A Generalized Gale-Shapley Algorithm for a Discrete-Concave Stable-Marriage Model*

Akinobu Eguchi[1], Satoru Fujishige[2], and Akihisa Tamura[2]

[1] Panasonic Mobile Communications Co., Ltd.
[2] Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan,
{fujishig,tamura}@kurims.kyoto-u.ac.jp

Abstract. The stable marriage model due to Gale and Shapley is one of the most fundamental two-sided matching models. Recently, Fleiner generalized the model in terms of matroids, and Eguchi and Fujishige extended the matroidal model to the framework of discrete convex analysis. In this paper, we extend their model to a vector version in which indifference on preferences is allowed, and show the existence of a stable solution by a generalization of the Gale-Shapley algorithm.

1 Introduction

The stable marriage model due to Gale and Shapley [7] is one of the most fundamental two-sided matching models. In the original stable marriage model, there are two sets of n men and n women, and each person arbitrarily gives a strict preference order on persons of the opposite gender. A matching is a set of n disjoint pairs of men and women, and is called stable if there is no pair whose members prefer each other to their partners in the matching. Gale and Shapley [7] gave a constructive proof of existence of a stable matching in 1962. Since the advent of their paper a lot of variations and extensions have been proposed in the literature. Recently, a remarkable extension has been made by Fleiner [3,5] (also see [4]). Fleiner [3] extended the stable marriage model to the framework of matroids, showed existence of a stable solution, and examined a lattice structure and a polyhedral characterization of stable solutions in his matroidal model. Fleiner [4] also gave a strong framework to show existence of a stable solution and a lattice structure of stable solutions by utilizing the Knaster-Tarski fixed point theorem. While in the model of Fleiner [3] preference of each person is described by a linear utility function on a matroidal domain, Eguchi and Fujishige [2] extended the matroidal model [3] to the framework of discrete convex analysis which was recently developed by Murota [8,9,10] as a unified framework of discrete optimization. In their model, each agent can express his/her preference by a discrete concave function, called an M$^\natural$-concave function.

* This work is supported by a Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

T. Ibaraki, N. Katoh, and H. Ono (Eds.): ISAAC 2003, LNCS 2906, pp. 495–504, 2003.

In this paper, we provide a general two-sided model including the Eguchi-Fujishige model as a special case (see Section 3). Our model has the following features (also see a model in Remark 1 in Section 2):

- the preference of agents on each side over the agents on the other side is expressed by an M$^\natural$-concave function, and indifference on preferences is allowed,
- each pair is permitted to form multiple partnerships.

We propose a generalization of the Gale-Shapley algorithm to show our main theorem claiming that there always exists a stable solution in our model.

This paper is organized as follows. Section 2 explains M$^\natural$-concavity together with its properties and describes our model based on discrete convex analysis. Section 3 gives several existing models that are special cases of our model. In Section 4 we propose an algorithm for finding a stable solution and prove its correctness, which shows our main theorem on existence of a stable solution in our general model. Section 5 gives remarks on time complexity.

2 A General Model

2.1 Preliminaries

We first introduce an M$^\natural$-concave function. Let V be a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We define the positive support and the negative support of $x = (x(v) : v \in V) \in \mathbf{Z}^V$, respectively, by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}. \tag{1}$$

For any $x, y \in \mathbf{Z}^V$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbf{Z}^V$ are defined by

$$x \wedge y(v) = \min\{x(v), y(v)\}, \quad x \vee y(v) = \max\{x(v), y(v)\} \quad (v \in V). \tag{2}$$

For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by $\chi_S(v) = 1$ if $v \in S$; otherwise 0, and simply write χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define the set of maximizers of f on $U \subseteq \mathbf{Z}^V$ by

$$\arg\max\{f(y) \mid y \in U\} = \{x \in U \mid \forall y \in U : \ f(x) \geq f(y)\}, \tag{3}$$

and the *effective domain* of f by

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) > -\infty\}. \tag{4}$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* [11] if it satisfies

(−M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \tag{5}$$

where χ_0 is a zero vector.

