$

7.6 Bob's Car Insurance

Suppose Bob is careful without insurance, so we know $\epsilon \leq 0.177$. Because the insurance company's lottery is fair, we have $x = 0.95(1200-z)$ if Bob is careful with insurance, and $x = 0.925(1200-z)$ if Bob is careless with insurance. We cannot assume he will be careless in this case, because the deductible might induce Bob to be careful.

If Bob is careless with insurance, the value of car plus insurance is $v = 0.925\ln(1201-x) + 0.075\ln(1201-z-x)$, and because the insurance is fair, we have $x = 0.075(1200-z)$. Subsituting the second expression for x in the first, and taking the derivative with respect to z, we find

$$\frac{dv}{dz} \approx \frac{z}{z^2 + 13612Z - 17792000},$$

which is negative for $z \in (0, 1200)$. Thus, zero deductible is optimal.

Now suppose Bob is careful. Then, the value of car plus insurance is $v = 0.95\ln(1201-x) + 0.05\ln(1201-z-x)$, and fair insurance implies $x = 0.05(1200-z)$. The derivative of this with respect to z is

$$\frac{dv}{dz} \approx \frac{z}{z^2 + 21618.9Z - 27408000},$$

which is also negative, so the optimal deductible for Bob is zero. However, in this case, z must be sufficiently large that Bob wants to be careful, or the insurance company will not be willing to issue the insurance at the low rate $x = 0.05(1200 - z)$. To make taking care worthwhile, we must have

$$0.95 \ln(1201 - z) + 0.05 \ln(1201 - z - x) - \epsilon \geq$$
$$0.925 \ln(1201 - z) + 0.075 \ln(1201 - z - x).$$

The minimum z satisfying this is when the equality holds. This equation cannot be solved analytically, but calculations show that there is no solution for $\epsilon > 0.0012$, and when $\epsilon = 0.001$, the deductible must be at least $z =$ \$625.

Signaling Games: Answers

8.3 Introductory Offers

If a high-quality firm sells to a consumer in the first period at some price p_1, then in the second period the consumer will be willing to pay $p_2 = h$, because he knows the product is of high quality. Knowing that it can make a profit $h - c_h$ from a customer in the second period, a high-quality firm might want to make a consumer an "introductory offer" at a price p_1 in the first period that would not be mimicked by the low-quality firm, in order to reap the second-period profit.

If $p_1 > c_l$, the low-quality firm could mimic the high-quality firm, so the best the high-quality firm can do is to charge $p_1 = c_l$, which the low-quality firm will not mimic, because the low-quality firm cannot profit by doing so (it cannot profit in the first period, and the consumer will not buy the low-quality product in the second period). In this case, the high-quality firm's profits are $(c_l - c_h) + \delta(h - c_h)$. As long as these profits are positive, which reduces to $h > c_h + \delta(c_h - c_l)$, the high-quality firm will stay in business.

8.6 The Shepherds Who Never Cry Wolf

The following payoffs are easy to derive:

$$\pi_1(N,N) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,N) = 1;$$
$$\pi_1(N,H) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,H) = 1;$$
$$\pi_1(N,A) = 1; \quad \pi_2(N,A) = 1-d;$$
$$\pi_1(H,N) = p(1-a) + (1-p)(1-b) - pc; \quad \pi_2(H,N) = 1;$$
$$\pi_1(H,H) = p(1-c) + (1-p)(1-b); \quad \pi_2(H,H) = p(1-d) + 1 - p;$$
$$\pi_1(H,A) = 1 - pc; \quad \pi_2(H,A) = 1-d;$$
$$\pi_1(A,N) = p(1-a) + (1-p)(1-b) - c; \quad \pi_2(A,N) = 1;$$
$$\pi_1(A,H) = 1-c; \quad \pi_2(A,H) = 1-d;$$
$$\pi_1(A,A) = 1-c; \quad \pi_2(A,A) = 1-d.$$

Now the total payoff for shepherd 1 is $\pi_1^t = \pi_1 + k\pi_2$, and the total payoff for shepherd 2 is $\pi_2^t = \pi_1 + k\pi_1$. Substituting in numbers and forming the normal form matrix for the game, we get

