

NP by Means of Lifts and Shadows^{*}

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Abstract. We show that every NP problem is polynomially equivalent to a simple combinatorial problem: the membership problem for a special class of digraphs. These classes are defined by means of shadows (projections) and by finitely many forbidden colored (lifted) subgraphs. Our characterization is motivated by the analysis of syntactical subclasses with the full computational power of NP, which were first studied by Feder and Vardi.

Our approach applies to many combinatorial problems and it induces the characterization of coloring problems (CSP) defined by means of shadows. This turns out to be related to homomorphism dualities. We prove that a class of digraphs (relational structures) defined by finitely many forbidden colored subgraphs (i.e. lifted substructures) is a CSP class if and only if all the forbidden structures are homomorphically equivalent to trees. We show a surprising richness of coloring problems when restricted to most frequent graph classes. Using results of Nešetřil and Ossona de Mendez for bounded expansion classes (which include bounded degree and proper minor closed classes) we prove that the restriction of every class defined as the shadow of finitely many colored subgraphs equals to the restriction of a coloring (CSP) class.

Keywords: Digraph, homomorphism, duality, NP, Constraint Satisfaction Problem.

1 Introduction, Background and Previous Work

Think of 3-colorability of a graph G . This is a well known hard (and a canonical NP-complete) problem. From the combinatorial point of view there is a standard way how to approach this problem (and monotone properties in general): investigate minimal graphs without this property, denote by $\mathcal{F}$ the language of

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all such critical graphs and define the set $Forb(\mathcal{F})$ of all structures which do not “contain” any $F \in \mathcal{F}$. Then the language $Forb(\mathcal{F})$ coincides with the language of 3-colorable graphs. Unfortunately, in the most cases the set $\mathcal{F}$ is infinite. However the properties characterized by a finite set $\mathcal{F}$ are very interesting if we allow *lifts* and *shadows*.

Let us briefly illustrate this by our example of 3-colorability. Instead of a graph $G = (V, E)$ we consider the graph G together with three unary relations C_1, C_2, C_3 (i.e. colors of vertices) which *cover* the vertex set V ; this structure will be denoted by G' and called a *lift* of G (thus G' has one binary and three unary relations). There are 3 *forbidden substructures*: For each $i = 1, 2, 3$ the single edge graph K_2 together with cover $C_i = \{1, 2\}$ and $C_j = \emptyset$ for $j \neq i$ form structure $\mathbf{F}'_i$ (where the signature of $\mathbf{F}'_i$ contains one binary and three unary relations). The language of all 3-colorable graphs is just the language $\Phi(Forb(\mathbf{F}'_1, \mathbf{F}'_2, \mathbf{F}'_3))$, where Φ is the forgetful functor which transforms G' to G . We call G the *shadow* of G' .

Clearly this situation can be generalized and one of the main results of this paper is Theorem 3 which states that every NP problem is polynomially equivalent to the membership problem for a class $\Phi(Forb(\mathcal{F}'))$. Here $\mathcal{F}'$ is a finite set of (vertex pair)-colored digraphs, $Forb(\mathcal{F}')$ is the class of all lifted graphs G' for which there is no homomorphism $F' \rightarrow G'$ for an $F' \in \mathcal{F}'$. Thus $Forb(\mathcal{F}')$ is the class of all graphs G' with *forbidden* homomorphisms from $\mathcal{F}'$. (See Section 2 for definitions.) Theorems 4 and 5 provide similar results for forbidden colored subgraphs and for forbidden induced subgraphs (in both cases vertex colorings suffice).

We should add one more remark. We of course do not only claim that every problem in NP can be polynomially *reduced* to a problem in any of these classes. This would only mean that each of these classes contains an NP-complete problem. What we claim is that these classes have the *computational power* of the whole NP class. More precisely, to each language L in NP there exists a language M in any of these three classes such that M is *polynomially equivalent* to L , i.e. there exist *polynomial reductions* of L to M and M to L . E.g. assuming $P \neq NP$ there is a language in any of these classes that is neither in P nor NP-complete, since there is such a language in NP by Ladner’s celebrated result [14].

