The Law of the Iterated Logarithm for Algorithmically Random Brownian Motion

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Abstract. Algorithmic randomness is most often studied in the setting of the fair-coin measure on the Cantor space, or equivalently Lebesgue measure on the unit interval. It has also been considered for the Wiener measure on the space of continuous functions. Answering a question of Fouché, we show that Khintchine's law of the iterated logarithm holds at almost all points for each *Martin-Löf random path of Brownian motion*. In the terminology of Fouché, these are the *complex oscillations*. The main new idea of the proof is to take advantage of the Wiener-Carathéodory measure algebra isomorphism theorem.

Keywords: Brownian motion, randomness, law of the iterated logarithm, Kolmogorov complexity.

1 Introduction

Algorithmic randomness for Brownian motion was introduced by Asarin and Pokrovskii. They defined what they called (according to the English translation [1]) *truly random* continuous functions. Fouché [3] called these functions *complex oscillations*.

In this article we answer a question of Fouché (see [5]) by showing that for each complex oscillation, Khintchine's law of the iterated logarithm holds at almost every point. To that end, in Section 2 we will borrow a construction from the proof of the Wiener-Carathéodory measure algebra isomorphism theorem. For the full statement of this theorem, the reader may consult for example Royden [6], Theorem 15.3.4; we shall not need it. We believe our method based on this isomorphism theorem can be used to yield other results than the one presented here. Namely, algorithmic randomness for the unit interval $[0,1]$ has been studied more extensively than algorithmic randomness for the space $C[0,1]$ of continuous functions, and the isomorphism theorem allows a transfer of some results. For a general introduction to computability theory and algorithmic randomness on $[0,1]$, the reader may consult [2].

Definition 1. *Suppose Ω is a set, $\mathscr{F} = \{T_i : i \in \mathbb{N}\}$ a countable Boolean algebra of subsets of Ω, μ a probability measure on the σ-algebra generated by $\mathscr{F}$. Let $t \in [0,1]$. Suppose $\phi : \mathbb{N}^2 \to \mathbb{N}$ is a total function Turing reducible to t.*

S. Artemov and A. Nerode (Eds.): LFCS 2007, LNCS 4514, pp. 310–317, 2007.

The sequence $U_n = \bigcup_m T_{\phi(n,m)}$, $n \in \mathbb{N}$ *is called a t-uniform sequence of* $\Sigma_1^t(\mathscr{F})$ *sets.*

A t-effective $\mathscr{F}$*-null set is a set* $A \subseteq \Omega$ *such that for some such* ϕ,

1. $A \subset \bigcap_n U_n$, *and*
2. μU_n *goes effectively to* 0 *as* $n \to \infty$. *That is, there is a computable function* ψ *such that whenever* $n \geq \psi(k)$, *we have* $\mu U_n \leq 2^{-k}$.

We review the Wiener measure W on $\Omega = C[0,1]$. It is such that for $\omega \in \Omega$, and $t_0 < t_1 < \cdots < t_n$, the values of $\omega(t_0)$ and $\omega(t_{i+1} - t_i)$ are independent random variables. Moreover, the probability that $\omega(s+t) - \omega(s) \in A$, where A is some set of reals, is $\int_A (2\pi t)^{-1/2} \exp(-x^2/2t)dx$. This says that $\omega(t)$ is normally distributed with standard deviation $\sqrt{t}$ (variance t) and mean 0. Informally, a sufficiently random member of Ω with respect to W is called a path of Brownian motion.

Let $\overline{G}$ denotes the closure of G, and let G^0 denotes the complement of G; moreover, $O_\epsilon(G)$ is the open ϵ-ball around G.