	N	H	A
N	$\frac{19}{24}, \frac{37}{32}$	$\frac{19}{24}, \frac{37}{32}$	$\frac{21}{16}, \frac{7}{6}$
H	$\frac{95}{192}, \frac{793}{768}$	$\frac{47}{48}, \frac{829}{768}$	$\frac{65}{64}, \frac{267}{256}$
A	$\frac{19}{48}, \frac{571}{576}$	$\frac{11}{12}, \frac{577}{768}$	$\frac{11}{12}, \frac{577}{576}$

It is easy to see that (H, H) and (N, A) are Nash equilibria, and you can check that there is a mixed-strategy equilibrium in which the threatened shepherd uses $\frac{1}{3}N + \frac{2}{3}H$ and the other shepherd uses $\frac{3}{5}H + \frac{2}{5}A$.

8.8 Honest Signaling among Partial Altruists

The payoff matrix for the encounter between a fisher observing a threatened fisher is as follows, where the first two lines are the payoffs to the individual players, and the third is the total payoff:

	Never Ask	Ask If Distressed	Always Ask
Never Help	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r(1-p)u-rpt$ $(1-p)u-pt$ $(1+r)[(1-p)u-pt]$	$r(1-p)u-rt$ $(1-p)u-t$ $(1+r)[(1-p)u-t]$
Help If Asked	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r[p(1-t)+(1-p)u]-pc$ $p(1-t)+(1-p)u$ $(1+r)[p(1-t)$ $+(1-p)u-pc]$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v$ $-t]-c$
Always Help	$r[p+(1-p)v]-c$ $p+(1-p)v$ $(1+r)[p+(1-p)v]-c$	$r[p(1-t)+(1-p)v]-c$ $p(1-t)+(1-p)v$ $(1+r)[p(1-t)+(1-p)v]$ $-c$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v-t]-c$

The answers to the problem can be obtained in a straightforward manner from this matrix.

8.10 Education as a Screening Device

a. Given the probabilities (c), the wages (b) follow from

$$w_k = \mathrm{P}[a_h|e_k]a_h + \mathrm{P}[a_l|e_k]a_l, \qquad k = h, l. \tag{A8}$$

Then, it is a best response for workers to choose low education whatever their ability type, so (a) follows. Because both types choose e_l, the conditional probability $P[a_l|e_l] = 1-\alpha$ is consistent with the behavior of the agents, and because e_h is off the path of play, any conditional for $P[a_l|e_h]$ is acceptable, so long as it induces a Nash equilibrium.

b. Assume the above conditions hold, and suppose c satisfies $a_l(a_h - a_l) < c < a_h(a_h - a_l)$. The wage conditions (b) follow from (A8) and (c). Also, $a_h - c/a_h > a_l$, so a high-ability worker prefers to choose $e = 1$ and signal his true type, rather than choose e_l and signal his type as low ability. Similarly, $a_l > a_h - c/a_l$, so a low-ability worker prefers to choose e_l and signal his true type, rather than choose e_h and signal his type as high ability.

c. The wage conditions (b) follow from (A8) and (c). Suppose $c < a_l(a_h - a_l)$. Then both high- and low-ability workers prefer to get education and the higher wage w_h rather than signal that they are low quality.

d. Let $\bar{e} = \alpha a_l(a_h - a_l)/c$, and choose $e^* \in [0, \bar{e}]$. Given the employer's wage offer, if a worker does not choose $e = e^*$ he might as well choose $e = 0$, because his wage in any case must be $w = a_l$. A low-ability worker then prefers to get education e^* rather than any other educational level, because $a_l \leq \alpha a_h + (1-\alpha)a_l - ce^*/a_l$. This is thus true for the high-ability worker, whose incentive compatibility constraint is not binding.

e. Consider the interval

$$\left[\frac{a_l(a_h - a_l)}{c}, \frac{a_h(a_h - a_l)}{c}\right].$$

If c is sufficiently large, this interval has a nonempty intersection with the unit interval $[0, 1]$. Suppose this intersection is $[e_{min}, e_{max}]$. Then, for $e^* \in [e_{min}, e_{max}]$, a high-ability worker prefers to acquire education e^* and receive the high wage $w = a_h$, whereas the low-ability worker prefers to receive $w = a_l$ with no education.