A simple example of an M$^\natural$-concave function is given as follows.

Example 1. Let $\mathcal{I}$ be the family of independent sets of a matroid on V and $w \in \mathbf{R}^V$. Then, a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f(x) = \begin{cases} \sum_{v \in I} w(v) & (\text{if } x = \chi_I \text{ for } I \in \mathcal{I}) \\ -\infty & (\text{otherwise}) \end{cases} \qquad (x \in \mathbf{Z}^V) \tag{6}$$

is M$^\natural$-concave.

An M$^\natural$-concave function has nice features as a utility function from the point of view of mathematical economics. A utility function is usually assumed to be concave in mathematical economics. For any M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, there exists a concave function $\bar{f} : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ with $\bar{f}(x) = f(x)$ for any $x \in \mathbf{Z}^V$ [8], that is, any M$^\natural$-concave function on $\mathbf{Z}^V$ has a concave extension on $\mathbf{R}^V$. A utility function usually has decreasing marginal returns, which is equivalent to submodularity in the discrete case. This is also the case for M$^\natural$-concave functions [12], i.e., any M$^\natural$-concave function f on $\mathbf{Z}^V$ satisfies

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \operatorname{dom} f). \tag{7}$$

M$^\natural$-concave functions enjoy some other combinatorially nice properties (see [6, 13]).

2.2 Model Description and the Main Theorem

Now we introduce our model. Let M and W denote two disjoint sets of agents and V be a finite set. In our model, utilities of M and W over V are described by M$^\natural$-concave functions $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, respectively. In the exemplary models described in Section 3, M and W denote disjoint sets of agents, and we have $V = M \times W$, where f_M and f_W can be regarded as aggregations of utilities of M-agents and W-agents in these models, respectively (see Remark 1 given below). Furthermore, we assume that f_M and f_W satisfy the following condition:

(A) Effective domains $\operatorname{dom} f_M$ and $\operatorname{dom} f_W$ are bounded and hereditary, and have a common minimum point $\mathbf{0}$,

where the heredity means that $\mathbf{0} \leq x_1 \leq x_2 \in \operatorname{dom} f_M$ (respectively $\operatorname{dom} f_W$) implies $x_1 \in \operatorname{dom} f_M$ (respectively $\operatorname{dom} f_W$).

We say that $x \in \operatorname{dom} f_M \cap \operatorname{dom} f_W$ is an $f_M f_W$*-stable solution* if there exist disjoint subsets V_M and V_W of V and vectors $z_M \in \mathbf{Z}^{V_M}$ and $z_W \in \mathbf{Z}^{V_W}$ such that

$$x \in \arg\max\{f_M(y) \mid y \in \mathbf{Z}^V,\ y|_{V_M} \leq z_M\}, \tag{8}$$

$$x \in \arg\max\{f_W(y) \mid y \in \mathbf{Z}^V,\ y|_{V_W} \leq z_W\}, \tag{9}$$

where $y|_{V_M}$ (resp. $y|_{V_W}$) denotes the restriction of y on V_M (resp. V_W). Since $\operatorname{dom} f_M$ and $\operatorname{dom} f_W$ are bounded due to Assumption (A), there exists $z \in \mathbf{Z}^V$

such that $y \leq z$ for all $y \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$. We see that $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an $f_M f_W$-stable solution if and only if there exist $z_M, z_W \in \mathbf{Z}^V$ satisfying the following (10)∼(12):

$$z = z_M \vee z_W, \tag{10}$$
$$x \in \arg\max\{f_M(y) \mid y \in \mathbf{Z}^V,\ y \leq z_M\}, \tag{11}$$
$$x \in \arg\max\{f_W(y) \mid y \in \mathbf{Z}^V,\ y \leq z_W\}. \tag{12}$$

In the sequel we will use (10)∼(12) instead of (8) and (9).

Our main result claims nonemptiness of the set of $f_M f_W$-stable solutions of our model.

