On (k, ℓ)-Leaf Powers

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Abstract. We say that, for $k \geq 2$ and $\ell > k$, a tree T is a (k, ℓ)-*leaf root of a graph* $G = (V_G, E_G)$ if V_G is the set of leaves of T, for all edges $xy \in E_G$, the distance $d_T(x, y)$ in T is at most k and, for all non-edges $xy \notin E_G$, $d_T(x, y)$ is at least ℓ. A graph G is a (k, ℓ)-*leaf power* if it has a (k, ℓ)-leaf root. This new notion modifies the concept of k-leaf power which was introduced and studied by Nishimura, Ragde and Thilikos motivated by the search for underlying phylogenetic trees. Recently, a lot of work has been done on k-leaf powers and roots as well as on their variants phylogenetic roots and Steiner roots. For $k = 3$ and $k = 4$, structural characterisations and linear time recognition algorithms of k-leaf powers are known, and, recently, a polynomial time recognition of 5-leaf powers was given. For larger k, the recognition problem is open.

We give structural characterisations of (k, ℓ)-leaf powers, for some k and ℓ, which also imply an efficient recognition of these classes, and in this way we also improve and extend a recent paper by Kennedy, Lin and Yan on strictly chordal graphs and leaf powers.

Keywords: (k, ℓ)-leaf powers, leaf powers, leaf roots, strictly chordal graphs, linear time algorithms.

1 Introduction

Nishimura, Ragde and Thilikos [20] introduced the notion of k-leaf power and k-leaf root, motivated by the following: "... a fundamental problem in computational biology is the reconstruction of the *phylogeny*, or evolutionary history, of a set of species or genes, typically represented as a *phylogenetic tree* ...". The species occur as leaves of the phylogenetic tree.

Let $G = (V_G, E_G)$ be a finite undirected graph. For $k \geq 2$, a tree T is a k-*leaf root* of G if V_G is the leaf set of T and two vertices $x, y \in V_G$ are adjacent in G if and only if their distance $d_T(x, y)$ in T is at most k, i.e., $xy \in E_G \iff d_T(x, y) \leq k$. The graph G is a k-*leaf power* if it has a k-leaf root. Obviously, a graph is a 2-leaf power if and only if it is the disjoint union of cliques, i.e., it contains no induced P_3. See [17] for the related notions of phylogenetic root and Steiner root and [2,3,4,6,9,11,14,15,16,21] for recent work on leaf powers (including characterisations of 3- and 4-leaf powers) and their variants. It is

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well known that every k-leaf power is a strongly chordal graph. For $k \geq 6$, no characterisation of k-leaf powers and no efficient recognition is known.

In this paper, we modify the notion of k-leaf power and k-leaf root in the following natural way: For $k \geq 2$ and $\ell > k$, a tree T is a *(k, ℓ)-leaf root of a graph* $G = (V_G, E_G)$ if V_G is the set of leaves of T, for all edges $xy \in E_G$, $d_T(x, y) \leq k$ and, for all non-edges $xy \notin E_G$, $d_T(x, y) \geq \ell$. A graph G is a *(k, ℓ)-leaf power* if it has a (k, ℓ)-leaf root. Thus, every k-leaf power is a $(k, k+1)$-leaf power, and every (k, ℓ)-leaf power is an (i, j)-leaf power, for all i and j with $k \leq i < j \leq \ell$. In particular, every (k, ℓ)-leaf power is simultaneously a k'-leaf power, for all k' with $k \leq k' \leq \ell - 1$. In a similar way, Steiner roots and powers can be generalised.