8.11 Capital as a Signaling Device

a. Given $p > 0$, choose k so that

$$1 > k(1 + \rho) > q + p(1 - q).$$

This is possible because $q + p(1-q) < 1$. Then it is clear that the fraction q of good projects is socially productive. The interest rate r that a producer must offer then must satisfy

$$k(1+\rho) = qk(1+r) + (1-q)kp(1+r) = k(1+r)[q + p(1-q)],$$

so

$$r = \frac{1+\rho}{q + p(1-q)} - 1. \tag{A9}$$

The net profit of a producer with a good project is then

$$1 - k(1+r) = \frac{q + p(1-q) - k(1+\rho)}{q + p(1-q)} < 0,$$

so such producers will be unwilling to offer lenders an interest rate they are willing to accept. The same is clearly true of bad projects, so no projects get funded. Note that bad projects are not socially productive in this case, because $p - k(1+\rho) < p - (q + p(1-q)) = -q(1-p) < 0$.

b. Choose k so that $p < k(1+\rho) < q + p(1-q)$, which is clearly always possible. Then the fraction $1-q$ of bad projects are socially unproductive. The interest rate r must still satisfy equation (A9), so the payoff to a successful project (good or bad) is

$$1 - k(1+r) = \frac{q + p(1-q) - k(1+\rho)}{q + p(1-q)} > 0,$$

so producers of both good and bad projects are willing to offer interest rate r, and lenders are willing to lend at this rate to all producers.

c. Let

$$k^p_{min} = \frac{p[1-k(1+\rho)]}{(1-p)(1+\rho)}.$$

Note that because good projects are socially productive, $k^p_{min} > 0$. Suppose all producers have wealth $k^p > k^p_{min}$, and lenders believe that only a producer with a good project will invest k^p in his project. Then lenders will be willing to lend at interest rate ρ. If a producer invests k^p in his project and borrows $k - k^p$, his return is 1 and his costs are forgone earnings $k^p(1+\rho)$ and capital costs $(k-k^p)(1+\rho)$. Thus, his profit is

$$1 - k^p(1+\rho) - (k - k^p)(1+\rho) = 1 - k(1+\rho) > 0,$$

so such a producer is willing to undertake this transaction. If the producer with a bad project invests his capital k^p, his return is

$$p[1 - k(1 + \rho)] - (1 - p)k^p(1 + \rho) < 0,$$

so he will not put up the equity. This proves the theorem.

Repeated Games : Answers

9.4 The Strategy of an Oil Cartel

Payoffs:

Low/Low	$(25-2)\times 2, (25-4)\times 2$	=	46,42
High/Low	$(15-2)\times 4, (15-4)\times 2$	=	52,22
Low/High	$(15-2)\times 2, (15-4)\times 4$	=	26,44
High/High	$(10-2)\times 4, (10-4)\times 4$	=	32,24

Normal Form Game:

	Low	High
Low	46,42	26,44
High	52,22	32,24

The condition for the cooperate payoff to be higher than the defect payoff for Iran is

$$\frac{46}{1-\delta} > 52 + \delta\frac{32}{1-\delta}.$$

We can solve this, getting $\delta > 0.3$, which corresponds to an interest rate r given by $r = (1-\delta)/\delta = 0.7/0.3 \approx 233.33\%$. The condition for cooperate to beat defect for Iraq is

$$\frac{42}{1-\delta} > 44 + \delta\frac{24}{1-\delta}.$$

We can solve this, getting $\delta > 0.1$, which corresponds to an interest rate r given by $r = 900\%$.

9.5 Reputational Equilibrium

If it is worthwhile for the firm to lie when it claims its product has quality $q > 0$, it might as well set its actual quality to 0, because the firm minimizes costs this way. Its profits are then

$$\pi_f = (4 + 6q_a - x - 2)x = (2 + 6q_a - x)x.$$

Profits are maximized when

$$\frac{d\pi_f}{dx} = 2 + 6q_a - 2x = 0,$$

so $x = 1 + 3q_a$, and $\pi_f = (1 + 3q_a)^2$.