The expressive power of classes $\Phi(Forb(\mathcal{F}'))$ corresponds to many combinatorially studied problems and presents a combinatorial counterpart to the celebrated result of Fagin [4] who expressed every NP problem in logical terms by means of an Existential Second Order formula.

The fact that the membership problem for classes $\Phi(Forb(\mathcal{F}'))$ and their injective and full variants $\Phi(Forb_{inj}(\mathcal{F}'))$ and $\Phi(Forb_{full}(\mathcal{F}'))$ have full computational power is pleasing from the combinatorial point of view as these classes cover well known examples of hard combinatorial problems: Ramsey type problems (where as in Theorem 3 we consider edge colored graphs), colorings of bounded degree graphs (defined by an injectivity condition as in Theorem 4) and structural partitions (studied e.g. in [8] as in Theorem 5). It follows that, in

the full generality, one cannot expect dichotomies here. On the other side of the spectrum, Feder and Vardi have formulated the celebrated *Dichotomy conjecture* for all coloring problems (CSP).

Our main result is Theorem 9: we give an easy characterization of those languages $\Phi(\text{Forb}(\mathcal{F}'))$ which are coloring problems (CSP). This can be viewed as an extension of the duality characterization theorem for structures [6]. We demonstrate the power of this theorem while reproving some theorems about the local chromatic number. In contrast with this we show that the shadow $\Phi(\text{Forb}(\mathcal{F}'))$ of a vertex colored class of digraphs $\text{Forb}(\mathcal{F}')$ is always a CSP language when restricted to a bounded expansion class (this notion generalizes bounded degree and proper minor closed classes) [20]. Our main tools are *finite dualities* [23,6], *restricted dualities* [21], and the *Sparse Incomparability Lemma* [22,9]. The detailed proofs can be found in the full version of this paper [13].

2 Preliminaries

We consider finite relational structures although in most of the paper we only deal with digraphs, i.e. relational structures with just one binary relation. This itself is one of the main features of this note: oriented graphs suffice. Digraphs will be denoted by $\mathbf{A}, \mathbf{B}, \dots$ (as we want to stress that they may be replaced by more general structures).

Let Γ denote a finite set we refer to as colors. A Γ -colored graph (structure) is a graph (or structure) together with either a coloring of its vertices or a coloring of all pairs of vertices by colors from Γ . Only in Theorem 3 we shall consider coloring of all pairs (but in Theorem 3 this will play an important role). Thus in the whole paper we shall understand by a colored graph a graph with colored vertices. We denote colored digraphs (relational structures) by $\mathbf{A}', \mathbf{B}'$ etc. Following the more general notions in category theory we call $\mathbf{A}'$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* of $\mathbf{A}'$. Thus (vertex-) colored digraphs (structures) can be also described as *monadic* lifts. A homomorphism of digraphs (relational structures) preserves all the edges (arcs). A homomorphism of colored digraphs (relational structures) preserves the color of vertices (pairs of vertices), too. The *Constraint Satisfaction Problem* corresponding to the graph (relational structure) $\mathbf{A}$ is the membership problem for the class of all graphs (structures) defined by $\{\mathbf{B} : \mathbf{B} \text{ is homomorphic to } \mathbf{A}\}$. We call a mapping between two (colored) digraphs a *full homomorphism* if in addition the preimage of an edge is an edge. Full homomorphisms have very easy structure, as every full homomorphism which is onto is a retraction. The other special homomorphisms we will be interested in are *injective* homomorphisms.

Let $\mathcal{F}'$ be a finite set of colored relational structures (digraphs). By $\text{Forb}(\mathcal{F}')$ we denote the set of all colored relational structures (digraphs) $\mathbf{A}'$ satisfying $\mathbf{F}' \not\rightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}'$. (If we use injective or full homomorphisms this will be denoted by $\text{Forb}_{inj}(\mathcal{F}')$ or $\text{Forb}_{full}(\mathcal{F}')$, respectively).