In [16], Kennedy, Lin and Yan study so-called strictly chordal graphs which were originally defined via (rather complicated) hypergraph properties but finally turn out to be exactly the (dart,gem)-free chordal graphs [14]. It is not hard to see that a graph is (dart,gem)-free chordal if and only if it results from substituting cliques into the vertices of a block graph (i.e., a graph whose blocks are cliques). We will show that these graphs are exactly the $(4, 6)$-leaf powers which explains various of their properties, and the same class appears as (k, ℓ)-leaf powers for infinitely many other values of k and ℓ, e.g., as $(6, 10)$-leaf powers and in general as $(4 + 2i, 6 + 4i)$-leaf powers, for $i \geq 0$. Moreover, it is known from [2,11,21] that 3-leaf powers (i.e., $(3, 4)$-leaf powers) are exactly the (bull,dart,gem)-free chordal graphs which in turn result from substituting cliques into the vertices of a tree. By a simple argument, every class of $(3 + 2i, 4 + 4i)$-leaf powers, for $i \geq 0$, is also exactly the same class of (bull,dart,gem)-free chordal graphs.

We give structural characterisations of (k, ℓ)-leaf powers, for some k and ℓ (and in particular for (8,11)-leaf powers) which also imply efficient recognition of these classes. Most of the proofs are omitted due to space constraints of this extended abstract.

2 Basic Notions and Results

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without loops and multiple edges and with vertex set V and edge set E, and let $|V| = n$, $|E| = m$. For a vertex $v \in V$, let $N_G(v) = N(v) = \{u \mid uv \in E\}$ denote the (*open*) *neighbourhood* of v in G, and let $N_G[v] = N[v] = \{v\} \cup \{u \mid uv \in E\}$ denote the *closed neighbourhood* of v in G. A *clique* is a set of vertices which are mutually adjacent. A *stable set* is a set of vertices which are mutually non-adjacent.

A vertex subset $U \subseteq V$ is a *module* in G if, for all $v \in V \setminus U$, either v is adjacent to all vertices of U or v is adjacent to none of them. A *clique module* in G is a module which induces a clique in G. A vertex $z \in V$ *distinguishes* $x, y \in V$ if $zx \in E$ and $zy \notin E$. Two vertices $x, y \in V$ are *true twins* in G if they have the same neighbors in G and are adjacent to each other.

Let $d_G(x, y)$ (or $d(x, y)$ for short if G is understood) be the length, i.e., number of edges, of a shortest path in G between x and y. Let $N_G^k(x) = \{y \mid d_G(x, y) = k\}$ and let $G^k = (V, E^k)$ with $xy \in E^k$ if and only if $d_G(x, y) \leq k$ denote the *k-th power of G*.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k-1$ edges, and, for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *diamond* (or $K_4 - e$, see Figure 1) consists of four vertices a, b, c, d and five edges ab, ac, bc, bd and cd.

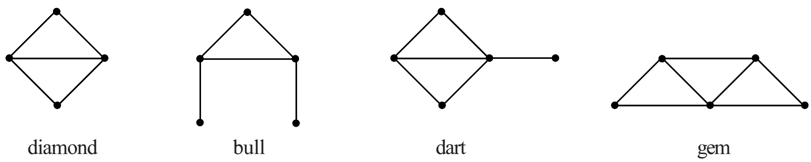

Fig. 1. Diamond, bull, dart and gem

For $k \geq 2$, let S_k denote the (*complete*) *sun* with $2k$ vertices $u_1, \ldots, u_k$ and $w_1, \ldots, w_k$ such that $u_1, \ldots, u_k$ is a clique, $w_1, \ldots, w_k$ is a stable set and, for $i \in \{1, \ldots, k\}$, w_i is adjacent to u_i and u_{i+1} (index arithmetic modulo k).

A graph is *chordal* if it contains no induced C_k, $k \geq 4$. A graph is *strongly chordal* if it is chordal and sun-free - see e.g. [5] for various characterisations of chordal and strongly chordal graphs. In particular, in a chordal graph G, the maximal cliques of G can be arranged in a tree T_G (a so-called *clique tree* of G) such that for every vertex v, the maximal cliques containing v form a subtree of T_G.