Theorem 1 (Main Theorem). *For any* $\mathrm{M}^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A)*, there always exists an* $f_M f_W$*-stable solution.*

A constructive proof of the main theorem will be given in Section 4 by using a generalized Gale-Shapley algorithm.

Remark 1. In our model given above each of M and W is regarded as a single aggregate agent but it can be interpreted as a set of agents as follows. Let $M = \{1, \cdots, m\}$, $W = \{1, \cdots, w\}$, and $V = M \times W$. Also define $V_i = \{i\} \times W$ $(i \in M)$ and $V_j = M \times \{j\}$ $(j \in W)$. Suppose that each agent $i \in M$ has an $\mathrm{M}^\natural$-concave utility function $f_i : \mathbf{Z}^{V_i} \to \mathbf{R} \cup \{-\infty\}$ on V_i and that each agent $j \in W$ has an $\mathrm{M}^\natural$-concave utility function $f_j : \mathbf{Z}^{V_j} \to \mathbf{R} \cup \{-\infty\}$ on V_j. Aggregations $f_M(x) = \sum_{i \in M} f_i(x|_{V_i})$ and $f_W(x) = \sum_{j \in W} f_j(x|_{V_j})$ are also $\mathrm{M}^\natural$-concave. It should be noted that this modified model is equivalent to our original model. □

3 Existing Special Models

In this section we explain some existing models that are special cases of our model. In these models there are two disjoint sets of agents $M = \{1, \cdots, m\}$ and $W = \{1, \cdots, w\}$. The pairs of agents in M and W may be recognized as those of men and women. We denote by V the set of all pairs of agents of M and W, i.e., $V = M \times W$. For each pair $(i, j) \in V$, a pair (a_{ij}, b_{ij}) is given, where a_{ij} and b_{ij} can be interpreted as utilities (or profits) of i and j, respectively, provided that they are paired. Here, we assume that either $a_{ij} \geq 0$ or $a_{ij} = -\infty$ and we say j is *acceptable* to i if $a_{ij} \geq 0$ and similarly, for b_{ij}.

Although there are several variations of the stable marriage model, we explain one of comprehensive variations. In this model each agent ranks the agents on the opposite side, where unacceptability and indifference are allowed. In our context, agent $i \in M$ *prefers* j_1 to j_2 if $a_{ij_1} > a_{ij_2}$, and j_1 and j_2 are *indifferent* for agent i if $a_{ij_1} = a_{ij_2}$ (similarly, preferences of each $j \in W$ are defined from b_{ij}'s). The model deals with the stability of matchings, where a matching is a subset of V such that every agent appears at most once in the subset. Given a matching X, $i \in M$ (resp. $j \in W$) is called *unmatched in* X if there exists no $j \in W$ (resp. $i \in M$) such that $(i, j) \in X$. A pair $(i, j) \notin X$ is said to be a *blocking pair* for X if i and j prefer each other to their partners or being alone in X. A matching X is called *stable* if each pair (i, j) in X is acceptable for i and j, and if there is no blocking pair for X. It is well-known that any instance of the above model has a stable matching, originally proved by Gale and Shapley [7].

Recently, Fleiner [3] has generalized the above model to matroids. A triple $\mathcal{M} = (V, \mathcal{I}, >)$ is called an *ordered matroid*, if $(V, \mathcal{I})$ is a matroid on ground set V with family $\mathcal{I}$ of independent sets and $>$ is a linear order on V. A subset X of V *dominates* element $v \in V$ if $v \in X$ or there exists an independent set $Y \subseteq X$ such that $\{v\} \cup Y \notin \mathcal{I}$ and $u > v$ for all $u \in Y$. The set of elements dominated by X is denoted by $D_{\mathcal{M}}(X)$. Given two ordered matroids $\mathcal{M}_M = (V, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (V, \mathcal{I}_W, >_W)$ on the same ground set V, a subset X of V is called an $\mathcal{M}_M\mathcal{M}_W$*-kernel* if X is a common independent set of $\mathcal{M}_M$ and $\mathcal{M}_W$, and if any element $v \in V$ is dominated by X in $\mathcal{M}_M$ or $\mathcal{M}_W$, that is, if the following condition holds:

$$X \in \mathcal{I}_M \cap \mathcal{I}_W \text{ and } D_{\mathcal{M}_M}(X) \cup D_{\mathcal{M}_W}(X) = V. \tag{13}$$

