Including the Past in 'Topologic'

Bernhard Heinemann

Fakultät für Mathematik und Informatik,
FernUniversität in Hagen,
58084 Hagen, Germany
bernhard.heinemann@fernuni-hagen.de

Abstract. In this paper, we extend Moss and Parikh's topo-logical view of knowledge. We incorporate a further modality, denoted P, into the original system. This operator describes the increase of sets. Regarding the usual logic of knowledge, P corresponds to no learning of agents. In the context of 'topologic', however, P represents the reverse effort operator and is related to the past therefore. It is our objective to prove nice properties of the accompanying logic like soundness and completeness with respect to the intended class of structures, or decidability. To this end, we take up a hybrid logic point of view, among other things. This not only yields the desired results, but also has some interesting consequences with regard to applications.

Keywords: modal logic, the logic of knowledge, topological reasoning, hybrid logic, temporal operators

1 Introduction

In the paper [1], Moss and Parikh presented a certain bi-modal logic of *knowledge and effort*. On the one hand, the language underlying that logic makes possible a qualitative description of procedures gaining knowledge, and on the other hand some expressive power concerning spatial concepts is provided. In fact, since knowledge is represented through *knowledge states,* which are the sets of states an agent in question considers possible at a time, knowledge acquisition appears as a *shrinking procedure* regarding the space of all such sets. Thus ideas from topology like *closeness* or *neighbourhood* turn up together with knowledge in a natural way.

Moss and Parikh suggestively called their system $\mathsf{topologic}$, and we adopt this naming here. In the following, we briefly recall the basics of the language of $\mathsf{topologic}$. As it has just been indicated, formulas may contain two one-place operators: a modality K describing the knowledge of an agent and another one, $\Box$, describing (e.g., computational) effort. The domains for evaluating formulas are *subset spaces* $(X, \mathcal{O}, V)$ consisting of a non-empty set X of states, a set $\mathcal{O}$ of subsets of X representing the knowledge states of the agent, and a valuation V determining the states where the atomic propositions are true. The

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operator K then quantifies across any $U \in \mathcal{O}$, whereas $\Box$ quantifies 'downward' across $\mathcal{O}$ since shrinking and acquiring knowledge correspond to each other.

After [1] was published, several classes of subset spaces, including the ordinary topological ones, could be characterized within the framework of topologic; cf [2,3,4,5]. Actually, the topological language proved to be quite suitable for dealing with 'local' properties of points and sets (whereas more expressive power is needed to capture 'non-local' notions); see the corresponding chapter of the handbook [6] for more details regarding this and an overview of the current state of the art.

In the present paper, we are less interested in spatial aspects than in those related to time. Note that a temporal dimension already inheres in the effort modality $\Box$ since $\Box$ quantifies across future knowledge states; this was made more explicit, e.g., in the paper [7]. The new operator we consider, P, is the converse of $\Box$. Thus P refers to properties in the past by quantifying 'upward' across $\mathcal{O}$, i.e., over all knowledge states that contain the actual one. We shall, therefore, call P the *past operator* more often than not, and only sometimes emphasize its connection with the increase of sets.[1]

The epistemic relevance of P (in case this operator is regarded independent of $\Box$) is worth mentioning. While the effort operator is associated with *no forgetting* of agents (also called *perfect recall* in the literature), the increase operator P comes along with *no learning;* cf [10].

Integrating the past operator into topologic seems to be very natural. The reader might wonder why this has not been done up to now. However, sometimes this modality turns out to be temperamental, and we must use methods going beyond ordinary modal logic for taming it. But it is worth doing this since we really get a useful means of expression; e.g., the *overlap operator* studied in the paper [11] will be definable then. This was the main motivation for us to examine the properties of P. In retrospect, the promising results we obtained justify our approach.

The subsequent technical part of the paper is organized as follows. In the next section we recapitulate the language of topologic from [12] (which is the journal version of [1]), and we define the semantics of the past operator at the same time. We then give some examples of valid formulas of the extended language. Moreover, we touch on the question of the expressiveness of P. In Section 3, we present a list of axioms for topologic including the past operator. Unfortunately, we do not have a corresponding completeness theorem. However, as the first of the main issues of this paper we prove that the resulting logic is at least decidable. In Section 4, we deal with a certain hybridization of topologic including the past. Concerning the concepts from basic hybrid logic we need for that, see, e.g., [13], Sec. 7.3. In this case we actually obtain the desired

[1] It should be noted that a related modality was examined in the paper [8] with regard to certain systems of *linear time*. However, the framework here is much more general, and the technical details are completely different from those there. – For a treatment of the past in temporal logic of concurrency, see the paper [9].

meta-theorems for the arising logic, i.e., soundness and completeness with respect to subset spaces, and decidability. Furthermore, we point to a nice application of that system. Finally, we give a brief summary of the paper and mention future research.

2 The Extended Modal Language

In this section, we first define the modal language, including the past operator, for subset spaces. Second, we give some examples of valid formulas. Finally, the expressive power of the new language is compared to that of a previous extension of topologic.