Now suppose the firm tells the truth. Then, if π_t is per-period profits, we have

$$\pi_t = (2 + 6q_a - 6q_a^2 - x)x,$$
$$\frac{d\pi_t}{dx} = 2 + 6q_a - 6q_a^2 - 2x = 0,$$

so $x = 1 + 3q_a - 3q_a^2$, and $\pi_t = (1 + 3q_a - 3q_a^2)^2$. But total profits Π from truth telling are π_t forever, discounted at rate $\delta = 0.9$, or

$$\Pi = \frac{\pi_t}{1-\delta} = 10(1 + 3q_a - 3q_a^2)^2.$$

Truth-telling is profitable then when $\Pi \geq \pi_f$, or when

$$10(1 + 3q_a - 3q_a^2)^2 > (1 + 3q_a)^2. \tag{A10}$$

Note that equation (A10) is true for very small q_a (that is, q_a near 0) and false for very large q_a (that is, q_a near 1).

Evolutionary Stable Strategies: Answers

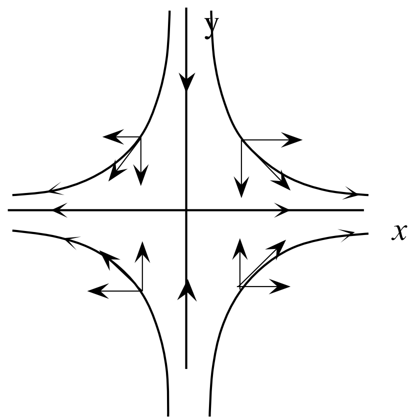
10.4 A Symmetric Coordination Game

a. Let s_a and s_b be the two strategies, and write $\sigma = \alpha s_a + (1-\alpha)s_b$ for the mixed strategy where s_a is played with probability α. If $\tau = \beta s_a + (1-\beta)s_b$, we have $\pi[\sigma,\tau] = \alpha\beta a + (1-\alpha)(1-\beta)b$. Suppose (σ,σ) is a Nash equilibrium. Then by the fundamental theorem (§3.6), $\pi[s_a,\sigma] = \pi[s_b,\sigma]$, which implies $\alpha = b/(a+b)$. Note that $\pi[\sigma,\sigma] = ab/(a+b)$, which is smaller than either a or b. We shall show that b can invade a population that plays σ. By the fundamental theorem, $\pi[s_b,\sigma] = \pi[\sigma,\sigma]$, because $\alpha < 1$. Thus σ is impervious to invasion by s_b only if $\pi[\sigma,s_b] > \pi[s_b,s_b]$, which reduces to $ab/(a+b) > b$, which is false.

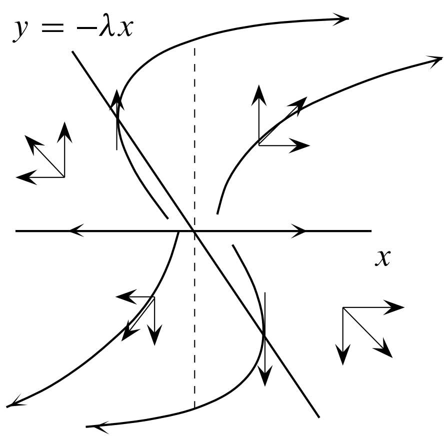
Dynamical Systems: Answers

11.10 Exercises in Two-Dimensional Linear Systems

(b)



(d)



Evolutionary Dynamics: Answers

12.5 Properties of the Replicator System

Only the last part of the question might not be obvious. Let $p(t)$ be a trajectory of (12.5) and define

$$b(t) = \int_0^t \frac{dt}{a(p(t),t)},$$

which is possible because $a(p,t) > 0$. Clearly, $b(t)$ is positive and increasing. Let $q(t) = p(b(t))$. Then, by the fundamental theorem of the calculus,

$$\begin{aligned}\dot{q}_i(t) &= \dot{b}(t)\dot{p}_i(b(t)) = \frac{1}{a(t)}a(t)p_i(b(t))(\pi_i(p(b(t))) - \bar{\pi}(p(b(t))))\\ &= q_i(t)(\pi_i(q(t)) - \bar{\pi}(q(t))). \blacksquare\end{aligned}$$

12.10 Trust in Networks III

You can check that the eigenvalues of the Jacobian at the equilibrium are given by

$$\begin{aligned}\lambda_1, \lambda_2 = &\frac{-2 + 5p - 4p^2 + p^3}{2(1+p)(3p-1)}\\ &\pm \frac{\sqrt{4 - 60p + 177p^2 - 116p^4 - 110p^4 + 104p^5 + p^6}}{2(1+p)(3p-1)}.\end{aligned}$$

This is pretty complicated, but you can check that the expression under the radical is negative for p near unity: factor out $(p-1)$ and show that the other factor has value 32 when $p = 1$. The rest of the expression is real and negative for p near unity, so the equilibrium is a stable focus.