Definition 2 (Fouché [3]). *A sequence* $\mathscr{F}_0 = (F_i : i < \omega)$ *of Borel subsets of* Ω *is a t-effective generating sequence if*

(1) for $F \in \mathscr{F}_0$, *for* $\epsilon > 0$ *and* $\delta \in \{0,1\}$, *if* $G = O_\epsilon(F^\delta)$ *or if* $G = F^\delta$, *then* $W(\overline{G}) = W(G)$;
(2) there is a t-effective procedure that yields, for each sequence $0 \leq i_1 < \cdots < i_n < \omega$ *and* $k < \omega$, *a binary number* β_k *such that* $|W(\bigcap_{1 \leq k \leq n} F_{i_k}) - \beta_k| < 2^{-k}$; *and*
(3) for $n, i < \omega$, *for rational numbers* $\epsilon > 0$ *and for* $x \in C_n$, *both the relations* $x \in O_\epsilon(F_i)$ *and* $x \in O_\epsilon(F_i^0)$ *are t-recursive in* x, ϵ, i *and* n.

A t-effectively generated algebra is the Boolean algebra generated from an t-effective generating sequence.

A set $A \subset C[0,1]$ *is of t-constructive measure 0 if, for some t-effectively generated algebra* $\mathscr{F}$, A *is a t-effective* $\mathscr{F}$*-null set.*

If ω *belongs to no t-effective* $\mathscr{F}$*-null set, then we say that* ω *is t-*$\mathscr{F}$*-random or* $\mathscr{F}$*-random relative to t. (This also applies when* $\mathscr{F}$ *is not effective.) If t is computable then we may omit mention of t.*

Finally, if there exists a t such that $\mathscr{F}_0$ *is a t-effective generating sequence, then* $\mathscr{F}_0$ *is called a* generating sequence. *The algebra it generates is similarly called a* generated algebra.

The precise definition of complex oscillations is immaterial to the present paper, but we include it for completeness.

Definition 3. *For* $n \geq 1$, *we write* C_n *for the class of continuous functions on* $[0,1]$ *that vanish at* 0 *and are linear with slope* $\pm\sqrt{n}$ *on the intervals* $[(i-1)/n, i/n]$, $i = 1, \ldots, n$.

To every $x \in C_n$ *one can associate a binary string in* $\{1,-1\}^*$, $a_1 \cdots a_n$, *of length n by setting* $a_i = 1$ *or* $a_i = -1$ *according to whether x increases or*

decreases on the interval $[(i-1)/n, i/n]$. *We call the word* $a_1 \cdots a_n$ *the* code *of* x *and denote it by* $c(x)$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* $C[0,1]$ *is* complex *if* $x_n \in C_n$ *for each* n *and there is some constant* $d > 0$ *such that* $K(c(x_n)) \geq n - d$ *for all* n, *where* K *denotes prefix-free Kolmogorov complexity.*

A function $x \in C[0,1]$ *is a complex oscillation if there is a complex sequence* $\{x_n\}_{n\in\mathbb{N}}$ *such that* $x_n - x$ *converges effectively to* 0 *as* $n \to \infty$, *in the uniform norm.*

The only use we make of Definition 3 is to quote the following result.

Theorem 1 (Fouché [3]; see also [4]). *Each complex oscillation is in the complement of each set of constructive measure 0.*

Let $LIL(\omega, t)$ be the statement that

$$\limsup_{h\to 0} \frac{|\omega(t+h) - \omega(t)|}{\sqrt{2|h| \log\log(1/|h|)}} = 1.$$

Thus $LIL(\omega, t)$ says that Khintchine's Law of the Iterated Logarithm holds for ω at t.

Theorem 2 (following Fouché [5]). *If* $t \in [0,1]$, *and* f *is* t-$\mathscr{F}$*-random for each* t*-effectively generated algebra* $\mathscr{F}$, *then the Law of the Iterated Logarithm holds for* f *at* t.

The proof is a straightforward relativization to t of Fouché's argument (which covers the case where t is computable).

2 Isomorphism Theorem

Let $\mathscr{A}_0$ be a generating sequence. Write $\mathscr{A}_0 = \{A_n\}_{n\in\mathbb{N}}$.