Similarly (well, dually), for the finite set of colored relational structures (digraphs) $\mathcal{D}'$ we denote by $CSP(\mathcal{D}')$ the class of all colored digraphs $\mathbf{A}'$ satisfying $\mathbf{A}' \rightarrow \mathbf{D}'$ for some $\mathbf{D}' \in \mathcal{D}'$. (This is sometimes denoted by $\rightarrow \mathcal{D}$.) Now suppose that the classes $Forb(\mathcal{F}')$ and $CSP(\mathcal{D}')$ are equal. Then we say that the pair $(\mathcal{F}', \mathcal{D}')$ is a *finite duality*. Explicitly, a finite duality means that the following equivalence holds for every (colored) relational structure (digraph):

$$\forall \mathbf{F}' \in \mathcal{F}' \quad \mathbf{F}' \not\rightarrow \mathbf{A}' \iff \exists \mathbf{D}' \in \mathcal{D}' \quad \mathbf{A}' \rightarrow \mathbf{D}'.$$

We say that the structure $\mathbf{A}$ is *core* if every homomorphism $\mathbf{A} \rightarrow \mathbf{A}$ is an automorphism. Every finite structure $\mathbf{A}$ contains (up to an isomorphism) a uniquely determined core substructure homomorphically equivalent to $\mathbf{A}$, see [23] [9]. The following result was recently proved in [6] and [23]. It characterizes finite dualities of digraphs (or more generally relational structures with a given signature).

Theorem 1. *For every finite set $\mathcal{F}$ of (relational) forests there exists (up to homomorphism equivalence) a finite uniquely determined set $\mathcal{D}$ of structures such that $(\mathcal{F}, \mathcal{D})$ forms a finite duality, i.e. $Forb(\mathcal{F}) = CSP(\mathcal{D})$. Up to homomorphism equivalence there are no other finite dualities.*

Let Φ denote the forgetful functor which corresponds to a Γ -colored relational structure (digraph) the uncolored one, i.e. it forgets about the coloring. We will investigate classes of the form $\Phi(Forb(\mathcal{F}'))$. We call the pair $(\mathcal{F}', \mathcal{D})$ *shadow duality* if $\Phi(Forb(\mathcal{F}')) = CSP(\mathcal{D})$. An example of shadow duality is the language of 3-colorable graphs discussed in the introduction (or, as can be seen easily, any CSP problem in general). Finite dualities became much more abundant when we demand the validity of the above formula just for all graphs from a given class $\mathcal{K}$. In such a case we speak about *$\mathcal{K}$ -restricted duality*. It has been proved in [21] that so called *Bounded Expansion* classes (which include both proper minor closed classes and classes of graphs with bounded degree) have a restricted duality for every choice of $\mathcal{F}'$.

The study of homomorphism properties of structures not containing short cycles (i.e. with a large girth) is a combinatorial problem studied intensively. The following result has proved particularly useful in various applications. It is often called the *Sparse Incomparability Lemma*:

Theorem 2. *Let k, ℓ be positive integers and let $\mathbf{A}$ be a structure. Then there exists a structure $\mathbf{B}$ with the following properties:*

1. *There exists a homomorphism $f : \mathbf{B} \rightarrow \mathbf{A}$;*
2. *For every structure $\mathbf{C}$ with at most k points the following holds: there exists a homomorphism $\mathbf{A} \rightarrow \mathbf{C}$ if and only if there exists a homomorphism $\mathbf{B} \rightarrow \mathbf{C}$;*
3. *$\mathbf{B}$ has girth $\geq \ell$.*

This result was proved by probabilistic method in [22] [24], see also [9]. The polynomial time construction of $\mathbf{B}$ is possible, too: in the case of binary relations (digraphs) this was done in [18] and for relational structures in [12].

3 Statement of Results

3.1 NP by Means of Finitely Many Forbidden Lifts

The class SNP consists of all problems expressible by an existential second-order formula with a universal first-order part [4]. The class SNP is computationally equivalent to NP. Feder and Vardi [5] have proved that three syntactically defined subclasses of the class SNP still have the full computational power of the class NP. We reformulate this result to our combinatorial setting of lifts and shadows.