12.13 A Generalization of Rock, Paper, and Scissors

Note first that no pure strategy is Nash. If one player randomizes between two pure strategies, the other can avoid the -1 payoff, so only strictly mixed solutions can be Nash. Check that the only such strategy σ that is Nash uses probabilities (1/3,1/3,1/3). This is not evolutionarily stable for $\alpha < 0$, however, because the pure strategy R has payoff $\alpha/3$ against σ, which is also the payoff to σ against σ, and has payoff α against itself.

The payoff of the strategies against $(x_1, x_2, 1 - x_1 - x_2)$ are

$$\begin{array}{rrcl} R: & \alpha x_1 + x_2 - (1 - x_1 - x_2) & = & (1+\alpha)x_1 + 2x_2 - 1 \\ P: & -x_1 + \alpha x_2 + (1 - x_1 - x_2) & = & -2x_1 - (1-\alpha)x_2 + 1 \\ S: & x_1 - x_2 + \alpha(1 - x_1 - x_2) & = & (1-\alpha)x_1 - (\alpha+1)x_2 + \alpha \end{array}$$

The average payoff is then $2\alpha(x_1^2 + x_1x_2 + x_2^2 - x_1 - x_2) + \alpha$, and the fitnesses of the three types are

$$\begin{array}{ll} f_1: & (1+3\alpha)x_1 + 2(1+\alpha)x_2 - (1+\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2) \\ f_2: & -2(1-\alpha)x_1 - (1-3\alpha)x_2 + (1-\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2) \\ f_3: & (1+\alpha)x_1 - (1-\alpha)x_2 - 2\alpha(x_1^2 + x_1x_2 + x_2^2). \end{array}$$

Note that $x_1 = x_2 = 1/3$ gives $f_1 = f_2 = f_3 = 0$, so this is our Nash equilibrium. For the replicator dynamic, we have $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$, so we need only the first two equations. Assuming $x_1, x_2 > 0$, we get

$$\frac{\dot{x}_1}{x_1} = -(2\alpha(x_1^2 + x_1x_2 + x_2^2) - (1+3\alpha)x_1 - 2(1+\alpha)x_2 + (1+\alpha))$$

$$\frac{\dot{x}_2}{x_2} = -(2\alpha(x_1^2 + x_1x_2 + x_2^2) + 2(1-\alpha)x_1 + (1-3\alpha)x_2 - (1-\alpha)).$$

It is straightforward to check that $x_1 = x_2 = 1/3$ is the only fixed point for this set of equations in the positive quadrant.

The Jacobian of this system at the Nash equilibrium is

$$\frac{1}{3}\begin{bmatrix} 1+\alpha & 2 \\ -2 & -1+\alpha \end{bmatrix}.$$

This has determinant $\beta = 1/3 + \alpha^2/9 > 0$, the trace is $\mathrm{Tr} = 2\alpha/3$ and the discriminant is $\gamma = \mathrm{Tr}^2/4 - \beta = -1/3$. The eigenvalues are thus $\alpha/3 \pm \sqrt{-3}/3$, which have nonzero real parts for $\alpha \neq 0$. Therefore, the system is hyperbolic. By theorem 11.5, the dynamical system is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$.

12.14 *Uta stansburiana* in Motion

It is easy to check that if the frequencies of orange-throats (rock), blue-throats (paper), and yellow-striped (scissors) are α, β, and $1-\alpha-\beta$, respectively, the payoffs to the three strategies are $1-\alpha-2\beta, 2\alpha+\beta-1$, and $\beta-\alpha$, respectively. The average payoff is zero (check this!), so the replicator dynamic equations are

$$\begin{aligned} \frac{d\alpha}{dt} &= \alpha(1-\alpha-2\beta) \\ \frac{d\beta}{dt} &= \beta(2\alpha+\beta-1). \end{aligned} \tag{A11}$$

The Jacobian matrix at the fixed point $\alpha = \beta = 1/3$ is given by

$$\begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

The trace of the Jacobian is thus zero, the determinant is $1/3 > 0$, and the discriminant is $-1/3 < 0$. By theorem 11.5 the eigenvalues are imaginary so the system is not hyperbolic. It is easy to solve for the trajectories of this system because, by theorem 11.5, they are closed orbits, and the fixed point is a center. But this tells us nothing about the original, nonlinear system (A11), because the fixed point is not hyperbolic (see theorem 11.3). So, back to the drawing board.