- Let $\mathfrak{A}_n$ be the Boolean algebra generated by $\{A_1, \ldots, A_n\}$.
- Let $\mathscr{A} = \mathfrak{A}_\infty = \bigcup_n \mathfrak{A}_n$, the Boolean algebra generated by $\mathscr{A}_0$.
- Let $\mathfrak{I}$ be the Boolean algebra of finite unions of half-open intervals $[a, b)$ in $[0, 1)$.

A *Boolean measure algebra homomorphism* is a map that preserves measure, unions, and complements.

Theorem 3 (Wiener, Carathéodory). *There is a Boolean measure algebra homomorphism* $\Phi : \mathscr{A} \to \mathfrak{I}$.

Proof. To start, since $\mathfrak{A}_1 = \{\emptyset, A_1, A_1', \Omega\}$ and $\mu A_1 + \mu A_1' = \mu\Omega = 1$, we let $\Phi(A_1) = [0, \mu A_1)$, $\Phi(A_1') = [\mu A_1, 1)$, $\Phi(\emptyset) = \emptyset$, and $\Phi(\Omega) = [0, 1)$. Then Φ is clearly a Boolean measure algebra homomorphism from $\mathfrak{A}_1$ into $\mathfrak{I}$.

Suppose now that Φ has been defined on $\mathfrak{A}_{n-1}$ so that it is a Boolean measure algebra homomorphism from $\mathfrak{A}_{n-1}$ onto the algebra generated by $k \in \mathbb{N}$ many half open intervals $[x_0, x_1), [x_1, x_2), \ldots, [x_{k-1}, x_k)$, where $x_0 = 0$ and $x_k = 1$.

We wish to extend the mapping Φ to $\mathfrak{A}_n$. Let B_i be the set in $\mathfrak{A}_{n-1}$ which is mapped onto the interval $[x_j, x_{j+1})$, for $j < k$. Then $\mathfrak{A}_{n-1}$ consists of all finite unions of the sets B_j, $j < k$, and $\mathfrak{A}_n$ consists of all finite unions from the $2k$ sets $A_n \cap B_j$, $A_n^0 \cap B_j$, $j < k$. Let

$$\Phi(A_n \cap B_j) = [x_j, x_j + \mu(A_n \cap B_j))$$
$$\Phi(A_n^0 \cap B_j) = [x_j + \mu(A_n \cap B_j), x_{j+1})$$

This might define Φ of some sets to be of the form $[x_j, x_j) = \emptyset$.

Clearly Φ as so defined is measure-preserving, $\Phi(A_n \cap B_j) \cup \Phi(A_n' \cap B_j) = [x_j, x_{j+1}) = \Phi(B_j)$, and $\mu(A_n \cap B_j) + \mu(A_n' \cap B_j) = \mu(B_j) = x_{j+1} - x_j$. From this it follows that we can extend Φ to all of $\mathfrak{A}_n$ so that it is a Boolean measure algebra homomorphism. Since $\mathfrak{A}_\infty = \bigcup_n \mathfrak{A}_n$, we have thus defined Φ on all of $\mathfrak{A}_\infty$.

Remark 1. The function Φ is effective in the following sense: if $\mathscr{F} = \{T_k : k \in \mathbb{N}\}$ is a t-effectively generated algebra, then the measure of $\Phi(T_k)$ can be computed t-effectively, uniformly in k.

Lemma 1. *Suppose $\mathcal{I}_n = (a_n, b_n)$, $n \in \mathbb{N}$, is a sequence of open intervals with $(a_{n+1}, b_{n+1}) \subseteq (a_n, b_n)$. Suppose $\bigcap_n (a_n, b_n) = \emptyset$. Then either $\{a_n\}_{n\in\mathbb{N}}$ or $\{b_n\}_{n\in\mathbb{N}}$ is an eventually constant sequence.*

The proof is routine. The set of Martin-Löf real numbers in $[0, 1]$ is denoted RAND, and relativized to t, RAND^t.