Theorem 3. *For every language $L \in NP$ there exist a finite set of colors Γ and a finite set of Γ -colored digraphs $\mathcal{F}'$, where we color all pairs of vertices such that L is computationally equivalent to the membership problem for $\Phi(\text{Forb}(\mathcal{F}'))$.*

Theorem 4. *For every language $L \in NP$ there exist a finite set of colors Γ and a finite set of Γ -colored digraphs $\mathcal{F}'$, (where we color the vertices) such that L is computationally equivalent to the membership problem for $\Phi(\text{Forb}_{\text{inj}}(\mathcal{F}'))$.*

Theorem 5. *For every language $L \in NP$ there exist a finite set of colors Γ and a finite set of Γ -colored digraphs $\mathcal{F}'$, (where we color the vertices) such that L is computationally equivalent to the membership problem for $\Phi(\text{Forb}_{\text{full}}(\mathcal{F}'))$.*

3.2 Lifts and Shadows of Dualities

It follows from Section 3.1 that shadows of Forb of a finite set of colored digraphs, this is classes $\Phi(\text{Forb}(\mathcal{F}'))$, where $\mathcal{F}'$ is a finite set, have the computational power of the whole NP. What about finite dualities? Are the shadow dualities also more frequent? The negative answer is expressed by Theorem 7 and shows a remarkable stability of dualities. Towards this end we first observe that every duality (of lifted structures) implies a shadow duality:

Theorem 6. *Let Γ be a finite set of colors and $\mathcal{F}'$ a finite set of Γ -colored digraphs (relational structures), where we color all of the vertices. Suppose that there exists a finite set of Γ -colored digraphs (relational structures) $\mathcal{D}'$ such that $\text{Forb}(\mathcal{F}') = \text{CSP}(\mathcal{D}')$. Then $\Phi(\text{Forb}(\mathcal{F}')) = \text{CSP}(\Phi(\mathcal{D}'))$.*

Theorem 6 may be sometimes reversed: Shadow dualities may be “lifted” in case that lifted graphs have colored vertices (this is sometimes described as *monadic lift*). This is non-trivial and in fact Theorem 7 may be seen as the core of this paper.

Theorem 7. *Let Γ be a finite set of colors and $\mathcal{F}'$ be a finite set of Γ -colored digraphs (relational structures), where we color all of the vertices. Suppose that $\Phi(\text{Forb}(\mathcal{F}')) = \text{CSP}(\mathcal{D})$ for a finite set $\mathcal{D}$ of digraphs (relational structures).*

Then there exists a finite set $\mathcal{D}'$ of Γ -colored digraphs (relational structures) such that $\text{Forb}(\mathcal{F}') = \text{CSP}(\mathcal{D}')$.

4 Proofs

The proofs of Theorems 3, 4 and 5 are in the full version of this paper [13]. We do not include them as they need some new definitions (and space) but nevertheless basically follow the strategy of [5].

Before proving Theorems 6 and 7 we formulate first a simple lemma which we shall use repeatedly:

Lemma 1. (*lifting*) Let $\mathbf{A}, \mathbf{B}$ relational structures, homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$, a finite set of colors Γ and $\Phi(\mathbf{B}') = \mathbf{B}$ be given. Then there exists a lift $\mathbf{A}'$, such that $\Phi(\mathbf{A}') = \mathbf{A}$ and the mapping f is a homomorphism $\mathbf{A}' \longrightarrow \mathbf{B}'$ (of colored structures).

Proof (of Theorem 6). Suppose that $\mathbf{A} \in \text{CSP}(\Phi(\mathcal{D}'))$, say $\mathbf{A} \in \text{CSP}(\Phi(\mathbf{D}'))$. Now for a homomorphism $f : \mathbf{A} \longrightarrow \Phi(\mathbf{D}')$ there is at least one lift $\mathbf{A}'$ of $\mathbf{A}$ such that the mapping f is a homomorphism $\mathbf{A}' \rightarrow \mathbf{D}'$ (here we use Lifting Lemma 1). Since the pair $(\mathcal{F}', \mathcal{D}')$ is a duality $\mathbf{F}' \dashv \mathbf{A}'$ holds for any $\mathbf{F}' \in \mathcal{F}'$ and thus in turn $\mathbf{A} \in \Phi(\text{Forb}(\mathcal{F}'))$.

Conversely, let us assume that $\mathbf{A}' \in \text{Forb}(\mathcal{F}')$ satisfies $\Phi(\mathbf{A}') = \mathbf{A}$. But then $\mathbf{A}' \in \text{CSP}(\mathcal{D}')$ and thus by the functorial property of Φ we have $\mathbf{A} = \Phi(\mathbf{A}') \in \text{CSP}(\Phi(\mathcal{D}'))$.