In [10,18,22], it is shown that the class of strongly chordal graphs is closed under powers:

Proposition 1 ([10,18,22]). *If G is strongly chordal then, for every $k \geq 1$, G^k is strongly chordal.*

A graph is a *block graph* if its 2-connected components (which are also called *blocks*) are cliques. It is well known that the following holds:

Proposition 2. *A graph G is a block graph if and only if G is diamond-free and chordal.*

In [17], the notion of k-th Steiner root of an undirected graph G, $k \geq 1$, is defined as follows: A tree $T = (V_T, E_T)$ is a *k-th Steiner root* of the graph $G = (V_G, E_G)$ if $V_G \subseteq V_T$ and $xy \in E_G$ if and only if $d_T(x, y) \leq k$. In this case, G is a *k-th Steiner power*. The vertices in $V_T \setminus V_G$ are called *Steiner nodes* in T.

In [6], we say that a graph G is a *basic k-leaf power* if G has a k-leaf root T such that no two leaves of T are attached to the same parent vertex in T (a so-called *basic k-leaf root*). Obviously, for $k \geq 2$, the set of leaves having the same parent node in T form a clique, and G is a k-leaf power if and only if G results from a basic k-leaf power by substituting cliques into its vertices. If T is a basic k-leaf root of G then T minus its leaves is a $(k-2)$-th Steiner root of G. Summarising, the following obvious equivalence holds:

Proposition 3. *For a graph G, the following conditions are equivalent for all $k \geq 2$:*

(i) *G has a k-th Steiner root.*
(ii) *G is an induced subgraph of the k-th power of a tree.*
(iii) *G is a basic $(k+2)$-leaf power.*

Similar to basic k-leaf roots, we say that a (k, l)-leaf root T is *basic* if no two leaves x and y of T have a distance satisfying $2 \leq d_T(x, y) \leq l - k + 1$.

3 Some Basic Facts on (k, ℓ)-Leaf Powers

The following facts are well known for k-leaf powers (see, e.g., [2]) and can easily be shown for (k, l)-leaf powers.

Proposition 4

(i) *Every induced subgraph of a (k, l)-leaf power is a (k, l)-leaf power.*
(ii) *A graph is a (k, l)-leaf power if and only if each of its connected components is a (k, l)-leaf power.*

Let T be a k-leaf root of a graph G. Then, by definition, G is isomorphic to the subgraph of T^k induced by the leaves of T. Since trees are strongly chordal and induced subgraphs of strongly chordal graphs are strongly chordal, Proposition 1 implies:

Proposition 5. *For every $k \geq 1$, k-leaf powers are strongly chordal.*

This strengthens the fact that k-leaf powers are chordal, which is observed in some previous papers dealing with k-leaf powers, and also implies that (k, ℓ)-leaf powers are strongly chordal. The converse implication is not true: In [1], based on [7], an example of a strongly chordal graph is given which is no k-leaf power, for any $k \geq 2$.

The following simple facts are helpful:

Proposition 6

(i) *For $k \leq k' < \ell$, if G is a (k, ℓ)-leaf power then it is a (k', ℓ)-leaf power.*
(ii) *For $k < \ell' \leq \ell$, if G is a (k, ℓ)-leaf power then it is a (k, ℓ')-leaf power.*
(iii) *If G is a (k, ℓ)-leaf power then it is a $(k+2i, \ell+2i)$-leaf power, for all $i \geq 1$.*
(iv) *If G is a (k, ℓ)-leaf power then it is a $(k + i(k-2), \ell + i(\ell-2))$-leaf power, for all $i \geq 1$.*

Proof. (i) and (ii) are obviously true, by definition. (iii) is shown by subdividing each edge of a (k, l)-leaf root T of G containing a leaf of T. (iv) is shown by subdividing each edge of T not containing a leaf of T. □

Thus, obviously every k-leaf power is also a $(k+2)$-leaf power but it is not known whether every k-leaf power is also a $(k+1)$-leaf power. For 3-leaf powers, however, it is noted in [2]:

Proposition 7. *Every 3-leaf power is a k-leaf power, for all $k \geq 3$.*

By the proof of Proposition 6 (*iv*), every 3-leaf power is a (4,6)-leaf power. By Proposition 6 (*iii*), we get the next proposition which also implies Proposition 7:

Proposition 8. *Every $(4,6)$-leaf power is a k-leaf power, for all $k \geq 4$, and, in general, every $(k,k+2)$-leaf power is an ℓ-leaf power, for all $k \leq \ell$.*

A graph $H = (V_H, E_H)$ results from a graph $G = (V_G, E_G)$ by *substituting* a clique Q into a vertex $v \in V_G$ (or *substituting a vertex v by a clique Q*), if V_H is the union of $V_G \setminus \{v\}$ and the vertices in Q, and E_H results from E_G by removing all edges containing v, adding all clique edges in Q and adding all edges between vertices in Q and in $N_G(v)$.