An effectively generated algebra $\mathscr{F} = \{T_k : k \in \mathbb{N}\}$ is *non-atomic* if for any $b : \mathbb{N} \to \{0, 1\}$, we have $W(\bigcap_k T_k^{b(k)}) = 0$. Here $T_k^1 = T_k$ and as before T_k^0 is the complement of T_k.

Lemma 2. *Let $t \in [0, 1]$ and let $\mathscr{F} = \{T_k : k \in \mathbb{N}\}$ be a non-atomic, t-effectively generated algebra. Let a function φ from $C[0, 1]$ to 2^ω be defined by: $\varphi(\omega) =$ the unique member of $\cap\{\Phi(T_k) : \omega \in T_k\}$, if it exists.*

(1) The domain of φ includes all t-$\mathscr{F}$-randoms.
(2) If $\varphi(\omega)$ is defined then for each k,

$$\omega \in T_k \leftrightarrow \varphi(\omega) \in \Phi(T_k).$$

Proof. (1): Suppose ω is not in the domain of φ. That is, $S = \cap\{\Phi(T_k) : \omega \in T_k\}$ does not have a unique element. It is clear that S is an interval. Since $\mathscr{F}$ is non-atomic, this interval must have measure zero. Thus, since S does not have exactly one element, S must be empty.

By Remark 1 and Lemma 1, there is a t-computable point a or b such that $a_n \to a$ or $b_n \to b$, where $(a_n, b_n) = \cap_{k\le n}\Phi(T_k)$. Using this point a or b one can t-effectively determine whether $\omega \in T_k$, given any $k \in \mathbb{N}$. Thus ω is not t-$\mathscr{F}$-random.

(2)$\leftarrow$: Since $\{T_k\}_{k\in\mathbb{N}}$ is a Boolean algebra and so closed under complements.

(2)$\rightarrow$: By definition of φ.

2.1 Effectiveness Lemmas

A *presentation* of a real number a is a sequence of open intervals I_n with rational endpoints, containing a, such that I_n has diameter $\leq 2^{-n}$.

Lemma 3. *There is a Turing machine which, given a presentation of $a = a_0 \oplus a_1$ as oracle, terminates iff $a_0 < a_1$.*

Proof (Proof sketch.). The presentation provides some information about a_0 and a_1. The machine terminates once enough information has been found to conclude that $a_0 < a_1$.

On the other hand, it is well known that if $a = b$ then no algorithm will be able to verify this in general. For intervals (a, b), (c, d), we say (a, b) is *bi-properly* contained in (c, d) if $c < a \leq b < d$.

Lemma 4. *(1) The set of pairs σ, k such that $\Phi(T_k)$ is bi-properly contained in $i([\sigma])$, is computably enumerable.*
(2) The set of pairs σ, k such that $i([\sigma])$ is bi-properly contained in $\Phi(T_k)$ is computably enumerable.

Proof. The endpoints of $\Phi(T_k)$ and $i([\sigma])$ have computable presentations. Thus the result follows from Lemma 3.

Lemma 5. *Let $t \in [0, 1]$ and let $\mathscr{F} = \{T_k : k \in \mathbb{N}\}$ be a t-effectively generated algebra.*

(1) If $\mathscr{F}$ is non-atomic, then for each t-ML-test $\{U_n\}_{n\in\mathbb{N}}$ there is a t-computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that

$$U_n \cap RAND = \bigcup_m \Phi(T_{f(n,m)}) \cap RAND.$$

(2) If for a t-computable function $f : \mathbb{N}^2 \to \mathbb{N}$, we have $U_n = \bigcup_m \Phi(T_{f(n,m)})$ then U_n has a subset U'_n such that $U_n \cap RAND = U'_n \cap RAND$ and $\{U'_n\}_{n\in\mathbb{N}}$ is uniformly $\Sigma^0_1(t)$.