Proof (of Theorem 7). Assume $\Phi(\text{Forb}(\mathcal{F}')) = \text{CSP}(\mathcal{D})$. Our goal is to find $\mathcal{D}'$ such that $\text{Forb}(\mathcal{F}') = \text{CSP}(\mathcal{D}')$. This will follow as a (non-trivial) combination of Theorems 1 and 2. By Theorem 1 we know that if $\mathcal{F}'$ is a set of (relational) forests then the set $\mathcal{F}'$ has a dual set $\mathcal{D}'$ (in the class of covering colored structures; we just list all covering colored substructures of the dual set guaranteed by Theorem 1). It is $\Phi(\mathcal{D}') = \mathcal{D}$ by Theorem 6. So assume to the contrary that one of the structures, say $\mathbf{F}'_0$, fails to be a forest (i.e. we assume that one of the components of $\mathbf{F}'_0$ has a cycle). We proceed by a refined induction (which will allow us to use more properties of $\mathbf{F}'_0$) to show that $\mathcal{D}'$ does not exist. Let us introduce carefully the setting of the induction.

We assume shadow duality $\Phi(\text{Forb}(\mathcal{F}')) = \text{CSP}(\mathcal{D})$. Let $\mathcal{D}$ be fixed throughout the proof. Clearly many sets $\mathcal{F}'$ will do the job and we select the set $\mathcal{F}'$ such that $\mathcal{F}'$ consists of cores of all homomorphic images (explicitly: we close $\mathcal{F}'$ under homomorphic images and then take the set of cores of all these structures). Among all such sets $\mathcal{F}'$ we take a set of minimal cardinality. It will be again denoted by $\mathcal{F}'$. We proceed by induction on the size $|\mathcal{F}'|$ of $\mathcal{F}'$.

The set $\text{Forb}(\mathcal{F}')$ is clearly determined by the minimal elements of $\mathcal{F}'$ (minimal in the homomorphism order). Thus let us assume that one of these minimal elements, say $\mathbf{F}'_0$, is not a forest. By the minimality of $\mathcal{F}'$ we see that we have a

proper inclusion $\Phi(\text{Forb}(\mathcal{F}' \setminus \{\mathbf{F}'_0\})) \supset CSP(\mathcal{D})$. Thus there exists a structure $\mathbf{S}$ in the difference. But this in turn means that there has to be a lift $\mathbf{S}'$ of $\mathbf{S}$ such that $\mathbf{F}'_0 \rightarrow \mathbf{S}'$ and $\mathbf{S} \not\rightarrow \mathbf{D}$ for every $\mathbf{D} \in \mathcal{D}$. In fact not only that: as $\mathbf{F}'_0$ is a core, as $\text{Forb}(\mathcal{F}')$ is homomorphism closed and as $\mathcal{F}'$ has minimal size we conclude that there exist $\mathbf{S}$ and $\mathbf{S}'$ such that any homomorphism $\mathbf{F}'_0 \rightarrow \mathbf{S}'$ is a monomorphism (i.e. one-to-one, otherwise we could replace $\mathbf{F}'_0$ by a set of all its homomorphic images - $\mathbf{F}'_0$ would not be needed).

Now we apply (the second non-trivial ingredient) Theorem 2 to structure $\mathbf{S}$ and an $\ell > |X(\mathbf{F}'_0)|$: we find a structure $\mathbf{S}_0$ with the following properties: $\mathbf{S}_0 \rightarrow \mathbf{S}$, $\mathbf{S}_0 \rightarrow \mathbf{D}$ if and only if $\mathbf{S} \rightarrow \mathbf{D}$ for every $\mathbf{D} \in \mathcal{D}$ and $\mathbf{S}_0$ contains no cycles of length $\leq \ell$. It follows that $\mathbf{S}_0 \notin CSP(\mathcal{D})$. Next we apply Lemma 1 to obtain a structure $\mathbf{S}'_0$ with $\mathbf{S}'_0 \rightarrow \mathbf{S}'$. Now we use that $\mathbf{S}'_0$ is a monadic lift and so does not contain cycles of length $\leq \ell$. Now for any $\mathbf{F}' \in \mathcal{F}'$, $\mathbf{F}' \neq \mathbf{F}'_0$ we have $\mathbf{F}' \rightarrow \mathbf{S}'_0$ as $\mathbf{S}'_0 \rightarrow \mathbf{S}'$ and $\mathbf{F}' \rightarrow \mathbf{S}'$. As the only homomorphism $\mathbf{F}'_0 \rightarrow \mathbf{S}'$ is a monomorphism the only (hypothetical) homomorphism $\mathbf{F}'_0 \rightarrow \mathbf{S}'$ is also monomorphism. But this is a contradiction as $\mathbf{F}'_0$ contains a cycle while $\mathbf{S}'_0$ has no cycles of length $\leq \ell$. This completes the proof.