Let $V(\alpha, \beta, \gamma) = \ln(\alpha) + \ln(\beta) + \ln(\gamma)$. Along a trajectory of the dynamical system, we have

$$\begin{aligned} \dot{V} &= \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} \\ &= (1-\alpha-2\beta) + (2\alpha+\beta-1) + (\beta-\alpha) = 0. \end{aligned}$$

Thus, V is constant on trajectories. This implies that trajectories are bounded and bounded away from $(0,0)$ so the set Γ of ω-limit points of a trajectory contains no fixed points, and hence by the Poincaré-Bendixson theorem (theorem 11.8), Γ is a periodic orbit. But then by theorem 11.9, Γ must contain $(0,0)$. Hence, trajectories also must spiral around $(0,0)$, and because V is increasing along a ray going northeast from the fixed point, trajectories must be closed orbits.

12.15 The Dynamics of Rock, Paper, and Scissors

Let π_α, π_β, and π_γ be the payoffs to the three strategies. Then, we have

$$\begin{aligned}\pi_\alpha &= \beta r + (1-\alpha-\beta)s = \beta(r-s) - \alpha s + s,\\ \pi_\beta &= \alpha s + (1-\alpha-\beta)r = \alpha(s-r) - \beta r + r,\\ \pi_\gamma &= \alpha r + \beta s.\end{aligned}$$

It is easy to check that the average payoff is then

$$\begin{aligned}\bar{\pi} &= \alpha\pi_\alpha + \beta\pi_\beta + (1-\alpha-\beta)\pi_\gamma\\ &= (r+s)(\alpha+\beta-\alpha^2-\alpha\beta-\beta^2).\end{aligned}$$

At any fixed point involving all three strategies with positive probability, we must have $\pi_\alpha = \pi_\beta = \pi_\gamma$. Solving these two equations, we find $\alpha = \beta = \gamma = 1/3$, which implies that $\bar{\pi} = (r+s)/3$.

In a replicator dynamic, we have

$$\begin{aligned}\dot{\alpha} &= \alpha(\pi_\alpha - \bar{\pi}),\\ \dot{\beta} &= \beta(\pi_\beta - \bar{\pi}).\end{aligned}$$

Expanding these equations, we get

$$\begin{aligned}\dot{\alpha} &= -2\alpha\beta s - (r+2s)\alpha^2 + \alpha s + \alpha p(\alpha,\beta),\\ \dot{\beta} &= -2\alpha\beta r - (2r+s)\beta^2 + \beta r + \beta p(\alpha,\beta),\end{aligned}$$

where $p(\alpha,\beta) = (r+s)(\alpha^2+\alpha\beta+\beta^2)$.

This is, of course, a nonlinear ordinary differential equation in two unknowns. It is easy to check that its unique fixed point for $\alpha,\beta > 0$ is $\alpha = \beta = 1/3$, the mixed-strategy Nash equilibrium for this game.

For the dynamics, we linearize the pair of differential equations by evaluating the Jacobian matrix of the right-hand sides at the fixed point. The Jacobian is

$$J(\alpha,\beta) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}a_{11} &= -2\beta s - 2\alpha(r+2s) + s + p(\alpha,\beta) + \alpha(2\alpha+\beta)(r+s),\\ a_{12} &= -2\alpha s + \alpha(\alpha+2\beta)(r+s),\\ a_{21} &= -2\beta r + \beta(2\alpha+\beta)(r+s),\\ a_{22} &= r - 2\alpha r - 2\beta(2r+s) + p(\alpha,\beta) + \beta(\alpha+2\beta)(r+s),\end{aligned}$$

so

$$J(1/3, 1/3) = \frac{1}{3}\begin{pmatrix} -s & r-s \\ s-r & -r \end{pmatrix}.$$

The eigenvalues of the linearized system are thus

$$\frac{1}{6}\left[-(r+s) \pm i\sqrt{3}(r-s)\right].$$

We prove the assertions as follows:

a. The determinant of the Jacobian is $(r^2 - rs + s^2)/9$. This has a minimum where $2s - r = 0$, with the value $r^2/12 > 0$. This shows that the system is hyperbolic, and because the determinant is positive, it is a node or a focus.