Let PROP $= \{A, B, \ldots\}$ be a denumerable set of symbols. The elements of PROP are called *proposition letters*. We define the set WFF of *well-formed formulas over* PROP by the rule

$$\alpha \ ::= \ A \mid \neg\alpha \mid \alpha \wedge \beta \mid \mathsf{K}\alpha \mid \Box\alpha \mid \mathsf{P}\alpha.$$

The operators K and $\Box$ represent *knowledge* and *effort,* respectively, as it is common for topologic. The operator P is the counterpart of $\Box$. Since P is, therefore, related to the past we call this modality the *past operator*. The duals of K, $\Box$ and P are denoted L, $\Diamond$ and $\langle\mathsf{P}\rangle$, respectively. The missing boolean connectives $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$ are treated as abbreviations, as needed.

We now turn to semantics. First, we define the domains for interpreting formulas. Given a set X, let $\mathcal{P}(X)$ be the powerset of X.

Definition 1 (Subset frames and subset spaces)

1. *Let $X \neq \emptyset$ be a set and $\mathcal{O} \subseteq \mathcal{P}(X)$ a set of subsets of X. Then, $\mathcal{F} := (X, \mathcal{O})$ is called a* (general) subset frame*.*
2. *A subset frame $\mathcal{F} = (X, \mathcal{O})$ satisfying $\{\emptyset, X\} \subseteq \mathcal{O}$ is called* special*.*
3. *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. The* set $\mathcal{N}_\mathcal{F}$ of neighbourhood situations *of $\mathcal{F}$ is defined by $\mathcal{N}_\mathcal{F} := \{(x, U) \mid x \in U$ and $U \in \mathcal{O}\}$. (Mostly, neighbourhood situations are written without brackets later on.)*
4. *Let $\mathcal{F}$ be a subset frame. A mapping V : PROP $\longrightarrow \mathcal{P}(X)$ is called an* $\mathcal{F}$–valuation*.*
5. *A* subset space *is a triple $\mathcal{M} := (X, \mathcal{O}, V)$, where $\mathcal{F} = (X, \mathcal{O})$ is a subset frame and V an $\mathcal{F}$–valuation; $\mathcal{M}$ is called* based on $\mathcal{F}$*.*

The requirement '$\{\emptyset, X\} \subseteq \mathcal{O}$' from item 2 of this definition is convenient, but in a sense insignificant for topologic because of the forward looking nature of the effort operator; cf [12], Sec. 1.1. This is no longer true for the extended system, as one will see later on. Nevertheless, we continue to deal with special subset frames in this paper as long as it is possible.

The next definition concerns the relation of satisfaction, which is defined with regard to subset spaces now.

Definition 2 (Satisfaction and validity). *Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space.*

1. *Let x, U be a neighbourhood situation of $\mathcal{F} = (X, \mathcal{O})$. Then*

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & x \in V(A) \\ x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : (x \in U' \subseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha) \\ x, U \models_{\mathcal{M}} \mathsf{P}\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : (U' \supseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha)\,, \end{array}$$

   where $A \in$ PROP and $\alpha, \beta \in$ WFF. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in $\mathcal{M}$ at *the neighbourhood situation x, U.*
2. *A formula α is called* valid in $\mathcal{M}$ *(written '$\mathcal{M} \models \alpha$'), iff it holds in $\mathcal{M}$ at every neighbourhood situation of the frame $\mathcal{M}$ is based on.*

Note that the meaning of proposition letters is independent of neighbourhoods by definition, thus 'stable' with respect to $\Box$ and P. This fact is reflected by a special axiom below; see Section 3.

We now look for the formulas which are valid in all subset spaces. It is a known fact that the schema

$$\mathsf{K}\Box\alpha \rightarrow \Box\mathsf{K}\alpha$$

is a typical validity of topologic. This schema was called the *Cross Axiom* in the paper [12] and plays a key role in the completeness and the decidability proof for that logic. The Cross Axiom describes the basic interaction between knowledge and effort. It is not very surprising that there is a complementary schema for P.

Proposition 1. *Let $\mathcal{M}$ be any subset space. Then, for all $\alpha \in$ WFF we have that*

$$\mathcal{M} \models \mathsf{PK}\alpha \rightarrow \mathsf{KP}\alpha.$$

The easy proof of Proposition 1 is omitted here. – Quite recently, a modal operator O was considered which describes *overlapping* of sets within the framework of topologic; cf [11]. With the aid of O, it is possible to access also points that are distant from the actual one. The precise semantics of O in subset spaces $\mathcal{M} = (X, \mathcal{O}, V)$ at neighbourhood situations is as follows:

$$x, U \models_{\mathcal{M}} \mathsf{O}\alpha :\Longleftrightarrow\ \forall\, U' \in \mathcal{O} :\ \text{if } x \in U',\ \text{then } x, U' \models_{\mathcal{M}} \alpha$$

where $\alpha \in$ WFF. Compared to the semantics of the effort operator, the condition $U' \subseteq U$ obviously was left out; cf Definition 2. Thus shrinking appears as a special case of overlapping. It turned out that O is rather strong. In fact, with the aid of O even the *global modality* A (cf [13], Sec. 7.1) is definable in subset spaces based on special subset frames, actually through

$$\mathsf{A}\alpha :\equiv \mathsf{OK}\Box\alpha.$$

Note that this is true since $X \in \mathcal{O}$. But now the overlap operator itself can be defined with regard to such subset spaces. The defining clause reads

$$\mathsf{O}\alpha :\equiv \mathsf{P}\Box\alpha.$$

Consequently, the approach from [11] is subsumed under the present one in a sense. (For a fair comparison, however, one should take into account the issues of the next section as well.)