Proposition 9. *For every graph G and for every $k \geq 2$ and $\ell > k$, G is a (k,ℓ)-leaf power if and only if every graph resulting from G by substituting its vertices by cliques is a (k,ℓ)-leaf power.*

Proof. If T is a (k,ℓ)-leaf root for the (k,ℓ)-leaf power $G = (V,E)$, and G' is the result of substituting a clique Q into a vertex $u \in V$, then attach all vertices in Q at the same parent in T as u and skip u; the resulting tree T' is a (k,ℓ)-leaf root for G'. The converse direction obviously holds. □

4 Metric Properties of (k,ℓ)-Leaf Powers

Obviously, a graph is P_3-free if and only if it is the disjoint union of cliques, and G is a 2-leaf power if and only if it is P_3-free.

Proposition 10. *Let G be a (k,ℓ)-leaf power.*

(i) *If $\ell > 2k-2$ then G is P_3-free.*
(ii) *If $\ell = 2k-2$ then P_3 has a unique (k,ℓ)-leaf root.*

Proof. Let T be a (k,ℓ)-leaf root of G, and suppose that G contains a P_3 with vertices a,b,c and edges ab and bc. Then $d_T(a,b) \leq k$ and $d_T(b,c) \leq k$, i.e., $d_T(a,c) \leq 2k-2$. On the other hand, $d_T(a,c) \geq \ell$ since $ac \notin E$. Thus, $\ell \leq 2k-2$. If $\ell = 2k-2$ then a (k,ℓ)-leaf root of the P_3 has distance exactly ℓ between a and c, and the leaf b is attached to the central vertex of the path between a and c. This is obviously the only (k,ℓ)-leaf root of the P_3 abc. □

A well known fact for distances in trees found by Buneman [8] (respectively, for block graphs found by Howorka [13]) is the following characterisation in terms of a four-point condition:

Theorem 1. *Let $G = (V,E)$ be a connected graph.*

(i) *G is a tree if and only if G contains no triangles and G satisfies the following four-point condition: For all $u,v,x,y \in V$,*
  (∗) $d_G(u,v) + d_G(x,y) \leq \max\{d_G(u,x) + d_G(v,y), d_G(u,y) + d_G(v,x)\}$.
(ii) *G is a block graph if and only if G satisfies (∗), for all $u,v,x,y \in V$.*

From now on, let G be a (k, ℓ)-leaf power with (k, ℓ)-leaf root T. We apply this four-point condition to various induced subgraphs of G such as P_4 as well as diamond, dart and gem (see Figure 1).

Proposition 11. *Let the four vertices a, b, c, d with non-edge ad induce a diamond in G. Then $d_T(b, c) \leq 2k - \ell$.*

Proof. According to condition $(*)$, $d_T(a, d) + d_T(b, c) \leq \max\{d_T(a, b) + d_T(c, d), d_T(a, c) + d_T(b, d)\} \leq 2k$ holds since $ab, cd, ac, bd \in E$. Since $ad \notin E$, we have $d_T(a, d) \geq \ell$. Thus $d_T(b, c) \leq 2k - \ell$. □

Proposition 12. *Let the four vertices a, b, c, d with edges ab, ac, bc, bd and cd induce a diamond in G such that b and c can be distinguished in G. Then $2\ell \leq 3k - 2$.*