Proof. (1): We can enumerate the cones $[\sigma]$ contained in U_n. Once we see some $[\sigma]$ get enumerated and then see (using Lemma 4(1)) that some $\Phi(T_k)$ is bi-properly contained in $[\sigma]$, we can enumerate $\Phi(T_k)$. Since $\mathscr{F}$ is non-atomic, we will gradually enumerate all of $[\sigma]$ except for possibly one or more of its computable endpoints.

(2): By Lemma 4(2). Given an enumeration of the sets $\Phi(T_{f(n,m)})$, $m \in \mathbb{N}$, we can enumerate sets $[\sigma_{n,m,p}]$ that are bi-properly contained in $\Phi(T_{f(n,m)})$ to ensure that $\bigcup_m \Phi(T_{f(n,m)} \cap \mathrm{RAND} = \bigcup_{p,m}[\sigma_{n,m,p}] \cap \mathrm{RAND}$. The endpoints of $\Phi(T_k)$ are computable and hence not in RAND.

The only result of this section that will be used in the next is the following:

Theorem 4. *Let $\omega \in \Omega$, $t \in [0,1]$, and let $\mathscr{F}_0 = \{T_k : k \in \mathbb{N}\}$ be a non-atomic t-effective generating sequence, and $\mathscr{F}$ its generated algebra. The following are equivalent:*

(1) ω is t-$\mathscr{F}$-random;
(2) $\varphi(\omega) \in RAND^t$.

Proof. (2) implies (1): Suppose ω is not t-$\mathscr{F}$-random, so $\omega \in \bigcap_n V_n$, a t-$\mathscr{F}$-null set. Then $V_n = \bigcup_m T_{f(n,m)}$ for some t-computable f.

Let $U_n = \bigcup_m \Phi(T_{f(n,m)})$. Note
(a) U_n is uniformly $\Sigma^0_1(t)$ by Lemma 5(2).
(b) Since Φ is measure preserving on $\mathscr{F}$ and is a Boolean algebra homomorphism by Theorem 3,

$$\mu(\cup_{i=1}^n \Phi(T_{a_i})) = \mu(\Phi(\cup_{i=1}^n T_{a_i})) = \mu(\cup_{i=1}^n T_{a_i})$$

Since the measure of a countable union is the limit of the measures of finite unions, $\mu U_n = \mu V_n \leq 2^{-n}$.
By (a) and (b), $\{U_n\}_{n\in\omega}$ is a t-ML-test.

If ω is not in the domain of φ then ω is not $\mathscr{F}$-random, by Lemma 2(1); so we may assume $\varphi(\omega)$ exists. Hence, since $\omega \in \bigcap_n V_n$, by definition of φ, we have $\varphi(\omega) \in \bigcap_n U_n$. Thus $\varphi(\omega) \notin \mathrm{RAND}^t$.

(1) implies (2): Suppose $\varphi(\omega)$ is not 1-t-random, so $\varphi(\omega) \in \bigcap_n U_n$, for some t-Martin-Löf test $\{U_n\}_{n\in\mathbb{N}}$. Let $V_n := \cup_m T_{f(n,m)}$ with f as in Lemma 5. So by its definition, V_n is uniformly $\Sigma^t_1(\mathscr{F}_0)$. As in the proof that (2) implies (1), V_n and U_n have the same measure. Since $\varphi(\omega) \in \bigcap_n U_n$, by Lemma 2(2) we have $\omega \in \bigcap_n V_n$.