For example, given a stable marriage instance $(M, W, \{a_{ij}\}, \{b_{ij}\})$ without indifferent preferences, we can construct an equivalent instance in terms of matroids as follows. Let V be the set of pairs (i, j) with $a_{ij}, b_{ij} > -\infty$. Assume that $(V, \mathcal{I}_M)$ is the partition matroid on V defined by disjoint sets $V_i = \{i\} \times W$ $(i \in M)$ and that $(V, \mathcal{I}_W)$ is the partition matroid on V defined by disjoint sets $V_j = M \times \{j\}$ $(j \in W)$. Thus, X is a matching if and only if $X \in \mathcal{I}_M \cap \mathcal{I}_W$. We next define linear orders $>_M$ and $>_W$ on V so that $(i, j_1) >_M (i, j_2)$ whenever $a_{ij_1} > a_{ij_2}$, and that $(i_1, j) >_W (i_2, j)$ whenever $b_{i_1 j} > b_{i_2 j}$. By the definitions of the linear orders, a matching X is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if for each pair $(i, j) \notin X$ there exists (i, j') in X such that $(i, j') >_M (i, j)$, or (i', j) in X such that $(i', j) >_W (i, j)$. Thus, the set of $\mathcal{M}_M\mathcal{M}_W$-kernels coincides with the set of stable matchings. The matroidal model also includes a many-to-many stable matching model, called stable b-matching model. We remark that the matroidal model can easily be modified so that indifference in preferences is admissible. Fleiner [3] showed that any instance of the matroidal model has an $\mathcal{M}_M\mathcal{M}_W$-kernel.

Quite recently, Eguchi and Fujishige [2] proposed a model in terms of M$^\natural$-concavity, which is a set version of our model in which $\mathrm{dom}\, f_M, \mathrm{dom}\, f_W \subseteq \{0,1\}^V$ and for any distinct $x, y \in \mathrm{dom}\, f_M$ (resp. $x, y \in \mathrm{dom}\, f_W$) $f_M(x) \neq f_M(y)$ (resp. $f_W(x) \neq f_W(y)$). For convenience, we identify a subset of V with its characteristic vector. The matroidal model described above can be recognized as a special case of this model with linear utility functions. Let $\mathcal{M}_M = (V, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (V, \mathcal{I}_W, >_W)$ be an instance of the matroidal model. We define linear orders $>_M$ and $>_W$ by positive numbers $\{a_v\}$ and $\{b_v\}$ as $a_u > a_v \iff u >_M v$ and $b_u > b_v \iff u >_W v$. Also define functions f_M and f_W by

$$f_M(X) = \begin{cases} \sum_{v \in X} a_v & (X \in \mathcal{I}_M) \\ -\infty & (X \notin \mathcal{I}_M), \end{cases} \qquad f_W(X) = \begin{cases} \sum_{v \in X} b_v & (X \in \mathcal{I}_W) \\ -\infty & (X \notin \mathcal{I}_W). \end{cases} \tag{14}$$

Then f_M and f_W are M$^\natural$-concave because these are linear on independence families of matroids. For an independent set X of $\mathcal{M}_M$ and $Z \subseteq V$ with $X \subseteq Z$, we have that $X \in \arg\max\{f_M(Y) \mid Y \subseteq Z\}$ if and only if $Z \subseteq D_{\mathcal{M}_M}(X)$, by the optimality criterion of maximum weight independent sets of a matroid (the same statement for $\mathcal{M}_W$ also holds). Thus, a subset X of V is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if it is $f_M f_W$-stable. Eguchi and Fujishige [2] showed that any instance of their model has an $f_M f_W$-stable solution.