Proof. According to Proposition 11, $d_T(b, c) \leq 2k - \ell$. Let z be a vertex which distinguishes b and c, say $bz \in E$ and $cz \notin E$. Then $d_T(b, z) \leq k$ and $d_T(c, z) \geq \ell$. The T-paths P_{bz} between b and z and P_{bc} between b and c have at least two vertices in common, namely b and its parent, say b'. Let x be the last common vertex of P_{bz} and P_{bc}. Then $d_T(x, b) + d_T(x, c) \leq 2k - \ell$, by Proposition 11, and $d_T(x, b) + d_T(x, z) \leq k$, which implies that $d_T(x, c) + d_T(x, z) + 2d_T(x, b) \leq 3k - \ell$. On the other hand, $2d_T(x, b) \geq 2$ and $d_T(x, c) + d_T(x, z) \geq \ell$, which implies $\ell + 2 \leq 3k - \ell$, i.e., $2\ell \leq 3k - 2$. □

Corollary 1. *If dart or gem is a (k, ℓ)-leaf power then $2\ell \leq 3k - 2$.*

Proposition 13. *Let the four vertices a, b, c, d with edges ab, bc and cd induce a P_4 in G. Then $d_T(a, d) \geq 2\ell - k$.*

Proof. According to condition $(*)$, $d_T(a, c) + d_T(b, d) \leq \max\{d_T(a, b) + d_T(c, d), d_T(a, d) + d_T(b, c)\}$ holds. Since $ac \notin E$ and $bd \notin E$ and T is a (k, ℓ)-leaf root of G, we have $d_T(a, c) + d_T(b, d) \geq 2\ell$. On the other hand, since $ab \in E$ and $cd \in E$, we have $d_T(a, b) + d_T(c, d) \leq 2k$, and this sum cannot be the maximum of the two sums on the right hand side of inequality $(*)$. Thus $d_T(a, c) + d_T(b, d) \leq d_T(a, d) + d_T(b, c)$ holds, which implies that $d_T(a, d) \geq 2\ell - k$. □

Proposition 14. *If $2\ell \leq 3k - 2$ then dart and gem are (k, ℓ)-leaf powers.*

5 Characterisations of $(4, 6)$-Leaf Powers

The characterisation of $(4, 6)$-leaf powers given in this section is very similar to the following one for 3-leaf (i.e., $(3, 4)$-leaf) powers:

Theorem 2 ([2,11,21]). *The following conditions are equivalent:*

(i) *G is a 3-leaf power.*
(ii) *G is (bull, dart, gem)-free chordal.*
(iii) *G results from substituting cliques into the vertices of a tree.*

Now we consider the class of $(4, 6)$-leaf powers. Recall that every $(4, 6)$-leaf power is a k-leaf power, for all $k \geq 4$. In [16], the authors study so-called strictly chordal graphs which are defined via (rather complicated) hypergraph properties but finally turn out to be exactly the (dart,gem)-free chordal graphs as Corollary 2.2.2. in [14] says:

Proposition 15 ([14]). *G is strictly chordal if and only if it is* $(dart, gem)$*-free chordal.*

The next theorem has been our motivation for defining and investigating the notion of (k, l)-leaf powers:

Theorem 3. *The following conditions are equivalent:*

(i) *G is a $(4, 6)$-leaf power.*
(ii) *G is $(dart, gem)$-free chordal (i.e., strictly chordal).*
(iii) *G results from substituting cliques into the vertices of a block graph.*

For the proof of Theorem 3 (which is omitted due to space constraints) we use:

Proposition 16. *Every block graph is a (basic) $(4, 6)$-leaf power, and a (basic) $(4, 6)$-leaf root of a given block graph can be determined in linear time.*

Now Theorem 3 together with Proposition 16 implies:

Corollary 2. *Strictly chordal graphs are k-leaf powers for all $k \geq 4$, and a k-leaf root of G can be determined in linear time.*

Corollary 2 is one of the main results (namely Theorem 4.1) in [16]. It has also been mentioned in [16] that strictly chordal graphs can be recognised in linear time. We give a simpler proof for it.