3 A Decidable Fragment of the Logic

Our starting point to this section is the system of axioms for topologic from [12]. We then add several schemata involving the new modality. After that we derive some auxiliary theorems of the resulting logic, topP, which are used for the proof of the decidability of topP. This proof makes up the main part of Section 3.

Unless otherwise noted, all subset spaces occurring in this section are based on special subset frames.

The complete list of axioms for topologic reads as follows:

1. All instances of propositional tautologies.
2. $\mathsf{K}(\alpha \to \beta) \to (\mathsf{K}\alpha \to \mathsf{K}\beta)$
3. $\mathsf{K}\alpha \to (\alpha \wedge \mathsf{K}\mathsf{K}\alpha)$
4. $\mathsf{L}\alpha \to \mathsf{K}\mathsf{L}\alpha$
5. $\Box\,(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
6. $(A \to \Box A) \wedge (\Diamond A \to A)$
7. $\Box\alpha \to (\alpha \wedge \Box\Box\alpha)$
8. $\mathsf{K}\Box\alpha \to \Box\mathsf{K}\alpha$,

where $A \in \mathrm{PROP}$ and $\alpha, \beta \in \mathrm{WFF}$. In this way, it is expressed that for every Kripke model M validating these axioms

- the accessibility relation $\xrightarrow{\mathsf{K}}$ of M belonging to the knowledge operator is an equivalence,
- the accessibility relation $\xrightarrow{\Box}$ of M belonging to the effort operator is reflexive and transitive,
- the composite relation $\xrightarrow{\Box} \circ \xrightarrow{\mathsf{K}}$ is contained in $\xrightarrow{\mathsf{K}} \circ \xrightarrow{\Box}$ (this is usually called the *cross property*), and
- the valuation of M is constant along every $\xrightarrow{\Box}$–path, for all propositional letters.

We now turn to the axioms in which P occurs:

9. $\mathsf{P}(\alpha \to \beta) \to (\mathsf{P}\alpha \to \mathsf{P}\beta)$
10. $\alpha \to \Box\langle\mathsf{P}\rangle\alpha$
11. $\alpha \to \mathsf{P}\Diamond\alpha$
12. $\langle\mathsf{P}\rangle\mathsf{P}\alpha \to \mathsf{P}\langle\mathsf{P}\rangle\mathsf{L}\alpha$,

where A and α, β are as above. – Some comments on these axioms seem to be appropriate. Item 9 contains the usual distribution schema being valid for every normal modality. The schemata 10 and 11 say that the accessibility relations belonging to $\Box$ and P, respectively, are inverse to each other. Note that the latter relation is reflexive and transitive, which follows logically from the respective property of the former one. All this is well-known from usual tense logic; cf, e.g., [14], § 6. The final schema is to capture the property '$X \in \mathcal{O}$' of subset spaces axiomatically; cf the remark following Definition 1.

We obtain a logical system by adding the standard proof rules of modal logic, i.e., *modus ponens* and *necessitation with respect to each modality.* We call this system topP, indicating both $\mathsf{topologic}$ and the past operator.

Definition 3 (The logic). *Let* topP *be the smallest set of formulas which contains the axiom schemata 1 – 12 and is closed under the application of the following rule schemata:*

$$\text{(MODUS PONENS)} \quad \frac{\alpha \to \beta, \alpha}{\beta} \qquad\qquad (\Delta\text{–NECESSITATION}) \quad \frac{\alpha}{\Delta\alpha},$$

where $\alpha, \beta \in \mathrm{WFF}$ *and* $\Delta \in \{\mathsf{K}, \Box, \mathsf{P}\}$. *Then we let* $\vdash$ *denote* topP*–derivability.*

Some useful derivations are contained in the following lemma.

Lemma 1. *For all* $A \in \mathrm{PROP}$ *and* $\alpha \in \mathrm{WFF}$, *we have that*

1. $\vdash (A \to \mathsf{P}A) \wedge (\langle\mathsf{P}\rangle A \to A)$
2. $\vdash \mathsf{LP}\alpha \to \mathsf{PL}\alpha$
3. $\vdash \mathsf{PK}\alpha \to \mathsf{KP}\alpha.$

Proof. 1. From the second conjunct of Axiom 6 we get $\vdash \mathsf{P}\Diamond A \to \mathsf{P}A$ with the aid of P–necessitation, Axiom 9 and propositional reasoning. Axiom 11 yields $\vdash A \to \mathsf{P}\Diamond A$. This gives us the first conjunct of the desired schema. From the first conjunct of Axiom 6 we infer $\vdash \langle\mathsf{P}\rangle A \to \langle\mathsf{P}\rangle\Box A$. The dual of Axiom 11 and propositional reasoning then imply that $\vdash \langle\mathsf{P}\rangle A \to A$, as desired.