Remark 2. Our identification of binary sequences with numbers in $[0,1]$ deserves some comment. A number $t \in [0,1]$ is a dyadic rational if it is of the form $\frac{p}{2^n}$, for $p, n \in \mathbb{N}$; otherwise, t is called a dyadic irrational. The set of dyadic irrationals can be identified with a full-measure subset of $2^{\mathbb{N}}$ via the map ι such that $\iota(\sum_{i\geq 1} b_i 2^{-i}) = \{b_i\}_{i\geq 1}$. This also gives an identification of cones $[\sigma] = \{A \in 2^{\mathbb{N}} : \forall n < |\sigma|\ A(n) = \sigma(n)\}$ for $\sigma \in \{0,1\}^*$ with intervals in the dyardic irrationals. Formally, we can let $i(2^\omega) = (0,1)$ and if $i([\sigma]) = (a,b)$ then $i([\sigma 0]) = (a, a+(b-a)/2)$ and $i([\sigma 1] = (a+(b-a)/2, b)$, but we leave i implicit. We only need φ restricted to the $\mathscr{F}$-random functions $\omega \in \Omega$. By Theorem 4, φ maps such functions ω into RAND and in particular into the dyadic irrationals.

3 Khintchine's Law for Complex Oscillations

It is common in probability theory to write, for $\omega \in \Omega$ and $x \in [0,1]$, $B_x(\omega) = \omega(x)$. This allows us to refer to the set $\{\omega \in \Omega : \omega(x) < y\}$, for example (where x, y are fixed rational numbers) as the event that $B_x < y$, and as a set this is written $[B_x < y]$. In words, the value of the Brownian motion at time x is less than y.

Let, for each $t \in [0,1]$, $\mathscr{F}_t$ be an t-effectively generated algebra that contains the one used in Theorem 2, and that moreover is non-atomic. The latter is achieved by including all events of the form $[B_x < y]$ for rational $x \in [0,1]$ and arbitrary rational y. Note that if $\mathscr{F}$ and $\mathscr{F}'$ are effectively generated algebras, and $\mathscr{F} \subseteq \mathscr{F}'$, then each $\mathscr{F}'$-random function $\omega \in \Omega$ is also $\mathscr{F}$-random, since adding elements to an effective generating sequence only adds new effective null sets.

Lemma 6. *For each $t \in [0,1]$ and each ω with $\varphi(\omega) \in RAND^t$, we have $LIL(\omega, t)$. In particular, for each $t \in RAND$ and each ω with $\varphi(\omega) \in RAND^t$, we have $LIL(\omega, t)$.*

Proof. Suppose $t \in [0,1]$ and $\varphi(\omega) \in \mathrm{RAND}^t$. By Theorem 4, ω belongs to no t-effective $\mathscr{F}_t$-null set. Hence by Theorem 2, $LIL(\omega, t)$.

The point now is that in the image of φ, we already know more of what is going on. Let $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$, for reals A, B (equivalently, $A, B \subseteq \mathbb{N}$).

Theorem 5 (van Lambalgen's Theorem). *Let A, B be reals. The following are equivalent.*

- *$A \in RAND$ and $B \in RAND^A$;*
- *$A \oplus B \in RAND$;*
- *$B \in RAND$ and $A \in RAND^B$.*

We can now approach our desired result:

Lemma 7. *If $\varphi(\omega) \in RAND$ and $t \in RAND^{\varphi(\omega)}$ then $LIL(\omega, t)$.*

Proof. Suppose $\varphi(\omega) \in \mathrm{RAND}$ and $t \in \mathrm{RAND}^{\varphi(\omega)}$. By Theorem 5 with $A = \varphi(\omega)$ and $B = t$, we have that $t \in \mathrm{RAND}$ and $\varphi(\omega) \in \mathrm{RAND}^t$. Hence by Lemma 6, we have $LIL(\omega, t)$.

Theorem 6. *If ω is a complex oscillation, then for almost all t, $LIL(\omega, t)$.*

Proof. Suppose ω is a complex oscillation. By Theorem 4, $\varphi(\omega) \in \mathrm{RAND}$. By Lemma 7, $LIL(\omega, t)$ holds for each $t \in \mathrm{RAND}^{\varphi(\omega)}$. Since RAND^A has measure 1 for each $A \in 2^\omega$, we are done.

Finally, we remark that our main result can be extended from Martin-Löf randomness to Schnorr randomness, using a weak version of van Lambalgen's theorem that holds for Schnorr randomness.

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