Corollary 3. *$(4, 6)$-leaf powers (and thus also strictly chordal graphs) can be recognised in linear time.*

Proof. By Theorem 3, we know that a graph G is a $(4, 6)$-leaf power if and only if G results from substituting cliques into the vertices of a block graph, and we check the last condition in the following way. For a given graph G, first check whether G is chordal. If not then G is not a $(4, 6)$-leaf power, else determine a clique tree of G (which can be done in linear time, see, e.g., [23]). It is well known (see, e.g., [19]) that the minimal separators of G are given as the intersections of cliques which are adjacent in the clique tree.

In a block graph, the minimal separators are the cut vertices. If G results from substituting cliques into the vertices of a block graph, then the cliques which replace cut vertices are pairwise disjoint minimal separators (which are also clique modules).

Thus, determine the minimal separators in G, check whether they are pairwise disjoint (if not, G is no $(4, 6)$-leaf power), shrink them to one vertex, respectively, and check whether the resulting graph G' is a block graph. If yes, G results from substituting cliques into the (cut) vertices of the block graph G', otherwise G is no $(4, 6)$-leaf power. $\square$

6 The Main Results

Very similar to Proposition 16, we obtain:

Proposition 17. *Every block graph is a $(5,7)$-leaf power, and a $(5,7)$-leaf root of a given block graph can be determined in linear time.*

Theorem 4

(i) *For all k, ℓ with $k \geq 2$ and $\ell > 2k-2$, the class of (k,ℓ)-leaf powers is the class of P_3-free graphs, i.e., disjoint unions of cliques.*
(ii) *For all k, ℓ with odd $k = 2i+1$, $i \geq 1$, and $\ell = 4i$, the class of (k,ℓ)-leaf powers is the class of 3-leaf powers, i.e., graphs obtained from substituting cliques into trees.*
(iii) *For all k, ℓ with $k \geq 2$ and $2\ell > 3k-2$ but $\ell \leq 2k-2$ and not the situation of (ii), the class of (k,ℓ)-leaf powers is the class of $(4,6)$-leaf powers, i.e., graphs obtained from substituting cliques into block graphs.*

Proof.
(i): This follows from Proposition 10 and the obvious fact that disjoint unions of cliques have (k,ℓ)-leaf roots with $k \geq 2$ and $\ell > 2k-2$ by simply connecting the central vertices of stars realising cliques by paths which are long enough.

(ii): The case $k = 3$ and thus $i = 1$ is the case of 3-leaf powers, and we can refer to Theorem 2. For larger odd k, we first have to show that every (3,4)-leaf power is a $(2i+1,4i)$-leaf power; to show this we subdivide the internal edges of a (3,4)-leaf root T, more precisely, we replace every internal edge of a (3,4)-leaf root T by a P_4 (P_6, respectively) and obtain a (5,8)-leaf root ((7,12)-leaf root, respectively), and we perform in the same way for larger k. Conversely, $(2i+1,4i)$-leaf powers are dart- and gem-free, by Corollary 1, since, for $k = 2i+1$ and $\ell = 4i$, the inequality $2\ell \leq 3k-2$ in Corollary 1 is not fulfilled. We claim that $(2i+1,4i)$-leaf powers are also bull-free: By Proposition 10, every P_3 has a unique $(2i+1,4i)$-leaf root since $\ell = 4i = 2(2i+1)-2 = 2k-2$. Let a, b, c, d, e induce a bull with the P_4 $abcd$ with edges ab, bc, cd and vertex e adjacent to b and c. Then the unique $(2i+1,4i)$-leaf root for the P_4 is a path of length $6i-1$ between a and d, and b and c are attached as leaves to this path such that $d_T(a,b) = 2i+1$, $d_T(b,c) = 2i+1$ and $d_T(c,d) = 2i+1$. Now a, b, e induce a P_3, and e, c, d induce a P_3, whose roots are unique and require that $d_T(b,e) = 2i+1$ and $d_T(e,c) = 2i+1$, which leads to a contradiction. Thus, $(2i+1,4i)$-leaf powers are bull-, dart- and gem-free chordal, and, by Theorem 2, they are 3-leaf powers.