2. The formula $\mathsf{LP}\alpha \to \mathsf{P}\Diamond\mathsf{LP}\alpha$ is an instance of Axiom 11. By using, among other things, the dual of Axiom 8 we get $\vdash \mathsf{P}\Diamond\mathsf{LP}\alpha \to \mathsf{PL}\Diamond\mathsf{P}\alpha$. As a consequence of the dual of Axiom 10, we have that $\vdash \mathsf{PL}\Diamond\mathsf{P}\alpha \to \mathsf{PL}\alpha$. Propositional reasoning now yields $\vdash \mathsf{LP}\alpha \to \mathsf{PL}\alpha$.
3. From Axiom 3 we infer $(*)$ $\vdash \mathsf{K}\alpha \to \alpha$, for all $\alpha \in \mathrm{WFF}$. Thus $\vdash \alpha \to \mathsf{L}\alpha$, for all formulas α. From that we get $\vdash \mathsf{PK}\alpha \to \mathsf{LPK}\alpha$ since this formula is an instance of the just derived schema. Axiom 4 implies $\vdash \mathsf{LPK}\alpha \to \mathsf{KLPK}\alpha$. With the aid of item 2 of this lemma we obtain $\vdash \mathsf{KLPK}\alpha \to \mathsf{KPLK}\alpha$. The dual of Axiom 4 gives us $\vdash \mathsf{KPLK}\alpha \to \mathsf{KPK}\alpha$, and $(*)$ finally implies $\vdash \mathsf{KPK}\alpha \to \mathsf{KP}\alpha$. It follows that $\vdash \mathsf{PK}\alpha \to \mathsf{KP}\alpha$, as desired.

The schema from item 1 of this lemma is complementary to Axiom 6. Both schemata together correspond to the stability condition mentioned in Section 2, right after Definition 2. The schema contained in item 3 is the one from Proposition 1 and will be called the *Reverse Cross Axiom;* see the discussion preceding Proposition 1.

The next proposition is quite obvious.

Proposition 2 (Soundness). *Let* $\alpha \in \mathrm{WFF}$ *be formula. If* α *is* topP*–derivable, then* α *is valid in all subset spaces.*

A possible proof of completeness must use the canonical model of topP in some way. We fix several notations concerning this model. Let $\mathcal{C}$ be the set of all maximal topP–consistent sets of formulas. Furthermore, let $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$ and $\xrightarrow{\mathsf{P}}$ be the accessibility relations on $\mathcal{C}$ induced by the modalities K, $\Box$ and P, respectively.

Three useful properties of the canonical model are listed in the subsequent lemma.

Lemma 2. 1. *For all* $\Psi, \Gamma, \Theta \in \mathcal{C}$ *satisfying* $\Psi \xrightarrow{\mathsf{K}} \Gamma \xrightarrow{\mathsf{P}} \Theta$ *there exists* $\Xi \in \mathcal{C}$ *such that* $\Psi \xrightarrow{\mathsf{P}} \Xi \xrightarrow{\mathsf{K}} \Theta$.

2. *For all* $\Psi, \Gamma, \Theta \in \mathcal{C}$ *satisfying* $\Psi \xrightarrow{\mathsf{P}} \Gamma$ *and* $\Psi \xrightarrow{\mathsf{P}} \Theta$ *there exist* $\Xi, \Phi \in \mathcal{C}$ *such that* $\Gamma \xrightarrow{\mathsf{P}} \Xi$ *and* $\Theta \xrightarrow{\mathsf{P}} \Phi \xrightarrow{\mathsf{L}} \Xi$.
3. $\left(\xrightarrow{\mathsf{K}} \cup \xrightarrow{\Box} \cup \xrightarrow{\mathsf{P}}\right)^* \subseteq \xrightarrow{\mathsf{P}} \circ \xrightarrow{\mathsf{K}} \circ \xrightarrow{\Box}$.

We only give some ideas, but not a detailed proof of Lemma 2 here. (The missing details will be contained in the full version of this paper.) The condition stated in item 1 follows from the Reverse Cross Axiom (see Lemma 1.3) and is, therefore, called the *reverse cross property.* The condition from item 2 is, among other things, a consequence of Axiom 12. Since a kind of confluence of the accessibility relation $\xrightarrow{\mathsf{P}}$ is forced, we call item 2 the *pseudo Church-Rosser property.* Quite some modal proof theory has to be applied in the proofs of items 1 and 2 of the lemma. Item 3 ensues from an iterated application of the previous items and the usual cross property, respectively.

Unfortunately, we do not have a proof of the completeness of topP with respect to the class of all subset spaces based on special frames. An adaption to the new system, including a suitable extension, of the corresponding proof for $\mathsf{topologic}$ (cf [12], Sec. 2.2) does not work anyhow.[2] This seems to be true even if the speciality condition (item 2 of Definition 1) is weakened a little, in the following way.

[2] If we drop Axiom 12, then completeness for subset spaces based on general frames can be proved in the just indicated way; cf [15]. The presence of that axiom, however, may be viewed as an indication of incompleteness. This is due to the formal similarity of Axiom 12 to the *Weak Directedness Axiom for intersection spaces* from [12], Sec. 2.4. It is a known fact that the latter is incomplete. In particular, the question arises whether all formulas of the form $\langle\mathsf{P}\rangle\mathsf{P}\alpha \rightarrow \mathsf{P}\langle\mathsf{P}\rangle\alpha$, which are sound for special subset spaces, are topP–derivable.

Definition 4 (Past-directed subset frames). *A (general) subset frame* $\mathcal{F} = (X, \mathcal{O})$ *is called* past-directed, *iff for all* $U_1, U_2 \in \mathcal{O}$ *there exists some* $U \in \mathcal{O}$ *such that* $U \supseteq U_1 \cup U_2$.

On the other hand, Lemma 2.2 suggests that completeness could hold for the larger class of all spaces based on past-directed frames. We will refer to this class of structures in the next section.

Contrasting those bad news, we at least can show that topP is decidable. This is done in the following.