Therefore, our model includes all of the above models. Moreover, our model admits multiplicity for each element of V. For example, our model naturally deals with the following problem. The same numbers of men and women attend a dance party at which each person dances a waltz k times and he/she can dance with the same person of the opposite gender time after time. The problem is to find an "agreeable" assignment of dance partners, in which each person is assigned at most k persons of the opposite gender with possible repetition. If preferences of assignments of dance partners for each person can be expressed by an M$^\natural$-concave function (see Remark 1 in Section 2), then our model gives a solution.

4 A Generalized Gale-Shapley Algorithm

In this section we prove our main theorem, Theorem 1, by giving an algorithm for finding $x, z_M, z_W \in \mathbf{Z}^V$ satisfying (10)$\sim$(12). This algorithm is a generalization of the Gale-Shapley algorithm.

Before describing the algorithm, we show two fundamental properties of M$^\natural$-concave functions as Lemmas 1 and 2, which hold without Assumption (A).

Lemma 1. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $z_1, z_2 \in \mathbf{Z}^V$ be such that $z_1 \geq z_2$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$.*

(a) *For any $x_1 \in \arg\max\{f(y) \mid y \leq z_1\}$, there exists x_2 such that*

$$x_2 \in \arg\max\{f(y) \mid y \leq z_2\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2. \tag{15}$$

(b) *For any $x_2 \in \arg\max\{f(y) \mid y \leq z_2\}$, there exists x_1 such that*

$$x_1 \in \arg\max\{f(y) \mid y \leq z_1\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2. \tag{16}$$

Proof. (a): Let x_2 be an element in $\arg\max\{f(y) \mid y \leq z_2\}$ that minimizes $\sum\{x_1(v) - x_2(v) \mid v \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that there exists $u \in V$ with $\min\{z_2(u), x_1(u)\} > x_2(u)$. Then $u \in \mathrm{supp}^+(x_1 - x_2)$. By ($-M^\natural$-EXC), there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{17}$$

If $v \neq 0$, then $x_1(v) < x_2(v) \leq z_2(v) \leq z_1(v)$. Hence we have $x_1 - \chi_u + \chi_v \leq z_1$, which implies $f(x_1) \geq f(x_1 - \chi_u + \chi_v)$. This together with (17) yields $f(x_2) \leq f(x_2 + \chi_u - \chi_v)$. Moreover, since $z_2(u) > x_2(u)$, we have $x_2' = x_2 + \chi_u - \chi_v \leq z_2$. It follows that $x_2' \in \arg\max\{f(y) \mid y \leq z_2\}$ and $x_2'(v) \geq \min\{z_2(v), x_1(v)\}$ if $v \neq 0$, which contradicts the minimality condition of x_2.

(b): Let x_1 be an element in $\arg\max\{f(y) \mid y \leq z_1\}$ that minimizes $\sum\{x_1(u) - x_2(u) \mid u \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that there exists $u \in V$ with $\min\{z_2(u), x_1(u)\} > x_2(u)$. Then $u \in \mathrm{supp}^+(x_1 - x_2)$. By (−M$^\natural$-EXC), there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{18}$$

Since $x_2(u) < z_2(u)$, we have $x_2 + \chi_u - \chi_v \leq z_2$, which implies $f(x_2) \geq f(x_2 + \chi_u - \chi_v)$. This together with (18) yields $f(x_1) \leq f(x_1 - \chi_u + \chi_v)$. Obviously $x_1' = x_1 - \chi_u + \chi_v \leq z_1$. However, this contradicts the minimality condition of x_1 because $x_2(v) \geq \min\{z_2(v), x_1'(v)\}$ if $v \neq 0$. □

Lemma 2. *For an M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and a vector $z_1 \in \mathbf{Z}^V$ suppose that $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$. For any $x \in \arg\max\{f(y) \mid y \leq z_1\}$ and any $z_2 \in \mathbf{Z}^V$ such that* (1) *$z_2 \geq z_1$ and* (2) *if $x(v) = z_1(v)$, then $z_2(v) = z_1(v)$, we have $x \in \arg\max\{f(y) \mid y \leq z_2\}$.*