5 Applications

5.1 Classes with Bounded Expansion

We study the restriction of classes $\Phi(\text{Forb}(\mathcal{F}'))$ to a class of digraphs with bounded expansion recently introduced in [20]. These classes are a generalization of proper minor closed and bounded degree classes of graphs. Using the decomposition technique of [20] [21] we can prove that any class $\Phi(\text{Forb}(\mathcal{F}'))$ (for a finite set $\mathcal{F}'$ of monadic lifts) when restricted to a bounded expansion class equals to a CSP class (when restricted to the same class).

Theorem 8. *Consider the finite set of colors Γ and the class $\Phi(\text{Forb}(\mathcal{F}'))$ for a finite set $\mathcal{F}'$ of Γ -colored digraphs. Let $\mathcal{C}$ be a class of digraphs of bounded expansion. Then there is a finite set of digraphs $\mathcal{D}$ such that $\Phi(\text{Forb}(\mathcal{F}')) \cap \mathcal{C} = CSP(\mathcal{D}) \cap \mathcal{C}$.*

Consider a monotone, first-order definable class of colored digraphs $\mathcal{C}$ which is closed under homomorphism and disjoint union. By a combination with recent results of [2] we also obtain (perhaps a bit surprisingly) that the shadow $\mathcal{C}$ is a CSP language of digraphs. It remains to be seen to which bounded expansion classes (of graphs and structures) this result generalizes.

5.2 The Classes MMSNP and FP - A Characterization

We conclude with an application to descriptive theory of complexity classes. Recall that the class of languages defined by monotone, monadic formulas without inequality is denoted by MMSNP (*Monotone Monadic Strict Nondeterministic*

Polynomial). (Feder and Vardi proved that the class MMSNP is computationally equivalent to the class CSP in a random sense [5], this was later derandomized by the first author [12].) Madeleine [16] introduced the class FP of languages defined similarly to our forbidden monadic lifts of structures.

It has been proved in [16] that the classes FP and MMSNP are equal. In fact the class MMSNP contains exactly the languages defined by forbidden monadic lifts.

Proposition 1. *A language of relational structures L is in the class MMSNP if and only if there is a finite set of colors Γ and a finite set of Γ -colored relational structures $\mathcal{F}'$ such that $L = \Phi(\text{Forb}(\mathcal{F}'))$.*

Madeline and Stewart [17] gave a long process to decide whether an FP language is a finite union of CSP languages. We use Theorems 6 and 7 and the description of dualities for relational structures [6] to give a short characterization of a more general class of languages.

Theorem 9. *Consider the finite set of colors Γ and the language $\Phi(\text{Forb}(\mathcal{F}'))$ for a finite set $\mathcal{F}'$ of Γ -colored digraphs (relational structures).*

If no $\mathbf{F}' \in \mathcal{F}'$ contains a cycle then there is a finite set of digraphs (relational structures) $\mathcal{D}$ such that $\Phi(\text{Forb}(\mathcal{F}')) = \text{CSP}(\mathcal{D})$. If one of the lifts $\mathbf{F}'$ in a minimal subfamily of $\mathcal{F}'$ contains a cycle in its core then the language $\Phi(\text{Forb}(\mathcal{F}'))$ is not a finite union of CSP languages.