Since topologic does not satisfy the finite model property with respect to subset spaces (cf [12], Sec. 1.3), topP too lacks this property. However, as in the former case this deficiency can be circumvented by a detour via suitable Kripke models.

Subsequently, let R and K, S and $\Box$, and T and P, respectively, correspond to each other.

Definition 5 (topP–model). *Let* $M := (W, \{R, S, T\}, V)$ *be a trimodal model, i.e.,* $R, S, T \subseteq W \times W$ *are binary relations and* V *is a valuation in the usual sense. Then,* M *is called a* topP–model *iff the following conditions are satisfied:*

1. *R is an equivalence relation, and S is reflexive and transitive,*
2. *S and T are inverse to each other,*
3. $S \circ R \subseteq R \circ S$,
4. *M satisfies the pseudo Church-Rosser property with respect to T and R,*
5. *for all* $w, w' \in W$ *and* $A \in \mathrm{PROP}$: *if* $w\, S\, w'$, *then* $w \in V(A)$ *iff* $w' \in V(A)$.

It is easy to see that all the axioms considered above are sound with respect to the class of all topP–models. Furthermore, the canonical model of topP is an example of a topP–model. (As to item 4 from Definition 5, cf Lemma 2.2). This gives us the following theorem.

Theorem 1 (Kripke completeness). *The logic* topP *is sound and complete with respect to the class of all* topP*–models.*

We now use the method of *filtration* for proving the finite model property of topP with respect to topP–models. The proceeding here follows the one from [12], Sec. 3.3, to a large degree. Thus we may be brief regarding this and stress the new aspects only.

For a given topP–consistent formula $\alpha \in \mathrm{WFF}$, we define a filter set $\Sigma \subseteq \mathrm{WFF}$ as follows. We first let

$$\Sigma_0 := \mathrm{sf}(\alpha) \cup \{\neg\beta \mid \beta \in \mathrm{sf}(\alpha)\},$$

where $\mathrm{sf}(\alpha)$ denotes the set of all subformulas of α. Second, we form the closure of Σ_0 under finite conjunctions of pairwise distinct elements of Σ_0. Third, we close under single applications of the operator L. And finally, we form the set of all subformulas of the formulas obtained up to now.[3] Let Σ denote the resulting set. Then Σ is finite and subformula closed.

[3] Note that this step is really necessary since L is an abbreviation.

We consider the respective *smallest* filtrations of the accessibility relations $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, and $\xrightarrow{\mathsf{P}}$, of the canonical model of topP; cf [14], § 4. Let

$$M := (W, \{R, S, T\}, V)$$

be the corresponding filtration of a suitably generated submodel M' of that model in which the valuation V assigns the empty set to all proposition letters not occurring in Σ. Then we have the following lemma.

Lemma 3. *The structure M is a finite* topP*–model of which the size depends computably on the length of α.*

Proof. Most of it is clear from the definitions and the proof of [12], Theorem 2.11. In particular, the finiteness of W follows from the finiteness of Σ. Only items 2 and 4 of Definition 5 have to be checked yet.

For item 2, let Γ, Θ be two points of the canonical model such that $[\Gamma]\ S\ [\Theta]$, where the brackets $[\ldots]$ indicate the respective classes. Due to the definition of the minimal filtration there are $\Gamma' \in [\Gamma]$ and $\Theta' \in [\Theta]$ such that $\Gamma' \xrightarrow{\Box} \Theta'$. Since $\Theta' \xrightarrow{\mathsf{P}} \Gamma'$ then holds because of the dual of Axiom 10, we conclude $[\Theta]\ T\ [\Gamma]$ from that with the aid of the first filtration condition (marked (F1) in [14], § 4). This shows $S^{-1} \subseteq T$. By interchanging the roles of S and T, the property $T^{-1} \subseteq S$ can be established in a similar manner. Now, the condition stated in item 2 of Definition 5 easily follows.

Item 4 is a bit harder to prove. We make use of the following property $(*)$ of the intermediate model M' introduced above:

$(*)$ For every finite set $\{\Gamma_1, \ldots, \Gamma_n\}$ of points of M' there exists a point Γ of M' such that $(\Gamma, \Gamma_i) \in \xrightarrow{\mathsf{K}} \circ \xrightarrow{\Box}$ for all $i = 1, \ldots, n$ (where $n \geq 1$).

This is consequence of Lemma 2.3 and the fact that M' is generated. Now, it is clear from the first filtration condition again that the property

$$(\Psi, \Theta) \in \xrightarrow{\mathsf{K}} \circ \xrightarrow{\Box}$$

passes down to the filtration (where Ψ, Θ are any maximal topP–consistent sets). Thus M is generated in the following sense. For all classes $[\Phi] \in W$ we have that

$$([\Gamma], [\Phi]) \in R \circ S,$$

where Γ is obtained according to $(*)$ after arbitrary representatives of the finitely many classes contained in W have been chosen. From that, the properties of R, and item 2, the validity of the pseudo Church-Rosser property for M is clear.

Since the topP–model M from Lemma 3 realizes α, the claimed decidability result follows readily.

Theorem 2 (Decidability). *The set of all* topP*–derivable formulas is decidable.*

Concerning the complexity of the topP–satisfiability problem, we only mention that this problem is very likely hard for EXPTIME. This is due to the (implicit) presence of the global modality (see Section 2); cf [16], Ch. 2.2.