Proof. Assume to the contrary that the assertion is not satisfied. Let x' be a point such that $x' \leq z_2$, $f(x') > f(x)$, and x' minimizes $\sum\{x'(v) - z_1(v) \mid v \in \mathrm{supp}^+(x' - z_1)\}$ among such points. By the assumption, there exists $u \in V$ with $x'(u) > z_1(u) > x(u)$. By (−M$^\natural$-EXC) for x', x, and u, there exists $v \in \mathrm{supp}^-(x' - x) \cup \{0\}$ such that

$$f(x') + f(x) \leq f(x' - \chi_u + \chi_v) + f(x + \chi_u - \chi_v). \tag{19}$$

Since $x + \chi_u - \chi_v \leq z_1$, we have $f(x) \geq f(x + \chi_u - \chi_v)$, which implies $f(x') \leq f(x' - \chi_u + \chi_v)$. Obviously, $x' - \chi_u + \chi_v \leq z_2$, However, this contradicts the minimality condition of x' because if $v \neq 0$, then $z_1(v) \geq x(v) > x'(v)$. □

It should be noted that Lemma 2 holds for any function f on $\mathbf{Z}^V$ that has a concave extension on $\mathbf{R}^V$.

To describe an algorithm for finding $x, z_M, z_W \in \mathbf{Z}^V$ satisfying (10)∼(12), we assume that we are initially given $x_M, x_W, z_M, z_W \in \mathbf{Z}^V$ satisfying the following:

$$z = z_M \vee z_W, \tag{20}$$
$$x_M \in \arg\max\{f_M(y) \mid y \leq z_M\}, \tag{21}$$
$$x_W \in \arg\max\{f_W(y) \mid y \leq z_W \vee x_M\}, \tag{22}$$
$$x_W \leq x_M. \tag{23}$$

We can easily compute such vectors by setting $z_M = z$, $z_W = \mathbf{0}$, and by finding x_M and x_W such that

$$x_M \in \arg\max\{f_M(y) \mid y \leq z_M\}, \quad x_W \in \arg\max\{f_W(y) \mid y \leq x_M\}. \tag{24}$$

The algorithm is given as follows.

Algorithm_GS$(f_M, f_W, x_M, x_W, z_M, z_W)$
Input: M$^\natural$-concave functions f_M, f_W and
x_M, x_W, z_M, z_W satisfying (20), (21), (22), (23) ;
repeat {
let x_M be any element in $\arg\max\{f_M(y) \mid x_W \leq y \leq z_M\}$;
let x_W be any element in $\arg\max\{f_W(y) \mid y \leq x_M\}$;
for each $v \in V$ with $x_M(v) > x_W(v)$ {
$z_M(v) \leftarrow x_W(v)$;
$z_W(v) \leftarrow z(v)$;
} ;
} **until** $x_M = x_W$;
return $(x_M, z_M, z_W \vee x_M)$.

It should be noted here that because of Assumption (A) x_M and x_W are well-defined within the effective domains and that Algorithm_GS terminates after at most $\sum_{v\in V} z(v)$ iterations, because $\sum_{v\in V} z_M(v)$ is strictly decreased at each iteration. In order to show that the outputs of Algorithm_GS satisfy (10)$\sim$(12), we will show two lemmas, Lemmas 3 and 4.

Let $x_M^{(i)}$, $x_W^{(i)}$, $z_M^{(i)}$, and $z_W^{(i)}$ be x_M, x_W, z_M, and z_W obtained after the ith iteration in Algorithm_GS for $i = 1, 2, \cdots, t$, where t is the last to get the outputs. For convenience, let us assume that $x_M^{(0)}$, $x_W^{(0)}$, $z_M^{(0)}$, and $z_W^{(0)}$ are the input vectors.