Proof. If no $\mathbf{F}' \in \text{Forb}(\mathcal{F}')$ contains a cycle then the set $\mathcal{F}'$ has a dual $\mathcal{D}'$ by Theorem 1, and the shadow of this set $\mathcal{D}'$ gives the dual set $\mathcal{D}$ of the set $\Phi(\text{Forb}(\mathcal{F}'))$ (by Theorem 6). On the other side if one $\mathbf{F}' \in \text{Forb}(\mathcal{F}')$ contains a cycle in its core and if $\mathcal{F}'$ is minimal (i.e. $\mathbf{F}'$ is needed) then $\text{Forb}(\mathcal{F}')$ does not have a dual. The shadow of the language $\text{Forb}(\mathcal{F}')$ is the language L and consequently this fails to be a finite union of CSP languages by Theorem 7.

Theorem 9 may be interpreted as stability of dualities for finite structures. While shadows of the classes $\text{Forb}(\mathcal{F}')$ are computationally equivalent to the whole NP, the shadow dualities are not bringing anything new: these are just shadows of dualities. In other words: the coloring problems in the class MMSNP are just shadow dualities. This holds for graphs as well for relational structures.

5.3 On the Local Chromatic Number

Now we apply Theorem 9 in the analysis of local chromatic number introduced in [3] (see also [26]): we say that a graph G is locally (a, b) -colorable if there exists a proper coloring of G by b colors so that every (closed) neighborhood of a vertex of G gets at most a colors. It follows from [3] that the class of all locally (a, b) -colorable graphs is of the form $\text{CSP}(U(a, b))$ for an explicitly constructed graph $U(a, b)$. We conclude this paper with an indirect proof of this result with an application to complexity:

Proposition 2. *Let a, b be integers and consider the membership problem for the class of locally (a, b) -colorable graphs. This is actually a Constraint Satisfaction Problem which is NP-complete if $a, b \geq 3$ and it is polynomial time solvable else.*

Proof. Consider the color set $\Gamma = \{1, \dots, b\}$ and the following set $\mathcal{F}'$ of Γ -colored undirected graphs. Let $\mathcal{F}'$ consist of all monochromatic edges (colored by any of the b colors) and all the stars with $a + 1$ vertices colored by at least $a + 1$ colors. The corresponding language is exactly the required one: a graph G is in the language iff it admits a proper Γ -coloring, this is no monochromatic edge is homomorphic to the colored graph, such that the neighbourhood of every vertex (including the vertex itself) has at most a different colors, i.e. no star with $a + 1$ vertices of different color is homomorphic to it. Since $\mathcal{F}'$ consists of colored trees this will be a CSP language by Theorem 9.

Hell and the second author proved that CSP problems defined by undirected graphs are in P if the graph is bipartite and NP-complete else [9]. We do not determine which graph defines this particular CSP problem (of locally (a, b) -colorable graphs). But if $a, b \geq 3$ then we know that it contains the triangle if, so the problem is NP-complete. It is easy to see that this membership problem is in P if $a < 3$ or $b < 3$.

6 Summary and Future Work

We found a computationally equivalent formulation of the class NP by means of finitely many forbidden lifts of very special type. An ambitious project would be to find an equivalent digraph coloring problem for a given NP language really effectively (in human sense, our results provide a polynomial time algorithm). For example it would be nice to exhibit a vertex coloring problem that is polynomially equivalent to the graph isomorphism problem. In general this mainly depends on how to express the problem in terms of logic. The next class we seem to be able to deal with are coloring problems of structures with an equivalence relation. Another good candidate are lifts using linear order. This promises several interesting applications which were studied earlier in a different setting.

We also proved that shadow dualities and lifted monadic dualities are in 1 – 1 correspondence. This abstract result has several consequences and streamlines some earlier results in descriptive complexity theory (related to MMSNP and CSP classes). The simplicity of this approach suggests some other problems. It is tempting to try to relate Ladner's diagonalization method [14] in this setting (as it was pioneered by Lovász and Gács [7] for $\text{NP} \cap \text{coNP}$ in a similar context). The characterization of Lifted Dualities is beyond reach but particular cases are interesting as they generalize results of [23] [6] and as the corresponding duals present polynomial instances of CSP.

But perhaps more importantly, our approach to the complexity subclasses of NP is based on lifts and shadows as a combination of algebra, combinatorics and logic. We believe that it has further applications and that it forms a useful paradigm.

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