4 Hybridization

Concluding the technical part of the paper, we develop a hybrid logic version of topP. This extension, HtopP, rectifies the shortcomings of the former system to some extent and simultaneously generalizes previous hybridizations of topologic (e.g., the one from [11]). By enriching the language once again we first obtain much more expressive power concerning properties of relations, and this can already be achieved by simply adding suitable sets of nominals to the ground language; see the just cited paper for some examples. Second, HtopP turns out to be sound and complete for subset spaces based on past-directed frames. And finally, HtopP is decidable, too. We outline only the completeness proof in this section.[4] Actually, we will have 'almost canonical' completeness. As an application, we show that the hybrid logic of *directed spaces,* cf [5], is finitely axiomatizable and decidable.[5] The first property is false for the modal fragment, and the second one apparently was unknown before the hybrid methods came into play.

We now carry out all these things. For a start, we define the extended language. As we already indicated above, we merely add two sets of nominals to the language underlying topP. If the denotation of a nominal is non-empty, then it should be either a unique state or a distinguished set of states. Let

$$\mathrm{N}_{stat} = \{i, j, \ldots\} \ \text{ and } \ \mathrm{N}_{sets} = \{I, J, \ldots\}$$

be the corresponding sets of symbols, which we call *names of states* and *names of sets,* respectively. We assume that PROP, N_{stat} and N_{sets} are mutually disjoint.

Definition 6 (Subset spaces with names)

1. *Let $\mathcal{F} = (X, \mathcal{O})$ be a past-directed subset frame. A* hybrid $\mathcal{F}$–valuation *is a mapping*

   $$V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(X)$$

   such that

   (a) $V(i)$ is either $\emptyset$ or a singleton subset of X, for every $i \in \mathrm{N}_{stat}$, and
   (b) $V(A) \in \mathcal{O}$ for every $A \in \mathrm{N}_{sets}$.
2. *A* subset space with names *(or, in short, an* SSN*) is a triple $(X, \mathcal{O}, V)$, where $\mathcal{F} = (X, \mathcal{O})$ is a subset frame as in item 1 and V a hybrid $\mathcal{F}$–valuation.*

Note that nominals may have an empty denotation. This is appropriate for our purposes since it simplifies the proof of the subsequent Theorem 3 a little, but not common in standard hybrid logic.

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[4] In order to establish decidability, the techniques from the previous section have to be tailored to the hybrid context. Due to space limitations we cannot give too many details regarding this here.

[5] A subset frame $\mathcal{F} = (X, \mathcal{O})$ is called *directed,* iff $\forall U_1, U_2 \in \mathcal{O}, \forall x \in X$: if $x \in U_1 \cap U_2$, then $\exists U \in \mathcal{O} : x \in U \subseteq U_1 \cap U_2$.

Definition 7 (Satisfaction for nominals). *Let $\mathcal{M} := (X, \mathcal{O}, V)$ be an SSN and x, U a neighbourhood situation of the underlying frame. Then*

$$x, U \models_{\mathcal{M}} i :\Longleftrightarrow x \in V(i)$$
$$x, U \models_{\mathcal{M}} I :\Longleftrightarrow V(I) = U,$$

for all $i \in \mathrm{N}_{stat}$ and $I \in \mathrm{N}_{sets}$.

We now turn to the axioms for nominals. These axioms are divided into two groups. The formulas of the first group provide for the right interpretation of the names of states and sets, respectively, in the canonical model which is used for the proof of completeness below.

13. $i \wedge \alpha \rightarrow \mathsf{K}(i \rightarrow \alpha)$
14. $I \rightarrow \mathsf{K} I$
15. $\mathsf{K}\Box\,(I \wedge \alpha \rightarrow \mathsf{L}\beta) \vee \mathsf{K}\Box\,(I \wedge \beta \rightarrow \mathsf{L}\alpha)\,,$

where $i \in \mathrm{N}_{stat}$, $I \in \mathrm{N}_{sets}$ and $\alpha, \beta \in \mathrm{WFF}$.[6] The schema 13 says that only one point of any $\xrightarrow{\mathsf{K}}$–equivalence class can be named by a fixed state name. And the schema 14 guarantees that a set name of such a class is valid across the whole class. Finally, Axiom 15 captures the property that every element of N_{sets} denotes at most one $\xrightarrow{\mathsf{K}}$–class. Note the formal similarity of this schema to the linearity schema L from classical modal logic (cf [14], p 22).

The following two axioms are responsible for the fact that really a structure of subset space can be ensured with the aid of that model.

16. $i \wedge I \rightarrow \Box\,(\Diamond(i \wedge I) \rightarrow i \wedge I)$
17. $\mathsf{K}(\Diamond J \rightarrow \Diamond I) \wedge \mathsf{L}\Diamond J \rightarrow \Box\,(I \rightarrow \mathsf{L}\Diamond J)\,,$

where $i, j \in \mathrm{N}_{stat}$ and $I, J \in \mathrm{N}_{sets}$. Axiom 16 corresponds to the antisymmetry of the relation $\xrightarrow{\Box}$ on the canonical model we construct later on. The part Axiom 17 plays is less obvious. Roughly speaking, the inclusion relation of subsets is arranged correctly by this axiom.

By fixing the hybrid proof rules we get to the logical system HtopP. The new rules are called NAME and ENRICHMENT, respectively. We comment on these rules right after the next definition.