Lemma 3. *For each* $i = 0, 1, \cdots, t$, *we have*

$$x_M^{(i+1)} \in \arg\max\left\{ f_M(y) \;\middle|\; y \leq z_M^{(i)} \right\}. \tag{25}$$

Proof. We prove (25) by induction on i. For $i = 0$, (25) holds from (21) and (23). We assume that for some l with $0 \leq l < t$ (25) holds for any $i \leq l$, and we show (25) for $i = l + 1$. Since $x_M^{(l+1)} \in \arg\max\{f_M(y) \mid y \leq z_M^{(l)}\}$ and $z_M^{(l)} \geq z_M^{(l+1)}$, Lemma 1 (a) guarantees the existence of an $x \in \arg\max\{f_M(y) \mid y \leq z_M^{(l+1)}\}$ with $z_M^{(l+1)} \wedge x_M^{(l+1)} \leq x$, which implies (25) for $i = l+1$ because $z_M^{(l+1)} \wedge x_M^{(l+1)} = x_W^{(l+1)}$ by the modification of z_M. □

Lemma 4. *For each* $i = 0, 1, \cdots, t$, *we have*

$$x_W^{(i)} \in \arg\max\left\{ f_W(y) \;\middle|\; y \leq z_W^{(i)} \vee x_M^{(i)} \right\}. \tag{26}$$

Proof. We show (26) by induction on i. For $i = 0$, (26) holds by (22). We assume that for some l with $0 \leq l < t$ (26) holds for any $i \leq l$, and we show (26) for $i = l + 1$. By the definition of x_M, we have

$$x_M^{(l+1)} \geq x_W^{(l)}. \tag{27}$$

By Lemma 1 (b) and the assumption, there exists x such that

$$x \in \arg\max\left\{ f_W(y) \;\middle|\; y \leq z_W^{(l)} \vee x_M^{(l)} \vee x_M^{(l+1)} \right\} \tag{28}$$

and

$$\left(z_W^{(l)} \vee x_M^{(l)}\right) \wedge x \leq x_W^{(l)}. \tag{29}$$

From (27), (28), and (29), we have $x \leq x_M^{(l+1)}$ and hence $f_W(x) = f_W(x_W^{(l+1)})$. If $z_W^{(l+1)} = z_W^{(l)}$, then we immediately obtain (26) for $i = l + 1$. So, we assume that $z_W^{(l+1)} \neq z_W^{(l)}$. By the modification of z_W, we have $x_W^{(l+1)}(v) < x_M^{(l+1)}(v)$ if $z_W^{(l)}(v) < z_W^{(l+1)}(v)$. Hence, by Lemma 2, (26) holds for $i = l + 1$. □

The correctness of Algorithm_GS follows from Lemmas 3 and 4.

Theorem 2. *The outputs of* Algorithm_GS *satisfy* (*10*) $\sim$ (*12*).

Proof. From Lemmas 3 and 4 we have for $i = t$

$$x_M \in \arg\max\left\{f_M(y) \;\middle|\; y \leq z_M^{(t)}\right\}, \tag{30}$$

$$x_W \in \arg\max\left\{f_W(y) \;\middle|\; y \leq z_W^{(t)} \vee x_M^{(t)}\right\}, \tag{31}$$

$$x_M = x_W. \tag{32}$$

By the way of modifying z_M, z_W, and x_M, we have

$$z_M^{(t)} \vee \left(z_W^{(t)} \vee x_M^{(t)}\right) = z. \tag{33}$$

This completes the proof of this theorem. □

Our main result, Theorem 1, is a direct consequence of Theorem 2.

5 Remarks on Time Complexity

We finally discuss the oracle complexities of the problems of finding an $f_M f_W$-stable solution and of checking whether a given point is $f_M f_W$-stable, provided that the function value $f(x)$ of a given M$^\natural$-concave function f can be calculated in constant time for each point x.