b. The real parts of the eigenvalues are negative if and only if $r + s > 0$ and are positive if and only if $r + s < 0$.

c. The eigenvalues are complex for $r \neq s$.

d. If $r + s = 0$, the eigenvalues are purely imaginary, so origin is a center. We thus cannot tell how the nonlinear system behaves using the linearization.

   However, we can show that the quantity $q(\alpha, \beta) = \alpha\beta(1 - \alpha - \beta)$ is constant along trajectories of the dynamical system. Assuming this (which we will prove in a moment), we argue as follows. Consider a ray R through the fixed point (1/3,1/3) pointing in the α-direction. Suppose $q(\alpha, \beta)$ is strictly decreasing along this ray (we will also prove this in a moment). Then, the trajectories of the dynamical system must be closed loops. To see this, note first that the fixed point cannot be a stable node, because if we start at a point on R near the fixed point, q decreases as we approach the fixed point, but q must be constant along trajectories, which is a contradiction. Thus, the trajectories of the system must be spirals or closed loops. But they cannot be spirals, because when they intersect R twice near the fixed point, the intersection points must be the same, because $q(\alpha, \beta)$ is constant on trajectories but decreasing on R near the fixed point.

   To see that q is decreasing along R near (1/3,1/3), note that

$$q(1/3 + t, 1/3) = \frac{1}{3}\left(\frac{1}{3} - t\right)^2,$$

which has a derivative with respect to t that evaluates to $-2/9 < 0$ at $t = 0$.

To see that $q(\alpha, \beta)$ is constant along trajectories, note that the differential equations for the dynamical system, assuming $r = 1$, $s = -1$, can be written as

$$\dot{\alpha} = 2\alpha\beta + \alpha^2 - \alpha \quad \text{(A12)}$$
$$\dot{\beta} = -2\alpha\beta - \beta^2 + \beta. \quad \text{(A13)}$$

Then,

$$\begin{aligned}\frac{d}{dt}q(\alpha, \beta) &= \frac{d}{dt}[\alpha\beta(1 - \alpha - \beta)] \\ &= \beta(1 - \alpha - \beta)\dot{\alpha} + \alpha(1 - \alpha - \beta)\dot{\beta} + \alpha\beta(-\dot{\alpha} - \dot{\beta}) \\ &= 0,\end{aligned}$$

where we get the last step by substituting the expressions for $\dot{\alpha}$ and $\dot{\beta}$ from (AA12) and (AA13).

12.16 The Lotka-Volterra Model and Biodiversity

a. This is simple algebra, though you should check that the restrictions on the signs of a, b, c,and d ensure that $p^* > 0$.

b. We have

$$\begin{aligned}\frac{\dot{p}}{p} &= \frac{\dot{u}}{u} - \left[\frac{\dot{u}}{w} + \frac{\dot{v}}{w}\right] \\ &= \frac{\dot{u}}{u} - \left[p\frac{\dot{u}}{u} + (1 - p)\frac{\dot{v}}{v}\right] \\ &= ap + b(1 - p) - kw \\ &\quad - [p[ap + b(1 - p) - kw] + (1 - p)[cp + d(1 - p) - kw]] \\ &= ap + b(1 - p) - [p[ap + b(1 - p)] + (1 - p)[cp + d(1 - p)]] \\ &= \pi_A - \bar{\pi}.\end{aligned}$$

c. We have

$$\begin{aligned}
\frac{\dot{p}_i}{p_i} &= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u} \\
&= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u_j} p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - kw - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} p_k - ku \right) p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - ku - \sum_{j,k=1}^{n} a_{jk} p_k p_j + ku \sum_{k=1}^{n} p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - \sum_{j,k=1}^{n} a_{jk} p_j p_k .
\end{aligned}$$

This proves the assertion, and the identification of the resulting equations as a replicator dynamic is clear from the derivation.

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