Definition 8 (The hybrid logic). *Let* HtopP *be the smallest set of formulas which contains the axiom schemata 1 – 17 and is closed under the application of the standard modal rule schemata and the following ones:*

$$(\mathrm{NAME}_{stat})\quad \frac{j \rightarrow \beta}{\beta} \qquad (\mathrm{NAME}_{sets})\quad \frac{J \rightarrow \beta}{\beta}$$

$$(\nabla\text{–ENRICHMENT})\quad \frac{\langle\mathsf{P}\rangle\mathsf{L}\Diamond\,(i \wedge I \wedge \nabla(j \wedge J \wedge \alpha)) \rightarrow \beta}{\langle\mathsf{P}\rangle\mathsf{L}\Diamond\,(i \wedge I \wedge \nabla\alpha) \rightarrow \beta}\,,$$

where $\alpha, \beta \in \mathrm{WFF}$, $i, j \in \mathrm{N}_{stat}$, $I, J \in \mathrm{N}_{sets}$, $\nabla \in \{\mathsf{L}, \Diamond, \langle\mathsf{P}\rangle\}$, and j, J are new *each time, i.e., do not occur in any other syntactic building block of the respective rule.*

[6] From now on, WFF denotes the set of formulas of the enriched language.

For the reader being not familiar with these unorthodox proof rules a 'contrapository' reading is suggested; e.g., the rule (NAME_{stat}) is to be read 'if β is satisfiable, then $j \wedge \beta$ is satisfiable, too' (provided that the nominal j does not occur in β). The soundness of the hybrid rules should be apparent from that now. Technically, the NAME and ENRICHMENT rules are used for establishing an appropriate *Lindenbaum Lemma,* which makes up the first step in the proof of the following theorem.

Theorem 3 (Hybrid completeness). HtopP *is sound and complete with respect to the class of all subset spaces with names.*

Proof. The proof of Theorem 3 goes along the lines of the proof of Theorem 3.2 from [17], but some modifications are required during the construction of the model refuting a given non-derivable formula α. We start off with the canonical model of HtopP. Let us retain the designations $\mathcal{C}$, $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, and $\xrightarrow{\mathsf{P}}$ from the previous section, and let Γ_0 be a maximal consistent set containing $\neg\alpha$. If Γ is any element of $\mathcal{C}$, then Γ is called

- *named* iff Γ contains some $i \in \mathrm{N}_{stat}$ and some $I \in \mathrm{N}_{sets}$, and
- *enriched* iff, for every $\nabla \in \{\mathsf{L}, \Diamond, \langle\mathsf{P}\rangle\}$, whenever $\langle\mathsf{P}\rangle\mathsf{L}\Diamond\,(i \wedge I \wedge \nabla\alpha) \in \Gamma$, then there are $j \in \mathrm{N}_{stat}$ and $J \in \mathrm{N}_{sets}$ such that

$$\langle\mathsf{P}\rangle\mathsf{L}\Diamond\,(i \wedge I \wedge \nabla(j \wedge J \wedge \alpha)) \in \Gamma.$$

Now, it is our first task to extend Γ_0 to a named and enriched maximal consistent set Γ' in the language to which enough new constants have been added. This can be achieved in the usual way by means of the Lindenbaum Lemma mentioned above.

Secondly, we consider the canonical model of HtopP which is formed with respect to the latter language. Let $\mathcal{M}_0$ be the submodel generated by Γ' of this model. We then have that the interpretation of the set names in $\mathcal{M}_0$ is uniquely determined.[7]

Lemma 4. *Let D be the domain of $\mathcal{M}_0$ and let I be a set name. Assume that both of $\Gamma, \Gamma' \in D$ contain I. Then $\Gamma \xrightarrow{\mathsf{K}} \Gamma'$.*

Proof. Suppose on the contrary that there are two points $\Gamma, \Gamma' \in D$ such that $I \in \Gamma \cap \Gamma'$, but not $\Gamma \xrightarrow{\mathsf{K}} \Gamma'$. Then there is some $n \in \mathbb{N}$ and a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ of elements of D such that $\Gamma_0 = \Gamma$, $\Gamma_n = \Gamma'$, and Γ_i, Γ_{i+1} are connected by either $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, or $\xrightarrow{\mathsf{P}}$, for all $i = 0 \ldots n-1$. This is true since $\mathcal{M}_0$ is generated. Now, Lemma 2.3 implies that there exist $\Psi, \Theta \in D$ such that

$$\Gamma \xrightarrow{\mathsf{P}} \Psi \xrightarrow{\mathsf{K}} \Theta \xrightarrow{\Box} \Gamma'.$$

The desired contradiction then follows with the aid of Axiom 15.

[7] We exemplarily focus on this point of the completeness proof a little more detailedly.

Let D' be the set of all points of D that are named, and let $\mathcal{M}'$ be the corresponding substructure of $\mathcal{M}_0$. $\mathcal{M}'$ is the model we want to operate on. The subsequent *Existence Lemma* is due to our special notion of enrichment.