Algorithm_GS solves the maximization problem of an M$^\natural$-concave function in each iteration. It is known that a maximizer of an M$^\natural$-concave function f on V can be found in polynomial time in n and $\log L$, where $n = |V|$ and $L = \max\{||x - y||_\infty \mid x, y \in \mathrm{dom}\, f\}$. For example, $\mathrm{O}(n^3 \log L)$-time algorithms are proposed in [14,15]. Since Algorithm_GS terminates after at most $\sum_{v \in V} z(v)$ iterations, the oracle time complexity of Algorithm_GS is $\mathrm{O}(\mathrm{poly}(n) \cdot L)$, where $L = ||z||_\infty$. Unfortunately, there exist a series of examples in which Algorithm_GS requires numbers of iterations proportional to L. While it is known that an $f_M f_W$-stable solution can be found in polynomial time in n for the special case where f_M and f_W are linear on rectangular effective domains [1], it is open whether an $f_M f_W$-stable solution for the general case can be found in polynomial time in n and $\log L$.

On the other hand, the problem of checking whether a given point $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is $f_M f_W$-stable, can be solved in $\mathrm{O}(n^2)$ time by using the following local criterion of the $f_M f_W$-stability.

Lemma 5. *A point $x \in \operatorname{dom} f_M \cap \operatorname{dom} f_W$ is $f_M f_W$-stable if and only if it satisfies the following conditions:*

$$\text{for each } u \in V,\ f_M(x) \geq f_M(x - \chi_u) \quad \text{and} \quad f_W(x) \geq f_W(x - \chi_u), \tag{34}$$

$$\begin{aligned} &\text{for each } u \in V,\ f_M(x) \geq f_M(x + \chi_u - \chi_v) \quad (\forall v \in V \cup \{0\}) \quad \text{or} \\ &\qquad\qquad\qquad f_W(x) \geq f_W(x + \chi_u - \chi_w) \quad (\forall w \in V \cup \{0\}). \end{aligned} \tag{35}$$

References

1. M. Baïou and M. Balinski, Erratum: The stable allocation (or ordinal transportation) problem, *Math. Oper. Res.* **27** (2002) 662–680.
2. A. Eguchi and S. Fujishige, An extension of the Gale-Shapley matching algorithm to a pair of M$^\natural$-concave functions, Discrete Mathematics and Systems Science Research Report 02-05, Osaka University, (2002).
3. T. Fleiner, A matroid generalization of the stable matching polytope, in: B. Gerards and K. Aardal, eds., *Proceedings of the 8th International IPCO Conference*, LNCS **2081** (Springer-Verlag, Berlin, 2001) 105–114.
4. T. Fleiner, Some results on stable matchings and fixed points, EGRES Technical Report 2002-08, (2002), `http://www.cs.elte.hu/egres`.
5. T. Fleiner, A fixed point approach to stable matchings and some applications, *Math. Oper. Res.* **28** (2003) 103–126.
6. S. Fujishige and Z. Yang, A note on Kelso and Crawford's gross substitutes condition, *Math. Oper. Res.* to appear.
7. D. Gale and L. S. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* **69** (1962) 9–15.
8. K. Murota, Convexity and Steinitz's exchange property, *Adv. Math.* **124** (1996) 272–311.
9. K. Murota, Discrete convex analysis, *Math. Programming* **83** (1998) 313–371.
10. K. Murota, *Discrete Convex Analysis*, (Society for Industrial and Applied Mathematics, Philadelphia, 2003).
11. K. Murota and A. Shioura, M-convex function on generalized polymatroid, *Math. Oper. Res.* **24** (1999) 95–105.
12. K. Murota and A. Shioura, Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella, *Discrete Appl. Math.* **115** (2001) 151–176.
13. K. Murota and A. Tamura, New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities, *Discrete Appl. Math.* to appear.
14. A. Shioura, Fast scaling algorithms for M-convex function minimization with application to the resource allocation problem, *Discrete Appl. Math.* to appear.
15. A. Tamura, Coordinatewise domain scaling algorithm for M-convex function minimization, in: W. J. Cook and A. S. Schulz, eds., *Proceedings of the 9th International IPCO Conference*, LNCS **2337**, (Springer-Verlag, Berlin, 2002) 21–35.