(iii): For $k = 2$ and $k = 3$ there is nothing to prove. As $2\ell > 3k-2$, by Corollary 1, (k,l)-leaf powers in this case are dart- and gem-free. As they are also chordal, by Theorem 3, they are $(4,6)$-leaf powers. For the other direction, again by Theorem 3, it suffices to show that every block graph has a basic (k,l)-leaf root. And, by Proposition 6, it suffices to show this for the largest possible l, for every k, i.e., for the (k,l)-pairs $(4,6), (5,7), (6,10), (7,11)$ and so on, i.e., for the (k,l)-pairs $(4+2i, 6+4i), (5+2i, 7+4i)$, for all $i \geq 0$. Proposition 16 and Proposition 17 together with their proofs deal with the case $i = 0$. In the case $i = 0$, we start with block roots which are stars whose edges are subdivided exactly once. For a general $i \geq 0$, we use the same construction with stars whose edges are subdivided exactly $i + 1$ times. $\square$

For the graphs $H_1, \ldots, H_8$ in Theorem 5 see Figure 2. Rautenbach [21] has shown that a graph without true twins is a 4-leaf power if and only if it is $(H_1, \ldots, H_8)$-free chordal. In [6], the following more detailed characterisation is shown:

Theorem 5. *G is a basic 4-leaf power if and only if G is $(H_1, \ldots, H_8)$-free chordal. In particular, G is the square of a tree if and only if G is 2-connected $(H_1, \ldots, H_5)$-free chordal.*

In fact, the forbidden subgraphs $H_1, \ldots, H_5$ are responsible for the blocks of a basic 4-leaf power, and H_6, H_7, H_8 represent the gluing conditions of blocks.

In Theorem 6 characterising the $(8,11)$-leaf powers, we additionally need graph H_9 which is given in Figure 2 and replaces the role of H_8 as a gluing condition.

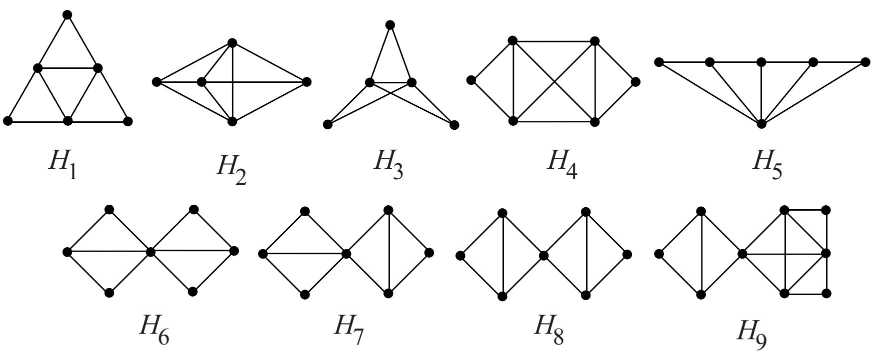

Fig. 2. Some forbidden subgraphs; the graphs $H_1, \ldots, H_8$ characterise basic 4-leaf powers.

Theorem 6. *The $(8,11)$-leaf powers are exactly the graphs obtained from substituting cliques into $(H_1, \ldots, H_7, H_9)$-free chordal graphs, i.e., the basic $(8,11)$-leaf powers are exactly the $(H_1, \ldots, H_7, H_9)$-free chordal graphs.*

Corollary 4. *$(8,11)$-leaf powers can be recognised in polynomial time.*

7 Conclusion

In this paper, we gave structural characterisations of (k,ℓ)-leaf powers, for some k and ℓ which imply efficient recognition of these classes, and in this way we improve and extend a recent paper [16] by Kennedy, Lin and Yan on strictly chordal graphs and leaf powers. Our main results are presented in Theorem 4 and 6. Other characterisations can be expected for "limit" classes (i.e., for every k, the largest l which is not yet covered by Theorem 4) such as (12,17)-leaf powers. We expect that our new notion of (k,ℓ)-leaf powers will shed new light on the open problem of characterising and recognising k-leaf powers for $k \geq 6$. We also have a characterisation of $(6,8)$-leaf powers in terms of induced subgraphs of squares of block graphs as well as in terms of forbidden induced subgraphs which will be described in a forthcoming paper.

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