Lemma 5. *Assume that $\Gamma \in D'$ contains $\nabla\alpha$, where $\nabla \in \{\mathsf{L}, \Diamond, \langle\mathsf{P}\rangle\}$ is the respective dual of $\Delta \in \{\mathsf{K}, \Box, \mathsf{P}\}$. Then there exists some $\Theta \in D'$ satisfying $\Gamma \xrightarrow{\Delta} \Theta$ and $\alpha \in \Theta$.*

The desired model falsifying α can be obtained as a certain space of partial functions, X, over the model $\mathcal{M}'$ now. The domain $\mathrm{dom}(h)$ of every such function $h \in X$ is a subset of the set

$$\mathcal{Q} := \{[\Gamma] \mid \Gamma \in D'\}$$

of all equivalence classes $[\Gamma] := \{\Gamma' \in D' \mid \Gamma \xrightarrow{\mathsf{K}} \Gamma'\}$ of the accessibility relation induced by the modality K. Actually, $\mathrm{dom}(h)$ is maximal in $\mathcal{Q}$ with regard to the following two conditions:

1. $h([\Gamma]) \in [\Gamma]$ for all $[\Gamma] \in \mathrm{dom}(h)$, and
2. $h([\Gamma]) \xrightarrow{\Box} h([\Theta])$ for all $[\Gamma], [\Theta] \in \mathrm{dom}(h)$ satisfying $[\Gamma] \preccurlyeq [\Theta]$;

the *precedence relation* $\preccurlyeq$ occurring in the second item is defined by

$$[\Gamma] \preccurlyeq [\Theta] :\Longleftrightarrow \exists\, \Gamma' \in [\Gamma], \Theta' \in [\Theta] : \Gamma' \xrightarrow{\Box} \Theta',$$

for all points $\Gamma, \Theta \in D'$.[8] We write $h_\Gamma := h([\Gamma])$ in case $h([\Gamma])$ exists. Furthermore, we let

– $U_{[\Gamma]} := \{h \in X \mid h_\Gamma \text{ exists}\}$, for all $\Gamma \in D'$,
– $\mathcal{O} := \{U_{[\Gamma]} \mid \Gamma \in D'\}$, and
– $V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(X)$ be defined by

$$h \in V(c) :\Longleftrightarrow c \in h_\Gamma \text{ for some } \Gamma \in D' \text{ for which } h_\Gamma \text{ exists},$$

for all $c \in \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets}$.

We then get that $\mathcal{M} := (X, \mathcal{O}, V)$ is a past-directed subset space for which the relevant *Truth Lemma* is valid. With that, the completeness of the system HtopP with respect to the intended class of structures can be concluded in a standard manner.

In addition, we obtain that the hybrid logic of past-directed subset spaces is decidable.

Theorem 4 (Hybrid decidability). *The set of all* HtopP*–derivable formulas is decidable.*

[8] Note that again quite some proof theory has to be used for establishing that h is well-defined.

As aforementioned, we have to skip the proof of this theorem here. For getting an idea of the proceeding the reader is referred to [17], proof of Theorem 4.4. But we at least want to emphasize the point crucial to hybrid decidability. This concerns the verification of the final group of axioms (16 and 17 above) with regard to a filtration. Actually, it is not necessary to establish the corresponding property on the filtrated model, but only to validate those instances of the schemata in which nominals from the filter set occur. This can be achieved by modifying that filter set appropriately.

At the end of this section, we revisit the 'guarded jump'–operators investigated in the paper [18]. For convenience, we remind the reader of the semantics of these operators. Let $\mathcal{M} = (X, \mathcal{O}, V)$ be an SSN and x, U a neighbourhood situation of the underlying frame. Then

$$x, U \models_{\mathcal{M}} [\epsilon_I]\alpha :\Longleftrightarrow \text{ if } x \in V(I), \text{ then } x, V(I) \models_{\mathcal{M}} \alpha,$$

for all $I \in \mathrm{N}_{sets}$ and $\alpha \in \mathrm{WFF}$. The 'guarded jump'–operators are, therefore, named variants of the overlap operator considered in Section 2. Thus it is hardly surprising $[\epsilon_I]$ can be defined as well:

$$[\epsilon_I]\alpha :\equiv \mathsf{P}\Box(I \to \alpha),$$

for all $\alpha \in \mathrm{WFF}$. As a consequence, we get that all the results obtained in the paper [18] can be inferred from the theorems of the present section. This applies to the hybrid logic of directed SSNs[9] and, in particular, the modal logic of directed spaces; cf [18], Theorem 20 and Corollary 21, and the discussion in the introduction to this section.

Theorem 5 (The logic of directed spaces)

1. *The hybrid logic of directed SSNs is finitely axiomatizable and decidable.*
2. *The topo-logic of directed spaces is decidable.*

More details concerning this application of the hybrid formalism will be contained in the full version of this paper.

5 Concluding Remarks

In this paper, Moss and Parikh's topologic was extended by a modality reversing the effort operator. While this goes smoothly for general subset frames, some complications arise if special frames form the semantic basis. In this case we could identify a natural fragment, topP, of the accompanying logic and prove its decidability. Incorporating concepts from hybrid logic then yielded more satisfactory results. We established the soundness, completeness, and decidability, of the hybrid logic of past-directed subset spaces. From that, the decidability of the logic of directed spaces was obtained as a corollary.

It remains to solve the completeness problem for topP. Moreover, the complexity of the logics must be determined each time. All this is postponed to future research.

[9] The axiom capturing directedness reads $\Diamond I \wedge \Diamond J \to \Diamond \mathsf{K}\left(\langle\epsilon_I\rangle\top \wedge \langle\epsilon_J\rangle\top\right)$, where $I, J \in \mathrm{N}_{sets}$ and $\langle\epsilon_I\rangle$ denotes the dual of $[\epsilon_I]$.

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